Infinitesimal Variational Calculus

H. Vic Dannon
vic0@comcast.net
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Abstract We define Variations as Infinitesimal Functions, and investigate their properties.

Variational Calculus is based on Euler's differential equation, Hence on differentials, and as such it is infinitesimal. But it is derived in an integral context, which imposes unnecessary global restrictions, requires unnecessary arguments, and is ill-equipped to resolve its over hundred-years-old problems.

Heuristic conception of Infinitesimals resulted in Ill-defined Variations. Texts of Variational Calculus apply variations inefficiently, or avoids them altogether.

We develop Variational Calculus in Infinitesimal context to

eliminate unnecessary arguments,

to eliminate restrictive global assumptions and to give meaning to the poorly understood, and thus, ill-defined variation δy, and variational derivative

$$\frac{\delta F}{\delta y}.$$

We supply a correct proof to the unresolved to date Weierstrass Sufficient condition for a Minimum, and prove its equivalence to Legendre Sufficient Condition for a Minimum. In non-infinitesimal context, the Legendre Sufficient Condition is misunderstood as being insufficient.

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Contents

Introduction

0.1 Global Solutions in the Calculus of Variations

The infinitesimal distance in the plane is

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (y')^2}dx.$$

The arc-length of $y(x)$ between $(a, y(a))$, and $(b, y(b))$ is

$$\int_{x=a}^{x=b} \sqrt{1 + (y')^2}dx,$$

and the curve $y(x)$ that minimizes the integral has the shortest arc-length.

That curve is the line over the interval $[a, b]$.

The speed of a bead falling a distance x is

$$\sqrt{2gx},$$

the infinitesimal descent time of a bead along a curve $y(x)$ is

$$dt = \frac{ds}{v} = \frac{\sqrt{1 + (y')^2}dx}{\sqrt{2gx}}$$

The descent time of the bead along a curve $y(x)$ between $(a, y(a))$, and $(b, y(b))$ is

$$\int_{x=a}^{x=b} \frac{\sqrt{1 + (y')^2}}{\sqrt{2gx}}dx,$$

and the curve $y(x)$ that minimizes the integral has the quickest descent.

That curve is an arc of a cycloid over the interval $[a,b]$.

In both problems, finding the curve of shortest path, and finding the curve of quickest descent, the domain of the solution is the interval $[a,b]$.

Thus, a basic problem of the Calculus of Variations is to find a curve

$$y(x) = y_0(x),$$

that will minimize the Functional

$$J(y) = \int_{x=a}^{x=b} F(x, y(x), y'(x))dx .$$

The solution has to hold globally, over the interval $[a,b]$.

But we are not required to find it by global methods.

Since the problem was constructed using infinitesimals, it is amenable to infinitesimal methods.

However, for lack of understanding of infinitesimals, variations were poorly understood, and their full power was not used. Instead, the analysis applied to the global problem.

We follow with the global derivation that has not been re-examined since Weierstrass time.

0.2 Global Derivation in the Calculus of Variations

For a curve,

$$y(x) = y_0(x) + \varepsilon\eta(x),$$

where

$\varepsilon > 0$ is very small,

$\eta(a) = \eta(b) = 0$,

the Taylor expansion to second order of

$$F(x, y_0(x) + \varepsilon\eta(x), y_0'(x) + \varepsilon\eta'(x)) - F(x, y_0(x), y_0'(x)),$$

about $y_0(x)$, is

$$\varepsilon\left\{\frac{\partial F}{\partial y}\bigg|_{y=y_0}\eta + \frac{\partial F}{\partial y'}\bigg|_{y=y_0}\eta'\right\} + O(\varepsilon^2).$$

Therefore,

$$I(\varepsilon) - I(0) = \varepsilon\int_{x=a}^{x=b}\left\{\frac{\partial F}{\partial y}\bigg|_{y=y_0}\eta + \frac{\partial F}{\partial y'}\bigg|_{y=y_0}\eta'\right\}dx + O(\varepsilon^2)$$

Integrating by parts,

$I(\varepsilon) - I(0) =$

$$= \varepsilon\int_{x=a}^{x=b}\left\{\frac{\partial F}{\partial y}\bigg|_{y=y_0} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\bigg|_{y=y_0}\right\}\eta(x)dx + \underbrace{\left[\eta(x)\frac{\partial F}{\partial y'}\right]_{x=a}^{x=b}}_{=0} + O(\varepsilon^2)$$

Hence,

$$\frac{I(\varepsilon) - I(0)}{\varepsilon} = \int_{x=a}^{x=b}\left\{\frac{\partial F}{\partial y}\bigg|_{y=y_0} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\bigg|_{y=y_0}\right\}\eta(x)dx + O(\varepsilon).$$

Letting $\varepsilon \downarrow 0$,

$$I'(0) = \int_{x=a}^{x=b} \left\{ \frac{\partial F}{\partial y}\bigg|_{y=y_0} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right)\bigg|_{y=y_0} \right\} \eta(x)dx .$$

Since $I(0) = J(y_0)$ is minimal, we have by Fermat Theorem, $I'(0) = 0$, and

$$\int_{x=a}^{x=b} \left\{ \frac{\partial F}{\partial y}\bigg|_{y=y_0} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right)\bigg|_{y=y_0} \right\} \eta(x)dx = 0 .$$

Since $\eta(x)$ is an arbitrary smooth enough function, it seems plausible that the integrand is identically zero.

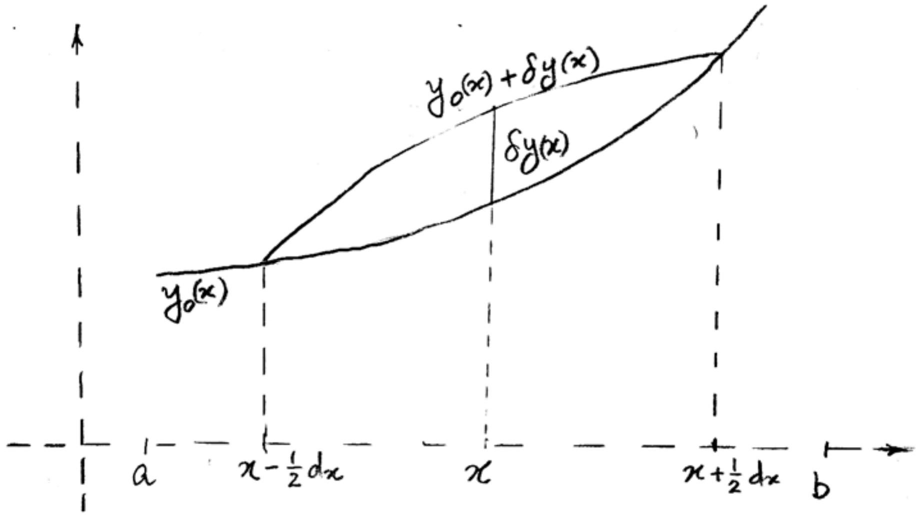
Namely,

$$\frac{\partial F}{\partial y}\bigg|_{y=y_0} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right)\bigg|_{y=y_0} = 0 .$$

That is the Euler equation for the extremal curve.

To that end, many texts apply the "Fundamental lemma of the Calculus of Variations".

But since the interval $[a,b]$ has no role in the derivation, it is not necessary to extend it beyond an infinitesimal. An infinitesimal interval $[x - \frac{1}{2}dx, x + \frac{1}{2}dx]$ will eliminate the need for the Fundamental lemma.



Clearly, the interval is irrelevant to the solution of the shortest distance problem:

Then, Euler's Variational Equation is

$$\frac{d}{dx}\frac{y'}{\sqrt{1+(y')^2}}=0,$$

$$(y')^2 = \text{constant},$$

$$y = Ax + B,$$

which is a line in the plane.

0.3 Weierstrass Sufficient Condition

In his Lectures on the Calculus of Variations, Weierstrass

observed that a sufficient condition for $\int_{x=a}^{x=b} F(x.y.y')dx$ to be minimal can be given.

Weierstrass Lectures were never published, and as the simple reason for the observation was not found, a whole theory was built to substantiate it.

That theory is based on the parametric family of extremals that solve the Euler Equation, and constitute a partition of a simply connected domain in the plane.

Hilbert claimed that in that domain, $\int_{x=a}^{x=b} F(x,y,y')dx$ is path independent, and based on that, a dizzying proof was given for Weierstrass observation.

<u>First</u>, path-independence of $\int_{x=a}^{x=b} F(x,y,y')dx$ defies the purpose of the Calculus of Variations. An integral that does not depend on the curve is irrelevant in the calculus of Variations.

<u>Second</u>, the disjoint extremals do not intersect at any point. For instance, the extremals of the shortest distance problem are parallel lines, each determined by its $(a, y(a))$, and $(b, y(b))$. Therefore, a closed path made of extremals is impossible in the extremals' field.

Third, not even a pencil of the disjoint extremals is possible. Pencil pictures that accompany that theory are delusionary.

We give here a correct proof, and show that Weierstrass sufficient condition, and Legendre sufficient condition are equivalent.

0.4 Functional Variational Derivatives

There is a false belief that Integrals may have a Variational Derivative, that is called a Functional Derivative.

But the first variation of an integral

$$\delta J = \varepsilon \int_{x=a}^{x=b} \left\{ \frac{\partial F}{\partial y}\bigg|_{y=y_0} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\bigg|_{y=y_0} \right\} \eta(x) dx,$$

does not allow the definition of a Variational Derivative $\frac{\delta J}{\delta y}$

of the Functional J, because it is not clear how $\delta y(x)$ may be pulled from under the integral sign.

We can only define the Variational Derivative, $\frac{\delta F}{\delta y}$ of the Function F.

We proceed with the infinitesimal treatment of variations.

1.

Infinitesimal Curve-Variation

The differential of the hyper-real function $f(x)$ is defined by

$$df(x) = f'(x)dx.$$

It is the product of the infinitesimal dx by the hyper-real function $f'(x)$, and it vanishes at the zeros of $f'(x)$.

Graphically, it is the rise of the tangent to $f(x)$ at x, over the infinitesimal run dx.

$df(x)$ is of infinitesimal size, but since it may vanish, it is not an infinitesimal.

We use

the same dx in the definition of all differentials.

For instance,

$$d(x^2) = 2xdx$$

$$d(e^x) = e^x dx$$

We shall call the product of a hyper-real function by an infinitesimal an **infinitesimal function.**

In Infinitesimal Variational Calculus, we define a variation as an infinitesimal function.

Denoting

the variation of the curve $y(x)$ by $\delta y(x)$,

an infinitesimal by o_δ,

and a smooth enough hyper-real function by $\eta(x)$

we define

1.1 The Infinitesimal Curve Variation

$$\delta y(x) = o_\delta \eta(x).$$

where,

1.2 *We use the same o_δ for all curve variations.*

the Greek omicron o, looks like a zero, and was used extensively by Riemann to denote an infinitesimal. In translations, it was sometimes confused with zero, and replaced by it, making Riemann's considerations evaporate.

That is why we subscripted it here with δ.

The Greek epsilon have been used extensively by Weierstrass to mean a non-imaginable very small positive quantity, not necessarily an infinitesimal, that may be diminished to a zero limit, while staying positive.

In other words, Weierstrass' epsilon is an infinitesimal on the real line.

But Infinitesimals exist on the Hyper-real line only. [Dan1]. Thus, there is no such ε, and to prevent confusion, we shall avoid using epsilon here.

From the definition 1.1, it is clear that

1.3 *The variation of* $y(x)$*, does not depend on the run* dx.

The infinitesimal o_δ endows $\delta y(x)$ with infinitesimal size, and $o_\delta \eta(x)$ shifts $y(x)$ to an infinitesimally close curve:

1.4 *The curve* $y(x) + \delta y(x)$ *is infinitesimally close to* $y(x)$.

In particular,

1.5 $\delta y(x)$ *does not depend on the size of the* x*-interval*

From the definition 1.1, we have

1.6 δ *is a linear operator*

Proof:

$$\delta[y_1(x) + y_2(x)] = o_\delta[\eta_1(x) + \eta_2(x)]$$

$$= o_\delta \eta_1(x) + o_\delta \eta_2(x)$$

$$= \delta[y_1(x)] + \delta[y_2(x)].$$

$$\delta[\alpha y(x)] = o_\delta \alpha \eta(x) = \alpha o_\delta \eta(x) = \alpha \delta[y(x)]. \square$$

Therefore, the operators d, and δ commute. Namely,

1.7 $$d(\delta y(x)) = \delta(dy(x))$$

Proof:

$$d[\delta y(x)] = \delta y(x+dx) - \delta y(x)$$

$$= \delta[y(x+dx) - y(x)]$$

$$= \delta[dy(x)].\square$$

Therefore, the operators $\frac{d}{dx}$, and δ commute. Namely,

1.8 $$\frac{d}{dx}(\delta y(x)) = \delta(\frac{d}{dx}y(x))$$

1.9 <u>If</u> $F(x, y(x), y'(x))$ *is a hyper-real function on an* x*-interval*

<u>Then</u>, *its' Taylor series to third order about* $y_0(x)$ *is*

$$F(x, y_0 + \delta y, y_0' + \delta y') = F(x, y_0, y_0') +$$

$$+\left\{\left.\frac{\partial F}{\partial y}\right|_{y=y_0}\delta y(x) + \left.\frac{\partial F}{\partial y'}\right|_{y=y_0}\delta y'(x)\right\}$$

$$+\left\{\left.\frac{\partial^2 F}{\partial y^2}\right|_{y_0}[\delta y(x)]^2 + 2\left.\frac{\partial^2 F}{\partial y\partial y'}\right|_{y_0}\delta y(x)\delta y'(x) + \left.\frac{\partial^2 F}{(\partial y')^2}\right|_{y_0}(\delta y'(x))^2\right\}$$

$$+O(o_y^3).$$

Define

1.10 the **First Variation** of $F(x,y,y')$

$$\delta F(x,y,y') = \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'$$

$$= \left(\delta y \frac{\partial}{\partial y} + \delta y' \frac{\partial}{\partial y'}\right) F$$

