Rationals Countability and Cantor’s Proof
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January, 2006

Abstract: Cantor’s proof that the rational numbers are countable uses a mapping that is not one-one. Thus, the countability of the rationals was not proved by Cantor.

We prove by cardinal number methods, using a result of Tarski, that the rationals are countable. We confirm this by exhibiting a one-one mapping from the rationals into the natural numbers.

Cantor’s mapping is not one-one.
Cantor’s proof appears in [1]. Cantor wrote

“By (6) of § 3, \(\aleph_0 \cdot \aleph_0\) is the cardinal number of the aggregate of bindings

\[\{(\mu, \nu)\},\]

where \(\mu\) and \(\nu\) are any finite cardinal numbers which are independent of one another. If also \(\lambda\) represents any finite cardinal number, so that \(\{\lambda\}\), \(\{\mu\}\), and \(\{\nu\}\) are only different notations for the same aggregate of all finite numbers, we have to show that

\[\{(\mu, \nu)\} \sim \{\lambda\}.\]

Let us denote \(\mu + \nu\) by \(\rho\); then \(\rho\) takes all the numerical values 2,3,4,...
and there are in all $\rho - 1$ elements $(\mu, \nu)$ for which $\mu + \nu = \rho$, namely:

$$(1, \rho - 1), (2, \rho - 2), \ldots, (\rho - 1, 1).$$

In this sequence imagine first the element $(1,1)$, for which $\rho = 2$, put , then the two elements for which $\rho = 3$, then the three elements for which $\rho = 4$, and so on. Thus, we get all the elements $(\mu, \nu)$ in a simple series:

$$(1,1); (1,2), (2,1); (1,3), (2,2), (3,1); (1,4), (2,3), \ldots,$$

and here, as we easily see, the element $(\mu, \nu)$ comes at the $\lambda$th place, where

$$(9) \quad \lambda = \mu + \frac{(\mu + \nu - 1)(\mu + \nu - 2)}{2}.$$

The variable $\lambda$ takes every numerical value $1, 2, 3, \ldots$, once. Consequently, by means of (9), a reciprocally univocal relation subsists between the aggregates $\{\nu\}$ and $\{(\mu, \nu)\}$.

Clearly, the variable $\lambda$ takes several times some of numerical value $1, 2, 3, \ldots$, The mapping is not one-one.

We have,

$\begin{align*}
\mu = 1, \quad \nu = 1, \quad \lambda = 1, \quad \text{and} \quad (1,1) & \rightarrow 1. \\
\mu = 1, \quad \nu = 2, \quad \lambda = 2, \quad \text{and} \quad (1,2) & \rightarrow 2. \\
\mu = 2, \quad \nu = 1, \quad \lambda = 2, \quad \text{and} \quad (2,1) & \rightarrow 2. \\
\mu = 1, \quad \nu = 3, \quad \lambda = 4, \quad \text{and} \quad (1,3) & \rightarrow 4. \\
\mu = 2, \quad \nu = 2, \quad \lambda = 4, \quad \text{and} \quad (2,2) & \rightarrow 4.
\end{align*}$
\( \mu = 3, \ \nu = 1, \ \lambda = 4, \ \text{and} \ (3,1) \to 4. \)
\( \mu = 1, \ \nu = 4, \ \lambda = 7, \ \text{and} \ (1,4) \to 7. \)
\( \mu = 2, \ \nu = 3, \ \lambda = 7, \ \text{and} \ (2,3) \to 7. \)
\( \mu = 3, \ \nu = 2, \ \lambda = 7, \ \text{and} \ (3,2) \to 7. \)
\( \mu = 4, \ \nu = 1, \ \lambda = 7, \ \text{and} \ (4,1) \to 7. \)

Apparently, Cantor expected his mapping to be a bijection, did not check it, and did not see that it is not even one-one.

It is well known [2] that a one-one mapping is required to establish that \( \text{card} \mathbb{Q} \leq \text{card} \mathbb{N} \). Thus, Cantor’s claim is unfounded, and his use of \( \aleph_0^\omega = \aleph_0 \), amounts to adding another axiom to his set theory.

