Infinitesimal Elasticity, Variational Principles, and Castigliano Theorems

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February, 2012

Abstract The Equilibrium Equations of Elasticity are never derived as Euler's Variational Equations. A Hamiltonian is not identified in Elasticity, and Hamilton's Equations are absent from Elasticity Theory.

Elasticity Theory includes Variational Principles known as Castigliano Theorems. Do the Castigliano Theorems of Elasticity Theory stem from Variational Principles that are unknown in Mechanics?

We show that Castigliano's Theorems are equivalent to Euler's Variational Equations, and to Hamilton's Variational Equations.

To that end we had to set up the theory of elasticity with infinitesimals, and give proofs to fundamental theorems of elasticity we found unproven.

The incomplete analysis that left Elasticity Theorems unproven, was not helped by concern about tensors in several Elasticity texts. For the sake of clarity, we avoided proofs with tensor indices here.

Keywords: Infinitesimal, Variation, Euler's Variational Equation, Variational Calculus, Variational Derivative, Legendre Sufficient Condition, Elasticity, Stress, Strain, Energy, Castigliano, Beam Bending, Shear.

2000 Mathematics Subject Classification; 49-02;26E30; 26E15; 26E20; 26A12; 46S20; 97I40;

Physics and Astronomy Classification Scheme ;45.20.Jj; 4510Db;

Contents

Introduction

0.1 Virtuals, and Infinitesimals

Elasticity Theory uses terms such as "Virtual Work", "Virtual displacement", "Virtual Strain", and "Virtual Stress" to describe Infinitesimal functions that in Calculus are known as Variations, and are not virtual.

The statements about infinitesimals are unclear, if not plain wrong. For instance, in [Chou, p.154] we have,

> " *Notice that in*
>
> $$\delta(\tfrac{1}{2}\sigma_x\varepsilon_x) = \sigma_x\delta\varepsilon_x,$$
>
> *no factor of* $\frac{1}{2}$ *is present, since the stress is constant during the virtual displacement.*"

In fact, the stress

$$\sigma_x = E\varepsilon_x$$

varies according to

$$\delta\sigma_x = E\delta\varepsilon_x,$$

and we have

$$\delta(\tfrac{1}{2}\sigma_x\varepsilon_x) = \tfrac{1}{2}\sigma_x\delta\varepsilon_x + \tfrac{1}{2}\varepsilon_x\underbrace{\delta\sigma_x}_{E\delta\varepsilon_x}$$

$$= \tfrac{1}{2}\sigma_x\delta\varepsilon_x + \tfrac{1}{2}\underbrace{E\varepsilon_x}_{\sigma_x}\delta\varepsilon_x$$

$$= \sigma_x \delta \varepsilon_x .$$

0.2 Infinitesimals and Limits

In terms of the Calculus of Limits, Stress is defined by

$$\lim_{\Delta A \to 0} \frac{\Delta P}{\Delta A},$$

where ΔP is the force on the area ΔA.

It is difficult to imagine what this limit amounts to when $\Delta A \to 0$.

It is easier to comprehend the quotient

$$\frac{dP}{dA},$$

where the infinitesimal dA is always positive, and never vanishes.

To avoid the fogginess of limits in Calculus, we have to use infinitesimals.

0.3 The Strain Energy

The reason why the infinitesimal strain energy density for a bar is

$$d\mu = \sigma_x d\varepsilon_x ,$$

is beyond all texts. The text by [Timoshenko] supplies only a plausibility argument for it.

It is even less obvious why the infinitesimal strain energy density of a plate is

$$d\mu = \sigma_{xx} d\varepsilon_{xx} + 2\sigma_{xy} d\varepsilon_{xy} + \sigma_{yy} d\varepsilon_{yy}.$$

Then, without any explanation, texts conclude that

$$\mu = \tfrac{1}{2}\sigma_{xx}\varepsilon_{xx} + \sigma_{xy}\varepsilon_{xy} + \tfrac{1}{2}\sigma_{yy}\varepsilon_{yy},$$

leaving unexplained the fact that the strain in the x direction on a plate depends on the stress in the y direction.

0.4 Variational Principles of Elasticity

The Principle of Virtual Work, and the Principle of Minimal Potential Energy at Equilibrium, are stand-alones, disconnected from mainstream Variational Principles of Mechanics.

The Principle of Virtual Work is equivalent to Elasticity's Equilibrium equations, which are hinted to be Euler's Variational Equations. But are never derived from Euler's Variational Theory.

The Principle of Minimal Potential Energy can be established only with the Calculus of Variations. But Variations are alien to Elasticity Theory.

Weinstock [Weinstock, pp. 199-260] derives the equations of Equilibrium in order to discuss transverse vibrations of bars, and plates. But once the kinematics is eliminated from his derivation, the Elasticity of structures remains, where Hooke's law is replaced by $f(x) = kx$, and a body is replaced by its center of mass.

0.5 Castigliano Theorems, and Hamilton's Equations

A Hamiltonian is not identified in Elasticity, and Hamilton's Equations are absent from Elasticity Theory.

Elasticity Theory of Structures includes Variational Principles known as Castigliano Theorems.

Do the Castigliano Theorems stem from Variational Principles that are unknown in Mechanics?

We show that Castigliano's Theorems are equivalent to Euler's Variational Equations, and to Hamilton's Variational Equations.

1.

Hyper-real Line

Each real number α can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant hyper-real.

In [Dan1] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of infinitesimal hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the infinite hyper-reals.
4. The infinite hyper-reals are greater than any real number, yet strictly smaller than infinity.
5. The infinite hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a

non-constant hyper-real.

7. The Hyper-reals are the totality of constant hyper-reals, a family of infinitesimals, a family of infinitesimals with negative sign, a family of infinite hyper-reals, a family of infinite hyper-reals with negative sign, and non-constant hyper-reals.
8. The hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.
9. That line includes the real numbers separated by the non-constant hyper-reals. Each real number is the center of an interval of hyper-reals, that includes no other real number.
10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.
11. Zero is not an infinitesimal, because zero is not strictly greater than zero.
12. We do not add infinity to the hyper-real line.
13. The infinitesimals, the infinitesimals with negative signs, the infinite hyper-reals, and the infinite hyper-reals with negative signs are semi-groups with

respect to addition. Neither set includes zero.

14. The hyper-real line is embedded in $\mathbb{R}^{\infty}$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the hyper-real onto the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal hyper-reals, or to the infinite hyper-reals, or to the non-constant hyper-reals.

16. No neighbourhood of a hyper-real is homeomorphic to an $\mathbb{R}^{n}$ ball. Therefore, the hyper-real line is not a manifold.

17. The hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

2.

Strain, and Stress on a Bar

2.1 Strain-Displacement Relation for a Bar

$$(dx')^2 - (dx)^2 \approx 2dxdu$$

$$= 2dx\,\underbrace{u'(x)}_{\varepsilon_x}\,dx\,.$$

The x*-gradient of the displacement is the* ***Axial Strain***

$\varepsilon_x \equiv u,_x =$ *elongation per unit length in* x *direction.*

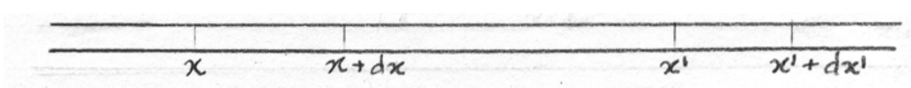
Proof: Elastic Deformation of a bar, transforms

the point x to x',

and the infinitesimally close $x + dx$, to $x' + dx'$.

$$x \to x',$$

$$x + dx \to x' + dx'.$$



The **Displacement** of x to x'

$$u(x) \equiv x' - x\,,$$

satisfies

$$du \equiv dx' - dx\,.$$

Therefore,

$$dx'dx' - dxdx = (dx+du)^2 - (dx)^2$$
$$= 2dxdu + dudu$$

Keeping first order in du, we have,

$$\approx 2dxdu$$
$$= 2(dx)\underbrace{u'(x)}_{u,_x}(dx).$$

And $(dx+du)^2 - (dx)^2$ is approximated by $2(dx)\varepsilon_x(dx)$.□

2.2 The Axial Stress σ_x

The Axial Stress $\sigma_x(x)$ is the force stretching the bar per unit area of the bar's cross section.

2.3 Hook's Law, Stress-Strain Relation in a Bar

The strain and the stress satisfy Hooke's Law

$$\boxed{\sigma_x \equiv E\varepsilon_x},$$

where the **Young Modulus** E is a constant that characterizes the Elastic material.

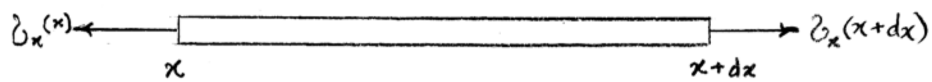
2.4 Body Force on a Bar $f_x(x)$

A body force is an exterior force $f_x(x)$, such as gravitational, or magnetic, per unit length.

2.5 The Equilibrium Equation of a Bar

$$\boxed{\sigma_{x,x} + f_x(x) = 0}.$$

Proof: If a body force $f_x(x)$ per unit length applies to the bar,



the static equilibrium over dx satisfies the equation

$$\underbrace{\sigma_x(x+dx) - \sigma_x(x)}_{\sigma_{x,x}dx} + f_x(x)dx = 0.$$

Hence, the Equilibrium Equation is

$$\sigma_{x,x} + f_x(x) = 0.\square$$

3.

Strain Energy of a Bar

3.1 Infinitesimal Strain Energy of a Bar is

$$d\mu = \sigma_x d\varepsilon_x,$$

$$\frac{d\mu}{d\varepsilon_x} = \sigma_x.$$

Proof: The infinitesimal work done by the axial stress $\sigma_x(x)$, along the displacement du_x is

$$(\sigma_x du_x)\Big|_{x+dx} - (\sigma_x du_x)\Big|_x = \partial_x(\sigma_x du_x)dx$$

$$= [\underbrace{\partial_x(\sigma_x)}_{\sigma_{x,x}} du_x + \sigma_x \underbrace{\partial_x du_x}_{du_{x,x}}]dx$$

$$= [\sigma_{x,x} du_x + \sigma_x d\varepsilon_x]dx$$

The infinitesimal work done by the body force $f_x(x)$ is

$$f_x(x)dxdu_x$$

Thus, the total infinitesimal work is the sum

$$[\sigma_{x,x} du_x + \sigma_x d\varepsilon_x]dx + f_x(x)dxdu_x = [\underbrace{\sigma_{x,x} + f_x(x)}_{=0}]du_x dx + \sigma_x d\varepsilon_x dx,$$

$$= \sigma_x d\varepsilon_x dx$$

The Infinitesimal work is stored in the bar as Strain Energy.

The Infinitesimal Axial Strain Energy per unit length is

$$d\mu = \sigma_x d\varepsilon_x.$$

That is,

$$\frac{d\mu}{d\varepsilon_x} = \sigma_x.\square$$

3.2 Axial Strain Energy density of a Bar is

$$\mu = \tfrac{1}{2}E\varepsilon_x^2.$$

Proof:

$$d\mu = \sigma_x d\varepsilon_x = E\varepsilon_x d\varepsilon_x = Ed(\tfrac{1}{2}\varepsilon_x^2)$$

$$\mu = \tfrac{1}{2}E\varepsilon_x^2.\square$$

3.3 Complementary Axial Strain Energy of a Bar is

$$\mu^* = \tfrac{1}{2E}\sigma_x^2,$$

$$\frac{d\mu^*}{d\sigma_x} = \varepsilon_x.$$

Proof: The complementary axial strain energy density is the axial strain energy density expressed in terms of the stress.

by substituting $\varepsilon_x = \frac{1}{E}\sigma_x$ into μ, we obtain

$$\mu^* = \frac{1}{2E}\sigma_x^2.$$

Although μ^* equals μ, it is denoted by μ^* to specify its dependence on the stress.

Differentiating both sides,

$$d\mu^* = \underbrace{\frac{1}{E}\sigma_x}_{\varepsilon_x} d\sigma_x .$$

$$\frac{d\mu^*}{d\sigma_x} = \varepsilon_x . \square$$

4.

Variational Principles of a Bar

4.1 Euler's Variational Equation

The deflection u *that minimize or maximizes the Bar's energy per unit length,*

$$F = F(x, u, u')$$

satisfies Euler's Variational Equation

$$\delta F(x, u, u') = 0\,,$$

which is,

$$\left(\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u'}\right)\delta u = 0\,.$$

or,

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u'} = 0\,.$$

Proof: The Bar's infinitesimal energy is

$$\int\limits_{\xi = x - \frac{1}{2}dx}^{\xi = x + \frac{1}{2}dx} F(\xi, u(\xi), u'(\xi)d\xi\,.$$

The deflection u that minimizes or maximizes the bar's energy satisfies Euler's Variational Equation

$$\delta F(x, u, u') = 0\,.$$

That is,

$$\frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial u'}\underbrace{\delta u'}_{D_x \delta u} = 0\,.$$

Integration by parts, where δu vanishes on the boundary, [Dan3] leads to the variational equation

$$\left(\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u'}\right)\delta u = 0\,.$$

Since this holds for any variation δu, the deflection u, that minimizes or maximizes the bar’s energy satisfies Euler’s Equation

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u'}.\Box$$

4.2 If *the body forces originate from a Potential* $\upsilon(u)$ *per unit length, so that*

$$-\frac{\partial \upsilon}{\partial u} = f_x$$

Then, *Euler’s Equation is the Equilibrium Equation*

$$\sigma_{x,x} + f_x = 0\,.$$

Proof: The Energy per unit length is

$$F(x, u(x), u'(x)) = \mu + \upsilon$$

$$= \tfrac{1}{2}\sigma_x \varepsilon_x + \upsilon\,.$$

Here,

$$\frac{\partial F}{\partial u} = \frac{\partial \upsilon}{\partial u} = -f_x$$

$$\frac{d}{dx}\frac{\partial F}{\partial u'} = \frac{d}{dx}\sigma_x$$

Euler's Equation is

$$\underbrace{\frac{\partial F}{\partial u}}_{-f_x} - \frac{d}{dx}\underbrace{\frac{\partial F}{\partial u'}}_{\sigma_x} = 0$$

That is, Euler's Equation is the equilibrium Equation

$$f_x + \partial_x \sigma_x = 0\,.\square$$

4.3 Principle of Virtual Work for a Bar

At Equilibrium, $\delta U = \delta W$

Proof: The equilibrium deflection u that minimizes or maximizes the bar's energy satisfies the Euler Variational Equation

$$\underbrace{\left(\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u}\right)\delta u}_{\delta F} = 0\,.$$

Thus, at equilibrium, the Variation of F, [Dan3], satisfies

$$\delta F = 0.$$

Therefore,

$$\delta U - \delta W = \delta(U - W)$$

$$= \delta \int_{x=0}^{x=l} F(x,u,u')dx$$

$$= \int_{x=0}^{x=l} \delta F(x,u,u')dx = 0$$

That is, $\delta U = \delta W$.□

4.4 Principle of Minimal Energy of a Bar

The Bar's Energy is Minimal at Equilibrium

Proof: In [Dan3], we established Legendre's claim that a Sufficient Condition for $\int_{x=a}^{x=b} F(x,u,u')dx$ to be Minimal is that

$$\left.\frac{\partial^2 F}{(\partial u')^2}\right|_{u_0} > 0.$$

Here,

$$\frac{\partial^2 F}{(\partial u')^2} = \frac{\partial^2}{(\partial u')^2}\left\{\tfrac{1}{2}E[u']^2\right\} = E > 0,$$

since E, Young's Elastic Modulus, is always positive.