1.11 $\delta x = 0$

Proof: $F(x,y,y') = x \Rightarrow \delta F = 0 .\square$

Define

1.12 the **Second Variation** of $F(x,y,y')$

$$\delta^2 F(x,y,y') = \left(\delta y \frac{\partial}{\partial y} + \delta y' \frac{\partial}{\partial y'}\right)^2 F$$

$$= \frac{\partial^2 F}{\partial y^2}(\delta y)^2 + 2\frac{\partial^2 F}{\partial y \partial y'}(\delta y)(\delta y') + \frac{\partial^2 F}{(\partial y')^2}(\delta y')^2$$

Just as $d^2 x = 0$, we have

1.13 $\delta^2 y(x) = 0$

Proof: $F(x,y,y') = y \Rightarrow \delta^2 F = 0 .\square$

Then, 1.8 becomes

1.14 If $F(x, y(x), y'(x))$ *is hyper-real function on an* x*-interval*

Then, *its' Taylor series to third order about* $y_0(x)$ *is*

$$F(x, y_0 + \delta y, y_0' + \delta y') = F(x, y_0, y_0') + +\delta F(x, y_0, y_0') + \delta^2 F(x, y_0, y_0') + O(o_y^3)$$

Therefore,

1.15 If $F(x, y(x), y'(x))$ *is hyper-real function on an* x*-interval,*

and $$J(y) = \int_{x=a}^{x=b} F(x, y(x), y'(x))dx$$

Then, *Taylor series to third order of* $J(y_0 + \delta y)$ *about*

$y_0(x)$ *is*

$$J(y_0 + \delta y) = \int_{x=a}^{x=b} F(x, y_0(x), y_0'(x))dx +$$

$$+\int_{x=a}^{x=b} \delta F(x, y_0, y_0')dx$$

$$+\int_{x=a}^{x=b} \delta^2 F(x, y_0, y_0')dx$$

$$+O(o_y^3).$$

Define

1.16 the **First Variation** of $\int\limits_{x=a}^{x=b} F(x, y(x), y'(x))dx$

$$\delta \int\limits_{x=a}^{x=b} F(x, y, y')dx = \int\limits_{x=a}^{x=b} \delta F(x, y, y')dx$$

Define

1.17 the **Second Variation** of $\int\limits_{x=a}^{x=b} F(x, y(x), y'(x))dx$

$$\delta^2 \int\limits_{x=a}^{x=b} F(x, y, y')dx = \int\limits_{x=a}^{x=b} \delta^2 F(x, y, y')dx\,.$$

Then, 1.15 becomes

1.18 <u>If</u> $F(x, y(x), y'(x))$ *is hyper-real function on an* x*-interval,*

and $J(y) = \int\limits_{x=a}^{x=b} F(x, y(x), y'(x))dx\,.$

<u>Then</u>, *Taylor series to third order of* $J(y)$ *about* $y_0(x)$ *is*

$$J(y_0 + \delta y) = J(y_0) + \delta J(y_0) + \delta^2 J(y_0) + O(o_y^3)\,.$$

2.

Euler's Equation for $F(x, y, y')$, and the Variational Derivative

We apply variations, over infinitesimal interval, to derive Euler's Variational Equation in Infinitesimal Variational Calculus.

2.1 Euler's Variational Equation

<u>If</u> $y_0(x)$ minimizes $J(y) = \int\limits_{x=a}^{x=b} F(x, y(x), y'(x))dx$

<u>Then</u> $$\left.\frac{\partial F}{\partial y}\right|_{y=y_0} - \frac{d}{dx}\left.\left(\frac{\partial F}{\partial y'}\right)\right|_{y=y_0} = 0\,.$$

Proof:

Since by 1.5, $\delta y(x)$ does not depend on the size of the x-interval, we will use an infinitesimal interval

$$[x - \tfrac{1}{2}dx, x + \tfrac{1}{2}dx], \qquad a < x < b\,.$$

Then, $y_0(x)$ minimizes

$$dI(o_\delta) = dJ(y_0 + \delta y) = \int_{\xi = x - \frac{1}{2}dx}^{\xi = x + \frac{1}{2}dx} F(\xi, y_0(\xi) + \delta y, y_0\,'(\xi) + \delta y\,')d\xi\,,$$

over all the curves $y_0 + \delta y$ that cross $y_0(\xi)$ at the endpoints of $[x - \frac{1}{2}dx, x + \frac{1}{2}dx]$. That is,

$$\delta y(x - \tfrac{1}{2}dx) = \delta y(x + \tfrac{1}{2}dx)\,.$$

The Taylor expansion to second order of

$$F(\xi, y_0(\xi) + \delta y(\xi), y_0\,'(\xi) + \delta y\,'(\xi)) - F(x, y_0(\xi), y_0\,'(\xi))\,,$$

about $y_0(\xi)$, is

$$\left.\frac{\partial F}{\partial y}\right|_{y=y_0} \delta y + \left.\frac{\partial F}{\partial y\,'}\right|_{y=y_0} \delta y\,' + O(o_\delta^2)\,.$$

Therefore,

$$dJ(y_0 + \delta y) - dJ(y_0) = \int_{\xi = x - \frac{1}{2}dx}^{\xi = x + \frac{1}{2}dx} \left\{ \left.\frac{\partial F}{\partial y}\right|_{y_0} \delta y + \left.\frac{\partial F}{\partial y\,'}\right|_{y_0} \delta y\,' \right\} d\xi + O(o_\delta^2)$$

By 1.8, $\delta y\,' = \dfrac{d}{dx}\delta y$, and integrating by parts,

$$dJ(y_0 + \delta y) - dJ(y_0) =$$

$$= \int_{\xi = x - \frac{1}{2}dx}^{\xi = x + \frac{1}{2}dx} \left\{ \left.\frac{\partial F}{\partial y}\right|_{y_0} - \frac{d}{d\xi}\left.\left(\frac{\partial F}{\partial y\,'}\right)\right|_{y_0} \right\} \delta y(\xi) d\xi + \underbrace{\left[\left.\left(\frac{\partial F}{\partial y\,'}\right)\right|_{y_0} \delta y(\xi) \right]_{\xi = x - \frac{1}{2}dx}^{\xi = x + \frac{1}{2}dx}}_{=0} + O(o_\delta^2)$$

The integration sum over the infinitesimal interval yields only the term, $\left\{ \frac{\partial F}{\partial y}\Big|_{y_0} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\Big|_{y_0} \right\}\delta y(x)dx$, and we obtain

$$dJ(y_0 + \delta y) - dJ(y_0) = \left\{ \frac{\partial F}{\partial y}\Big|_{y_0} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\Big|_{y_0} \right\}\delta y(x)dx + O(o_\delta^2)$$

Substituting by 1.1, $\delta y(x) = o_\delta \eta(x)$,

$$dI(o_\delta) - dI(0) = \left\{ \frac{\partial F}{\partial y}\Big|_{y_0} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\Big|_{y_0} \right\} o_\delta \eta(x)dx + O(o_\delta^2),$$

$$\frac{dI(o_\delta) - dI(0)}{o_\delta} = \left\{ \frac{\partial F}{\partial y}\Big|_{y_0} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\Big|_{y_0} \right\}\eta(x)dx + O(o_\delta),$$

Since this hold for any infinitesimal o_δ, $dI(\varepsilon)$ has a derivative at $\varepsilon = 0$, which is,

$$dI'(0) = \left\{ \frac{\partial F}{\partial y}\Big|_{y_0} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\Big|_{y_0} \right\}\eta(x)dx.$$

Since $dI(0) = dJ(y_0)$ is minimal, we have by Fermat Theorem, $dI'(0) = 0$, and since $dx > 0$,

$$\left\{ \frac{\partial F}{\partial y}\Big|_{y_0} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\Big|_{y_0} \right\}\eta(x) = 0.$$

Since $\eta(x)$ is an arbitrary smooth enough function,

$$\frac{\partial F}{\partial y}\Big|_{y=y_0} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\Big|_{y=y_0} = 0.\square$$

From the derivation of Euler's equation,

2.2 The First variation of $F(x,y,y')$ **is**

$$\delta F(x,y,y') = \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'$$

$$= \left\{\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right\}\delta y\,.$$

$$= (\delta y)(\partial_y - D_x \partial_{y'})F$$

Therefore, we define

2.3 The First Variational Derivative of $F(x,y,y')$

$$\frac{\delta F}{\delta y} = \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$$

$$= (\partial_y - D_x \partial_{y'})F$$

2.4 $$\delta \int_{\xi=a}^{\xi=x} F(\xi, y(\xi), y'(\xi))d\xi = \int_{\xi=a}^{\xi=x} \delta F(\xi, y(\xi), y'(\xi))dx\,.$$

2.5 <u>If</u> $y_0(x)$ minimizes $J(y) = \int_{x=a}^{x=b} F(x, y(x), y'(x))dx$

<u>Then</u> $$\left.\frac{\delta F}{\delta y}\right|_{y_0} = 0\,,$$

$$\delta F\Big|_{y_0} = 0\,,$$

$$\delta J\Big|_{y_0} = 0\,.$$

2.6 The Second Variation of $F(x,y,y')$

$$\delta^2 F(x,y,y') = \left(\delta y\frac{\partial}{\partial y} + \delta y'\frac{\partial}{\partial y'}\right)^2 F$$

$$= \left((\delta y)^2\frac{\partial^2}{(\partial y)^2} + 2(\delta y)(\delta y')\frac{\partial^2}{\partial y\partial y'} + (\delta y')^2\frac{\partial^2}{(\partial y')^2}\right)F$$

2.7 Second Variational Derivative of $F(x,y,y')$

By 2.3,

$$\frac{\delta}{\delta y} = \partial_y - D_x\partial_{y'},$$

Applying this operator consecutively

$$\frac{\delta^2}{(\delta y)^2} = (\partial_y - D_x\partial_{y'})^2$$

and we may define the second Variational Derivative of $F(x,y,y')$ by

$$\frac{\delta^2 F}{(\delta y)^2} = (\partial_y^2 - 2D_x\partial_y\partial_{y'} + D_x^2\partial_{y'}^2)F$$

But we know of no use to this proposition.

3.

Euler's Equation, and Strong Variations

In the introduction, we presented the textbooks' derivation of Euler's equation, with variations defined as

$$\varepsilon\eta(x),$$

where $\varepsilon > 0$ is very small, and is allowed to decrease to zero. Such an epsilon is, in fact, an infinitesimal on the real line, and no such infinitesimal exists.

Calculus of Variations Texts call such variations "Weak Variations".

We defined these variations correctly, as hyper-real infinitesimal functions, [Dan2], and applied them to derive Euler's equation in the Infinitesimal Calculus of Variations.

For instance, if

$$F(x,y,y') = (y')^2 + (y')^3, \quad 0 \le x \le 1,$$

Euler's Equation is

$$\frac{d}{dx}\left(2y' + 3(y')^2\right) = 0,$$

$$y_0(x) = 0\,, \qquad\qquad 0 \le x \le 1$$

is a solution, hence, an extremal, and

$$I(o_\delta) = J(\delta y) = \int_{x=0}^{x=1} \left\{ (\delta y')^2 + (\delta y')^3 \right\} dx$$

$$= o_\delta^2 \int_{x=0}^{x=1} (\eta')^2 dx + o_\delta^3 \int_{x=0}^{x=1} (\eta')^3 dx$$

Therefore,

$$I''(o_\delta) = 2\int_{x=0}^{x=1} (\eta')^2 dx + 6o_\delta \int_{x=0}^{x=1} (\eta')^3 dx\,.$$

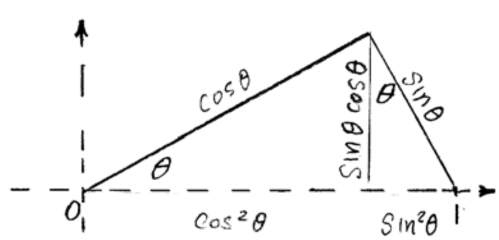
Hence,

$$I''(0) = 2\int_{x=0}^{x=1} (\eta')^2 dx > 0\,,$$

and $y_0(x) = 0$, minimizes $J(y) = \int_{x=0}^{x=1} \left\{ (y')^2 + (y')^3 \right\} dx\,.$

Calculus of variations Texts consider “Strong Variations”, that have derivatives that are infinite hyper-reals [Dan2].

Such strong variation is the saw-tooth $y(\theta)$ [Pars, p.46]



It is bounded by

$$\sin\theta\cos\theta\,,$$

that is infinitesimal for θ infinitesimal.

But for θ infinitesimal, y' close to $x = 1$, has an infinite hyper-real jump, and the integral is an infinite hyper-real.

Indeed,

$$J(y) = (\tan^2\theta + \tan^3\theta)\cos^2\theta + (\cot^2\theta - \cot^3\theta)\sin^2\theta$$

$$= 1 - 2\cot 2\theta$$

is a negative infinite hyper-real for θ infinitesimal.

The purpose of using such $y(\theta)$ is unclear, because we define variations in a form that will be useful to us.

In particular, Euler's Equation is derived only with variations that are infinitesimal functions. Strong variations are inapplicable to the derivation of Euler's Equation, and Hence irrelevant to the calculus of variations.

To date, no other derivation of the Euler Equation is known except the one that uses infinitesimal variations.

3.1 *The derivation of Euler's Variational Equation requires infinitesimal variations, and the Equation holds only if derived with infinitesimal variations*.

Consequently,

3.2 *Euler's Variational Equation prohibits arbitrary, non-infinitesimal variations*.

But Euler's Variational Equation is the main result of the Calculus of Variations, and without Euler's equation, there is no Calculus of Variations.

The mention of "strong variations" is pointless because with such variations, Euler's Equation disappears, and the Calculus of variations has no results.

3.3 *Euler's Equation, and hence, the Calculus of Variations, are incompatible with strong variations*

Thus, when we talk about variations we mean only the infinitesimal variations.

The irrelevant Strong Variations, defy the fundamental role of Euler's Variational Equation in the foundations of Dynamics, and Theoretical Physics.

4.

First Integral of Euler's Equation

Forsyth presents a derivation of the first Integral of the Euler's equation for $F(x, y, y')$.

