

Rationals Countability and Cantor's Proof

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January, 2006

Abstract: Cantor's proof that the rational numbers are countable uses a mapping that is not one-one. Thus, the countability of the rationals was not proved by Cantor.

We prove by cardinal number methods, using a result of Tarski, that the rationals are countable. We confirm this by exhibiting a one-one mapping from the rationals into the natural numbers.

Cantor's mapping is not one-one.

Cantor's proof appears in [1]. Cantor wrote

“By (6) of § 3, $\aleph_0 \cdot \aleph_0$ is the cardinal number of the aggregate of bindings

$$\{(\mu, \nu)\},$$

where μ and ν are any finite cardinal numbers which are independent of one another. If also λ represents any finite cardinal number, so that $\{\lambda\}$, $\{\mu\}$, and $\{\nu\}$ are only different notations for the same aggregate of all finite numbers, we have to show that

$$\{(\mu, \nu)\} \sim \{\lambda\}.$$

Let us denote $\mu + \nu$ by ρ ; then ρ takes all the numerical values 2, 3, 4, ...

and there are in all $\rho - 1$ elements (μ, ν) for which $\mu + \nu = \rho$, namely:

$$(1, \rho - 1), (2, \rho - 2), \dots, (\rho - 1, 1).$$

In this sequence imagine first the element $(1, 1)$, for which $\rho = 2$, put , then the two elements for which $\rho = 3$, then the three elements for which $\rho = 4$,

and so on. Thus, we get all the elements (μ, ν) in a simple series:

$$(1, 1); (1, 2), (2, 1); (1, 3), (2, 2), (3, 1); (1, 4), (2, 3), \dots,$$

and here, as we easily see, the element (μ, ν) comes at the λ th place, where

$$(9) \quad \lambda = \mu + \frac{(\mu + \nu - 1)(\mu + \nu - 2)}{2}.$$

The variable λ takes every numerical value $1, 2, 3, \dots$, once. Consequently, by means of (9), a reciprocally univocal relation subsists between the aggregates $\{\nu\}$ and $\{(\mu, \nu)\}$."

Clearly, the variable λ takes several times some of numerical value $1, 2, 3, \dots$, The mapping is not one-one.

We have,

$$\mu = 1, \quad \nu = 1, \quad \lambda = 1, \quad \text{and } (1, 1) \rightarrow 1.$$

$$\mu = 1, \quad \nu = 2, \quad \lambda = 2, \quad \text{and } (1, 2) \rightarrow 2.$$

$$\mu = 2, \quad \nu = 1, \quad \lambda = 2, \quad \text{and } (2, 1) \rightarrow 2.$$

$$\mu = 1, \quad \nu = 3, \quad \lambda = 4, \quad \text{and } (1, 3) \rightarrow 4.$$

$$\mu = 2, \quad \nu = 2, \quad \lambda = 4, \quad \text{and } (2, 2) \rightarrow 4.$$

$$\mu = 3, \quad \nu = 1, \quad \lambda = 4, \quad \text{and } (3,1) \rightarrow 4.$$

$$\mu = 1, \quad \nu = 4, \quad \lambda = 7, \quad \text{and } (1,4) \rightarrow 7.$$

$$\mu = 2, \quad \nu = 3, \quad \lambda = 7, \quad \text{and } (2,3) \rightarrow 7.$$

$$\mu = 3, \quad \nu = 2, \quad \lambda = 7, \quad \text{and } (3,2) \rightarrow 7.$$

$$\mu = 4, \quad \nu = 1, \quad \lambda = 7, \quad \text{and } (4,1) \rightarrow 7.$$

Apparently, Cantor expected his mapping to be a bijection, did not check it, and did not see that it is not even one-one.

It is well known [2] that a one-one mapping is required to establish that $\text{card}Q \leq \text{card}N$. Thus, Cantor's claim is unfounded, and his use of $\aleph_0^2 = \aleph_0$, amounts to adding another axiom to his set theory.

Proof of Rationals Countability by Tarski result.

We use the graphical interpretation of Cantor's proof by a zig-zag through an infinite triangular matrix of rationals. While the zig-zag by itself does not prove the countability, it is useful to clarify our argument:

The first line in the zig-zag has one rational

$$1/1.$$

The second line has two rationals,

$$1/2, \text{ and } 2/1.$$

.....

The n-th line has n rationals,

$$1/n, 2/(n-1), \dots, (n-1)/2, n/1,$$

.....