Therefore, the Energy of the bar is minimal at the equilibrium.□

5.

Strain on a Plate

5.1 Strain-Displacement Relations

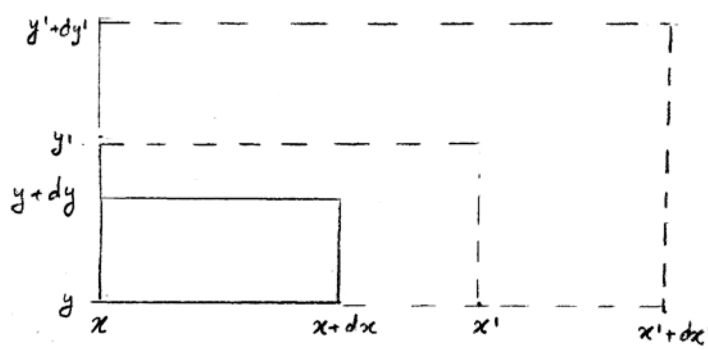
$$(dx')^2 + (dy')^2 - (dx)^2 - dy)^2 \approx 2[\varepsilon_{xx}(dx)^2 + 2\varepsilon_{xy}dxdy + \varepsilon_{yy}(dy)^2]$$

$$= 2\begin{bmatrix} dx, dy \end{bmatrix}\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix}\begin{bmatrix} dx \\ dy \end{bmatrix}.$$

Proof: Elastic Deformation of a plate, transforms the point (x, y) to (x', y'), and the infinitesimally close point $(x + dx, y + dy)$, to $(x' + dx', y' + dy')$.

$$x \to x', \quad y \to y',$$

$$x + dx \to x' + dx', \quad y + dy \to y' + dy'.$$



The **Displacements**

$$u_x(x, y) \equiv x' - x,$$

$$u_y(x,y) \equiv y' - y$$

satisfy

$$du_x = dx' - dx,$$

$$du_y = dy' - dy.$$

Therefore,

$$(dx')^2 + (dy')^2 - [(dx)^2 + (dy)^2] = (dx + du_x)^2 - (dx)^2 + (dy + du_y)^2 - (dy)^2$$

$$= 2dxdu_x + 2dydu_y + (du_x)^2 + (du_y)^2$$

Keeping first order in du_x, and in du_y,

$$\approx 2dxdu_x + 2dydu_y$$

$$= 2\left\{dx[u_{x,x}dx + u_{x,y}dy] + dy[u_{y,x}dx + u_{y,y}dy]\right\}.$$

$$= 2[dx\underbrace{(u_{x,x})}_{\varepsilon_{xx}}dx + dx\underbrace{(u_{x,y} + u_{y,x})}_{2\varepsilon_{xy}=2\varepsilon_{yx}}dy + dy\underbrace{(u_{y,y})}_{\varepsilon_{yy}}dy]$$

$$= 2\begin{bmatrix} dx, dy \end{bmatrix}\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix}\begin{bmatrix} dx \\ dy \end{bmatrix}.$$

5.2 The Axial Strains on a Plate are

$\varepsilon_{xx} \equiv u_{x,x}$ = *elongation per unit length in* x *direction,*

$\varepsilon_{yy} \equiv u_{y,y}$ = *elongation per unit length in* y *direction.*

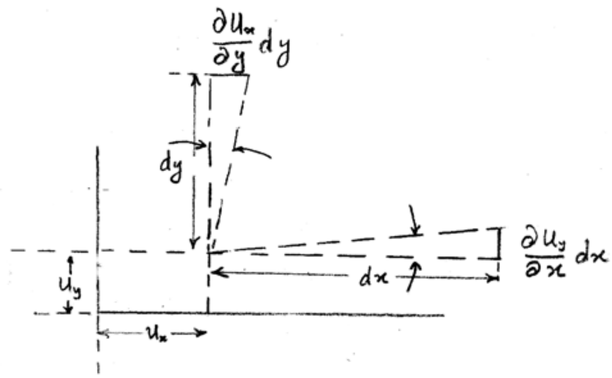
5.3 Shear Strains on a Plate are

$$\varepsilon_{xy} = \varepsilon_{yx} \equiv \tfrac{1}{2}(u_{x,y} + u_{y,x}).$$

As the drawing shows,

$$u_{y,x}dx = \textit{deflection at length } dx \Rightarrow u_{y,x} = \textit{angle at radius } dx$$

$$u_{x,y}dy = \textit{deflection at length } dy \Rightarrow u_{x,y} = \textit{angle at radius } dy$$

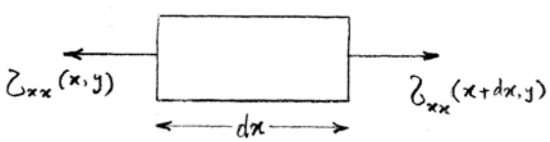


6.

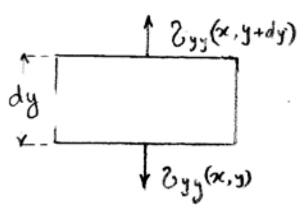
Stress on a Plate

6.1 The Axial Stresses σ_{xx}, σ_{yy}

The Axial Stress $\sigma_{xx}(x, y)$ is the force per unit length of the y side, stretching the plate on both sides in the x direction.



The Axial Stress $\sigma_{yy}(x, y)$ is the force per unit length of the x side , stretching the plate on both sides in the y direction.



6.2 Hook's Law for Axial Stresses on a Plate

E = *Young's Elastic Modulus*

ν = *Poisson Ratio*

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \end{bmatrix}$$

Proof: The stretching along y, by $\frac{1}{E}\sigma_{yy}$, contracts the plate along x by $-\nu\frac{1}{E}\sigma_{yy}$. Then,

$$\varepsilon_{xx} = \tfrac{1}{E}\sigma_{xx} - \nu\tfrac{1}{E}\sigma_{yy} \Rightarrow \; E\varepsilon_{xx} = \sigma_{xx} - \nu\sigma_{yy}.$$

Similarly,

$$\varepsilon_{yy} = \tfrac{1}{E}\sigma_{yy} - \nu\tfrac{1}{E}\sigma_{xx} \Rightarrow \; E\varepsilon_{yy} = -\nu\sigma_{xx} + \sigma_{yy}.$$

That is,

$$\begin{bmatrix} E\varepsilon_{xx} \\ E\varepsilon_{yy} \end{bmatrix} = \begin{bmatrix} 1 & -\nu \\ -\nu & 1 \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \end{bmatrix}.$$

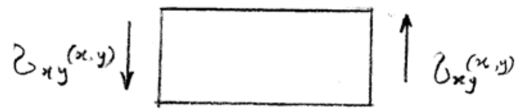
Hence,

$$\sigma_{xx} = \frac{\begin{vmatrix} E\varepsilon_{xx} & -\nu \\ E\varepsilon_{yy} & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -\nu \\ -\nu & 1 \end{vmatrix}} = \tfrac{E}{1-\nu^2}(\varepsilon_{xx} + \nu\varepsilon_{yy}),$$

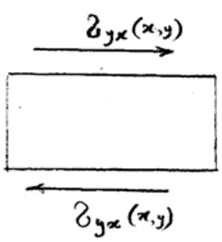
$$\sigma_{yy} = \frac{\begin{vmatrix} 1 & E\varepsilon_{xx} \\ -\nu & E\varepsilon_{yy} \end{vmatrix}}{\begin{vmatrix} 1 & -\nu \\ -\nu & 1 \end{vmatrix}} = \tfrac{E}{1-\nu^2}(\nu\varepsilon_{xx} + \varepsilon_{yy}).\square$$

6.3 The Shear Stresses, σ_{xy}, σ_{yx}

The Shear Stress $\sigma_{xy}(x,y)$ is the force per unit length, along the y sides,



The Shear Stress $\sigma_{yx}(x,y)$ is the force per unit length, along the x sides,



6.4 Hook's Law for a Shear Stresses on a Plate

The shear stress σ_{xy} satisfies Hooke's Law.

$$\sigma_{xy} = 2G\varepsilon_{xy},$$

where $G =$ **Shear Modulus.**

Similarly, in isotropic material,

$$\sigma_{yx} = 2G\varepsilon_{yx}.$$

Since $\varepsilon_{xy} = \varepsilon_{yx}$, we have

6.5 $$\sigma_{xy} = \sigma_{yx}.$$

6.6 Hook's Law for a Plate

$E = $ *Young's Elastic Modulus*

$\nu = $ *Poisson Ratio*

$G = $ *Shear Modulus* $= g\dfrac{E}{1-\nu^2}$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 2g \end{bmatrix}\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix}$$

6.7 $$2G = \frac{E}{1+\nu}$$

$$\sigma_{xy} = 2G\varepsilon_{xy} = \frac{E}{1+\nu}\varepsilon_{xy}$$

Proof: By 6.2

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix}\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \end{bmatrix}.$$

The matrix

$$\begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix}$$

has the eigen values $1-\nu$, and $1+\nu$, and can be rotated so that

$$\begin{bmatrix}\sigma'_{xx}\\ \sigma'_{yy}\end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1-\nu & 0\\ 0 & 1+\nu\end{bmatrix}\begin{bmatrix}\varepsilon'_{xx}\\ \varepsilon'_{yy}\end{bmatrix} = E\begin{bmatrix}\frac{1}{1+\nu} & 0\\ 0 & \frac{1}{1-\nu}\end{bmatrix}\begin{bmatrix}\varepsilon'_{xx}\\ \varepsilon'_{yy}\end{bmatrix},$$

That rotation diagonalizes also the matrix

$$\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & 2g\end{bmatrix}.$$

The equation for its eigen values yields

$$\lambda = 2g; \quad \lambda = 1+\nu; \ \lambda = 1-\nu.$$

For $\lambda = 1-\nu$,

$$2g = 1-\nu,$$

$$2G = \frac{E}{1-\nu^2}(1-\nu) = \frac{E}{1+\nu},$$

$$\sigma_{xy} = \frac{E}{1+\nu}\varepsilon_{xy}.\square$$

6.8 Body Forces $f_x(x,y)$, $f_y(x,y)$

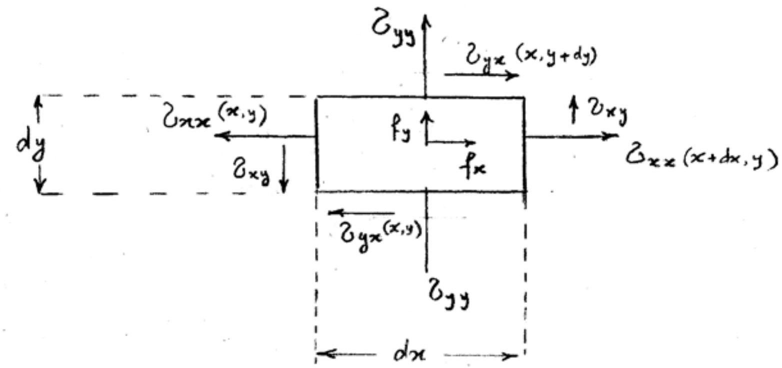
Body forces are exterior forces $f_x(x,y)$, and $f_y(x,y)$ such as gravitational, or magnetic, per unit area.

6.9 The Equilibrium Equations for a Plate

$$\sigma_{xx,x} + \sigma_{xy,y} + f_x = 0,$$

$$\sigma_{yx,x} + \sigma_{yy,y} + f_y = 0.$$

Proof: If body forces $f_x(x,y)$, and $f_y(x,y)$ per unit area apply to the plate,



<u>In the x direction</u>,

the force due to σ_{xx} is

$$\underbrace{[\sigma_{xx}(x+dx,y) - \sigma_{xx}(x,y)]}_{=\sigma_{xx,x}dx} dy = \sigma_{xx,x} dxdy,$$

the force due to σ_{yx} is

$$\underbrace{[\sigma_{yx}(x,y+dy) - \sigma_{yx}(x,y)]}_{=\sigma_{yx,y}dy=\sigma_{xy,y}dy} dx = \sigma_{xy,y} dxdy,$$

and the force due to f_x is $f_x(x,y)dxdy$,

Therefore, the Equilibrium of forces in the x direction is,

$$\sigma_{xx,x} + \sigma_{xy,y} + f_x = 0.$$

Similarly, the Equilibrium of forces in the y direction is,

$$\sigma_{yy,y} + \sigma_{yx,x} + f_y = 0.\square$$

7.