To be well defined, the extremals that solve the Euler equation for $F(x, y, y')$, need to be disjoint curves, infinitesimally close to each other, but non-intersecting.

We parametrize the family of extremals by α, and denote them by

$$y_\alpha(x).$$

At each plane point (x, y), the slope of an extremal y_α is a function of (x, y). That is

$$\frac{dy_\alpha}{dx} = p(x, y)$$

Thus,

$$F(x, y, y') = \Phi(x, y, p)$$

and the Euler Equation transforms into a first order equation for $p(x, y)$

5.1 First Integral of Euler's equation

$y' = p(x,y)$ transforms Euler's equation for y, into a first order equation for p

$$\frac{\partial p}{\partial x} + p\frac{\partial p}{\partial y} = \frac{1}{\partial_p^2\Phi}\left(\frac{\partial\Phi}{\partial y} - \frac{\partial^2\Phi}{\partial x\partial p} - \frac{\partial^2\Phi}{\partial y\partial p}p\right)$$

which characteristic equations are

$$dx = \frac{dy}{p} = \frac{dp}{\frac{1}{\partial_p^2\Phi}\left(\frac{\partial\Phi}{\partial y} - \frac{\partial^2\Phi}{\partial x\partial p} - \frac{\partial^2\Phi}{\partial y\partial p}p\right)}.$$

Proof:

Euler's Equation for y is

$$\frac{\partial\Phi}{\partial y} = \frac{d}{dx}\frac{\partial\Phi(x,y,p)}{\partial p}$$

$$= \frac{\partial^2\Phi}{\partial x\partial p}\frac{dx}{dx} + \frac{\partial^2\Phi}{\partial y\partial p}\frac{dy}{dx} + \frac{\partial^2\Phi}{(\partial p)^2}\frac{dp}{dx}$$

Substituting

$$\frac{dy}{dx} = p\,,$$

$$\frac{dp(x,y)}{dx} = \frac{\partial p}{\partial x}\frac{dx}{dx} + \frac{\partial p}{\partial y}\frac{dy}{dx},$$

$$= \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y}p\,,$$

we obtain Euler's equation for p,

$$\frac{\partial\Phi}{\partial y} = \frac{\partial^2\Phi}{\partial x\partial p} + \frac{\partial^2\Phi}{\partial y\partial p}p + \frac{\partial^2\Phi}{(\partial p)^2}\left(\frac{\partial p}{\partial x} + \frac{\partial p}{\partial y}p\right).$$

Reorganizing it as a first order equation for $p(x,y)$, we obtain

$$\frac{\partial p}{\partial x} + p\frac{\partial p}{\partial y} = \frac{1}{\partial_p^2\Phi}\left(\frac{\partial\Phi}{\partial y} - \frac{\partial^2\Phi}{\partial x\partial p} - \frac{\partial^2\Phi}{\partial y\partial p}p\right).$$

The characteristic equations are

$$\frac{dx}{1} = \frac{dy}{p} = \frac{dp}{\frac{1}{\partial_p^2\Phi}\left(\frac{\partial\Phi}{\partial y} - \frac{\partial^2\Phi}{\partial x\partial p} - \frac{\partial^2\Phi}{\partial y\partial p}p\right)}.\square$$

5.2 Surface of Revolution

The area of the surface generated by the revolution of $y = y(x)$ about the x axis between $x = 0$, and $x = 1$, is

$$2\pi\int_{x=0}^{x=1} y\sqrt{1+(y')^2}\,dx,$$

Taking

$$F(x,y,y') = y\sqrt{1+(y')^2},$$

we have

$$\Phi(x,y,p) = y\sqrt{1+p^2},$$

$$\frac{\partial\Phi}{\partial y} = (1+p^2)^{\frac{1}{2}},$$

$$\frac{\partial \Phi}{\partial p} = y \frac{p}{(1 + p^2)^{\frac{1}{2}}},$$

$$\frac{\partial^2 \Phi}{\partial x \partial p} = 0,$$

$$\frac{\partial^2 \Phi}{\partial y \partial p} = \frac{p}{(1 + p^2)^{\frac{1}{2}}},$$

$$\frac{\partial^2 \Phi}{(\partial p)^2} = y\left(\frac{1}{(1 + p^2)^{\frac{1}{2}}} - \frac{p^2}{(1 + p^2)^{\frac{3}{2}}}\right) = \frac{y}{(1 + p^2)^{\frac{3}{2}}},$$

$$\frac{1}{\partial_p^2 \Phi}\left(\frac{\partial \Phi}{\partial y} - \frac{\partial^2 \Phi}{\partial x \partial p} - \frac{\partial^2 \Phi}{\partial y \partial p} p\right) = \frac{(1 + p^2)^{\frac{3}{2}}}{y}\left((1 + p^2)^{\frac{1}{2}} - \frac{p}{(1 + p^2)^{\frac{1}{2}}} p\right)$$

$$= \frac{1 + p^2}{y}.$$

Thus, the characteristic equations are

$$dx = \frac{dy}{p} = y \frac{dp}{1 + p^2}$$

Hence,

$$\frac{1}{y} dy = \frac{p}{1 + p^2} dp,$$

$$\log y = \tfrac{1}{2} \log(1 + p^2) + \log \alpha,$$

$$y = \alpha(1 + p^2)^{\frac{1}{2}},$$

$$y = \alpha(1 + (\tfrac{dy}{dx})^2)^{\frac{1}{2}},$$

$$\frac{dy}{\sqrt{y^2-\alpha^2}} = \frac{1}{\alpha}dx,$$

$$\log(y+\sqrt{y^2-\alpha^2}) = \frac{1}{\alpha}x+\beta,$$

$$y+\sqrt{y^2-\alpha^2} = Ce^{\frac{1}{\alpha}x}.$$

Otherwise, solving Euler's equation directly,

$$\frac{\partial F}{\partial y} = [1+(y')^2]^{\frac{1}{2}},$$

$$\frac{\partial F}{\partial y'} = y\frac{y'}{[1+(y')^2]^{\frac{1}{2}}},$$

$$\frac{d}{dx}\left(y\frac{y'}{[1+(y')^2]^{\frac{1}{2}}}\right) = \frac{(y')^2+yy''}{[1+(y')^2]^{\frac{1}{2}}} - \frac{y(y')^2}{[1+(y')^2]^{\frac{3}{2}}},$$

Euler's equation is

$$[1+(y')^2]^{\frac{1}{2}} = \frac{(y')^2+yy''}{[1+(y')^2]^{\frac{1}{2}}} - \frac{y(y')^2}{[1+(y')^2]^{\frac{3}{2}}},$$

$$1+(y')^2 = (y')^2+yy'' - \frac{y(y')^2}{1+(y')^2},$$

$$1 = yy'' - \frac{y(y')^2}{1+(y')^2},$$

$$1+(y')^2 = yy''[1+(y')^2] - y(y')^2,$$

which seems to be more difficult to solve.

5.

Legendre Sufficient Condition for $\int_{x=a}^{x=b} F(x,y,y')dx$ to be Minimal

5.1 $\delta^2 F\Big|_{y_0} > 0$ is sufficient for a minimum

Proof: Since by 1.18

$$J(y_0 + \delta y) = J(y_0) + \delta J(y_0) + \delta^2 J(y_0) + O(o_y^3),$$

and since at the minimum

$$\delta J(y_0) = 0,$$

a sufficient condition for $J(y_0 + \delta y) > J(y_0)$, is

$$\delta^2 J(y_0) > 0.$$

This condition can be guaranteed by

$$\delta^2 F\Big|_{y_0} > 0.$$

5.2 Legendre Sufficient Condition for a Minimum

$$\left.\frac{\partial^2 F}{(\partial y')^2}\right|_{y_0} > 0\,.$$

Proof:

$$\delta^2 F(\xi, y, y')dx = \int_{\xi = x-\frac{1}{2}dx}^{\xi = x+\frac{1}{2}dx} \delta^2 F(\xi, y, y')d\xi$$

$$= \int_{x-\frac{1}{2}dx}^{x+\frac{1}{2}dx} \left\{(\delta y)^2 \frac{\partial^2 F}{(\partial y)^2} + 2(\delta y)(\delta y')\frac{\partial^2 F}{\partial y \partial y'} + (\delta y')^2 \frac{\partial^2 F}{(\partial y')^2}\right\} d\xi\,,$$

where δy vanishes at the integration end points.

Since $2(\delta y)(\delta y') = 2(\delta y)(\delta y)' = \frac{d}{d\xi}(\delta y)^2$, we have

$$= \int_{x-\frac{1}{2}dx}^{x+\frac{1}{2}dx} (\delta y)^2 \frac{\partial^2 F}{(\partial y)^2} d\xi + \int_{x-\frac{1}{2}dx}^{x+\frac{1}{2}dx} \frac{d(\delta y)^2}{d\xi}\frac{\partial^2 F}{\partial y \partial y'} d\xi + \int_{x-\frac{1}{2}dx}^{x+\frac{1}{2}dx} (\delta y')^2 \frac{\partial^2 F}{(\partial y')^2} d\xi$$

Integration by parts gives

$$\int_{x-\frac{1}{2}dx}^{x+\frac{1}{2}dx} \frac{d(\delta y)^2}{d\xi}\frac{\partial^2 F}{\partial y \partial y'} d\xi = \underbrace{\left[(\delta y)^2 \frac{\partial^2 F}{\partial y \partial y'}\right]_{x-\frac{1}{2}dx}^{x+\frac{1}{2}dx}}_{=0} - \int_{x-\frac{1}{2}dx}^{x+\frac{1}{2}dx} (\delta y)^2 \frac{d}{d\xi}\frac{\partial^2 F}{\partial y \partial y'} d\xi$$

Hence,

$$\delta^2 F(\xi, y, y')dx = \int_{x-\frac{1}{2}dx}^{x+\frac{1}{2}dx} (\delta y)^2 \left[\frac{\partial^2 F}{(\partial y)^2} - \frac{d}{d\xi}\frac{\partial^2 F}{\partial y \partial y'}\right] d\xi + \int_{x-\frac{1}{2}dx}^{x+\frac{1}{2}dx} (\delta y')^2 \frac{\partial^2 F}{(\partial y')^2} d\xi$$

$$= \int_{x-\frac{1}{2}dx}^{x+\frac{1}{2}dx} (\delta y)^2 \frac{\partial}{\partial y}\left[\frac{\partial F}{\partial y} - \frac{d}{d\xi}\frac{\partial F}{\partial y'}\right] d\xi + \int_{x-\frac{1}{2}dx}^{x+\frac{1}{2}dx} (\delta y')^2 \frac{\partial^2 F}{(\partial y')^2} d\xi$$

$$= (\delta y)^2 \frac{\partial}{\partial y}\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right] dx + (\delta y')^2 \frac{\partial^2 F}{(\partial y')^2} dx$$

At $y = y_0$, $\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$, and we have,

$$\delta^2 F(\xi, y_0, y_0')dx = (\delta y')^2 \left[\frac{\partial^2 F}{(\partial y')^2}\right]_{y_0} dx\,.$$

Since $dx > 0$, it cancels on both sides, and

$$\delta^2 F(\xi, y_0, y_0') = (\delta y')^2 \left[\frac{\partial^2 F}{(\partial y')^2}\right]_{y_0}.$$

Since $(\delta y')^2 > 0$ in the infinitesimal integration interval,

$$\left[\frac{\partial^2 F}{(\partial y')^2}\right]_{y_0} > 0,$$

guarantees $\delta^2 F(\xi, y_0, y_0') > 0$.$\square$

5.3 Shortest Distance

For

$$F(x, y, y') = \sqrt{1 + (y')^2}\,,$$

the Euler equation solution is a line with slope $y' = m$, and applying Legendre Sufficient Condition

$$\left[\frac{\partial^2}{(\partial y')^2}\sqrt{1+(y')^2}\right]_{y_0} = \left[\frac{1}{[1+(y')^2]^{\frac{3}{2}}}\right]_{y'=m} = \frac{1}{[1+m^2]^{\frac{3}{2}}} > 0 .$$

That is, the line is the shortest distance.

6.

Weierstrass Sufficient Condition, for $\int_{x=a}^{x=b} F(x,y,y')dx$ to be Minimal

In his Lectures on the Calculus of Variations, Weierstrass observed that a sufficient condition for $\int_{x=a}^{x=b} F(x,y,y')dx$ to be minimal can be given.

A bizarre theory to substantiate Weierstrass observation, did not produce a correct proof, and to date Weierstrass result remains unproved.

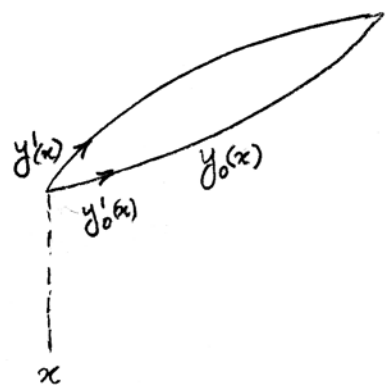
We give here a one line proof.

6.1 Weierstrass Sufficient Condition for a Minimum

$$F(x,y(x),y'(x)) - F(x,y_0(x),y'(x)) - \left.\frac{\partial F}{\partial y'}\right|_{y_0}(y'-y_0') > 0\,.$$

Proof:

Let $y_0(\xi)$, the extremal that solves the Euler equation, intersect the curve $y(\xi)$ at x.



