Rationals Countability and Cantor's Proof

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Abstract: Cantor's proof that the rational numbers are countable uses a mapping that is not one-one. Thus, the countability of the rationals was not proved by Cantor.

We prove by cardinal number methods, using a result of Tarski, that the rationals are countable. We confirm this by exhibiting a one-one mapping from the rationals into the natural numbers.

Cantor's mapping is not one-one.

Cantor's proof appears in [1]. Cantor wrote

"*By (6) of* § *3,* $\aleph_0 \bullet \aleph_0$ *is the cardinal number of the aggregate of bindings*

$$\{(\mu, \nu)\},$$

where μ *and* ν *are any finite cardinal numbers which are independent of one another. If also* λ *represents any finite cardinal number, so that* $\{\lambda\}$, $\{\mu\}$, *and* $\{\nu\}$ *are only different notations for the same aggregate of all finite numbers, we have to show that*

$$\{(\mu, \nu)\} \sim \{\lambda\}.$$

Let us denote $\mu + \nu$ *by* ρ*; then* ρ *takes all the numerical values* $2, 3, 4, \ldots$

and there are in all $\rho - 1$ *elements* (μ, ν) *for which* $\mu + \nu = \rho$, *namely:*

$$(1, \rho - 1),\ (2, \rho - 2), \dots, (\rho - 1, 1).$$

In this sequence imagine first the element $(1,1)$, *for which* $\rho = 2$, *put , then the two elements for which* $\rho = 3$, *then the three elements for which* $\rho = 4$,

and so on. Thus, we get all the elements (μ, ν) *in a simple series:*

$$(1,1);\ (1,2),\ (2,1);\ (1,3),\ (2,2),\ (3,1);\ (1,4),\ (2,3), \dots,$$

and here, as we easily see, the element (μ, ν) *comes at the* λ*th place, where*

$$(9) \qquad \lambda = \mu + \frac{(\mu + \nu - 1)(\mu + \nu - 2)}{2}.$$

The variable λ *takes every numerical value* $1, 2, 3, \dots$, *once. Consequently, by means of (9), a reciprocally univocal relation subsists between the aggregates* $\{\nu\}$ *and* $\{(\mu, \nu)\}$*."*

Clearly, the variable λ takes several times some of numerical value $1, 2, 3, \dots$, The mapping is not one-one.

We have,

$\mu = 1$, $\nu = 1$, $\lambda = 1$, and $(1,1) \to 1$.

$\mu = 1$, $\nu = 2$, $\lambda = 2$, and $(1,2) \to 2$.

$\mu = 2$, $\nu = 1$, $\lambda = 2$, and $(2,1) \to 2$.

$\mu = 1$, $\nu = 3$, $\lambda = 4$, and $(1,3) \to 4$.

$\mu = 2$, $\nu = 2$, $\lambda = 4$, and $(2,2) \to 4$.

$\mu = 3$, $\nu = 1$, $\lambda = 4$, and $(3,1) \to 4$.

$\mu = 1$, $\nu = 4$, $\lambda = 7$, and $(1,4) \to 7$.

$\mu = 2$, $\nu = 3$, $\lambda = 7$, and $(2,3) \to 7$.

$\mu = 3$, $\nu = 2$, $\lambda = 7$, and $(3,2) \to 7$.

$\mu = 4$, $\nu = 1$, $\lambda = 7$, and $(4,1) \to 7$.

Apparently, Cantor expected his mapping to be a bijection, did not check it, and did not see that it is not even one-one.

It is well known [2] that a one-one mapping is required to establish that $cardQ \le cardN$. Thus, Cantor's claim is unfounded, and his use of $\aleph_0^2 = \aleph_0$, amounts to adding another axiom to his set theory.

Proof of Rationals Countability by Tarski result.

We use the graphical interpretation of Cantor's proof by a zig-zag through an infinite triangular matrix of rationals. While the zig-zag by itself does not prove the countability, it is useful to clarify our argument:

The first line in the zig-zag has one rational

1/1.

The second line has two rationals,

1/2, and 2/1.

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The n-th line has n rationals,

1/n, 2/(n-1),…,(n-1)/2, n/1,

…………………………………………………..

$$
\begin{array}{ccccccccccccccc}
\tfrac{1}{1} & \to & \tfrac{1}{2} & & \tfrac{1}{3} & \to & \tfrac{1}{4} & & \tfrac{1}{5} & \to & \ldots & \ldots & * & \ldots & * \\
& \swarrow & & \nearrow & & \swarrow & & \nearrow & \ldots & \ldots & \ldots & \nearrow & \ldots & \swarrow & \\
\tfrac{2}{1} & & \tfrac{2}{2} & & \tfrac{2}{3} & & \tfrac{2}{4} & \ldots & \ldots & \ldots & * & \ldots & * & & \\
\downarrow & \nearrow & & \swarrow & & \nearrow & \ldots & \ldots & \ldots & \nearrow & \ldots & \swarrow & & & \\
\tfrac{3}{1} & & \tfrac{3}{2} & & \tfrac{3}{3} & \ldots & \ldots & \ldots & * & \ldots & * & & & & \\
& \swarrow & & \nearrow & \ldots & \ldots & \ldots & \nearrow & \ldots & \swarrow & & & & & \\
\tfrac{4}{1} & & \tfrac{4}{2} & \ldots & \ldots & \ldots. & * & \ldots & * & & & & & & \\
\downarrow & \nearrow & \ldots & \ldots & \ldots & \nearrow & \ldots & \swarrow & & & & & & & \\
\tfrac{5}{1} & \ldots & \ldots & \ldots & * & \ldots & * & & & & & & & & \\
\ldots & \ldots & \ldots & \nearrow & \ldots & \swarrow & & & & & & & & & \\
* & .. & * & .. & * & & & & & & & & & & \\
\downarrow & \nearrow & \ldots & \swarrow & & & & & & & & & & & \\
* & \ldots & * & & & & & & & & & & & & \\
\ldots & \swarrow & & & & & & & & & & & & & \\
* & \to & & & & & & & & & & & & &
\end{array}
$$