Strain Energy of a Plate

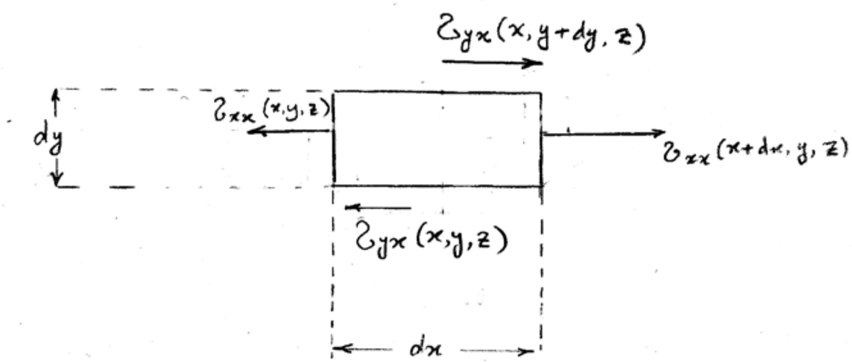
7.1 Infinitesimal Strain Energy per unit area

$$d\mu = \sigma_{xx} d\varepsilon_{xx} + 2\sigma_{xy} d\varepsilon_{xy} + \sigma_{yy} d\varepsilon_{yy}$$

$$= \sigma_{ij} d\varepsilon_{ij}. \quad \text{(summation over } i = 1,2;\ j = 1,2\text{)}$$

$$\frac{\partial \mu}{\partial \varepsilon_{xx}} = \sigma_{xx}; \quad \frac{\partial \mu}{\partial \varepsilon_{xy}} = 2\sigma_{xy}; \quad \frac{\partial \mu}{\partial \varepsilon_{yy}} = \sigma_{yy}$$

Proof: In the x direction,



The infinitesimal work done by $\sigma_{xx} dy$, along du_x is

$$(\sigma_{xx} dy du_x)\Big|_{(x+dx,y)} - (\sigma_{xx} dy du_x)\Big|_{(x,y)} = \partial_x(\sigma_{xx} du_x) dxdy$$

$$= [\underbrace{\partial_x(\sigma_{xx})}_{\sigma_{xx,x}} du_x + \sigma_{xx} \underbrace{\partial_x du_x}_{du_{x,x}}] dxdy$$

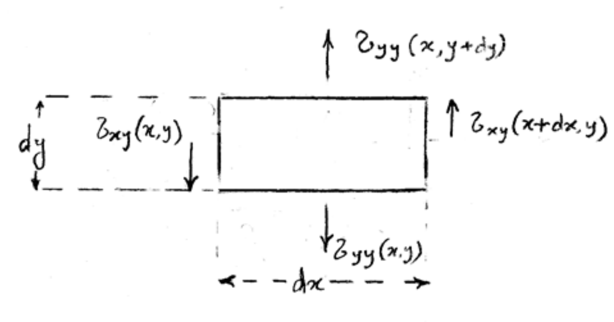
The infinitesimal work done by $\sigma_{yx} dx$, along du_x is

$$(\sigma_{yx} dx du_x)\Big|_{(x,y+dy)} - (\sigma_{yx} dx du_x)\Big|_{(x,y)} = \partial_y(\sigma_{yx} du_x) dy dx$$

$$= [\underbrace{\partial_y(\sigma_{yx})}_{\sigma_{xy,y}} du_x + \sigma_{yx} \underbrace{\partial_y du_x}_{du_{x,y}}] dx dy$$

The infinitesimal work by $f_x dx dy$ is $f_x(x,y) dx dy du_x$

<u>In the y direction</u>,



The infinitesimal work done by $\sigma_{yy} dx$, along du_y is

$$(\sigma_{yy} dx du_y)\Big|_{(x,y+dy)} - (\sigma_{yy} dx du_y)\Big|_{(x,y)} = \partial_y(\sigma_{yy} du_y) dx dy$$

$$= [\underbrace{\partial_y(\sigma_{yy})}_{\sigma_{yy,y}} du_y + \sigma_{yy} \underbrace{\partial_y du_y}_{du_{y,y}}] dx dy$$

The infinitesimal work done by $\sigma_{xy} dy$, along du_y is

$$(\sigma_{xy} dy du_y)\Big|_{(x+dx,y)} - (\sigma_{xy} dy du_y)\Big|_{(x,y)} = \partial_x(\sigma_{xy} du_y) dx dy$$

$$= [\underbrace{\partial_x(\sigma_{yx})}_{\sigma_{xy,x}} du_y + \sigma_{yx} \underbrace{\partial_x du_y}_{du_{y,x}}] dx dy$$

The infinitesimal work by f_y is $f_y(x,y) dx dy du_y$

Thus, the total infinitesimal work is the sum

$$\{\underbrace{[\sigma_{xx,x} + \sigma_{xy,y} + f_x(x,y)]}_{0} du_x + \sigma_{xx} d\underbrace{u_{x,x}}_{\varepsilon_{xx}} + \sigma_{xy} du_{x,y}\} dxdy +$$

$$+\{\underbrace{[\sigma_{yx,x} + \sigma_{yy,y} + f_y(x,y)]}_{0} du_y + \sigma_{yy} d\underbrace{u_{y,y}}_{\varepsilon_{yy}} + \sigma_{xy} du_{y,x}\} dxdy$$

$$= \{\sigma_{xx} d\varepsilon_{xx} + \underbrace{\sigma_{xy} du_{x,y} + \sigma_{yx} du_{y,x}}_{2\sigma_{xy} d\varepsilon_{xy}} + \sigma_{yy} d\varepsilon_{yy}\} dxdy$$

The Infinitesimal work is stored in the plate as Strain Energy. The Infinitesimal Strain Energy per unit area is

$$d\mu = \sigma_{xx} d\varepsilon_{xx} + 2\sigma_{xy} d\varepsilon_{xy} + \sigma_{yy} d\varepsilon_{yy}.\square$$

Hence,

$$\frac{\partial \mu}{\partial \varepsilon_{xx}} = \sigma_{xx}; \quad \frac{\partial \mu}{\partial \varepsilon_{xy}} = 2\sigma_{xy}; \quad \frac{\partial \mu}{\partial \varepsilon_{yy}} = \sigma_{yy}.\square$$

7.2 Strain Energy density of a Plate

$$\mu = \tfrac{1}{2}\sigma_{xx}\varepsilon_{xx} + \sigma_{xy}\varepsilon_{xy} + \tfrac{1}{2}\sigma_{yy}\varepsilon_{yy}$$

$$= \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij} \quad \text{(summation over } i, j = 1, 2)$$

$$= \tfrac{1}{2}\tfrac{E}{1-\nu^2}[\varepsilon_{xx}^2 + 2\nu\varepsilon_{yy}\varepsilon_{xx} + \varepsilon_{yy}^2] + 2G\varepsilon_{xy}^2$$

Proof:

$$d\mu = \sigma_{xx} d\varepsilon_{xx} + 2\sigma_{xy} d\varepsilon_{xy} + \sigma_{yy} d\varepsilon_{yy}$$

Substituting

$$\sigma_{xx} = \frac{E}{1-\nu^2}(\varepsilon_{xx} + \nu\varepsilon_{yy}),$$

$$\sigma_{yy} = \frac{E}{1-\nu^2}(\nu\varepsilon_{xx} + \varepsilon_{yy}),$$

$$\sigma_{xy} = \frac{E}{1+\nu}\varepsilon_{xy},$$

$$d\mu = \frac{E}{1-\nu^2}(\varepsilon_{xx} + \nu\varepsilon_{yy})d\varepsilon_{xx} + 2\frac{E}{1+\nu}\varepsilon_{xy}d\varepsilon_{xy} + \frac{E}{1-\nu^2}(\nu\varepsilon_{xx} + \varepsilon_{yy})d\varepsilon_{yy}$$

$$= \frac{1}{2}\frac{E}{1-\nu^2}d\varepsilon_{xx}^2 + \frac{E}{1-\nu^2}\nu\varepsilon_{yy}d\varepsilon_{xx} + \frac{E}{1+\nu}d(\varepsilon_{xy}^2) + \frac{E}{1-\nu^2}\nu\varepsilon_{xx}d\varepsilon_{yy} + \frac{1}{2}\frac{E}{1-\nu^2}d\varepsilon_{yy}^2$$

$$= \frac{1}{2}\frac{E}{1-\nu^2}d(\varepsilon_{xx}^2 + \varepsilon_{yy}^2) + \frac{E}{1-\nu^2}\nu[\varepsilon_{yy}d\varepsilon_{xx} + \varepsilon_{xx}d\varepsilon_{yy}] + \frac{E}{1+\nu}d(\varepsilon_{xy}^2)$$

$$= \frac{1}{2}\frac{E}{1-\nu^2}d(\varepsilon_{xx}^2 + \varepsilon_{yy}^2) + \frac{E}{1-\nu^2}\nu d(\varepsilon_{yy}\varepsilon_{xx}) + \underbrace{\frac{E}{1+\nu}}_{2G}d(\varepsilon_{xy}^2)$$

Integrating,

$$\mu = \frac{1}{2}\frac{E}{1-\nu^2}(\varepsilon_{xx}^2 + \varepsilon_{yy}^2) + \frac{E}{1-\nu^2}\nu\varepsilon_{xx}\varepsilon_{yy} + 2G\varepsilon_{xy}^2$$

$$= \frac{1}{2}\frac{E}{1-\nu^2}[\varepsilon_{xx}^2 + 2\nu\varepsilon_{yy}\varepsilon_{xx} + \varepsilon_{yy}^2] + 2G\varepsilon_{xy}^2$$

$$= \frac{1}{2}\{\underbrace{\frac{E}{1-\nu^2}[\varepsilon_{xx} + \nu\varepsilon_{yy}]}_{\sigma_{xx}}\varepsilon_{xx} + \underbrace{\frac{E}{1-\nu^2}[\nu\varepsilon_{xx} + \varepsilon_{yy}]}_{\sigma_{yy}}\varepsilon_{yy}\} + \sigma_{xy}\varepsilon_{xy}.\square$$

7.3 Complementary Strain Energy of a Plate

$$\mu^* = \frac{1}{2}\sigma_{xx}\varepsilon_{xx} + \sigma_{xy}\varepsilon_{xy} + \frac{1}{2}\sigma_{yy}\varepsilon_{yy}$$

$$= \frac{1}{2E}(\sigma_{xx}^2 - 2\nu\sigma_{xx}\sigma_{yy} + \sigma_{yy}^2) + \frac{1}{2G}\sigma_{xy}^2$$

$$\frac{\partial \mu^*}{\partial \sigma_{xx}} = \varepsilon_{xx}; \quad \frac{\partial \mu^*}{\partial \sigma_{xy}} = 2\varepsilon_{xy}; \quad \frac{\partial \mu^*}{\partial \sigma_{yy}} = \varepsilon_{yy}$$

Proof: The complementary strain energy density is the strain energy density expressed in terms of the stresses.

Substituting

$$\varepsilon_{xx} = \tfrac{1}{E}\sigma_{xx} - \nu\tfrac{1}{E}\sigma_{yy}$$

$$\varepsilon_{yy} = \tfrac{1}{E}\sigma_{yy} - \nu\tfrac{1}{E}\sigma_{xx}$$

$$\varepsilon_{xy} = \tfrac{1}{2G}\sigma_{xy}$$

into μ,

$$\mu = \tfrac{1}{2}\sigma_{xx}\varepsilon_{xx} + \sigma_{xy}\varepsilon_{xy} + \tfrac{1}{2}\sigma_{yy}\varepsilon_{yy}$$

$$= \tfrac{1}{2E}\sigma_{xx}(\sigma_{xx} - \nu\sigma_{yy}) + \sigma_{xy}\tfrac{1}{2G}\sigma_{xy} + \tfrac{1}{2E}\sigma_{yy}(-\nu\sigma_{xx} + \sigma_{yy})$$

$$= \tfrac{1}{2E}(\sigma_{xx}^2 - 2\nu\sigma_{xx}\sigma_{yy} + \sigma_{yy}^2) + \tfrac{1}{2G}\sigma_{xy}^2$$

Although μ^* equals μ, it is denoted by μ^* to specify its dependence on the stress.

$$\frac{\partial \mu^*}{\partial \sigma_{xx}} = \tfrac{1}{E}\sigma_{xx} - \nu\tfrac{1}{E}\sigma_{yy} = \varepsilon_{xx}.$$

$$\frac{\partial \mu^*}{\partial \sigma_{yy}} = \tfrac{1}{E}\sigma_{yy} - \nu\tfrac{1}{E}\sigma_{xx} = \varepsilon_{yy}$$

$$\frac{\partial \mu^*}{\partial \sigma_{xy}} = \tfrac{1}{G}\sigma_{xy} = 2\varepsilon_{xy}.\Box$$

8.

Variational Principles on a Plate

8.1 Euler's Variational Equations

The deflections u_x, and u_y that minimize or maximize the plate's energy per unit area,

$$F = F(x, y, u_x, u_y, \varepsilon_{xx}, \varepsilon_{xy}, \varepsilon_{yy})$$

satisfy Euler's Variational Equation

$$\delta F(x, y, u_x, u_y, \varepsilon_{xx}, \varepsilon_{xy}, \varepsilon_{yy}) = 0,$$

which is,

$$\left(\frac{\partial F}{\partial u_x} - \partial_x \frac{\partial F}{\partial \varepsilon_{xx}} - \tfrac{1}{2}\partial_y \frac{\partial F}{\partial \varepsilon_{xy}}\right)\delta u_x + \left(\frac{\partial F}{\partial u_y} - \tfrac{1}{2}\partial_x \frac{\partial F}{\partial \varepsilon_{xy}} - \partial_y \frac{\partial F}{\partial \varepsilon_{yy}}\right)\delta u_y = 0.$$

or,

$$\frac{\partial F}{\partial u_x} - \partial_x \frac{\partial F}{\partial \varepsilon_{xx}} - \tfrac{1}{2}\partial_y \frac{\partial F}{\partial \varepsilon_{xy}} = 0,$$

$$\frac{\partial F}{\partial u_y} - \tfrac{1}{2}\partial_x \frac{\partial F}{\partial \varepsilon_{xy}} - \partial_y \frac{\partial F}{\partial \varepsilon_{yy}} = 0.$$

Proof: The Plate's infinitesimal energy is

$$\int_{\xi=x-\frac{1}{2}dx}^{\xi=x+\frac{1}{2}dx}\int_{\eta=y-\frac{1}{2}dy}^{\eta=y+\frac{1}{2}dy} F(\xi,\eta,u_\xi,u_\eta,\varepsilon_{\xi\xi},\varepsilon_{\xi\eta},\varepsilon_{\eta\eta})d\xi d\eta\,.$$

The deflections u_x, and u_y that minimizes or maximize the plate's energy, satisfy Euler's Variational Equation

$$\delta F(x,y,u_x,u_y,\varepsilon_{xx},\varepsilon_{xy},\varepsilon_{yy}) = 0\,.$$

That is,

$$\frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial \varepsilon_{xx}}\underbrace{\delta\varepsilon_{xx}}_{\partial_x \delta u_x} + \frac{\partial F}{\partial \varepsilon_{xy}}\underbrace{\delta\varepsilon_{xy}}_{\frac{1}{2}\partial_y\delta u_x + \frac{1}{2}\partial_x \delta u_y} + \frac{\partial F}{\partial u_y}\delta u_y + \frac{\partial F}{\partial \varepsilon_{yy}}\underbrace{\delta\varepsilon_{yy}}_{\partial_y \delta u_y} = 0$$

$$\frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial \varepsilon_{xx}}\partial_x \delta u_x + \tfrac{1}{2}\frac{\partial F}{\partial \varepsilon_{xy}}\partial_y \delta u_x + \frac{\partial F}{\partial u_y}\delta u_y + \tfrac{1}{2}\frac{\partial F}{\partial \varepsilon_{xy}}\partial_x \delta u_y + \frac{\partial F}{\partial \varepsilon_{yy}}\partial_y \delta u_y = 0$$