Since $y(x) = y_0(x)$, the Taylor expansion of $F(x, y, y')$ to second order about $(x, y_0(x), y_0'(x))$ is

$$F(x, y(x), y'(x)) - F(x, y_0(x), y'(x)) - \left.\frac{\partial F}{\partial y'}\right|_{y_0}(y' - y_0') =$$

$$= \left.\frac{\partial^2 F}{(\partial y')^2}\right|_{y_0}(y' - y_0')^2 + O(o_\delta^3).$$

Therefore, Legendre's sufficient condition $\left.\frac{\partial^2 F}{(\partial y')^2}\right|_{y_0} > 0$, is equivalent to Weierstrass sufficient Condition

$$F(x, y(x), y'(x)) - F(x, y_0(x), y'(x)) - \left.\frac{\partial F}{\partial y'}\right|_{y_0}(y' - y_0') > 0.\square$$

From the proof of 6.1, we have

6.2 *Weierstrass' and Legendre's conditions are equivalent*

6.3 The Shortest Distance

For

$$F(x, y, y') = \sqrt{1 + (y')^2},$$

the Euler equation solution is a line with slope $y' = m$, and applying Weierstrass Sufficient Condition

$$F(x, y(x), y'(x)) - F(x, y_0(x), y'(x)) - \left.\frac{\partial F}{\partial y'}\right|_{y_0} (y' - y_0') =$$

$$= \sqrt{1 + (y')^2} - \sqrt{1 + (y_0')^2} - \left[\frac{\partial\sqrt{1 + (y')^2}}{\partial y'}\right]_{y_0} (y' - y_0')$$

$$= \sqrt{1 + (y')^2} - \sqrt{1 + (y_0')^2} - \frac{y_0'}{[1 + (y_0')^2]^{\frac{1}{2}}}(y' - y_0')$$

$$= \sqrt{1 + (y')^2} - \frac{1 + y'y_0'}{[1 + (y_0')^2]^{\frac{1}{2}}},$$

which is > 0, for $y \neq y_0$, because $(y' - y_0')^2 > 0$.□

That is, the line is the shortest distance.

7.

Euler's Equation for $F(x, y, y'y'')$ and the Variational Derivative

We apply variations, over infinitesimal interval, to derive Euler's Variational Equation for $F(x, y, y'y'')$in Infinitesimal Variational Calculus.

7.1 Euler's Variational Equation for $F(x, y, y'y'')$

<u>If</u> $y_0(x)$ minimizes $J(y) = \int_{x=a}^{x=b} F(x, y, y'y'')dx$

<u>Then</u> $$\left.\frac{\partial F}{\partial y}\right|_{y_0} - \frac{d}{dx}\left.\left(\frac{\partial F}{\partial y'}\right)\right|_{y_0} + \frac{d^2}{(dx)^2}\left.\left(\frac{\partial F}{\partial y''}\right)\right|_{y_0} = 0.$$

Proof:

Since by 1.5, $\delta y(x)$ does not depend on the size of the x-interval, we will use an infinitesimal interval

$$[x - \tfrac{1}{2}dx, x + \tfrac{1}{2}dx], \qquad a < x < b.$$

Then, $y_0(x)$ minimizes

$$dI(o_\delta) = dJ(y_0 + \delta y) = \int_{\xi = x - \frac{1}{2}dx}^{\xi = x + \frac{1}{2}dx} F(\xi, y_0 + \delta y, y_0{'} + \delta y{'}, y_0{''} + \delta y{''}) d\xi \,,$$

over all the curves $y_0 + \delta y$ with

$$\delta y(x - \tfrac{1}{2}dx) = \delta y(x + \tfrac{1}{2}dx) \,,$$

$$\delta y{'}(x - \tfrac{1}{2}dx) = \delta y{'}(x + \tfrac{1}{2}dx)$$

The Taylor expansion to second order of

$$F(\xi, y_0(\xi) + \delta y(\xi), y_0{'}(\xi) + \delta y{'}(\xi), y_0{''}(\xi) + \delta y{''}(\xi)) - F(x, y_0(\xi), y_0{'}(\xi), y_0{''}(\xi)) \,,$$

about $y_0(\xi)$, is

$$\left.\frac{\partial F}{\partial y}\right|_{y_0} \delta y + \left.\frac{\partial F}{\partial y{'}}\right|_{y_0} \delta y{'} + \left.\frac{\partial F}{\partial y{''}}\right|_{y_0} \delta y{''} + O(o_\delta^2) \,.$$

Therefore,

$dJ(y_0 + \delta y) - dJ(y_0) =$

$$= \int_{\xi = x - \frac{1}{2}dx}^{\xi = x + \frac{1}{2}dx} \left\{ \left.\frac{\partial F}{\partial y}\right|_{y_0} \delta y + \left.\frac{\partial F}{\partial y{'}}\right|_{y_0} \delta y{'} + \left.\frac{\partial F}{\partial y{''}}\right|_{y_0} \delta y{''} \right\} d\xi + O(o_\delta^2)$$

Since $\delta y{'} = \dfrac{d}{dx}\delta y$, and $\delta y(x - \tfrac{1}{2}dx) = \delta y(x + \tfrac{1}{2}dx)$,

integrating by parts,

$$\int_{\xi = x - \frac{1}{2}dx}^{\xi = x + \frac{1}{2}dx} \left.\frac{\partial F}{\partial y{'}}\right|_{y_0} \delta y{'} d\xi = \underbrace{\left[\left.\left(\frac{\partial F}{\partial y{'}}\right)\right|_{y_0} \delta y(\xi) \right]_{x - \frac{1}{2}dx}^{x + \frac{1}{2}dx}}_{=0} - \int_{\xi = x - \frac{1}{2}dx}^{\xi = x + \frac{1}{2}dx} \left.\frac{d}{d\xi}\left(\frac{\partial F}{\partial y{'}}\right)\right|_{y_0} \delta y(\xi) d\xi$$

Since $\delta y'' = \dfrac{d^2}{(dx)^2}\delta y$, and $\delta y'(x - \frac{1}{2}dx) = \delta y'(x + \frac{1}{2}dx)$, two integrations by parts give,

$$\int_{\xi=x-\frac{1}{2}dx}^{\xi=x+\frac{1}{2}dx} \frac{\partial F}{\partial y''}\bigg|_{y_0} \delta y''\, d\xi = \underbrace{\left[\left(\frac{\partial F}{\partial y''}\right)\bigg|_{y_0} \delta y'(\xi)\right]_{x-\frac{1}{2}dx}^{x+\frac{1}{2}dx}}_{=0} - \int_{\xi=x-\frac{1}{2}dx}^{\xi=x+\frac{1}{2}dx} \frac{d}{d\xi}\left(\frac{\partial F}{\partial y''}\right)\bigg|_{y_0} \delta y'(\xi) d\xi$$

$$= -\underbrace{\left[\left(\frac{\partial F}{\partial y''}\right)\bigg|_{y_0} \delta y(\xi)\right]_{x-\frac{1}{2}dx}^{x+\frac{1}{2}dx}}_{=0} + \int_{\xi=x-\frac{1}{2}dx}^{\xi=x+\frac{1}{2}dx} \frac{d^2}{(d\xi)^2}\left(\frac{\partial F}{\partial y''}\right)\bigg|_{y_0} \delta y(\xi) d\xi$$

Hence,

$$dJ(y_0 + \delta y) - dJ(y_0) =$$

$$= \int_{\xi=x-\frac{1}{2}dx}^{\xi=x+\frac{1}{2}dx} \left\{\frac{\partial F}{\partial y}\bigg|_{y_0} - \frac{d}{d\xi}\left(\frac{\partial F}{\partial y'}\right)\bigg|_{y_0} + \frac{d^2}{(d\xi)^2}\left(\frac{\partial F}{\partial y''}\right)\bigg|_{y_0}\right\}\delta y(\xi) d\xi + O(o_\delta^2).$$

The integration sum over the infinitesimal interval yields $\left\{\frac{\partial F}{\partial y}\bigg|_{y_0} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\bigg|_{y_0} + \frac{d^2}{(d\xi)^2}\left(\frac{\partial F}{\partial y''}\right)\bigg|_{y_0}\right\}\delta y(x)dx$, and we obtain

$$dJ(y_0 + \delta y) - dJ(y_0) = \left\{\frac{\partial F}{\partial y}\bigg|_{y_0} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\bigg|_{y_0} + \frac{d^2}{(d\xi)^2}\left(\frac{\partial F}{\partial y''}\right)\bigg|_{y_0}\right\}\delta y(x)dx + O(o_\delta^2)$$

Substituting by 1.1, $\delta y(x) = o_\delta \eta(x)$,

$$dI(o_\delta) - dI(0) = \left\{\frac{\partial F}{\partial y}\bigg|_{y_0} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\bigg|_{y_0} + \frac{d^2}{(d\xi)^2}\left(\frac{\partial F}{\partial y''}\right)\bigg|_{y_0}\right\}o_\delta \eta(x)dx + O(o_\delta^2),$$

$$\frac{dI(o_\delta) - dI(0)}{o_\delta} = \left\{ \frac{\partial F}{\partial y}\bigg|_{y_0} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\bigg|_{y_0} + \frac{d^2}{(d\xi)^2}\left(\frac{\partial F}{\partial y''}\right)\bigg|_{y_0} \right\} \eta(x)dx + O(o_\delta).$$

Since this hold for any infinitesimal o_δ, $dI(\varepsilon)$ has a derivative at $\varepsilon = 0$, which is,

$$dI'(0) = \left\{ \frac{\partial F}{\partial y}\bigg|_{y_0} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\bigg|_{y_0} + \frac{d^2}{(d\xi)^2}\left(\frac{\partial F}{\partial y''}\right)\bigg|_{y_0} \right\} \eta(x)dx.$$

Since $dI(0) = dJ(y_0)$ is minimal over the infinitesimal interval, we have by Fermat Theorem, $dI'(0) = 0$, and since $dx > 0$,

$$\left\{ \frac{\partial F}{\partial y}\bigg|_{y_0} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\bigg|_{y_0} + \frac{d^2}{(d\xi)^2}\left(\frac{\partial F}{\partial y''}\right)\bigg|_{y_0} \right\} \eta(x) = 0.$$

Since $\eta(x)$ is an arbitrary smooth enough function,

$$\frac{\partial F}{\partial y}\bigg|_{y=y_0} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\bigg|_{y=y_0} + \frac{d^2}{(d\xi)^2}\left(\frac{\partial F}{\partial y''}\right)\bigg|_{y_0} = 0.\square$$

From the derivation of the Euler's equation,

7.2 The First variation of $F(x,y,y',y'')$ is

$$\delta F(x,y,y',y'') = \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' + \frac{\partial F}{\partial y''}\delta y''$$

$$= \delta y \left\{ \frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} + \frac{d^2}{(dx)^2}\left(\frac{\partial F}{\partial y''} \right) \right\}$$

$$= (\delta y)(\partial_y - D_x \partial_{y'} + D_x^2 \partial_{y''})F .$$

Therefore, we define

7.3 The First Variational Derivative of $F(x, y, y', y'')$

$$\frac{\delta F}{\delta y} = \frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} + \frac{d^2}{(dx)^2}\left(\frac{\partial F}{\partial y''} \right)$$

$$= (\partial_y - D_x \partial_{y'} + D_x^2 \partial_{y''})F$$

7.4 $$\delta \int_{\xi=a}^{\xi=x} F(\xi, y, y', y'') d\xi = \int_{\xi=a}^{\xi=x} \delta F(\xi, y, y', y'') dx .$$

7.5 <u>If</u> $y_0(x)$ minimizes $\displaystyle\int_{x=a}^{x=b} F(x, y(x), y'(x), y''(x))dx$

<u>Then</u> $$\left.\frac{\delta F}{\delta y}\right|_{y_0} = 0 ,$$

$$\left.\delta F\right|_{y_0} = 0 ,$$

$$\left.\delta J\right|_{y_0} = 0 .$$

7.6 The Second Variation of $F(x, y, y', y'')$

$$\delta^2 F(x,y,y',y'') = \left((\delta y)\frac{\partial}{\partial y} + (\delta y')\frac{\partial}{\partial y'} + (\delta y'')\frac{\partial}{\partial y''} \right)^2 F$$

$$= (\delta y)^2 \frac{\partial^2 F}{(\partial y)^2} + (\delta y')^2 \frac{\partial^2 F}{(\partial y')^2} + (\delta y'')^2 \frac{\partial^2 F}{(\partial y'')^2} +$$

$$+2(\delta y)(\delta y')\frac{\partial^2 F}{(\partial y)(\partial y')} + 2(\delta y)(\delta y'')\frac{\partial^2 F}{(\partial y)(\partial y'')} + 2(\delta y')(\delta y'')\frac{\partial^2 F}{(\partial y')(\partial y'')}$$

8.

Euler's Equation for $F(x, y_1, y_2, y_1{}', y_2{}')$ and the Variational Derivative

The infinitesimal distance in 3 dimensional space is

$$ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{1 + (y')^2 + (z')^2}dx\,.$$

The arc-length of the curves $y(x)$, and $z(x)$, between $(a, y(a), z(a))$, and $(b, y(b), z(b))$ is

$$\int_{x=a}^{x=b} \sqrt{1 + (y')^2 + (z')^2}dx\,,$$

and the curves $y(x)$, and $z(x)$ that minimize the integral have the shortest arc-length.

These curves are the line between $(a, y(a), z(a))$, and $(b, y(b), z(b))$, over the interval $[a, b]$.

Then, the problem of the Calculus of Variations is to find curves

$$y_1(x) = y_{1_0}(x),$$

$$y_2(x) = y_{2_0}(x)$$

that will minimize the Functional

$$J(y_1, y_2) = \int_{x=a}^{x=b} F(x, y_1, y_2, y_1', y_2')dx .$$

We apply variations, over infinitesimal interval, to derive Euler's Variational Equation in Infinitesimal Variational Calculus.