Summing the number of the rationals along the zig-zag, for $n = 1,2,3....$,

$$1 + 2 + 3 + ... + n < \aleph_0 . \quad (1)$$

Thus,

$$2(1 + 2 + 3 + ... + n) < \aleph_0 .$$

That is,

$$n + n^2 < \aleph_0 . \qquad (2)$$

Tarski ([3], or [4, p.174]) proved that

for any sequence of cardinal numbers, m_1, m_2, m_3,..., and a cardinal m, the partial sums inequalities

$$m_1 + m_2 + ... + m_n \leq m, \qquad \textbf{\textit{for}} \quad n = 1,2,3....$$

imply the series inequality

$$m_1 + m_2 + ... + m_n + ... \leq m.$$

Applying to (1),

$$1+2+3+....+n+...\leq \aleph_0 .$$

Regarding (2), this says

$$\aleph_0 + \aleph_0^2 \leq \aleph_0 .$$

Since $\aleph_0^2 \leq \aleph_0 + \aleph_0^2$, by transitivity of cardinal inequalities [3, p. 147],

$$\aleph_0^2 \leq \aleph_0 .$$

Since $\aleph_0 \leq \aleph_0^2$,

$$\aleph_0^2 = \aleph_0 .$$

Proof of Rationals Countability by a one-one mapping.

Aided by the zig-zag listing of the rationals, we produce a one-one mapping from the rationals into the natural numbers. We construct our mapping with numerical examples. Then we give the general formula.

The first line in the zig-zag has one rational

$$1/1$$

which we assign as follows

$$\frac{1}{1} \rightarrow 1+2^1 = 3 .$$

The second line in the zig-zag has two rationals,

1/2, and 2/1,

which we assign as follows

$$\frac{1}{2} \rightarrow 1+2^2 = 5 ,$$

$$\frac{2}{1} \to 2 + 2^2 = 6.$$

The third line in the zig-zag has three rationals

3/1, 2/2, 1/3,

which are assigned as follows

$$\frac{3}{1} \to 1 + 2^3 = 9,$$

$$\frac{2}{2} \to 2 + 2^3 = 10,$$

$$\frac{1}{3} \to 3 + 2^3 = 11.$$

The fourth line in the zig-zag has four rationals

1/4, 2/3, 3/2, 4/1

which are assigned as follows

$$\frac{1}{4} \to 1 + 2^4 = 17,$$

$$\frac{2}{3} \to 2 + 2^4 = 18,$$

$$\frac{3}{2} \to 3 + 2^4 = 19.$$

$$\frac{4}{1} \to 4 + 2^4 = 20.$$

The fifth line in the zig-zag has five rationals

5/1, 4/2, 3/3, 2/4, 1/5,

which are assigned as follows

$$\frac{5}{1} \to 1 + 2^5 = 33,$$

$$\frac{2}{4} \to 2 + 2^5 = 34,$$

$$\frac{3}{3} \to 3 + 2^5 = 35.$$

$$\frac{2}{4} \to 4 + 2^5 = 36.$$

$$\frac{1}{5} \to 5 + 2^5 = 37.$$

If

$$m + n - 1 = even = 2k,$$

the $m + n - 1 = 2k$ zig-zag line has the $m + n - 1 = 2k$ rationals

1/n, 2/(n-1),…(n-1)/2, n/1,

which are assigned as follows

$$\frac{1}{n} \to 1 + 2^{2k},$$

$$\frac{2}{n-1} \to 2 + 2^{2k},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{n-1}{2} \to 2k - 1 + 2^{2k},$$

$$\frac{n}{1} \to 2k + 2^{2k}.$$

If

$$m + n - 1 = odd = 2k + 1,$$

the $m + n - 1 = 2k + 1$ zig-zag line has the $m + n - 1 = 2k + 1$ rationals

m/1, (m-1)/2,…, 2/(m-1),1/m,

which are assigned as follows

$$\frac{m}{1} \to 1 + 2^{2k+1},$$

$$\frac{m-1}{2} \to 2 + 2^{2k+1},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\frac{2}{m-1} \to 2k + 2^{2k+1},$$

$$\frac{1}{m} \to 2k + 1 + 2^{2k+1}.$$

This defines a one-one function from the rationals into the natural numbers.

References

[1] Cantor, Georg, *Contributions to the founding of the theory of Transfinite Numbers,* p.107. Open Court Publishing 1915, Dover 1955.

[2] Lipschutz, Seymour, *Theory and problems of Set Theory and Related Topics*, McGraw-Hill, 1964.

[3] Tarski, Alfred, *Axiomatic and algebraic aspects of two theorems on sums of cardinals*, Fundamenta Mathematicae, Volume 35, 1948, p. 79-104. Reprinted in *Alfred Tarski Collected papers*, edited by Steven R. Givant and Ralph N. McKenzie, Volume 3, 1945-1957, p. 173, Birkhauser, 1986.

[4] Sierpinski, Waclaw, *Cardinal and Ordinal Numbers.* Warszawa, 2nd edition, 1965, (Also in the 1958, 1st edition).