Integration by parts, where δu_x, and δu_y vanish on the boundary, [Dan3] leads to the variational equation

$$\left(\frac{\partial F}{\partial u_x} - \partial_x \frac{\partial F}{\partial \varepsilon_{xx}} - \tfrac{1}{2}\partial_y \frac{\partial F}{\partial \varepsilon_{xy}}\right)\delta u_x + \left(\frac{\partial F}{\partial u_y} - \tfrac{1}{2}\partial_x \frac{\partial F}{\partial \varepsilon_{xy}} - \partial_y \frac{\partial F}{\partial \varepsilon_{yy}}\right)\delta u_y = 0\,.$$

Since the variations δu_x, and δu_y are independent, the deflections u_x, and u_y that minimizes or maximize the plates's energy satisfy Euler's Variational Equations

$$\frac{\partial F}{\partial u_x} - \partial_x \frac{\partial F}{\partial \varepsilon_{xx}} - \tfrac{1}{2}\partial_y \frac{\partial F}{\partial \varepsilon_{xy}} = 0$$

$$\frac{\partial F}{\partial u_y} - \tfrac{1}{2}\partial_x \frac{\partial F}{\partial \varepsilon_{xy}} - \partial_y \frac{\partial F}{\partial \varepsilon_{yy}} = 0$$

8.2 If *the body forces originate from a Potential* $\upsilon(u_x, u_y)$ *per unit area, so that*

$$-\frac{\partial \upsilon}{\partial u_x} = f_x$$

$$-\frac{\partial \upsilon}{\partial u_y} = f_y$$

Then, *Euler's Equations are the Equilibrium Equations*

$$\sigma_{xx,x} + \sigma_{xy,y} + f_x = 0$$

$$\sigma_{yx,x} + \sigma_{yy,y} + f_y = 0\,.$$

Proof: The Energy per unit area is

$$F(x, y, u_x, u_y, \varepsilon_{xx}, \varepsilon_{xy}, \varepsilon_{yy}) = \mu + \upsilon$$

$$= \tfrac{1}{2}\sigma_{xx}\varepsilon_{xx} + \sigma_{xy}\varepsilon_{xy} + \tfrac{1}{2}\sigma_{yy}\varepsilon_{yy} + \upsilon\,.$$

The Euler's Variational Equations are

$$\underbrace{\frac{\partial F}{\partial u_x}}_{-f_x} - \partial_x \underbrace{\frac{\partial F}{\partial \varepsilon_{xx}}}_{\sigma_{xx}} - \tfrac{1}{2}\partial_y \underbrace{\frac{\partial F}{\partial \varepsilon_{xy}}}_{2\sigma_{xy}} = 0$$

$$\underbrace{\frac{\partial F}{\partial u_y}}_{-f_y} - \tfrac{1}{2}\partial_x \underbrace{\frac{\partial F}{\partial \varepsilon_{xy}}}_{\sigma_{yx}} - \partial_y \underbrace{\frac{\partial F}{\partial \varepsilon_{yy}}}_{\sigma_{yy}} = 0$$

That is, Euler's Equations are the equilibrium Equations

$$f_x + \partial_x \sigma_{xx} + \partial_y \sigma_{xy} = 0,$$

$$f_y + \partial_x \sigma_{yx} + \partial_y \sigma_{yy} = 0. \square$$

8.3 Principle of Virtual Work

At Equilibrium, $\delta U = \delta W$

Proof: The equilibrium deflections u_x, and u_y that minimize the bar's energy satisfy the Euler Variational Equation

$$\underbrace{\left(\frac{\partial F}{\partial u_x} - \partial_x \frac{\partial F}{\partial \varepsilon_{xx}} - \tfrac{1}{2}\partial_y \frac{\partial F}{\partial \varepsilon_{xy}}\right)\delta u_x + \left(\frac{\partial F}{\partial u_y} - \tfrac{1}{2}\partial_x \frac{\partial F}{\partial \varepsilon_{xy}} - \partial_y \frac{\partial F}{\partial \varepsilon_{yy}}\right)\delta u_y}_{\delta F} = 0.$$

Thus, at equilibrium, the Variation of F, [Dan3], satisfies

$$\delta F = 0.$$

Therefore,

$$\delta U - \delta W = \delta(U - W)$$

$$= \delta \int_{x=0}^{x=l_x} \int_{y=0}^{y=l_y} F(x, y, u_x, u_y, \varepsilon_{xx}, \varepsilon_{xy}, \varepsilon_{yy}) dxdy$$

$$= \int_{x=0}^{x=l_x} \int_{y=0}^{y=l_y} \delta F(x, y, u_x, u_y, \varepsilon_{xx}, \varepsilon_{xy}, \varepsilon_{yy}) dxdy = 0$$

That is, $\delta U = \delta W$.□

8.4 Principle of Minimal Energy for a Plate

The Plate's Energy is Minimal at Equilibrium

Proof: Legendre's Sufficient Condition for

$$\int_{x=0}^{x=l_x} \int_{y=0}^{y=l_y} F(x, y, u_x, u_y, \varepsilon_{xx}, \varepsilon_{xy}, \varepsilon_{yy}) dxdy$$

to be Minimal at the equilibrium generalizes to

$$\left. \frac{\partial^2 F}{(\partial \varepsilon_{ij})^2} \right|_{\text{equilibrium } u_x; u_y} > 0 .$$

By 7.2,

$$F = \tfrac{1}{2} \tfrac{E}{1-\nu^2} [\varepsilon_{xx}^2 + 2\nu \varepsilon_{yy} \varepsilon_{xx} + \varepsilon_{yy}^2] + 2G\varepsilon_{xy}^2 + \upsilon ,$$

where $E > 0$ is Young's Elastic Modulus,

$\nu =$ Poisson's Ratio, and $0 < \nu < 1$

$G = \frac{E}{1+\nu} =$ Shear Modulus.

Hence,

$$\frac{\partial^2 F}{(\partial \varepsilon_{xx})^2} = \tfrac{E}{1-\nu^2} > 0\,,$$

$$\frac{\partial^2 F}{(\partial \varepsilon_{yy})^2} = \tfrac{E}{1-\nu^2} > 0\,,$$

$$\frac{\partial^2 F}{(\partial \varepsilon_{xy})^2} = 4G = 2\tfrac{E}{1+\nu} > 0$$

Therefore, the Total Energy of the plate is minimal at the equilibrium.□

9.

Strain on a Box

5.1 Strain-Displacement Relations for a box

$$(dx')^2 + (dy')^2 + (dz')^2 - (dx)^2 - dy)^2 + (dz)^2 \approx$$

$$\approx 2[\varepsilon_{xx}(dx)^2 + \varepsilon_{yy}(dy)^2 + \varepsilon_{zz}(dz)^2 + 2\varepsilon_{xy}dxdy + 2\varepsilon_{yz}dydz + 2\varepsilon_{zx}dzdx]$$

$$= 2\begin{bmatrix} dx, dy, dz \end{bmatrix}\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}.$$

Proof: Elastic Deformation of a body, transforms the point (x, y, z) to (x', y', z'), and $(x + dx, y + dy, z + dz)$, to $(x' + dx', y' + dy', z + dz')$.

The **Displacements**

$$u_x(x, y, z) \equiv x' - x,$$
$$u_y(x, y, z) \equiv y' - y,$$
$$u_z(x, y, z) \equiv z' - z$$

satisfy

$$du_x = dx' - dx,$$
$$du_y = dy' - dy,$$
$$du_z = dz' - dz$$

Therefore,

$$(dx')^2 + (dy')^2 + (dz')^2 - [(dx)^2 + (dy)^2 + (dz)^2] =$$

$$= 2dxdu_x + 2dydu_y + 2dzdu_z + (du_x)^2 + (du_y)^2 + (du_z)^2$$

Keeping first order in du_x, du_y, and du_z

$$\approx 2dxdu_x + 2dydu_y + 2dzdu_z$$

$$= 2\{dx[u_{x,x}dx + u_{x,y}dy + u_{x,z}dz] +$$

$$+dy[u_{y,x}dx + u_{y,y}dy + u_{y,z}dz] +$$

$$+dz[u_{z,x}dx + u_{z,y}dy + u_{z,z}dz]\}.$$

$$= 2[dx\underbrace{(u_{x,x})}_{\varepsilon_{xx}}dx + dy\underbrace{(u_{y,y})}_{\varepsilon_{yy}}dy + dz\underbrace{(u_{z,z})}_{\varepsilon_{zz}}dz]$$

$$+2[dx\underbrace{(u_{x,y} + u_{y,x})}_{2\varepsilon_{xy}=2\varepsilon_{yx}}dy + dy\underbrace{(u_{y,z} + u_{z,y})}_{2\varepsilon_{yz}=2\varepsilon_{zy}}dz + dz\underbrace{(u_{z,x} + u_{x,z})}_{2\varepsilon_{zx}=2\varepsilon_{xz}}dx]$$

$$= 2\left[dx, dy, dz\right]\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz}\\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz}\\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}\end{bmatrix}\begin{bmatrix}dx\\ dy\\ dz\end{bmatrix}.$$

5.2 The Axial Strains on a Box are

$\varepsilon_{xx} \equiv u_{x,x}(x,y,z) =$ *elongation per unit length in* x *direction,*

$\varepsilon_{yy} \equiv u_{y,y}(x,y,z) =$ *elongation per unit length in* y *direction,*

$\varepsilon_{zz} \equiv u_{z,z}(x,y,z) =$ *elongation per unit length in* y *direction.*

5.3 Shear Strains on the faces of a Box are

$$\varepsilon_{xy} = \varepsilon_{yx} \equiv \tfrac{1}{2}[u_{x,y}(x,y,z) + u_{y,x}(x,y,z)],$$

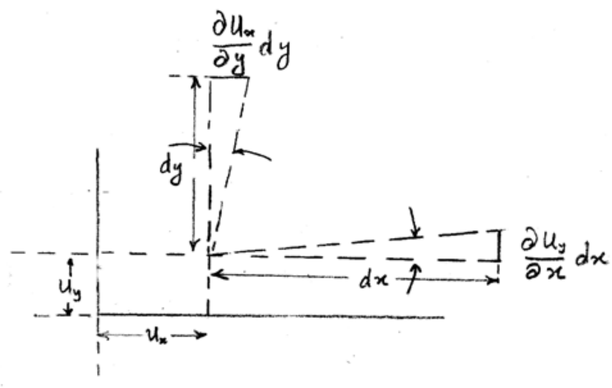
$$\varepsilon_{yz} = \varepsilon_{zy} \equiv \tfrac{1}{2}[u_{y,z}(x,y,z) + u_{z,y}(x,y,z)],$$

$$\varepsilon_{zx} = \varepsilon_{xz} \equiv \tfrac{1}{2}[u_{z,x}(x,y,z) + u_{x,z}(x,y,z)]$$

As the drawing shows,

$u_{y,x}dx =$ ***deflection at length*** $dx \Rightarrow u_{y,x} =$ ***angle at radius*** dx

$u_{x,y}dy =$ ***deflection at length*** $dy \Rightarrow u_{x,y} =$ ***angle at radius*** dy

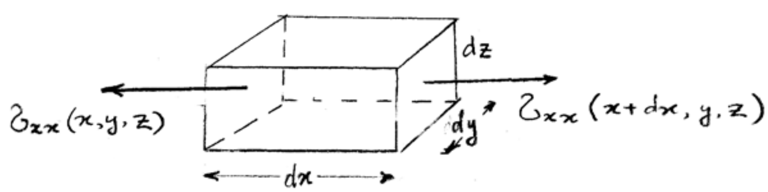


10.

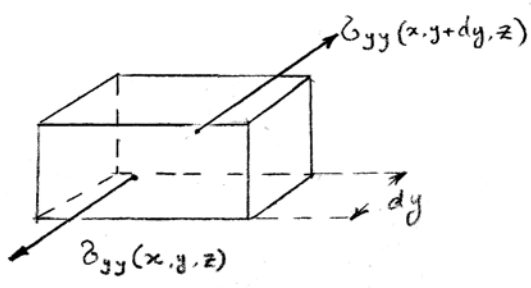
Stress on a Box

10.1 The Axial Stresses σ_{xx}, σ_{yy}, σ_{zz}

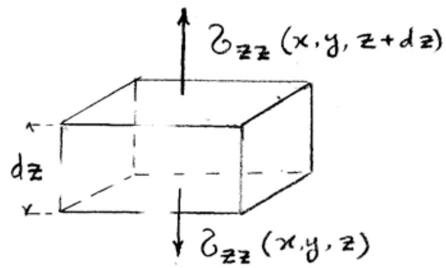
The Axial Stress $\sigma_{xx}(x,y,z)$ is the force per unit area of the yz face, stretching the box in the x direction.



The Axial Stress $\sigma_{yy}(x,y,z)$is the force per unit area of the xz face , stretching the box in the y direction.



The Axial Stress $\sigma_{zz}(x,y,z)$is the force per unit area of the xy face , stretching the box in the z direction.