8.1 Euler's Equation for $F(x, y_1, y_2, y_1', y_2')$

<u>If</u> $y_{1_0}(x)$, and $y_{2_0}(x)$ minimize $J(y_1, y_2) = \int_{x=a}^{x=b} F(x, y_1, y_2, y_1', y_2')dx$

<u>Then</u>
$$\left.\frac{\partial F}{\partial y_1}\right|_{y_{1_0}} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_1'}\right)\Bigg|_{y_{1_0}} = 0,$$

$$\left.\frac{\partial F}{\partial y_2}\right|_{y_{2_0}} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_2'}\right)\Bigg|_{y_{2_0}} = 0$$

Proof:

Since by 1.5, $\delta y_1(x)$, and $\delta y_2(x)$ do not depend on the size of the x-interval, we will use an infinitesimal interval

$$[x - \tfrac{1}{2}dx, x + \tfrac{1}{2}dx], \qquad a < x < b .$$

Then, $y_{1_0}(x)$, and $y_{2_0}(x)$ minimize

$$dI(o_\delta) = dJ(y_{1_0} + \delta y_1, y_{2_0} + \delta y_2) =$$

$$= \int_{\xi=x-\frac{1}{2}dx}^{\xi=x+\frac{1}{2}dx} F(\xi, y_{1_0} + \delta y_1, y_{2_0} + \delta y_2, y_{1_0}' + \delta y_1', y_{2_0}' + \delta y_2')d\xi,$$

over all the curves $y_{1_0} + \delta y_1$, and $y_{2_0} + \delta y_2$, that cross $y_{1_0}(\xi)$ and $y_{2_0}(\xi)$ at the endpoints of $[x - \frac{1}{2}dx, x + \frac{1}{2}dx]$. That is,

$$\delta y_1(x - \tfrac{1}{2}dx) = \delta y_1(x + \tfrac{1}{2}dx),$$

$$\delta y_2(x - \tfrac{1}{2}dx) = \delta y_2(x + \tfrac{1}{2}dx).$$

The Taylor expansion to second order of

$$F(\xi, y_{1_0}(\xi) + \delta y_1, y_{2_0}(\xi) + \delta y_2, y_{1_0}'(\xi) + \delta y_1', y_{2_0}'(\xi) + \delta y_2'),$$

about the curves $y_{1_0}(\xi)$, and $y_{2_0}(\xi)$ is

$$\left.\frac{\partial F}{\partial y_1}\right|_{y_{1_0}} \delta y_1 + \left.\frac{\partial F}{\partial y_1'}\right|_{y_{1_0}} \delta y_1' + \left.\frac{\partial F}{\partial y_2}\right|_{y_{2_0}} \delta y_2 + \left.\frac{\partial F}{\partial y_2'}\right|_{y_{2_0}} \delta y_2' + O(o_\delta^2).$$

Therefore,

$$dJ(y_{1_0} + \delta y_1, y_{2_0} + \delta y_2) - dJ(y_{1_0}, y_{2_0}) =$$

$$= \int_{\xi=x-\frac{1}{2}dx}^{\xi=x+\frac{1}{2}dx} \left\{ \left.\frac{\partial F}{\partial y_1}\right|_{y_{1_0}} \delta y_1 + \left.\frac{\partial F}{\partial y_1'}\right|_{y_{1_0}} \delta y_1' + \left.\frac{\partial F}{\partial y_2}\right|_{y_{2_0}} \delta y_2 + \left.\frac{\partial F}{\partial y_2'}\right|_{y_{2_0}} \delta y_2' \right\} dx + O(o_\delta^2)$$

By 1.8, $\delta y' = \dfrac{d}{dx}\delta y$, and integrating by parts,

$$dJ(y_{1_0} + \delta y_1, y_{2_0} + \delta y_2) - dJ(y_{1_0}, y_{2_0}) =$$

$$= \int_{\xi=x-\frac{1}{2}dx}^{\xi=x+\frac{1}{2}dx} \left\{ \frac{\partial F}{\partial y_1}\bigg|_{y_{1_0}} - \frac{d}{d\xi}\left(\frac{\partial F}{\partial y_1 '}\right)\bigg|_{y_{1_0}} \right\} \delta y_1(\xi) d\xi + \underbrace{\left[\left(\frac{\partial F}{\partial y_1 '}\right)\bigg|_{y_{1_0}} \delta y_1(\xi) \right]_{x-\frac{1}{2}dx}^{x+\frac{1}{2}dx}}_{=0} +$$

$$+ \int_{\xi=x-\frac{1}{2}dx}^{\xi=x+\frac{1}{2}dx} \left\{ \frac{\partial F}{\partial y_2}\bigg|_{y_{2_0}} - \frac{d}{d\xi}\left(\frac{\partial F}{\partial y_2 '}\right)\bigg|_{y_{2_0}} \right\} \delta y_2(\xi) d\xi + \underbrace{\left[\left(\frac{\partial F}{\partial y_2 '}\right)\bigg|_{y_{2_0}} \delta y_1(\xi) \right]_{x-\frac{1}{2}dx}^{x+\frac{1}{2}dx}}_{=0} + O(o_\delta^2)$$

The integration over the infinitesimal interval yields only the terms,

$$\left\{ \frac{\partial F}{\partial y_1}\bigg|_{y_{1_0}} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_1 '}\right)\bigg|_{y_{1_0}} \right\} \delta y_1(x) dx + \left\{ \frac{\partial F}{\partial y_2}\bigg|_{y_{2_0}} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_2 '}\right)\bigg|_{y_{2_0}} \right\} \delta y_2(x) dx .$$

Substituting by 1.1,

$$\delta y_1(x) = o_\delta \eta_1(x),$$

$$\delta y_2(x) = o_\delta \eta_2(x),$$

$$dI(o_\delta) - dI(0) =$$

$$= \left(\left\{ \frac{\partial F}{\partial y_1}\bigg|_{y_{1_0}} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_1 '}\right)\bigg|_{y_{1_0}} \right\} \eta_1(x) + \left\{ \frac{\partial F}{\partial y_2}\bigg|_{y_{2_0}} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_2 '}\right)\bigg|_{y_{2_0}} \right\} \eta_2(x) \right) o_\delta dx + O(o_\delta^2)$$

$$\frac{dI(o_\delta) - dI(0)}{o_\delta} =$$

$$= \left(\left\{ \frac{\partial F}{\partial y_1}\bigg|_{y_{1_0}} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_1 '}\right)\bigg|_{y_{1_0}} \right\} \eta_1(x) + \left\{ \frac{\partial F}{\partial y_2}\bigg|_{y_{2_0}} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_2 '}\right)\bigg|_{y_{2_0}} \right\} \eta_2(x) \right) dx + O(o_\delta)$$

Since this hold for any infinitesimal o_δ, $dI(\xi)$ has a derivative at $\xi = 0$, which is,

$$dI'(0) = \left(\left\{ \left.\frac{\partial F}{\partial y_1}\right|_{y_{1_0}} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_1'} \right)\right|_{y_{1_0}} \right\} \eta_1(x) + \left\{ \left.\frac{\partial F}{\partial y_2}\right|_{y_{2_0}} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_2'} \right)\right|_{y_{2_0}} \right\} \eta_2(x) \right) dx\,.$$

Since $dI(0) = dJ(y_{1_0}, y_{2_0})$ is minimal, we have by Fermat Theorem, $dI'(0) = 0$, and since $dx > 0$,

$$\left\{ \left.\frac{\partial F}{\partial y_1}\right|_{y_{1_0}} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_1'} \right)\right|_{y_{1_0}} \right\} \eta_1(x) + \left\{ \left.\frac{\partial F}{\partial y_2}\right|_{y_{2_0}} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_2'} \right)\right|_{y_{2_0}} \right\} \eta_2(x) = 0\,.$$

Since $\eta_1(x)$ and $\eta_2(x)$ arbitrary smooth enough functions, independent of each other,

$$\left.\frac{\partial F}{\partial y_1}\right|_{y_{1_0}} - \left.\frac{d}{dx}\left(\frac{\partial F}{\partial y_1'} \right)\right|_{y_{1_0}} = 0,$$

$$\left.\frac{\partial F}{\partial y_2}\right|_{y_{2_0}} - \left.\frac{d}{dx}\left(\frac{\partial F}{\partial y_2'} \right)\right|_{y_{2_0}} = 0\,.\square$$

For the shortest distance in 3 dimensional space, the equations are

$$\frac{y'}{\sqrt{1 + (y')^2 + (z')^2}} = C_1,$$

$$\frac{z'}{\sqrt{1 + (y')^2 + (z')^2}} = C_2.$$

Hence,

$$\frac{z'}{y'} = C,$$

$$dz = Cdy,$$

which yields a line.

From the derivation of Euler's equation,

8.2 The First Variation of $F(x, y_1, y_2, y_1', y_2')$ is

$$\delta F = \frac{\partial F}{\partial y_1}\delta y_1 + \frac{\partial F}{\partial y_1'}\delta y_1' + \frac{\partial F}{\partial y_2}\delta y_2 + \frac{\partial F}{\partial y_2'}\delta y_2'$$

$$= \left\{\frac{\partial F}{\partial y_1} - \frac{d}{dx}\frac{\partial F}{\partial y_1'}\right\}\delta y_1 + \left\{\frac{\partial F}{\partial y_2} - \frac{d}{dx}\frac{\partial F}{\partial y_2'}\right\}\delta y_2.$$

$$= \left\{\delta y_1(\partial_{y_1} - D_x\partial_{y_1'}) + \delta y_2(\partial_{y_2} - D_x\partial_{y_2'})\right\}F.$$

Therefore, we define

8.3 First Variational Derivatives of $F(x, y_1, y_2, y_1', y_2')$

$$\frac{\delta F}{\delta y_1} = \frac{\partial F}{\partial y_1} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_1'}\right),$$

$$\frac{\delta F}{\delta y_2} = \frac{\partial F}{\partial y_2} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_2'}\right).$$

8.4 $$\delta \int_{\xi=a}^{\xi=x} F(\xi, y_1, y_2, y_1', y_2')dx = \int_{\xi=a}^{\xi=x} \delta F(\xi, y_1, y_2, y_1', y_2')d\xi .$$

8.5 <u>If</u> $y_{1_0}(x)$, and $y_{2_0}(x)$ minimize

$$J(y_1, y_2) = \int_{x=a}^{x=b} F(x, y_1, y_2, y_1', y_2')dx$$

<u>Then</u>
$$\left.\frac{\delta F}{\delta y_1}\right|_{y_{1_0}, y_{2_0}} = 0, \quad \left.\frac{\delta F}{\delta y_2}\right|_{y_{1_0}, y_{2_0}} = 0$$

$$\left.\delta F\right|_{y_{1_0}, y_{2_0}} = 0,$$

$$\left.\delta J\right|_{y_{1_0}, y_{2_0}} = 0.$$

8.6 The Second Variation of $F(x, y_1, y_2, y_1', y_2')$

$$\delta^2 F = \left\{\left[(\delta y_1)\partial_{y_1} + (\delta y_1')\partial_{y_1'}\right]^2 + \left[(\delta y_2)\partial_{y_2} + (\delta y_2')\partial_{y_2'}\right]^2\right\}F$$

$$= \left\{(\delta y_1)^2 \partial_{y_1}^2 + 2(\delta y_1)(\delta y_1')\partial_{y_1}\partial_{y_1'} + (\delta y_1')^2 \partial_{y_1'}^2\right\}F$$

$$+\left\{(\delta y_2)^2 \partial_{y_2}^2 + 2(\delta y_2)(\delta y_2')\partial_{y_2}\partial_{y_2'} + (\delta y_2')^2 \partial_{y_2'}^2\right\}F$$

9.

Euler's Equation for $f(t,x,y,\dot{x},\dot{y})$ and the Variational Derivative

The infinitesimal distance in the plane is

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(\tfrac{dx}{dt})^2 + (\tfrac{dy}{dt})^2}\,dt\,.$$

The arc-length of the curve with coordinates $x(t)$, and $y(t)$, between $(x(t_0), y(t_0))$, and $(x(t_1), y(t_1))$ is

$$\int_{t=t_0}^{t=t_1} \sqrt{(\tfrac{dx}{dt})^2 + (\tfrac{dy}{dt})^2}\,dt\,,$$

and the curve with $x(t)$, and $y(t)$ that minimize the integral has the shortest arc-length, between $(x(t_0), y(t_0))$, and $(x(t_1), y(t_1))$, over the interval $[t_0, t_1]$.

The Parametric problem is to find the curve with

$$x(t) = x_0(t)\,,$$

$$y(t) = y_0(t)$$

that will minimize the Functional $K(x,y) = \int_{t=t_0}^{t=t_1} f(t,x,y,\dot{x},\dot{y})dt$.

By 8.1,

9.1 Euler's Equation for $f(t,x,y,\dot{x},\dot{y})$

<u>If</u> $x_0(t)$, and $y_0(t)$ minimize $K(x,y) = \int_{t=t_0}^{t=t_1} f(t,x,y,\dot{x},\dot{y})dt$

<u>Then</u>
$$\left.\frac{\partial f}{\partial x}\right|_{x_0,y_0} - \frac{d}{dt}\left.\left(\frac{\partial f}{\partial \dot{x}}\right)\right|_{x_0,y_0} = 0,$$

$$\left.\frac{\partial f}{\partial y}\right|_{x_0,y_0} - \frac{d}{dt}\left.\left(\frac{\partial f}{\partial \dot{y}}\right)\right|_{x_0,y_0} = 0$$

For the shortest distance in the plane, the equations are

$$\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = C_1,$$

$$\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = C_1.$$

Hence,

$$\frac{\dot{y}}{\dot{x}} = C,$$

$$dy = Cdx,$$

which yields a line.