Applying to (1),

$$1 + 2 + 3 + \dots + n + \dots \leq \aleph_0.$$

Regarding (2), this says

$$\aleph_0 + \aleph_0^2 \leq \aleph_0.$$

Since $\aleph_0^2 \leq \aleph_0 + \aleph_0^2$, by transitivity of cardinal inequalities [3, p. 147],

$$\aleph_0^2 \leq \aleph_0.$$

Since $\aleph_0 \leq \aleph_0^2$,

$$\aleph_0^2 = \aleph_0.$$

Proof of Rationals Countability by a one-one mapping.

Aided by the zig-zag listing of the rationals, we produce a one-one mapping from the rationals into the natural numbers. We construct our mapping with numerical examples. Then we give the general formula.

The first line in the zig-zag has one rational

$$1/1$$

which we assign as follows

$$\frac{1}{1} \rightarrow 1 + 2^1 = 3.$$

The second line in the zig-zag has two rationals,

$$1/2, \text{ and } 2/1,$$

which we assign as follows

$$\frac{1}{2} \rightarrow 1 + 2^2 = 5,$$

$$\frac{2}{1} \rightarrow 2 + 2^2 = 6.$$

The third line in the zig-zag has three rationals

$$3/1, 2/2, 1/3,$$

which are assigned as follows

$$\frac{3}{1} \rightarrow 1 + 2^3 = 9,$$

$$\frac{2}{2} \rightarrow 2 + 2^3 = 10,$$

$$\frac{1}{3} \rightarrow 3 + 2^3 = 11.$$

The fourth line in the zig-zag has four rationals

$$1/4, 2/3, 3/2, 4/1$$

which are assigned as follows

$$\frac{1}{4} \rightarrow 1 + 2^4 = 17,$$

$$\frac{2}{3} \rightarrow 2 + 2^4 = 18,$$

$$\frac{3}{2} \rightarrow 3 + 2^4 = 19.$$

$$\frac{4}{1} \rightarrow 4 + 2^4 = 20.$$

The fifth line in the zig-zag has five rationals

$$5/1, 4/2, 3/3, 2/4, 1/5,$$

which are assigned as follows

$$\frac{5}{1} \rightarrow 1 + 2^5 = 33,$$

$$\frac{2}{4} \rightarrow 2 + 2^5 = 34,$$

$$\frac{3}{3} \rightarrow 3 + 2^5 = 35.$$

$$\frac{2}{4} \rightarrow 4 + 2^5 = 36.$$

$$\frac{1}{5} \rightarrow 5 + 2^5 = 37.$$

If

$$m + n - 1 = \text{even} = 2k,$$

the $m + n - 1 = 2k$ zig-zag line has the $m + n - 1 = 2k$ rationals

$$1/n, 2/(n-1), \dots, (n-1)/2, n/1,$$

which are assigned as follows

$$\frac{1}{n} \rightarrow 1 + 2^{2k},$$

$$\frac{2}{n-1} \rightarrow 2 + 2^{2k},$$

.....

$$\frac{n-1}{2} \rightarrow 2k - 1 + 2^{2k},$$

$$\frac{n}{1} \rightarrow 2k + 2^{2k}.$$

If

$$m + n - 1 = \text{odd} = 2k + 1,$$

the $m + n - 1 = 2k + 1$ zig-zag line has the $m + n - 1 = 2k + 1$ rationals

$$m/1, (m-1)/2, \dots, 2/(m-1), 1/m,$$

which are assigned as follows

$$\frac{m}{1} \rightarrow 1 + 2^{2k+1},$$

$$\frac{m-1}{2} \rightarrow 2 + 2^{2k+1},$$

.....

$$\frac{2}{m-1} \rightarrow 2k + 2^{2k+1},$$

$$\frac{1}{m} \rightarrow 2k + 1 + 2^{2k+1}.$$

This defines a one-one function from the rationals into the natural numbers.

References

- [1] Cantor, Georg, *Contributions to the founding of the theory of Transfinite Numbers*, p.107. Open Court Publishing 1915, Dover 1955.
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- [3] Tarski, Alfred, *Axiomatic and algebraic aspects of two theorems on sums of cardinals*, *Fundamenta Mathematicae*, Volume 35, 1948, p. 79-104. Reprinted in *Alfred Tarski Collected papers*, edited by Steven R. Givant and Ralph N. McKenzie, Volume 3, 1945-1957, p. 173, Birkhauser, 1986.
- [4] Sierpinski, Waclaw, *Cardinal and Ordinal Numbers*. Warszawa, 2nd edition, 1965, (Also in the 1958, 1st edition).