**Proof of Rationals Countability by Tarski result.**

We use the graphical interpretation of Cantor’s proof by a zig-zag through an infinite triangular matrix of rationals. While the zig-zag by itself does not prove the countability, it is useful to clarify our argument:

The first line in the zig-zag has one rational

\[ \frac{1}{1} \]

The second line has two rationals,

\[ \frac{1}{2}, \text{ and } \frac{2}{1} \]

………………………………………………

The n-th line has n rationals,

\[ \frac{1}{n}, \frac{2}{n-1}, \ldots, \frac{(n-1)}{2}, \frac{n}{1}, \]

………………………………………………
Summing the number of the rationals along the zig-zag, for
\( n = 1, 2, 3, \ldots \),
\[
1 + 2 + 3 + \ldots + n < \aleph_0. \quad (1)
\]
Thus,
\[
2(1 + 2 + 3 + \ldots + n) < \aleph_0.
\]
That is,
\[
n + n^2 < \aleph_0. \quad (2)
\]
Tarski ([3], or [4, p.174]) proved that
for any sequence of cardinal numbers, \( m_1, m_2, m_3, \ldots \), and a
cardinal \( m \), the partial sums inequalities
\[
m_1 + m_2 + \ldots + m_n \leq m, \quad \text{for } n = 1, 2, 3, \ldots
\]
imply the series inequality
\[
m_1 + m_2 + \ldots + m_n + \ldots \leq m.
\]
Applying to (1),

\[ 1 + 2 + 3 + \ldots + n + \ldots \leq \aleph_0. \]

Regarding (2), this says

\[ \aleph_0 + \aleph_0^2 \leq \aleph_0. \]

Since \( \aleph_0^2 \leq \aleph_0 + \aleph_0^2 \), by transitivity of cardinal inequalities [3, p. 147],

\[ \aleph_0^2 \leq \aleph_0. \]

Since \( \aleph_0 \leq \aleph_0^2 \),

\[ \aleph_0^2 = \aleph_0. \]

**Proof of Rationals Countability by a one-one mapping.**

Aided by the zig-zag listing of the rationals, we produce a one-one mapping from the rationals into the natural numbers. We construct our mapping with numerical examples. Then we give the general formula.

The first line in the zig-zag has one rational

\[ 1/1 \]

which we assign as follows

\[ \frac{1}{1} \rightarrow 1 + 2^1 = 3. \]

The second line in the zig-zag has two rationals,

\[ 1/2, \text{and} \ 2/1, \]

which we assign as follows

\[ \frac{1}{2} \rightarrow 1 + 2^2 = 5, \]
\[
\frac{2}{1} \rightarrow 2 + 2^2 = 6.
\]

The third line in the zig-zag has three rationals
\(3/1, 2/2, 1/3,\)
which are assigned as follows
\[
\begin{align*}
\frac{3}{1} & \rightarrow 1 + 2^1 = 9, \\
\frac{2}{2} & \rightarrow 2 + 2^3 = 10, \\
\frac{1}{3} & \rightarrow 3 + 2^3 = 11.
\end{align*}
\]

The fourth line in the zig-zag has four rationals
\(1/4, 2/3, 3/2, 4/1\)
which are assigned as follows
\[
\begin{align*}
\frac{1}{4} & \rightarrow 1 + 2^4 = 17, \\
\frac{2}{3} & \rightarrow 2 + 2^4 = 18, \\
\frac{3}{2} & \rightarrow 3 + 2^4 = 19, \\
\frac{4}{1} & \rightarrow 4 + 2^4 = 20.
\end{align*}
\]

The fifth line in the zig-zag has five rationals
\(5/1, 4/2, 3/3, 2/4, 1/5,\)
which are assigned as follows
\[
\begin{align*}
\frac{5}{1} & \rightarrow 1 + 2^5 = 33, \\
\frac{2}{4} & \rightarrow 2 + 2^5 = 34,
\end{align*}
\]
\[
\frac{3}{3} \rightarrow 3 + 2^5 = 35. \\
\frac{2}{4} \rightarrow 4 + 2^5 = 36. \\
\frac{1}{5} \rightarrow 5 + 2^5 = 37.
\]

If \( m+n-1 = \text{even} = 2k \),
the \( m+n-1 = 2k \) zig-zag line has the \( m+n-1 = 2k \) rationals
\[1/n, 2/(n-1), \ldots (n-1)/2, n/1,\]
which are assigned as follows
\[
\frac{1}{n} \rightarrow 1 + 2^{2k}, \\
\frac{2}{n-1} \rightarrow 2 + 2^{2k}, \\
\ldots \\
\frac{n-1}{2} \rightarrow 2k - 1 + 2^{2k}, \\
\frac{n}{1} \rightarrow 2k + 2^{2k}.
\]

If \( m+n-1 = \text{odd} = 2k+1 \),
the \( m+n-1 = 2k + 1 \) zig-zag line has the \( m+n-1 = 2k + 1 \) rationals
\[m/1, (m-1)/2, \ldots, 2/(m-1), 1/m,\]
which are assigned as follows
\[
\frac{m}{1} \rightarrow 1 + 2^{2k+1}, \\
\frac{m-1}{2} \rightarrow 2 + 2^{2k+1},
\]
\[
\begin{align*}
\frac{2}{m-1} & \rightarrow 2k + 2^{k+1}, \\
\frac{1}{m} & \rightarrow 2k + 1 + 2^{k+1}.
\end{align*}
\]

This defines a one-one function from the rationals into the natural numbers.

References