From the derivation of Euler's equation,

8.2 The First Variation of $f(t,x,y,\dot{x},\dot{y})$ **is**

$$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial \dot{x}}\delta \dot{x} + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial \dot{y}}\delta \dot{y}$$

$$= \left\{\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}}\right\}\delta x + \left\{\frac{\partial f}{\partial y} - \frac{d}{dt}\frac{\partial f}{\partial \dot{y}}\right\}\delta y\,.$$

$$= \left\{\delta x(\partial_x - D_t\partial_{\dot{x}}) + \delta y(\partial_y - D_t\partial_{\dot{y}})\right\} f\,.$$

Therefore, we define

9.3 First Variational Derivatives of $f(t, x, y, \dot{x}, \dot{y})$

$$\frac{\delta f}{\delta x} = \frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}},$$

$$\frac{\delta f}{\delta y} = \frac{\partial f}{\partial y} - \frac{d}{dt}\frac{\partial f}{\partial \dot{y}}.$$

9.4 $$\delta \int_{\tau=t_0}^{\tau=t} f(\tau, x(\tau), y(\tau), \dot{x}(\tau), \dot{y}(\tau)d\tau = \int_{\tau=t_0}^{\tau=t} \delta f(\tau, x(\tau), y(\tau), \dot{x}(\tau), \dot{y}(\tau))d\tau\,.$$

9.5 <u>If</u> $x_0(t)$, $y_0(t)$ **minimize** $K(x, y) = \int_{t=t_0}^{t=t_1} f(t, x, y, \dot{x}, \dot{y})dt$

<u>Then</u> $$\left.\frac{\delta f}{\delta x}\right|_{x_0, y_0} = 0\,, \quad \left.\frac{\delta f}{\delta y}\right|_{x_0, y_0} = 0$$

$$\left.\delta f\right|_{x_0, y_0} = 0\,,$$

$$\delta K\Big|_{x_0, y_0} = 0\,.$$

9.6 The Second Variation of $f(t, x, y, \dot{x}, \dot{y})$

$$\delta^2 f = \left\{\left[(\delta x)\partial_x + (\delta \dot{x})\partial_{\dot{x}}\right]^2 + \left[(\delta y)\partial_y + (\delta \dot{y})\partial_{\dot{y}}\right]^2\right\} f$$

$$= \left\{(\delta x)^2 \partial_x^2 + 2(\delta x)(\delta \dot{x})\partial_x \partial_{\dot{x}} + (\delta \dot{x})^2 \partial_{\dot{x}}^2\right\} f$$

$$+ \left\{(\delta y)^2 \partial_y^2 + 2(\delta y)(\delta \dot{y})\partial_y \partial_{\dot{y}} + (\delta \dot{y})^2 \partial_{\dot{y}}^2\right\} f$$

10.

Weierstrass Sufficient Condition, for $\int_{t=t_0}^{t=t_1} f(t,x,y,\dot{x},\dot{y})dt$ to be Minimal

Weierstrass sufficient condition for a parametric curve, can be derived by translation of the condition for the curve $y(x)$.

But it follows immediately from the Euler Identity for $f(t,x,y,\dot{x},\dot{y})$.

For $t = \lambda\tau$,

$$f(x(t),y(t),\lambda\dot{x}(t),\lambda\dot{y}(t))dt = \lambda f(x(\tau),y(\tau),\dot{x}(\tau),\dot{y}(\tau))d\tau$$

Differentiating both sides with respect to λ,

$$\frac{\partial f}{\partial(\lambda\dot{x}(t))}\frac{\partial(\lambda\dot{x}(t))}{\partial\lambda}dt + \frac{\partial f}{\partial(\lambda\dot{y}(t))}\frac{\partial(\lambda\dot{y}(t))}{\partial\lambda}dt = f(x(\tau),y(\tau),\dot{x}(\tau),\dot{y}(\tau))d\tau$$

$$\underbrace{\frac{\partial f}{\partial(\lambda\dot{x}(t))}}_{\frac{1}{\lambda}\frac{\partial f}{\partial\dot{x}(t)}}\underbrace{\frac{\partial(\lambda\dot{x}(t))}{\partial\lambda}}_{\dot{x}(t)}\lambda d\tau + \underbrace{\frac{\partial f}{\partial(\lambda\dot{y}(t))}}_{\frac{1}{\lambda}\frac{\partial f}{\partial\dot{y}(t)}}\underbrace{\frac{\partial(\lambda\dot{y}(t))}{\partial\lambda}}_{\dot{y}(t)}\lambda d\tau = f(x(\tau),y(\tau),\dot{x}(\tau),\dot{y}(\tau))d\tau$$

and we obtain

10.1 Euler's Identity

$$t = \lambda\tau \Rightarrow \frac{\partial f}{\partial \dot{x}(t)}\dot{x}(t) + \frac{\partial f}{\partial \dot{y}(t)}\dot{y}(t) = f(x(\tau), y(\tau), \dot{x}(\tau), \dot{y}(\tau))$$

10.2 Weierstrass Sufficient Condition for Minimum

$$f(x_0, y_0, \dot{x}, \dot{y}) - \dot{x}\left.\frac{\partial f}{\partial \dot{x}}\right|_{x_0, y_0, \dot{x}_0, \dot{y}_0} - \dot{y}\left.\frac{\partial f}{\partial \dot{y}}\right|_{x_0, y_0, \dot{x}_0, \dot{y}_0} > 0,$$

or,

$$\dot{x}\left(\left.\frac{\partial f}{\partial \dot{x}}\right|_{x_0, y_0, \dot{x}, \dot{y}} - \left.\frac{\partial f}{\partial \dot{x}}\right|_{x_0, y_0, \dot{x}_0, \dot{y}_0}\right) + \dot{y}\left(\left.\frac{\partial f}{\partial \dot{y}}\right|_{x_0, y_0, \dot{x}, \dot{y}} - \left.\frac{\partial f}{\partial \dot{y}}\right|_{x_0, y_0, \dot{x}_0, \dot{y}_0}\right) > 0.$$

Proof:

Let $x_0(\tau), y_0(\tau)$, the parametric extremal that solves the Euler equation, intersect the parametric curve $x(\tau), y(\tau)$ at τ, so that $x(\tau) = x_0(\tau)$, and $y(\tau) = y_0(\tau)$.

Using the Euler Identity in 10.1, either inequality ensures

$$f(x_0(\tau), y_0(\tau), \dot{x}(\tau), \dot{y}(\tau)) - f(x_0(\tau), y_0(\tau), \dot{x}_0(\tau), \dot{y}_0(\tau)) > 0 . \square$$

10.3 The shortest distance

$$f(x, y, \dot{x}, \dot{y}) = \sqrt{\dot{x}^2 + \dot{y}^2}$$

Weierstrass sufficient condition

$$\sqrt{\dot{x}^2 + \dot{y}^2} - \dot{x}\frac{\dot{x}_0}{\sqrt{\dot{x}_0^2 + \dot{y}_0^2}} - \dot{y}\frac{\dot{y}_0}{\sqrt{\dot{x}_0^2 + \dot{y}_0^2}} > 0\,,$$

holds if

$$(\dot{x}^2 + \dot{y}^2)(\dot{x}_0^2 + \dot{y}_0^2) > (\dot{x}\dot{x}_0 + \dot{y}\dot{y}_0)^2$$

which is guaranteed to hold since

$$(\dot{x}_0\dot{y} - \dot{x}\dot{y}_0)^2 > 0\,.\square$$

11.

Euler-Lagrange Equations for $\mathcal{L}(t, q_1(t), q_2(t), \dot{q}_1(t), \dot{q}_2(t))$, and the Variational Derivative

At time t, a particle with mass m, and speed $(\dot{x}, \dot{y}, \dot{z})$, has kinetic energy $\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$, potential energy $U(x, y, z)$, and Lagrangian $\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z)$.

Its path of least action is the curve $(x_0(t), y_0(t), z_0(t))$ that minimizes the Action Integral, $\int\limits_{t=t_0}^{t=t_1} \mathcal{L}\, dt$.

The Euler equation of 7.1 with respect to x is

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}},$$

$$-\frac{\partial U}{\partial x} = \frac{d}{dt}(m\dot{x}),$$

$$F_x = \frac{dp_x}{dt},$$

Similarly, $F_y = \dfrac{dp_y}{dt}$, and $F_z = \dfrac{dp_z}{dt}$.

Therefore, the path of least action satisfies

$$\vec{F} = \frac{d\vec{p}}{dt}$$

which is Newton's Force Law.

For two masses, m_1 with kinetic energy $\frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2)$, and m_2 with kinetic energy $\frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2)$, in a potential $U(x_1, y_1, z_1, x_2, y_2, z_2)$, the Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \tfrac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2) - U(x_1, y_1, z_1, x_2, y_2, z_2).$$

The possible paths may depend on say, two generalized coordinates q_1, q_2, such as distances, and angles, so that

$$x_1 = x_1(q_1, q_2); \quad x_2 = x_2(q_1, q_2)$$

$$y_1 = y_1(q_1, q_2); \quad y_2 = y_2(q_1, q_2)$$

$$z_1 = z_1(q_1, q_2); \quad z_2 = z_2(q_1, q_2).$$

Then,

$$\frac{dx_1}{dt} = \frac{\partial x_1}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial x_1}{\partial q_2}\frac{dq_2}{dt}; \qquad \frac{dx_2}{dt} = \frac{\partial x_2}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial x_2}{\partial q_2}\frac{dq_2}{dt}$$

$$\frac{dy_1}{dt} = \frac{\partial y_1}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial y_1}{\partial q_2}\frac{dq_2}{dt}; \qquad \frac{dy_2}{dt} = \frac{\partial y_2}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial y_2}{\partial q_2}\frac{dq_2}{dt}$$

$$\frac{dz_1}{dt} = \frac{\partial z_1}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial z_1}{\partial q_2}\frac{dq_2}{dt}; \qquad \frac{dz_2}{dt} = \frac{\partial z_2}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial z_2}{\partial q_2}\frac{dq_2}{dt}$$

and the Lagrangian is

$$\mathcal{L} = \left[m_1\left\{\left(\frac{\partial x_1}{\partial q_1}\right)^2 + \left(\frac{\partial y_1}{\partial q_1}\right)^2 + \left(\frac{\partial z_1}{\partial q_1}\right)^2\right\} + m_2\left\{\left(\frac{\partial x_2}{\partial q_1}\right)^2 + \left(\frac{\partial y_2}{\partial q_1}\right)^2 + \left(\frac{\partial z_2}{\partial q_1}\right)^2\right\}\right]\dot{q}_1^2$$

$$+\left[m_1\left\{\left(\frac{\partial x_1}{\partial q_2}\right)^2 + \left(\frac{\partial y_1}{\partial q_2}\right)^2 + \left(\frac{\partial z_1}{\partial q_2}\right)^2\right\} + m_2\left\{\left(\frac{\partial x_2}{\partial q_2}\right)^2 + \left(\frac{\partial y_2}{\partial q_2}\right)^2 + \left(\frac{\partial z_2}{\partial q_2}\right)^2\right\}\right]\dot{q}_2^2$$

$$-U(q_1, q_2).$$

Thus, the Lagrangian depends on $q_1, q_2, \dot{q}_1, \dot{q}_2$. That is,

$$\mathcal{L} = \mathcal{L}(t, q_1(t), q_2(t), \dot{q}_1(t), \dot{q}_2(t)).$$

The Euler-Lagrange problem is to find the curve $q_{1_0}(t), q_{2_0}(t)$,

that minimizes the Action $\int_{t=t_0}^{t=t_1} \mathcal{L}\,dt$, over the interval $[t_0, t_1]$.

By 9.1, we have,

11.1 Euler-Lagrange Equation

If $q_{1_0}(t), q_{2_0}(t)$ minimizes $\int_{t=t_0}^{t=t_1} \mathcal{L}(t, q_1(t), q_2(t), \dot{q}_1(t), \dot{q}_2(t))dt$

Then
$$\left.\frac{\partial \mathcal{L}}{\partial q_1}\right|_{q_{1_0}, q_{2_0}} - \frac{d}{dt}\left.\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right)\right|_{q_{1_0}, q_{2_0}} = 0,$$

$$\left.\frac{\partial \mathcal{L}}{\partial q_2}\right|_{q_{1_0},q_{2_0}} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_2}\right)\Bigg|_{q_{1_0},q_{2_0}} = 0$$

By 9.2,

11.2 The First Variation of $\mathcal{L}(t, q_1, q_2, \dot{q}_1, \dot{q}_2)$

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial q_1}\delta q_1 + \frac{\partial \mathcal{L}}{\partial \dot{q}_1}\delta \dot{q}_1 + \frac{\partial \mathcal{L}}{\partial q_2}\delta q_2 + \frac{\partial \mathcal{L}}{\partial \dot{q}_2}\delta \dot{q}_2$$

$$= \delta q_1\left\{\frac{\partial \mathcal{L}}{\partial q_1} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right\} + \delta q_2\left\{\frac{\partial \mathcal{L}}{\partial q_2} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2}\right\}.$$

$$= \left\{\delta q_1(\partial_{q_1} - D_t\partial_{\dot{q}_1}) + \delta q_2(\partial_{q_2} - D_t\partial_{\dot{q}_2})\right\}\mathcal{L}\,.$$

Therefore, we define

11.3 First Variational Derivatives of $\mathcal{L}(t, q_1, q_2, \dot{q}_1, \dot{q}_2)$

$$\frac{\delta \mathcal{L}}{\delta q_1} = \frac{\partial \mathcal{L}}{\partial q_1} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1},$$

$$\frac{\delta \mathcal{L}}{\delta q_2} = \frac{\partial \mathcal{L}}{\partial q_2} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_{12}}.$$

11.4 $\delta\int\limits_{\tau=t_0}^{\tau=t}\mathcal{L}(\tau, q_1(\tau), q_2(\tau), \dot{q}_1(\tau), \dot{q}_2(\tau))d\tau = \int\limits_{\tau=t_0}^{\tau=t}\delta\mathcal{L}(\tau, q_1(\tau), q_2(\tau), \dot{q}_1(\tau), \dot{q}_2(\tau))d\tau\,.$

11.5 <u>If</u> $q_{1_0}(t), q_{2_0}(t)$ minimizes $K(q_1, q_2) = \int\limits_{t=t_0}^{t=t_1} \mathcal{L}(t, q_1(t), q_2(t), \dot{q}_1(t), \dot{q}_2(t))dt$

<u>Then</u>
$$\left.\frac{\delta \mathcal{L}}{\delta q_1}\right|_{q_{1_0}, q_{2_0}} = 0, \qquad \left.\frac{\delta \mathcal{L}}{\delta q_2}\right|_{q_{1_0}, q_{2_0}} = 0$$

$$\left.\delta \mathcal{L}\right|_{q_{1_0}, q_{2_0}} = 0,$$

$$\left.\delta \int\limits_{t=t_0}^{t=t_1} \mathcal{L} dt\right|_{q_{1_0}, q_{2_0}} = 0.$$

11.6 The Second Variation of $\mathcal{L}(t, q_1, q_2, \dot{q}_1, \dot{q}_2)$

$$\delta^2 \mathcal{L} = \left\{\left[(\delta q_1)\partial_{q_1} + (\delta \dot{q}_1)\partial_{\dot{q}_1}\right]^2 + \left[(\delta q_2)\partial_{q_2} + (\delta \dot{q}_2)\partial_{\dot{q}_2}\right]^2\right\}\mathcal{L}$$

$$= \left\{(\delta q_1)^2 \partial_{q_1}^2 + 2(\delta q_1)(\delta \dot{q}_1)\partial_{q_1}\partial_{\dot{q}_1} + (\delta \dot{q}_1)^2 \partial_{\dot{q}_1}^2\right\}\mathcal{L}$$

$$+ \left\{(\delta q_2)^2 \partial_{q_2}^2 + 2(\delta q_2)(\delta \dot{q}_2)\partial_{q_2}\partial_{\dot{q}_2} + (\delta \dot{q}_2)^2 \partial_{\dot{q}_2}^2\right\}\mathcal{L}$$

12.