10.2 Hooke's Law for Axial Stresses on a Box

$E =$ *Young's Elastic Modulus*

$\nu =$ *Poisson Ratio*

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{bmatrix}$$

Proof: Stretching along y, by $\frac{1}{E}\sigma_{yy}$, and along z by $\frac{1}{E}\sigma_{zz}$, contracts the box along x by $-\nu\frac{1}{E}\sigma_{yy} - \nu\frac{1}{E}\sigma_{zz}$. Then,

$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx} - \nu\frac{1}{E}\sigma_{yy} - \nu\frac{1}{E}\sigma_{zz} \Rightarrow E\varepsilon_{xx} = \sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz}.$$

Similarly,

$$\varepsilon_{yy} = \frac{1}{E}\sigma_{yy} - \nu\frac{1}{E}\sigma_{xx} - \nu\frac{1}{E}\sigma_{zz} \Rightarrow E\varepsilon_{yy} = -\nu\sigma_{xx} + \sigma_{yy} - \nu\sigma_{zz}.$$

$$\varepsilon_{zz} = \frac{1}{E}\sigma_{zz} - \nu\frac{1}{E}\sigma_{xx} - \nu\frac{1}{E}\sigma_{yy} \Rightarrow E\varepsilon_{yy} = -\nu\sigma_{xx} - \nu\sigma_{yy} + \sigma_{zz}.$$

That is,

$$\begin{bmatrix} E\varepsilon_{xx} \\ E\varepsilon_{yy} \\ E\varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{bmatrix}.$$

Since

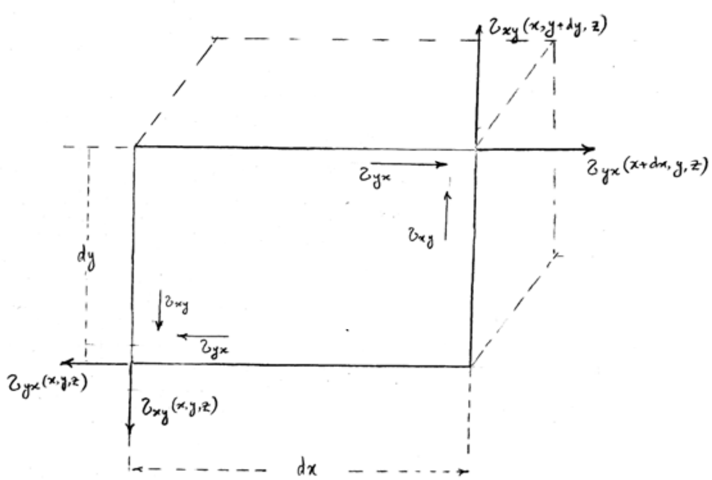
$$\begin{vmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{vmatrix} = (1+\nu)^2(1-2\nu),$$

$$\sigma_{xx} = \frac{\begin{vmatrix} E\varepsilon_{xx} & -\nu & -\nu \\ E\varepsilon_{yy} & 1 & -\nu \\ E\varepsilon_{zz} & -\nu & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{vmatrix}} = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy} + \nu\varepsilon_{zz}],$$

$$\sigma_{yy} = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_{xx} + (1-\nu)\varepsilon_{yy} + \nu\varepsilon_{zz}],$$

$$\sigma_{zz} = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_{xx} + \nu\varepsilon_{yy} + (1-\nu)\varepsilon_{zz}].\square$$

10.3 The Shear Stresses, σ_{xy}, σ_{yx}, on box faces



The Shear Stress $\sigma_{xy}(x,y,z)$ is the force per unit area in the xy plane, along y,

The Shear Stress $\sigma_{yx}(x,y,z)$ is the force per unit area in yx plane, along x,

10.4 Hooke's Law for a Shear Stresses on box faces

The shear stress σ_{xy} satisfies Hooke's Law.

$$\sigma_{xy} = 2G\varepsilon_{xy},$$

$$\sigma_{yz} = 2G\varepsilon_{yz},$$

$$\sigma_{zx} = 2G\varepsilon_{zx}.$$

where $G =$ **Shear Modulus.**

Similarly, in isotropic material,

$$\sigma_{yx} = 2G\varepsilon_{yx},$$

$$\sigma_{zy} = 2G\varepsilon_{zy},$$

$$\sigma_{xz} = 2G\varepsilon_{xz}.$$

10.5 $\sigma_{xy} = \sigma_{yx}; \quad \sigma_{yz} = \sigma_{zy}; \quad \sigma_{xy} = \sigma_{yx}.$

Proof: Since $\varepsilon_{xy} = \varepsilon_{yx}$, $\varepsilon_{yz} = \varepsilon_{zy}$, $\varepsilon_{zx} = \varepsilon_{xz}$.$\square$

10.6 Hook's Law for a Box

$$E = \textit{Young's Elastic Modulus}$$

$$\nu = \textit{Poisson Ratio}$$

$$G = \textit{Shear Modulus} = g\frac{E}{1-\nu^2}$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & & & \\ \nu & 1-\nu & \nu & & & \\ \nu & 0 & 1-\nu & & & \\ & & & 2g & & \\ & & & & 2g & \\ & & & & & 2g \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{bmatrix}$$

10.7

$$2G = \frac{E}{1+\nu}$$

$$\sigma_{xy} = 2G\varepsilon_{xy} = \frac{E}{1+\nu}\varepsilon_{xy}$$

$$\sigma_{yz} = 2G\varepsilon_{yz} = \frac{E}{1+\nu}\varepsilon_{yz}$$

$$\sigma_{zx} = 2G\varepsilon_{zx} = \frac{E}{1+\nu}\varepsilon_{zx}$$

Proof: By 10.2

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{bmatrix}.$$

The matrix

$$\begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix}$$

has the eigen values $1-2\nu$, and $1+\nu$, and can be rotated so that

$$\begin{bmatrix} \sigma'_{xx} \\ \sigma'_{yy} \\ \sigma'_{zz} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-2\nu & 0 & 0 \\ 0 & 1-2\nu & 0 \\ 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \varepsilon'_{xx} \\ \varepsilon'_{yy} \\ \varepsilon'_{zz} \end{bmatrix} = E \begin{bmatrix} \frac{1}{1+\nu} & 0 & 0 \\ 0 & \frac{1}{1+\nu} & 0 \\ 0 & 0 & \frac{1}{1-2\nu} \end{bmatrix} \begin{bmatrix} \varepsilon'_{xx} \\ \varepsilon'_{yy} \\ \varepsilon'_{zz} \end{bmatrix}.$$

That rotation diagonalizes also the matrix

$$\begin{bmatrix} 1-\nu & \nu & \nu & & & \\ \nu & 1-\nu & \nu & & & \\ \nu & 0 & 1-\nu & & & \\ & & & 2g & & \\ & & & & 2g & \\ & & & & & 2g \end{bmatrix}.$$

The equation for its eigen values yields

$$\lambda = 2g; \quad \lambda = 1+\nu; \; \lambda = 1-2\nu.$$

For $\lambda = 1-2\nu$,

$$2g = 1 - 2\nu\,,$$

$$2G = \tfrac{E}{(1+\nu)(1-2\nu)}(1 - 2\nu) = \tfrac{E}{(1+\nu)}\,,$$

$$\sigma_{xy} = \tfrac{E}{(1+\nu)}\varepsilon_{xy}$$

$$\sigma_{yz} = \tfrac{E}{(1+\nu)}\varepsilon_{yz}$$

$$\sigma_{zx} = \tfrac{E}{(1+\nu)}\varepsilon_{zx}\,.\square$$

10.8 Body Forces $f_x(x,y,z)$, $f_y(x,y,z)$, $f_z(x,y,z)$

Body forces are exterior forces such as gravitational, or magnetic, per unit volume.

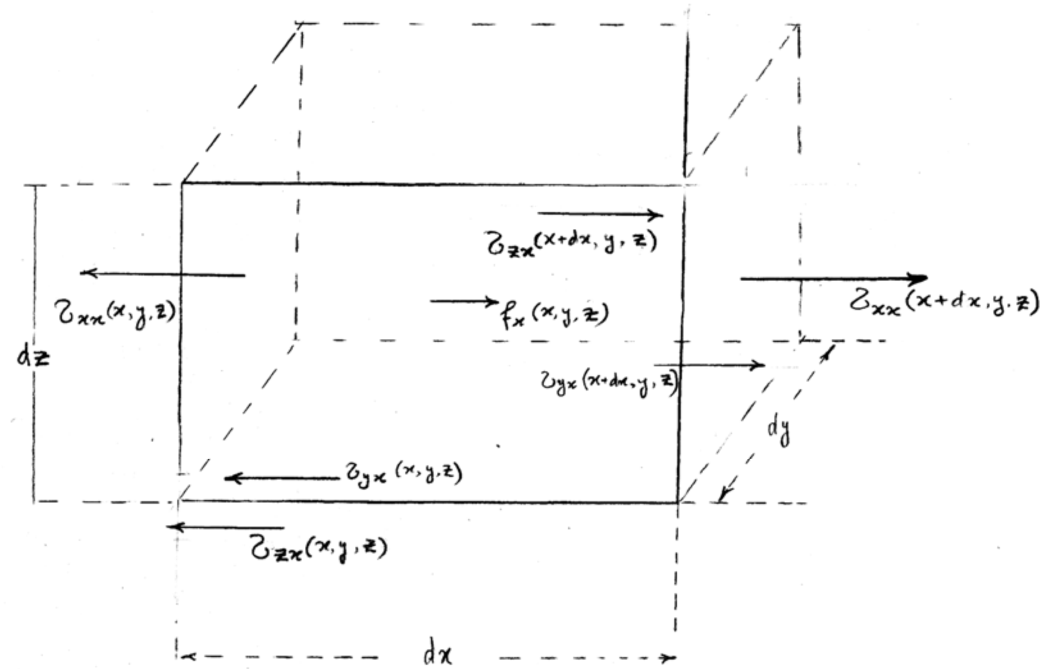
10.9 The Equilibrium Equations for a Box

$$\sigma_{xx,x} + \sigma_{xy,y} + \sigma z_{xz,z} + f_x = 0\,,$$

$$\sigma_{yx,x} + \sigma_{yy,y} + \sigma_{yz,z} + f_y = 0\,,$$

$$\sigma_{zx,x} + \sigma_{zy,y} + \sigma_{zz,z} + f_z = 0\,.$$

Proof: If body forces $f_x(x,y,z)$, $f_y(x,y,z)$, and $f_z(x,y,z)$ per unit volume apply to the box,



In the x direction,

the force due to σ_{xx} is

$$\underbrace{[\sigma_{xx}(x+dx,y,z)-\sigma_{xx}(x,y,z)]}_{=\sigma_{xx,x}dx}dydz = \sigma_{xx,x}dxdydz\,,$$

the force due to σ_{yx} is

$$\underbrace{[\sigma_{yx}(x,y+dy,z)-\sigma_{yx}(x,y,z)]}_{=\sigma_{yx,y}dy=\sigma_{xy,y}dy}dxdz = \sigma_{xy,y}dxdydz\,,$$

the force due to σ_{zx} is

$$\underbrace{[\sigma_{zx}(x,y,z+dz)-\sigma_{zx}(x,y,z)]}_{=\sigma_{zx,z}dy=\sigma_{xz,z}dz}dxdy = \sigma_{xz,z}dxdydz\,,$$

and the force due to f_x is $f_x(x,y,z)dxdydz$.

Therefore, the Equilibrium of forces in the x direction is,

$$\sigma_{xx,x} + \sigma_{xy,y} + \sigma z_{xz,z} + f_x = 0.$$

Similarly,

<u>in the y direction</u>

the Equilibrium of forces is

$$\sigma_{yx,x} + \sigma_{yy,y} + \sigma_{yz,z} + f_y = 0,$$

<u>and in the z direction</u>

the Equilibrium of forces is

$$\sigma_{zx,x} + \sigma_{zy,y} + \sigma_{zz,z} + f_z = 0. \square$$

11.

Strain Energy of a Box

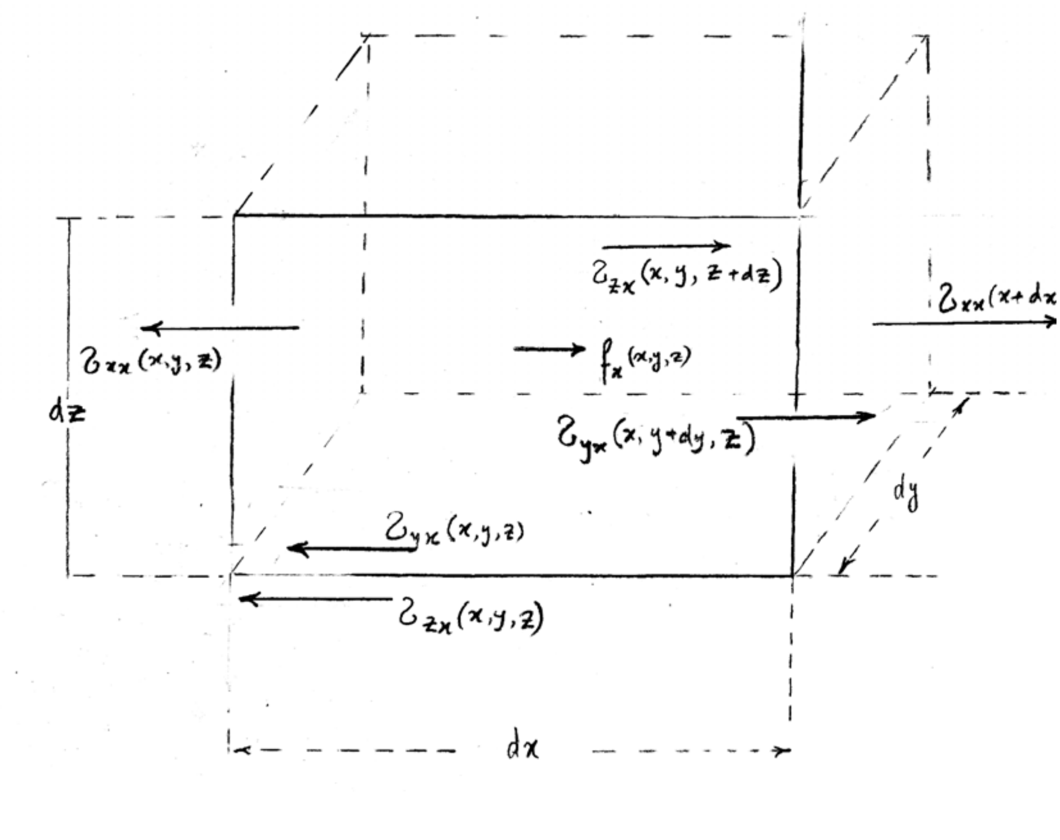
11.1 Infinitesimal Strain Energy of a Box

$$d\mu = \sigma_{xx} d\varepsilon_{xx} + \sigma_{yy} d\varepsilon_{yy} + \sigma_{zz} d\varepsilon_{zz} + 2\sigma_{xy} d\varepsilon_{xy} + 2\sigma_{zy} d\varepsilon_{zy} + 2\sigma_{zx} d\varepsilon_{zx}$$

$$= \sigma_{ij} d\varepsilon_{ij}. \text{ (summation over } i = 1,2,3;\ j = 1,2,3\text{).}$$

$$\frac{\partial \mu}{\partial \varepsilon_{xx}} = \sigma_{xx};\ \frac{\partial \mu}{\partial \varepsilon_{yy}} = \sigma_{yy};\ \frac{\partial \mu}{\partial \varepsilon_{zz}} = \sigma_{zz};\ \frac{\partial \mu}{\partial \varepsilon_{xy}} = 2\sigma_{xy};\ \frac{\partial \mu}{\partial \varepsilon_{yz}} = 2\sigma_{yz};\ \frac{\partial \mu}{\partial \varepsilon_{zx}} = 2\sigma_{xz}$$

Proof: In the x direction,



The infinitesimal work done by $\sigma_{xx}dydz$, along du_x is

$$(\sigma_{xx}dydzdu_x)\Big|_{(x+dx,y,z)} - (\sigma_{xx}dydzdu_x)\Big|_{(x,y,z)} = \partial_x(\sigma_{xx}du_x)dxdydz$$

$$= [\underbrace{\partial_x(\sigma_{xx})}_{\sigma_{xx,x}}du_x + \sigma_{xx}\underbrace{\partial_x du_x}_{du_{x,x}}]dxdydz$$

The infinitesimal work done by $\sigma_{yx}dydz$, along du_x is

$$(\sigma_{yx}dydzdu_x)\Big|_{(x,y+dy,z)} - (\sigma_{yx}dxdzdu_x)\Big|_{(x,y,z)} = \partial_y(\sigma_{yx}du_x)dydxdz$$

$$= [\underbrace{\partial_y(\sigma_{yx})}_{\sigma_{xy,y}}du_x + \sigma_{yx}\underbrace{\partial_y du_x}_{du_{x,y}}]dxdydz$$

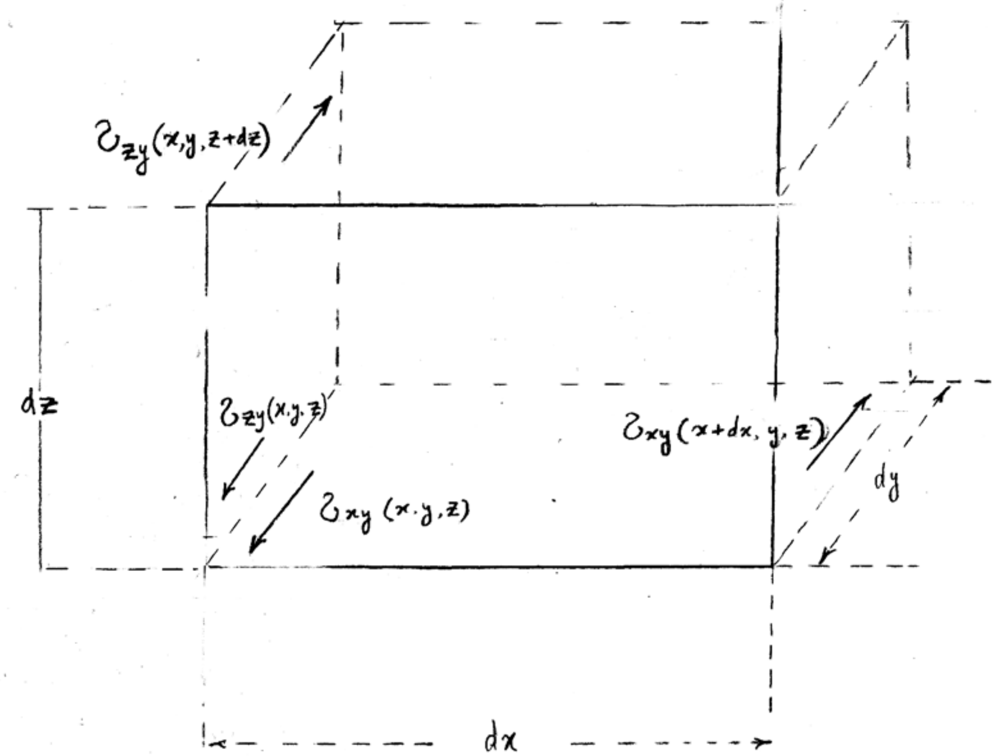
The infinitesimal work done by $\sigma_{zx}dxdy$, along du_x is

$$(\sigma_{zx}dxdydu_x)\Big|_{(x,y,z+dz)} - (\sigma_{zx}dxdydu_x)\Big|_{(x,y,z)} = \partial_z(\sigma_{zx}du_x)dydxdz$$

$$= [\underbrace{\partial_z(\sigma_{zx})}_{\sigma_{xz,z}}du_x + \sigma_{zx}\underbrace{\partial_z du_x}_{du_{x,z}}]dxdydz$$

The infinitesimal work done by $f_x(x,y,z)dxdydz$ is $f_x(x,y,z)dxdydzdu_x$.

<u>In the y direction</u>,



The infinitesimal work done by $\sigma_{yy}dxdz$, along du_y is

$$(\sigma_{yy}dxdzdu_y)\Big|_{(x,y+dy,z)} - (\sigma_{yy}dxdzdu_y)\Big|_{(x,y,z)} = \partial_y(\sigma_{yy}du_y)dxdydz$$

$$= [\underbrace{\partial_y(\sigma_{yy})}_{\sigma_{yy,y}}du_y + \sigma_{yy}\underbrace{\partial_y du_y}_{du_{y,y}}]dxdydz$$

The infinitesimal work done by $\sigma_{xy}dydz$, along du_y is

$$(\sigma_{xy}dydzu_y)\Big|_{(x+dx,y,z)} - (\sigma_{xy}dydzdu_y)\Big|_{(x,y,z)} = \partial_x(\sigma_{xy}du_y)dxdydz$$

$$= [\underbrace{\partial_x(\sigma_{xy})}_{\sigma_{yx,x}}du_y + \sigma_{yx}\underbrace{\partial_x du_y}_{du_{y,x}}]dxdydz$$

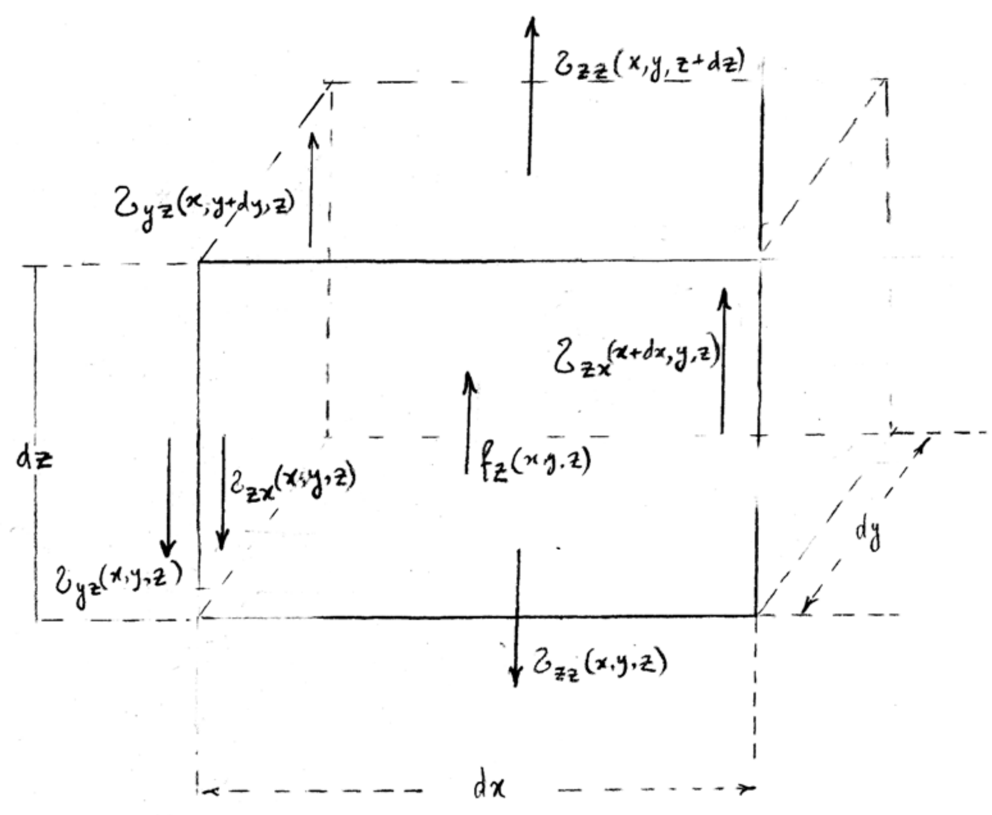
The infinitesimal work done by $\sigma_{zy}dxdy$, along du_y is

$$(\sigma_{zy}dxdydu_y)\Big|_{(x,y,z+dz)} - (\sigma_{zy}dxdydu_y)\Big|_{(x,y,z)} = \partial_z(\sigma_{zy}du_y)dxdydz$$

$$= [\underbrace{\partial_z(\sigma_{zy})}_{\sigma_{yz,z}}du_y + \sigma_{yz}\underbrace{\partial_z du_y}_{du_{y,z}}]dxdydz$$

The work by $f_y(x,y,z)dxdydz$ is $f_y(x,y,z)dxdydzdu_y$.

<u>In the z direction</u>,



The infinitesimal work done by $\sigma_{zz}dxdy$, along du_z is

$$(\sigma_{zz}dxdydu_z)\Big|_{(x,y,z+dz)} - (\sigma_{zz}dxdzdu_z)\Big|_{(x,y,z)} = \partial_z(\sigma_{zz}du_z)dxdydz$$

$$= [\underbrace{\partial_z(\sigma_{zz})}_{\sigma_{zz,z}} du_z + \sigma_{zz} \underbrace{\partial_z du_z}_{du_{z,z}}]dxdydz$$

The infinitesimal work done by $\sigma_{zx}dydz$, along du_z is

$$(\sigma_{zx}dydzu_z)\Big|_{(x+dx,y,z)} - (\sigma_{zx}dydzdu_z)\Big|_{(x,y,z)} = \partial_x(\sigma_{zx}du_z)dxdydz$$

$$= [\underbrace{\partial_x(\sigma_{zx})}_{\sigma_{zx,x}} du_z + \sigma_{zx} \underbrace{\partial_x du_z}_{du_{z,x}}]dxdydz$$

The infinitesimal work done by $\sigma_{zy}dxdz$, along du_z is

$$(\sigma_{zy}dxdzdu_z)\Big|_{(x,y+dy,z)} - (\sigma_{zy}dxdzdu_z)\Big|_{(x,y,z)} = \partial_y(\sigma_{zy}du_z)dxdydz$$

$$= [\underbrace{\partial_y(\sigma_{zy})}_{\sigma_{yz,z}} du_y + \sigma_{yz} \underbrace{\partial_y du_z}_{du_{z,y}}]dxdydz$$

The work by $f_z(x,y,z)dxdydz$ is $f_z(x,y,z)dxdydzdu_z$.

The total infinitesimal work is the sum

$$\{\underbrace{[\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} + f_x(x,y,z)]}_{0} du_x\}dxdydz +$$

$$+\{\underbrace{[\sigma_{yx,x} + \sigma_{yy,y} + \sigma_{yz,z} + f_y(x,y,z)]}_{0} du_y\}dxdydz +$$

$$+\{\underbrace{[\sigma_{zx,x} + \sigma_{zy,y} + \sigma_{zz,z} + f_z(x,y,z)]}_{0} du_z\}dxdydz +$$

$$+\{\sigma_{xx}d\underbrace{u_{x,x}}_{\varepsilon_{xx}} + \sigma_{xy}du_{x,y} + \sigma_{xz}du_{x,z}\}dxdydz +$$

$$+\{\sigma_{yy}d\underbrace{u_{y,y}}_{\varepsilon_{yy}} + \sigma_{yx}du_{y,x} + \sigma_{yz}du_{y,z}\}dxdydz +$$

$$+\{\sigma_{zz}d\underbrace{u_{z,z}}_{\varepsilon_{zz}} + \sigma_{zy}du_{z,y} + \sigma_{z,x}du_{z,x}\}dxdydz$$

$$= \{\sigma_{xx}d\varepsilon_{xx} + \sigma_{yy}d\varepsilon_{yy} + \sigma_{zz}d\varepsilon_{zz}\}dxdydz$$

$$+\{\underbrace{\sigma_{xy}du_{x,y} + \sigma_{yx}du_{y,x}}_{2\sigma_{xy}d\varepsilon_{xy}} + \underbrace{\sigma_{yz}du_{y,z} + \sigma_{zy}du_{z,y}}_{2\sigma_{yz}d\varepsilon_{yz}} + \underbrace{\sigma_{zx}du_{z,x} + \sigma_{xz}du_{x,z}}_{2\sigma_{zx}d\varepsilon_{zx}}\}dxdydz$$

The Infinitesimal work is stored in the box as Strain Energy.