Infinitesimal Surface Variation

Denoting

the variation of the surface $y(x_1, x_2)$ by $\delta y(x_1, x_2)$,

an infinitesimal by o_δ,

and a smooth enough hyper-real function by $\eta(x_1, x_2)$

we define

12.1 The Infinitesimal Surface variation

$$\delta y(x_1, x_2) = o_\delta \eta(x_1, x_2).$$

where,

12.2 *We use the same* o_δ *for all surface variations.*

From the definition 8.1, it is clear that

12.3 *Surface variation does not depend on* dx_1, *or* dx_2

12.4 *The surface* $y(x_1, x_2) + \delta y(x_1, x_2)$ *is infinitesimally-close to the surface* $y(x_1, x_2)$.

In particular,

12.5 $\delta y(x_1, x_2)$ *does not depend on* dx_1 *or* dx_2

Similarly to 1.6,

12.6 δ *is a linear operator*

Similarly to 1.7,

12.7 d*, and* δ *commute*

Therefore,

12.8 ∂_{x_1}*, and* δ *commute.*

∂_{x_2}*, and* δ *commute.*

12.9 <u>If</u> $F(x_1, x_2, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2})$ *is a hyper-real function on a surface* $y(x_1, x_2)$

<u>Then</u>, *its' Taylor series to third order about the surface* $y_0(x_1, x_2)$ *is*

$$F(x_1, x_2, y_0 + \delta y, \frac{\partial y_0}{\partial x_1} + \delta \frac{\partial y}{\partial x_1}, \frac{\partial y_0}{\partial x_2} + \delta \frac{\partial y}{\partial x_2}) =$$

$$= F(x_1, x_2, y_0, \frac{\partial y_0}{\partial x_1}, \frac{\partial y_0}{\partial x_2}) +$$

$$+\left\{ \left.\frac{\partial F}{\partial y}\right|_{y_0} \delta y + \left.\frac{\partial F}{\partial \frac{\partial y}{\partial x_1}}\right|_{y_0} \delta \frac{\partial y}{\partial x_1} + \left.\frac{\partial F}{\partial \frac{\partial y}{\partial x_2}}\right|_{y_0} \delta \frac{\partial y}{\partial x_2} \right\}$$

$$+\left\{ \left.\frac{\partial^2 F}{(\partial y)^2}\right|_{y_0} (\delta y)^2 + 2\left.\frac{\partial^2 F}{\partial y \partial \frac{\partial y}{\partial x_1}}\right|_{y_0} \delta y \delta \frac{\partial y}{\partial x_1} + \left.\frac{\partial^2 F}{(\partial \frac{\partial y}{\partial x_1})^2}\right|_{y_0} (\delta \frac{\partial y}{\partial x_1})^2 \right.$$

$$\left. +2\left.\frac{\partial^2 F}{\partial y \partial \frac{\partial y}{\partial x_2}}\right|_{y_0} \delta y \delta (\frac{\partial y}{\partial x_2}) + 2\left.\frac{\partial^2 F}{\partial \frac{\partial y}{\partial x_1} \partial \frac{\partial y}{\partial x_2}}\right|_{y_0} \delta \frac{\partial y}{\partial x_1} \delta \frac{\partial y}{\partial x_2} + \left.\frac{\partial^2 F}{(\partial \frac{\partial y}{\partial x_2})^2}\right|_{y_0} (\delta \frac{\partial y}{\partial x_2})^2 \right\}$$

$$+O(o_y^3).$$

Define

12.10 the **First Variation** of $F(x_1, x_2, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2})$

$$\delta F = \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial \frac{\partial y}{\partial x_1}} \delta \frac{\partial y}{\partial x_1} + \frac{\partial F}{\partial \frac{\partial y}{\partial x_2}} \delta \frac{\partial y}{\partial x_2}$$

$$= \left\{ \delta y \frac{\partial}{\partial y} + \delta \frac{\partial y}{\partial x_1} \frac{\partial}{\partial \frac{\partial y}{\partial x_1}} + \delta \frac{\partial y}{\partial x_2} \frac{\partial}{\partial \frac{\partial y}{\partial x_2}} \right\} F$$

12.11 $\delta x_1 = \delta x_2 = 0$

Proof: $F(x_1, x_2, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}) = x_1 \Rightarrow \delta F = 0. \square$

Define

12.12 The Second Variation of $F(x_1, x_2, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2})$

$$\delta^2 F = \frac{\partial^2 F}{(\partial y)^2}\bigg|_{y_0} (\delta y)^2 + 2\frac{\partial^2 F}{\partial y \partial \frac{\partial y}{\partial x_1}}\bigg|_{y_0} \delta y \delta \frac{\partial y}{\partial x_1} + \frac{\partial^2 F}{(\partial \frac{\partial y}{\partial x_1})^2}\bigg|_{y_0} (\delta \frac{\partial y}{\partial x_1})^2 +$$

$$+2\frac{\partial^2 F}{\partial y \partial \frac{\partial y}{\partial x_2}}\bigg|_{y_0} \delta y \delta (\frac{\partial y}{\partial x_2}) + 2\frac{\partial^2 F}{\partial \frac{\partial y}{\partial x_1} \partial \frac{\partial y}{\partial x_2}}\bigg|_{y_0} \delta \frac{\partial y}{\partial x_1} \delta \frac{\partial y}{\partial x_2} + \frac{\partial^2 F}{(\partial \frac{\partial y}{\partial x_2})^2}\bigg|_{y_0} (\delta \frac{\partial y}{\partial x_2})^2$$

Just as $d^2 x = 0$, we have

12.13 $$\delta^2 y(x_1, x_2) = 0$$

Proof: $F(x_1, x_2, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}) = y \Rightarrow \delta^2 F = 0$.□

Then, 12.9 becomes

12.14 <u>If</u> $F(x_1, x_2, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2})$ *is hyper-real function on a surface*

$y(x_1, x_2)$

<u>Then</u>, *its' Taylor series to third order about the surface* $y_0(x_1, x_2)$ *is*

$$F(x_1, x_2, y_0 + \delta y, \frac{\partial y_0}{\partial x_1} + \delta\frac{\partial y}{\partial x_1}, \frac{\partial y_0}{\partial x_2} + \delta\frac{\partial y}{\partial x_2}) =$$

$$= F(x_1, x_2, y_0, \frac{\partial y_0}{\partial x_1}, \frac{\partial y_0}{\partial x_2}) + \delta F + \delta^2 F + O(o_y^3)$$

Therefore,

12.15 <u>If</u> $F(x_1, x_2, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2})$ *is hyper-real function on the surface*

$$y(x_1, x_2) \quad \textit{and} \quad J(y) = \int\limits_{x_2=a_2}^{x_2=b_2}\int\limits_{x_1=a_1}^{x_1=b_1} F(x_1, x_2, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2})dx_1dx_2$$

<u>Then</u>, *Taylor series to third order of* $J(y_0 + \delta y)$ *about the surface* $y_0(x_1, x_2)$ *is*

$$J(y_0 + \delta y) = \int\limits_{x_2=a_2}^{x_2=b_2}\int\limits_{x_1=a_1}^{x_1=b_1} F(x_1, x_2, y_0, \frac{\partial y_0}{\partial x_1}, \frac{\partial y_0}{\partial x_2})dx_1dx_2 +$$

$$+\int\limits_{x_2=a_2}^{x_2=b_2}\int\limits_{x_1=a_1}^{x_1=b_1} \delta F(x_1, x_2, y_0, \frac{\partial y_0}{\partial x_1}, \frac{\partial y_0}{\partial x_2})dx_1dx_2$$

$$+\int\limits_{x_2=a_2}^{x_2=b_2}\int\limits_{x_1=a_1}^{x_1=b_1} \delta^2 F(x_1, x_2, y_0, \frac{\partial y_0}{\partial x_1}, \frac{\partial y_0}{\partial x_2})dx_1dx_2$$

$$+O(o_y^3).$$

Define

12.16 The First Variation of $\int\limits_{x_2=a_2}^{x_2=b_2}\int\limits_{x_1=a_1}^{x_1=b_1} F(x_1, x_2, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2})dx_1dx_2$

$$\delta\int\limits_{x_2=a_2}^{x_2=b_2}\int\limits_{x_1=a_1}^{x_1=b_1} F(x_1, x_2, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2})dx_1dx_2 = \int\limits_{x_2=a_2}^{x_2=b_2}\int\limits_{x_1=a_1}^{x_1=b_1} \delta F(x_1, x_2, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2})dx_1dx_2$$

Define

12.17 The Second Variation of $\int\limits_{x_2=a_2}^{x_2=b_2}\int\limits_{x_1=a_1}^{x_1=b_1} F(x_1, x_2, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2})dx_1dx_2$

$$\delta^2\int\limits_{x_2=a_2}^{x_2=b_2}\int\limits_{x_1=a_1}^{x_1=b_1} F(x_1, x_2, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2})dx_1dx_2 = \int\limits_{x_2=a_2}^{x_2=b_2}\int\limits_{x_1=a_1}^{x_1=b_1} \delta^2 F(x_1, x_2, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2})dx_1dx_2 .$$

Then, 12.15 becomes

12.18 <u>If</u> $F(x_1, x_2, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2})$ *is hyper-real function on the surface*

$$y(x_1, x_2) \quad \textit{and} \quad J(y) = \int\limits_{x_2=a_2}^{x_2=b_2}\int\limits_{x_1=a_1}^{x_1=b_1} F(x_1, x_2, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2})dx_1dx_2$$

<u>Then</u>, Taylor series to third order of $J(y)$ about $y_0(x)$ is

$$J(y_0 + \delta y) = J(y_0) + \delta J(y_0) + \delta^2 J(y_0) + O(o_y^3) .$$

13.

Euler's Equation for $F(x_1, x_2, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2})$ and the Variational Derivative

At distance x, and at time t, let $u(x,t)$ be the amplitude of a vibrating string, over the x-interval $[0,l]$, and the time interval $[0,T]$.

The density of the string at x is $\rho(x)$.

At time t, the material point at x has mass $\rho(x)dx$, speed $\frac{\partial u}{\partial t}$, and kinetic energy $\frac{1}{2}\rho(x)dx\left(\frac{\partial u}{\partial t}\right)^2$.

The potential energy density at x is $\frac{1}{2}k(x)\left(\frac{\partial u}{\partial x}\right)^2 dx$.

Thus, the Lagrangian density is

$$\mathcal{L}(x,t,u,\tfrac{\partial u}{\partial x},\tfrac{\partial u}{\partial t}) = \tfrac{1}{2}\rho(x)dx\left(\tfrac{\partial u}{\partial t}\right)^2 - \tfrac{1}{2}k(x)\left(\tfrac{\partial u}{\partial x}\right)^2 dx.$$

Then, the surface $u(x,t)$ that minimizes the Action integral

$$\int_{t=0}^{t=T}\int_{x=0}^{x=l} \mathcal{L}(x,t,u,\tfrac{\partial u}{\partial x},\tfrac{\partial u}{\partial t})dxdt$$

satisfies Euler's equation of 9.1,which is the wave equation

$$\partial_x(k\tfrac{\partial u}{\partial x}) - \partial_t(\rho\tfrac{\partial u}{\partial t}) = 0\,.$$

13.1 Euler's Variational Equation for $F(x_1, x_2, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2})$

<u>If</u> $y_0(x)$ minimizes $J(y) = \int\limits_{x_2=a_2}^{x_2=b_2}\int\limits_{x_1=a_1}^{x_1=b_1} F(x_1, x_2, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2})dx_1dx_2$

<u>Then</u>
$$\frac{\partial F}{\partial y}\Big|_{y_0} - \frac{\partial}{\partial x_1}\left(\frac{\partial F}{\partial \frac{\partial y}{\partial x_1}}\right)\Bigg|_{y_0} - \frac{\partial}{\partial x_2}\left(\frac{\partial F}{\partial \frac{\partial y}{\partial x_2}}\right)\Bigg|_{y_0} = 0\,.$$

Proof:

Since by 12.5, $\delta y(x_1, x_2)$ does not depend on the x_1, or the x_2 intervals, we will use infinitesimal intervals

$$[x_1 - \tfrac{1}{2}dx_1, x_1 + \tfrac{1}{2}dx_1], \qquad a_1 < x_1 < b_1.$$

$$[x_2 - \tfrac{1}{2}dx_2, x_2 + \tfrac{1}{2}dx_2], \qquad a_2 < x_2 < b_2.$$

Then, $y_0(x)$ minimizes

$dI(o_\delta) = dJ(y_0 + \delta y) =$

$$= \int\limits_{\xi_2=x_2-\frac{1}{2}dx_2}^{\xi_2=x_2+\frac{1}{2}dx_2}\int\limits_{\xi_1=x_1-\frac{1}{2}dx_1}^{\xi_1=x_1+\frac{1}{2}dx_1} F(\xi_1, \xi_2, y_0 + \delta y, \frac{\partial y}{\partial \xi_1}\Big|_{y_0} + \delta\frac{\partial y}{\partial \xi_1}, \frac{\partial y}{\partial \xi_2}\Big|_{y_0} + \delta\frac{\partial y}{\partial \xi_2})d\xi_1 d\xi_2\,,$$

over all the surfaces $y_0 + \delta y$ that cross $y_0(\xi_1, \xi_2)$ at the curves through the endpoints of the infinitesimal intervals. That is,

$$\delta y(x_1 - \tfrac{1}{2}dx_1, \xi_2) = \delta y(x_2 + \tfrac{1}{2}dx_2, \xi_2)\,,\ x_2 - \tfrac{1}{2}dx_2 < \xi_2 < x_2 + \tfrac{1}{2}dx_2$$

$$\delta y(\xi_1, x_2 - \tfrac{1}{2}dx_2) = \delta y(\xi_2, x_2 + \tfrac{1}{2}dx_2),\ x_1 - \tfrac{1}{2}dx_1 < \xi_1 < x_1 + \tfrac{1}{2}dx_1$$