The Infinitesimal Strain Energy per unit volume is

$$d\mu = \sigma_{xx}d\varepsilon_{xx} + \sigma_{yy}d\varepsilon_{yy} + \sigma_{zz}d\varepsilon_{zz} + 2\sigma_{xy}d\varepsilon_{xy} + 2\sigma_{yz}d\varepsilon_{yz} + 2\sigma_{zx}d\varepsilon_{zx}.\square$$

Hence,

$$\frac{\partial\mu}{\partial\varepsilon_{xx}} = \sigma_{xx}; \qquad \frac{\partial\mu}{\partial\varepsilon_{yy}} = \sigma_{yy}; \qquad \frac{\partial\mu}{\partial\varepsilon_{zz}} = \sigma_{zz};$$

$$\frac{\partial\mu}{\partial\varepsilon_{xy}} = 2\sigma_{xy}; \qquad \frac{\partial\mu}{\partial\varepsilon_{yz}} = 2\sigma_{yz}; \qquad \frac{\partial\mu}{\partial\varepsilon_{zx}} = 2\sigma_{xz}.\square$$

11.2 Strain Energy density of a Box

$$\mu = \tfrac{1}{2}\sigma_{xx}\varepsilon_{xx} + \tfrac{1}{2}\sigma_{yy}\varepsilon_{yy} + \tfrac{1}{2}\sigma_{zz}\varepsilon_{zz} + \sigma_{xy}\varepsilon_{xy} + \sigma_{yz}\varepsilon_{yz} + \sigma_{zx}\varepsilon_{zx}.$$

$= \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij}$. (summation over $i, j = 1,2,3$).

$$= \tfrac{1}{2}\tfrac{E(1-\nu)}{(1+\nu)(1-2\nu)}(\varepsilon_{xx}^2 + \varepsilon_{yy}^2 + \varepsilon_{zz}^2) + \tfrac{E\nu}{(1+\nu)(1-2\nu)}\{\varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{yy}\varepsilon_{zz} + \varepsilon_{zz}\varepsilon_{xx}\} +$$

$$+2G(\varepsilon_{xy}^2 + \varepsilon_{yz}^2 + \varepsilon_{zx}^2).$$

Proof:

$$d\mu = \sigma_{xx}d\varepsilon_{xx} + \sigma_{yy}d\varepsilon_{yy} + \sigma_{zz}d\varepsilon_{zz} + 2\sigma_{xy}d\varepsilon_{xy} + 2\sigma_{yz}d\varepsilon_{yz} + 2\sigma_{zx}d\varepsilon_{zx}$$

Substituting

$$\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy} + \nu\varepsilon_{zz}],$$

$$\sigma_{yy} = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_{xx} + (1-\nu)\varepsilon_{yy} + \nu\varepsilon_{zz}],$$

$$\sigma_{zz} = \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_{xx} + \nu\varepsilon_{yy} + (1-\nu)\varepsilon_{zz}],$$

$$\sigma_{xy} = \frac{E}{1+\nu}\varepsilon_{xy},$$

$$\sigma_{yz} = \frac{E}{1+\nu}\varepsilon_{yz},$$

$$\sigma_{zx} = \frac{E}{1+\nu}\varepsilon_{zx}.$$

$$d\mu = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy} + \nu\varepsilon_{zz}]d\varepsilon_{xx} +$$

$$+\frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_{xx} + (1-\nu)\varepsilon_{yy} + \nu\varepsilon_{zz}]d\varepsilon_{yy} +$$

$$+\frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_{xx} + \nu\varepsilon_{yy} + (1-\nu)\varepsilon_{zz}]d\varepsilon_{zz} +$$

$$+2\frac{E}{1+\nu}\varepsilon_{xy}d\varepsilon_{xy} + 2\frac{E}{1+\nu}\varepsilon_{yz}d\varepsilon_{yz} + 2\frac{E}{1+\nu}\varepsilon_{zx}d\varepsilon_{zx}.$$

$$= \frac{1}{2}\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}d(\varepsilon_{xx}^2 + \varepsilon_{yy}^2 + \varepsilon_{zz}^2) +$$

$$+\frac{E\nu}{(1+\nu)(1-2\nu)}\{\underbrace{[\varepsilon_{yy}d\varepsilon_{xx} + \varepsilon_{xx}d\varepsilon_{yy}]}_{d(\varepsilon_{xx}\varepsilon_{yy})} + \underbrace{[\varepsilon_{yy}d\varepsilon_{zz} + \varepsilon_{zz}d\varepsilon_{yy}]}_{d(\varepsilon_{yy}\varepsilon_{zz})} + \underbrace{[\varepsilon_{zz}d\varepsilon_{xx} + \varepsilon_{xx}d\varepsilon_{zz}]}_{d(\varepsilon_{zz}\varepsilon_{xx})}\} +$$

$$+\underbrace{\frac{E}{1+\nu}}_{2G}d(\varepsilon_{xy}^2 + \varepsilon_{yz}^2 + \varepsilon_{zx}^2).$$

Integrating,

$$\mu = \frac{1}{2}\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}(\varepsilon_{xx}^2 + \varepsilon_{yy}^2 + \varepsilon_{zz}^2) + \frac{E\nu}{(1+\nu)(1-2\nu)}\{\varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{yy}\varepsilon_{zz} + \varepsilon_{zz}\varepsilon_{xx}\} +$$

$$+2G(\varepsilon_{xy}^2 + \varepsilon_{yz}^2 + \varepsilon_{zx}^2).$$

$$= \frac{1}{2}\underbrace{\frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_{xx} + (1-\nu)\varepsilon_{yy} + \nu\varepsilon_{zz}]}_{\sigma_{yy}}\varepsilon_{yy} +$$

$$+\frac{1}{2}\underbrace{\frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_{xx} + \nu\varepsilon_{yy} + (1-\nu)\varepsilon_{zz}]}_{\sigma_{zz}}\varepsilon_{zz} +$$

$$+\underbrace{2G\varepsilon_{xy}}_{\sigma_{xy}}\varepsilon_{xy} + \underbrace{2G\varepsilon_{yz}}_{\sigma_{yz}}\varepsilon_{yz} + \underbrace{2G\varepsilon_{zx}}_{\sigma_{zx}}\varepsilon_{zx}$$

$$= \frac{1}{2}\{\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz}\} + \sigma_{xy}\varepsilon_{xy} + \sigma_{yz}\varepsilon_{yz} + \sigma_{zx}\varepsilon_{zx}.\square$$

11.3 Complementary Strain Energy of a Box

$$\mu^* = \frac{1}{2}\sigma_{xx}\varepsilon_{xx} + \frac{1}{2}\sigma_{yy}\varepsilon_{yy} + \frac{1}{2}\sigma_{zz}\varepsilon_{zz} + \sigma_{xy}\varepsilon_{xy} + \sigma_{yz}\varepsilon_{yz} + \sigma_{zx}\varepsilon_{zx}$$

$$= \frac{1}{2E}(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2) - \frac{\nu}{E}(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx}) +$$

$$+\frac{1}{2G}(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2)$$

$$\frac{\partial\mu^*}{\partial\sigma_{xx}} = \varepsilon_{xx}; \quad \frac{\partial\mu^*}{\partial\sigma_{yy}} = \varepsilon_{yy}; \quad \frac{\partial\mu^*}{\partial\sigma_{zz}} = \varepsilon_{zz};$$

$$\frac{\partial\mu^*}{\partial\sigma_{xy}} = 2\varepsilon_{xy}; \quad \frac{\partial\mu^*}{\partial\sigma_{yz}} = 2\varepsilon_{yz}; \quad \frac{\partial\mu^*}{\partial\sigma_{zx}} = 2\varepsilon_{zx}$$

Proof: The complementary strain energy density is the strain energy density expressed in terms of the stresses.

Substituting

$$\varepsilon_{xx} = \tfrac{1}{E}\sigma_{xx} - \nu\tfrac{1}{E}\sigma_{yy} - \nu\tfrac{1}{E}\sigma_{zz},$$

$$\varepsilon_{yy} = -\nu\tfrac{1}{E}\sigma_{xx} + \tfrac{1}{E}\sigma_{yy} - \nu\tfrac{1}{E}\sigma_{zz},$$

$$\varepsilon_{zz} = -\nu\tfrac{1}{E}\sigma_{xx} - \nu\tfrac{1}{E}\sigma_{yy} + \tfrac{1}{E}\sigma_{zz},$$

$$\varepsilon_{xy} = \tfrac{1}{2G}\sigma_{xy},$$

$$\varepsilon_{yz} = \tfrac{1}{2G}\sigma_{yz},$$

$$\varepsilon_{zx} = \tfrac{1}{2G}\sigma_{zx},$$

into μ,

$$\mu = \tfrac{1}{2}\sigma_{xx}\varepsilon_{xx} + \tfrac{1}{2}\sigma_{yy}\varepsilon_{yy} + \tfrac{1}{2}\sigma_{zz}\varepsilon_{zz} + \sigma_{xy}\varepsilon_{xy} + \sigma_{yz}\varepsilon_{yz} + \sigma_{zx}\varepsilon_{zx},$$

$$= \tfrac{1}{2E}\sigma_{xx}(\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz}) +$$

$$+\tfrac{1}{2E}\sigma_{yy}(-\nu\sigma_{xx} + \sigma_{yy} - \nu\sigma_{zz}) +$$

$$+\tfrac{1}{2E}\sigma_{zz}(-\nu\sigma_{xx} - \nu\sigma_{yy} + \sigma_{zz}) +$$

$$+\tfrac{1}{2G}\sigma_{xy}^2 + \tfrac{1}{2G}\sigma_{yz}^2 + \tfrac{1}{2G}\sigma_{zx}^2,$$

$$= \tfrac{1}{2E}(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2) - \tfrac{\nu}{E}(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx}) +$$

$$+\tfrac{1}{2G}(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2).$$

Although μ^* equals μ, it is denoted by μ^* to specify its dependence on the stress.

$$\frac{\partial \mu^*}{\partial \sigma_{xx}} = \tfrac{1}{E}\sigma_{xx} - \nu \tfrac{1}{E}\sigma_{yy} - \nu \tfrac{1}{E}\sigma_{zz} = \varepsilon_{xx},$$

$$\frac{\partial \mu^*}{\partial \sigma_{yy}} = \tfrac{1}{E}\sigma_{yy} - \nu \tfrac{1}{E}\sigma_{xx} - \nu \tfrac{1}{E}\sigma_{zz} = \varepsilon_{yy},$$

$$\frac{\partial \mu^*}{\partial \sigma_{zz}} = \tfrac{1}{E}\sigma_{zz} - \nu \tfrac{1}{E}\sigma_{xx} - \nu \tfrac{1}{E}\sigma_{yy} = \varepsilon_{zz},$$

$$\frac{\partial \mu^*}{\partial \sigma_{xy}} = \tfrac{1}{G}\sigma_{xy} = 2\varepsilon_{xy},$$

$$\frac{\partial \mu^*}{\partial \sigma_{yz}} = \tfrac{1}{G}\sigma_{yz} = 2\varepsilon_{yz},$$

$$\frac{\partial \mu^*}{\partial \sigma_{zx}} = \tfrac{1}{G}\sigma_{zx} = 2\varepsilon_{zx}.\Box$$

12.

Variational Principles on a Box

12.1 Euler's Variational Equations

The deflections u_x, u_y, *and* u_z *th at minimize or maximize*

the box energy per unit volume,

$$F = F(x, y, z, u_x, u_y, u_z, \varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx})$$

satisfy Euler's Variational Equation

$$\delta F(x, y, z, u_x, u_y, u_z, \varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx}) = 0,$$

which is,

$$\left(\frac{\partial F}{\partial u_x} - \partial_x \frac{\partial F}{\partial \varepsilon_{xx}} - \tfrac{1}{2}\partial_y \frac{\partial F}{\partial \varepsilon_{xy}} - \tfrac{1}{2}\partial_z \frac{\partial F}{\partial \varepsilon_{xz}}\right)\delta u_x +$$

$$+\left(\frac{\partial F}{\partial u_y} - \tfrac{1}{2}\partial_x \frac{\partial F}{\partial \varepsilon_{yx}} - \partial_y \frac{\partial F}{\partial \varepsilon_{yy}} - \tfrac{1}{2}\partial_z \frac{\partial F}{\partial \varepsilon_{yz}}\right)\delta u_y +$$

$$+\left(\frac{\partial F}{\partial u_z} - \tfrac{1}{2}\partial_x \frac{\partial F}{\partial \varepsilon_{zx}} - \tfrac{1}{2}\partial_y \frac{\partial F}{\partial \varepsilon_{zy}} - \partial_z \frac{\partial F}{\partial \varepsilon_{zz}}\right)\delta u_z = 0.$$

or,

$$\frac{\partial F}{\partial u_x} - \partial_x \frac{\partial F}{\partial \varepsilon_{xx}} - \tfrac{1}{2}\partial_y \frac{\partial F}{\partial \varepsilon_{xy}} - \tfrac{1}{2}\partial_z \frac{\partial F}{\partial \varepsilon_{xz}} = 0,$$

$$\frac{\partial F}{\partial u_y} - \tfrac{1}{2}\partial_x \frac{\partial F}{\partial \varepsilon_{yx}} - \partial_y \frac{\partial F}{\partial \varepsilon_{yy}} - \tfrac{1}{2}\partial_z \frac{\partial F}{\partial \varepsilon_{yz}} = 0,$$

$$\frac{\partial F}{\partial u_z} - \tfrac{1}{2}\partial_x \frac{\partial F}{\partial \varepsilon_{zx}} - \tfrac{1}{2}\partial_y \frac{\partial F}{\partial \varepsilon_{zy}} - \partial_z \frac{\partial F}{\partial \varepsilon_{zz}} = 0.$$

Proof: The Box infinitesimal energy is

$$\int_{\xi=x-\frac{1}{2}dx}^{\xi=x+\frac{1}{2}dx} \int_{\eta=y-\frac{1}{2}dy}^{\eta=y+\frac{1}{2}dy} \int_{\zeta=z-\frac{1}{2}dz}^{\zeta=z+\frac{1}{2}dz} F(\xi,\eta,\zeta,u_\xi,u_\eta,u_\zeta,\varepsilon_{\xi\xi},\varepsilon_{\eta\eta},\varepsilon_{\zeta\zeta},\varepsilon_{\xi\eta},\varepsilon_{\eta\zeta},\varepsilon_{\zeta\xi})d\xi d\eta d\zeta.$$

The deflections u_x, u_y, and u_z that minimizes or maximize the box energy, satisfy Euler's Variational Equation

$$\delta F(x,y,z,u_x,u_y,u_z,\varepsilon_{xx},\varepsilon_{yy},\varepsilon_{zz},\varepsilon_{xy},\varepsilon_{yz},\varepsilon_{zx}) = 0.$$

That is,

$$\frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial \varepsilon_{xx}}\underbrace{\delta\varepsilon_{xx}}_{\partial_x \delta u_x} + \frac{\partial F}{\partial u_y}\delta u_y + \frac{\partial F}{\partial \varepsilon_{yy}}\underbrace{\delta\varepsilon_{yy}}_{\partial_y \delta u_y} + + \frac{\partial F}{\partial u_z}\delta u_z + \frac{\partial F}{\partial \varepsilon_{zz}}\underbrace{\delta\varepsilon_{zz}}_{\partial_z \delta u_z} +$$

$$+ \frac{\partial F}{\partial \varepsilon_{xy}}\underbrace{\delta\varepsilon_{xy}}_{\frac{1}{2}\partial_x\delta u_y + \frac{1}{2}\partial_y \delta u_x} + \frac{\partial F}{\partial \varepsilon_{yz}}\underbrace{\delta\varepsilon_{yz}}_{\frac{1}{2}\partial_z\delta u_y + \frac{1}{2}\partial_y \delta u_z} + \frac{\partial F}{\partial \varepsilon_{xy}}\underbrace{\delta\varepsilon_{zx}}_{\frac{1}{2}\partial_x\delta u_z + \frac{1}{2}\partial_z \delta u_x} = 0.$$

$$\frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial \varepsilon_{xx}}\partial_x \delta u_x + \tfrac{1}{2}\frac{\partial F}{\partial \varepsilon_{xy}}\partial_y \delta u_x + \tfrac{1}{2}\frac{\partial F}{\partial \varepsilon_{xz}}\partial_z \delta u_x +$$

$$+\frac{\partial F}{\partial u_y}\delta u_y + \frac{\partial F}{\partial \varepsilon_{yy}}\partial_y \delta u_y + \tfrac{1}{2}\frac{\partial F}{\partial \varepsilon_{yx}}\partial_x \delta u_y + \tfrac{1}{2}\frac{\partial F}{\partial \varepsilon_{yz}}\partial_z \delta u_y +$$

$$+\frac{\partial F}{\partial u_z}\delta u_z + \frac{\partial F}{\partial \varepsilon_{zz}}\partial_z \delta u_z + \tfrac{1}{2}\frac{\partial F}{\partial \varepsilon_{zx}}\partial_x \delta u_z + \tfrac{1}{2}\frac{\partial F}{\partial \varepsilon_{zy}}\partial_y \delta u_z = 0$$