The Taylor expansion to second order of

$$F(\xi_1, \xi_2, y_0 + \delta y, \frac{\partial y}{\partial \xi_1}\Big|_{y_0} + \delta\frac{\partial y}{\partial \xi_1}, \frac{\partial y}{\partial \xi_2}\Big|_{y_0} + \delta\frac{\partial y}{\partial \xi_2}) - F(\xi_1, \xi_2, y_0, \frac{\partial y}{\partial \xi_1}\Big|_{y_0}, \frac{\partial y}{\partial \xi_2}\Big|_{y_0}),$$

about $y_0(\xi_1, \xi_2)$, is

$$\frac{\partial F}{\partial y}\Big|_{y_0}\delta y + \frac{\partial F}{\partial \frac{\partial y}{\partial \xi_1}}\Bigg|_{y_0}\delta\frac{\partial y}{\partial \xi_1} + \frac{\partial F}{\partial \frac{\partial y}{\partial \xi_2}}\Bigg|_{y_0}\delta\frac{\partial y}{\partial \xi_2} + O(o_\delta^2).$$

Therefore,

$$dJ(y_0 + \delta y) - dJ(y_0) =$$

$$= \int_{x_2 - \frac{1}{2}dx_2}^{x_2 + \frac{1}{2}dx_2}\int_{x_1 - \frac{1}{2}dx_1}^{x_1 + \frac{1}{2}dx_1}\left\{\frac{\partial F}{\partial y}\Big|_{y_0}\delta y + \frac{\partial F}{\partial \frac{\partial y}{\partial \xi_1}}\Bigg|_{y_0}\delta\frac{\partial y}{\partial \xi_1} + \frac{\partial F}{\partial \frac{\partial y}{\partial \xi_2}}\Bigg|_{y_0}\delta\frac{\partial y}{\partial \xi_2}\right\}d\xi_1 d\xi_2 + O(o_\delta^2)$$

By 12.8, δ, and ∂_{ξ_1} commute. Thus, ξ_1-integrating by parts,

$$\int_{x_1 - \frac{1}{2}dx_1}^{x_1 + \frac{1}{2}dx_1}\left\{\frac{\partial F}{\partial y}\Big|_{y_0}\delta y + \frac{\partial F}{\partial \frac{\partial y}{\partial \xi_1}}\Bigg|_{y_0}\delta\frac{\partial y}{\partial \xi_1} + \frac{\partial F}{\partial \frac{\partial y}{\partial \xi_2}}\Bigg|_{y_0}\delta\frac{\partial y}{\partial \xi_2}\right\}d\xi_1 =$$

$$= \int_{x_1 - \frac{1}{2}dx_1}^{x_1 + \frac{1}{2}dx_1}\left\{\frac{\partial F}{\partial y}\Big|_{y_0}\delta y - \frac{\partial}{\partial \xi_1}\frac{\partial F}{\partial \frac{\partial y}{\partial \xi_1}}\Bigg|_{y_0}\delta y + \frac{\partial F}{\partial \frac{\partial y}{\partial \xi_2}}\Bigg|_{y_0}\delta\frac{\partial y}{\partial \xi_2}\right\}d\xi_1 + \underbrace{\left[\left(\frac{\partial F}{\partial \frac{\partial y}{\partial \xi_1}}\right)\Bigg|_{y_0}\delta y(\xi_1, \xi_2)\right]_{x_1 - \frac{1}{2}dx_1}^{x_1 + \frac{1}{2}dx_1}}_{=0}$$

Interchanging the order of integration,

$$dJ(y_0 + \delta y) - dJ(y_0) =$$

$$= \int_{x_1-\frac{1}{2}dx_1}^{x_1+\frac{1}{2}dx_1} \int_{x_2-\frac{1}{2}dx_2}^{x_2+\frac{1}{2}dx_2} \left\{ \frac{\partial F}{\partial y}\bigg|_{y_0} \delta y - \frac{\partial}{\partial \xi_1} \frac{\partial F}{\partial \frac{\partial y}{\partial \xi_1}}\bigg|_{y_0} \delta y + \frac{\partial F}{\partial \frac{\partial y}{\partial \xi_2}}\bigg|_{y_0} \delta \frac{\partial y}{\partial \xi_2} \right\} d\xi_2 d\xi_1 + O(o_\delta^2)$$

By 12.8, δ, and ∂_{ξ_2} commute. Thus, ξ_2-integrating by parts,

$$\int_{x_2-\frac{1}{2}dx_2}^{x_2+\frac{1}{2}dx_2} \left\{ \frac{\partial F}{\partial y}\bigg|_{y_0} \delta y - \frac{\partial}{\partial \xi_1} \frac{\partial F}{\partial \frac{\partial y}{\partial \xi_1}}\bigg|_{y_0} \delta y + \frac{\partial F}{\partial \frac{\partial y}{\partial \xi_2}}\bigg|_{y_0} \delta \frac{\partial y}{\partial \xi_2} \right\} d\xi_2 =$$

$$= \int_{x_2-\frac{1}{2}dx_2}^{x_2+\frac{1}{2}dx_2} \left\{ \frac{\partial F}{\partial y}\bigg|_{y_0} - \frac{\partial}{\partial \xi_1} \frac{\partial F}{\partial \frac{\partial y}{\partial \xi_1}}\bigg|_{y_0} - \frac{\partial}{\partial \xi_2} \frac{\partial F}{\partial \frac{\partial y}{\partial \xi_2}}\bigg|_{y_0} \right\} \delta y d\xi_2 + \underbrace{\left[\left(\frac{\partial F}{\partial \frac{\partial y}{\partial \xi_2}} \right)\bigg|_{y_0} \delta y(\xi_1, \xi_2) \right]_{x_2-\frac{1}{2}dx_2}^{x_2+\frac{1}{2}dx_2}}_{=0}$$

Thus,

$$dJ(y_0 + \delta y) - dJ(y_0) =$$

$$= \int_{x_1-\frac{1}{2}dx_1}^{x_1+\frac{1}{2}dx_1} \int_{x_2-\frac{1}{2}dx_2}^{x_2+\frac{1}{2}dx_2} \left\{ \frac{\partial F}{\partial y}\bigg|_{y_0} - \frac{\partial}{\partial \xi_1} \frac{\partial F}{\partial \frac{\partial y}{\partial \xi_1}}\bigg|_{y_0} - \frac{\partial}{\partial \xi_2} \frac{\partial F}{\partial \frac{\partial y}{\partial \xi_2}}\bigg|_{y_0} \right\} \delta y d\xi_1 d\xi_2 + O(o_\delta^2)$$

The integration sum over the infinitesimal interval yields

only the term, $\left\{ \frac{\partial F}{\partial y}\bigg|_{y_0} - \frac{\partial}{\partial x_1} \frac{\partial F}{\partial \frac{\partial y}{\partial x_1}}\bigg|_{y_0} - \frac{\partial}{\partial x_2} \frac{\partial F}{\partial \frac{\partial y}{\partial x_2}}\bigg|_{y_0} \right\} \delta y dx_1 dx_2$,

and we obtain

$$dJ(y_0 + \delta y) - dJ(y_0) =$$

$$= \left\{ \frac{\partial F}{\partial y}\bigg|_{y_0} - \frac{\partial}{\partial x_1}\frac{\partial F}{\partial \frac{\partial y}{\partial x_1}}\Bigg|_{y_0} - \frac{\partial}{\partial x_2}\frac{\partial F}{\partial \frac{\partial y}{\partial x_2}}\Bigg|_{y_0} \right\} \delta y dx_1 dx_2 + O(o_\delta^2)$$

Substituting by 12.1, $\delta y(x_1, x_2) = o_\delta \eta(x_1, x_2)$,

$dI(o_\delta) - dI(0) =$

$$= \left\{ \frac{\partial F}{\partial y}\bigg|_{y_0} - \frac{\partial}{\partial x_1}\frac{\partial F}{\partial \frac{\partial y}{\partial x_1}}\Bigg|_{y_0} - \frac{\partial}{\partial x_2}\frac{\partial F}{\partial \frac{\partial y}{\partial x_2}}\Bigg|_{y_0} \right\} o_\delta \eta(x_1, x_2) dx_1 dx_2 + O(o_\delta^2),$$

$$\frac{dI(o_\delta) - dI(0)}{o_\delta} = \left\{ \frac{\partial F}{\partial y}\bigg|_{y_0} - \frac{\partial}{\partial x_1}\frac{\partial F}{\partial \frac{\partial y}{\partial x_1}}\Bigg|_{y_0} - \frac{\partial}{\partial x_2}\frac{\partial F}{\partial \frac{\partial y}{\partial x_2}}\Bigg|_{y_0} \right\} \eta(x_1, x_2) dx_1 dx_2 + O(o_\delta),$$

Since this hold for any infinitesimal o_δ, $dI(\varepsilon)$ has a derivative at $\varepsilon = 0$, which is,

$$dI'(0) = \left\{ \frac{\partial F}{\partial y}\bigg|_{y_0} - \frac{\partial}{\partial x_1}\frac{\partial F}{\partial \frac{\partial y}{\partial x_1}}\Bigg|_{y_0} - \frac{\partial}{\partial x_2}\frac{\partial F}{\partial \frac{\partial y}{\partial x_2}}\Bigg|_{y_0} \right\} \eta(x_1, x_2) dx_1 dx_2.$$

Since $dI(0) = dJ(y_0)$ is minimal, we have by Fermat Theorem, $dI'(0) = 0$, and since $dx > 0$,

$$\left(\frac{\partial F}{\partial y}\bigg|_{y_0} - \frac{\partial}{\partial x_1}\frac{\partial F}{\partial \frac{\partial y}{\partial x_1}}\Bigg|_{y_0} - \frac{\partial}{\partial x_2}\frac{\partial F}{\partial \frac{\partial y}{\partial x_2}}\Bigg|_{y_0} \right) \eta(x_1, x_2) = 0.$$

Since $\eta(x_1, x_2)$ is an arbitrary smooth enough function,

$$\left.\frac{\partial F}{\partial y}\right|_{y_0} - \frac{\partial}{\partial x_1}\left.\frac{\partial F}{\partial \frac{\partial y}{\partial x_1}}\right|_{y_0} - \frac{\partial}{\partial x_2}\left.\frac{\partial F}{\partial \frac{\partial y}{\partial x_2}}\right|_{y_0} = 0 . \square$$

From the derivation of Euler's equation,

13.2 The First variation of $F(x_1, x_2, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2})$ **is**

$$\delta F(x_1, x_2, y, \tfrac{\partial y}{\partial x_1}, \tfrac{\partial y}{\partial x_2}) = \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial \frac{\partial y}{\partial x_1}}\delta\frac{\partial y}{\partial x_1} + \frac{\partial F}{\partial \frac{\partial y}{\partial x_2}}\delta\frac{\partial y}{\partial x_2}$$

$$= \left(\frac{\partial F}{\partial y} - \frac{\partial}{\partial x_1}\frac{\partial F}{\partial \frac{\partial y}{\partial x_1}} - \frac{\partial}{\partial x_2}\frac{\partial F}{\partial \frac{\partial y}{\partial x_2}}\right)\delta y$$

$$= \delta y\left(\partial_y - \partial_{x_1}\partial_{\frac{\partial y}{\partial x_1}} - \partial_{x_2}\partial_{\frac{\partial y}{\partial x_2}}\right)F$$

Thus, we define

13.3 The First Variational Derivative of $F(x_1, x_2, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2})$

$$\frac{\delta F}{\delta y} = \frac{\partial F}{\partial y} - \frac{\partial}{\partial x_1}\frac{\partial F}{\partial \frac{\partial y}{\partial x_1}} - \frac{\partial}{\partial x_2}\frac{\partial F}{\partial \frac{\partial y}{\partial x_2}}$$

$$= \left(\partial_y - \partial_{x_1}\partial_{\frac{\partial y}{\partial x_1}} - \partial_{x_2}\partial_{\frac{\partial y}{\partial x_2}}\right)F$$

13.4 $\delta \int_{\xi_2=a_2}^{\xi_{12}=x_2} \int_{\xi_1=a_1}^{\xi_1=x_1} F(\xi_1, \xi_2, y, \frac{\partial y}{\partial \xi_1}, \frac{\partial y}{\partial \xi_2}) d\xi_1 d\xi_2 = \int_{\xi_2=a_2}^{\xi_{12}=x_2} \int_{\xi_1=a_1}^{\xi_1=x_1} \delta F(\xi_1, \xi_2, y, \frac{\partial y}{\partial \xi_1}, \frac{\partial y}{\partial \xi_2}) d\xi_1 d\xi_2$.

13.5 If $y_0(x_1, x_2)$ minimizes

$$J(x_1, x_2, y) = \int_{\xi_2=a_2}^{\xi_{12}=x_2} \int_{\xi_1=a_1}^{\xi_1=x_1} F(\xi_1, \xi_2, y, \frac{\partial y}{\partial \xi_1}, \frac{\partial y}{\partial \xi_2}) d\xi_1 d\xi_2$$

Then $\left.\frac{\delta F}{\delta y}\right|_{y_0} = 0$,

$\left.\delta F\right|_{y_0} = 0$,

$\left.\delta J\right|_{y_0} = 0$.

13.6 The Second Variation of $F(x_1, x_2, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2})$ **is**

$$\delta^2 F = \left((\delta y)\partial_y + (\delta \frac{\partial y}{\partial x_1})\partial_{\frac{\partial y}{\partial x_1}} + \delta \frac{\partial y}{\partial x_2} \partial_{\frac{\partial y}{\partial x_2}} \right)^2 F$$

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