Integration by parts, where δu_x, δu_y, and δu_z vanish on the boundary, [Dan3] leads to the variational equation

$$\left(\frac{\partial F}{\partial u_x} - \partial_x \frac{\partial F}{\partial \varepsilon_{xx}} - \tfrac{1}{2}\partial_y \frac{\partial F}{\partial \varepsilon_{xy}} - \tfrac{1}{2}\partial_z \frac{\partial F}{\partial \varepsilon_{xz}}\right)\delta u_x +$$

$$+\left(\frac{\partial F}{\partial u_y} - \tfrac{1}{2}\partial_x \frac{\partial F}{\partial \varepsilon_{yx}} - \partial_y \frac{\partial F}{\partial \varepsilon_{yy}} - \tfrac{1}{2}\partial_z \frac{\partial F}{\partial \varepsilon_{yz}}\right)\delta u_y +$$

$$+\left(\frac{\partial F}{\partial u_z} - \tfrac{1}{2}\partial_x \frac{\partial F}{\partial \varepsilon_{zx}} - \tfrac{1}{2}\partial_y \frac{\partial F}{\partial \varepsilon_{zy}} - \partial_z \frac{\partial F}{\partial \varepsilon_{zz}}\right)\delta u_z = 0.$$

Since the variations δu_x, δu_y, and δu_z are independent, the deflections u_x, u_y, and u_z that minimizes or maximize the box energy satisfy Euler's Variational Equations

$$\frac{\partial F}{\partial u_x} - \partial_x \frac{\partial F}{\partial \varepsilon_{xx}} - \tfrac{1}{2}\partial_y \frac{\partial F}{\partial \varepsilon_{xy}} - \tfrac{1}{2}\partial_z \frac{\partial F}{\partial \varepsilon_{xz}} = 0,$$

$$\frac{\partial F}{\partial u_y} - \tfrac{1}{2}\partial_x \frac{\partial F}{\partial \varepsilon_{yx}} - \partial_y \frac{\partial F}{\partial \varepsilon_{yy}} - \tfrac{1}{2}\partial_z \frac{\partial F}{\partial \varepsilon_{yz}} = 0,$$

$$\frac{\partial F}{\partial u_z} - \tfrac{1}{2}\partial_x \frac{\partial F}{\partial \varepsilon_{zx}} - \tfrac{1}{2}\partial_y \frac{\partial F}{\partial \varepsilon_{zy}} - \partial_z \frac{\partial F}{\partial \varepsilon_{zz}} = 0 . \Box$$

12.2 <u>If</u> *the body forces originate from a Potential* $\upsilon(u_x, u_y, u_z)$ *per unit volume, so that*

$$-\frac{\partial \upsilon}{\partial u_x} = f_x ,$$

$$-\frac{\partial \upsilon}{\partial u_y} = f_y ,$$

$$-\frac{\partial \upsilon}{\partial u_z} = f_z ,$$

<u>Then,</u> *Euler's Equations are the Equilibrium Equations*

$$\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} + f_x = 0 ,$$

$$\sigma_{yx,x} + \sigma_{yy,y} + \sigma_{yz,z} + f_y = 0 ,$$

$$\sigma_{zx,x} + \sigma_{zy,y} + \sigma_{zz,z} + f_z = 0 .$$

Proof: The Energy per unit volume is

$$F(x, y, z, u_x, u_y, u_z, \varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx}) = \mu + \upsilon$$

$$= \tfrac{1}{2}\sigma_{xx}\varepsilon_{xx} + \tfrac{1}{2}\sigma_{yy}\varepsilon_{yy} + \tfrac{1}{2}\sigma_{zz}\varepsilon_{zz} + \sigma_{xy}\varepsilon_{xy} + \sigma_{yz}\varepsilon_{yz} + \sigma_{zx}\varepsilon_{zx} + \upsilon .$$

The Euler's Variational Equations are

$$\frac{\partial F}{\partial u_x} - \partial_x \underbrace{\frac{\partial F}{\partial \varepsilon_{xx}}}_{\sigma_{xx}} - \tfrac{1}{2}\partial_y \underbrace{\frac{\partial F}{\partial \varepsilon_{xy}}}_{2\sigma_{xy}} - \tfrac{1}{2}\partial_z \underbrace{\frac{\partial F}{\partial \varepsilon_{xz}}}_{2\sigma_{xz}} = 0,$$

$$\frac{\partial F}{\partial u_y} - \tfrac{1}{2}\partial_x \underbrace{\frac{\partial F}{\partial \varepsilon_{yx}}}_{2\sigma_{xy}} - \partial_y \underbrace{\frac{\partial F}{\partial \varepsilon_{yy}}}_{\sigma_{yy}} - \tfrac{1}{2}\partial_z \underbrace{\frac{\partial F}{\partial \varepsilon_{yz}}}_{2\sigma_{yz}} = 0,$$

$$\frac{\partial F}{\partial u_z} - \tfrac{1}{2}\partial_x \underbrace{\frac{\partial F}{\partial \varepsilon_{zx}}}_{2\sigma_{xz}} - \tfrac{1}{2}\partial_y \underbrace{\frac{\partial F}{\partial \varepsilon_{zy}}}_{2\sigma_{zy}} - \partial_z \underbrace{\frac{\partial F}{\partial \varepsilon_{zz}}}_{\sigma_{zz}} = 0.$$

That is, Euler's Equations are the equilibrium Equations

$$\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} + f_x = 0,$$

$$\sigma_{yx,x} + \sigma_{yy,y} + \sigma_{yz,z} + f_y = 0,$$

$$\sigma_{zx,x} + \sigma_{zy,y} + \sigma_{zz,z} + f_z = 0.\square$$

12.3 Principle of Virtual Work

At Equilibrium, $\delta U = \delta W$

Proof: The equilibrium deflections u_x, and u_y that minimize the box energy per unit volume satisfy the Euler Variational Equation

$$\delta F = \left(\frac{\partial F}{\partial u_x} - \partial_x \frac{\partial F}{\partial \varepsilon_{xx}} - \tfrac{1}{2}\partial_y \frac{\partial F}{\partial \varepsilon_{xy}} - \tfrac{1}{2}\partial_z \frac{\partial F}{\partial \varepsilon_{xz}} \right) \delta u_x +$$

$$+ \left(\frac{\partial F}{\partial u_y} - \tfrac{1}{2}\partial_x \frac{\partial F}{\partial \varepsilon_{yx}} - \partial_y \frac{\partial F}{\partial \varepsilon_{yy}} - \tfrac{1}{2}\partial_z \frac{\partial F}{\partial \varepsilon_{yz}} \right) \delta u_y +$$

$$+ \left(\frac{\partial F}{\partial u_z} - \tfrac{1}{2}\partial_x \frac{\partial F}{\partial \varepsilon_{zx}} - \tfrac{1}{2}\partial_y \frac{\partial F}{\partial \varepsilon_{zy}} - \partial_z \frac{\partial F}{\partial \varepsilon_{zz}} \right) \delta u_z = 0.$$

Thus, at equilibrium, the Variation of F, [Dan3], satisfies

$$\delta F = 0.$$

Therefore,

$$\delta U - \delta W = \delta(U - W)$$

$$= \delta \int_{x=0}^{x=l_x} \int_{y=0}^{y=l_y} \int_{z=0}^{z=l_z} F(x, y, z, u_x, u_y, u_z, \varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx}) dxdydz$$

$$= \int_{x=0}^{x=l_x} \int_{y=0}^{y=l_y} \int_{z=0}^{z=l_z} \delta F(x, y, z, u_x, u_y, u_z, \varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx}) dxdydz$$

$$= 0.$$

That is, $\delta U = \delta W$.□

12.4 Principle of Minimal Energy for a Plate

The Box Energy is Minimal at Equilibrium

Proof: Legendre's Sufficient Condition for

$$\int_{x=0}^{x=l_x}\int_{y=0}^{y=l_y}\int_{z=0}^{z=l_z} F(x,y,z,u_x,u_y,u_z,\varepsilon_{xx},\varepsilon_{yy},\varepsilon_{zz},\varepsilon_{xy},\varepsilon_{yz},\varepsilon_{zx})dxdydz$$

to be Minimal at the equilibrium generalizes to

$$\left.\frac{\partial^2 F}{(\partial\varepsilon_{ij})^2}\right|_{\text{equilibrium } u_x;u_y;u_z} > 0\,.$$

By 11.2,

$$F = \tfrac{1}{2}\tfrac{E(1-\nu)}{(1+\nu)(1-2\nu)}(\varepsilon_{xx}^2 + \varepsilon_{yy}^2 + \varepsilon_{zz}^2) +$$

$$+\tfrac{E\nu}{(1+\nu)(1-2\nu)}\{\varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{yy}\varepsilon_{zz} + \varepsilon_{zz}\varepsilon_{xx}\} +$$

$$+2G(\varepsilon_{xy}^2 + \varepsilon_{yz}^2 + \varepsilon_{zx}^2) + \upsilon\,.$$

where $E > 0$ is Young's Elastic Modulus,

$\nu =$ Poisson's Ratio, and $0 < \nu < 1$

$G = \frac{E}{1+\nu} =$ Shear Modulus.

Hence,

$$\frac{\partial^2 F}{(\partial\varepsilon_{xx})^2} = \tfrac{E(1-\nu)}{(1+\nu)(1-2\nu)} > 0\,,$$

$$\frac{\partial^2 F}{(\partial\varepsilon_{yy})^2} = \tfrac{E(1-\nu)}{(1+\nu)(1-2\nu)} > 0\,,$$

$$\frac{\partial^2 F}{(\partial\varepsilon_{zz})^2} = \tfrac{E(1-\nu)}{(1+\nu)(1-2\nu)} > 0$$

$$\frac{\partial^2 F}{(\partial \varepsilon_{xy})^2} = 4G = 2\frac{E}{1+\nu} > 0\,,$$

$$\frac{\partial^2 F}{(\partial \varepsilon_{yz})^2} = 4G = 2\frac{E}{1+\nu} > 0\,,$$

$$\frac{\partial^2 F}{(\partial \varepsilon_{zx})^2} = 4G = 2\frac{E}{1+\nu} > 0$$

Therefore, the Total Energy of the bar is minimal at the equilibrium.□

13.

Castigliano Theorems

There are infinitely many body forces applying to the infinitely many material particles of an elastic structure.

In Engineering, those many forces are replaced by several forces concentrated and at their points of application in the structure.

Then, Castigliano Theorem for the Displacements yields the displacements as the derivatives of the complementary strain energy with respect to the forces at the application points.

And Castigliano Theorem for the Forces yields the forces as the derivatives of the strain energy with respect to the displacements at the application points.

13.1 Castigliano Theorem for the Displacements

If *an elastic structure is at equilibrium under concentrated Body Forces*

$$P_1, P_2, \ldots$$

applied at several points

Then *the equilibrium displacements at the application points are*

$$u_1 = \frac{\partial U^*}{\partial P_1}, \quad u_2 = \frac{\partial U^*}{\partial P_2}, \ldots,$$

where $U^*(P_1, \ldots P_n)$ *is the complementary strain energy of the structure*.

Proof: in chapters 15, and 16.

13.2 Castigliano Theorem for the Forces

If *an elastic structure is at equilibrium under concentrated Body Forces that cause displacements*

$$u_1,\ u_2, \ldots$$

at the application points

Then *the equilibrium forces at the application points are*

$$P_1 = \frac{\partial U}{\partial u_1}, \quad P_2 = \frac{\partial U}{\partial u_2}, \ldots,$$

where $U(u_{1,}, \ldots u_n)$ *is the strain energy of the structure*.

Proof: in chapters 15, and 16.

14. Erroneous Derivations of Castigliano Theorems

Castigliano Theorems follow from the Variational Principles of Elasticity. They can be derived from the Principle of Virtual Work, or from the Principle of Minimal Potential.

Attempting to prove them otherwise may fail.

The following erroneous derivation appears in [Timoshenko], and is followed by [Mura].

[Mura, p.124] has

> *"The variation of the strain energy of an elastic body of volume G is*
>
> $$\delta U = \iiint_G \underbrace{\sigma_{ij}\delta u_{i,j}}_{\partial_j(\sigma_{ij}\delta u_i) - \sigma_{ij,j}\delta u_i} dv \qquad (12.7)$$
>
> $$= \underbrace{\iiint_G \partial_j(\sigma_{ij}\delta u_i)dv}_{\iint_{\partial G} \sigma_{ij}n_j\delta u_i ds} - \iiint_G \sigma_{ij,j}\delta u_i dv$$
>
> *where*
>
> ∂G *is the surface that bounds* G,

n_j *is direction cosines so that* $n_j ds = ds_j$

If

$$\sigma_{ij,j} = 0, \quad in \quad G$$

$$\sigma_{ij} n_j = F_i \quad on \quad \partial G$$

we have Castigliano's Theorem

$$\delta U[u_i] = \iint_{\partial G} F_i \delta u_i ds \text{."}$$

Clearly, there is no way to avoid the surface integration, and pull δu_i from under the surface integral, in order to find

$$\frac{\partial U}{\partial u_i}.$$

But even more puzzling is the requirement $\sigma_{ij,j} = 0, \quad in \quad G$.

By the equilibrium equations, this means that the body forces, f_x, f_y, and f_z vanish in the elastic body.

In other words, this requires bridges subject to no gravitational forces.

Timoshenko's derivation [Timoshenko, p.255] makes a similarly puzzling claim, just wrapped in variational notation, He has

$$\partial_x \delta\sigma_{xx} + \partial_y \delta\sigma_{xy} + \partial_z \delta\sigma_{xz} = 0.$$

That means,

$$\delta\partial_x\sigma_{xx} + \delta\partial_y\sigma_{xy} + \delta\partial_z\sigma_{xz} = 0,$$

$$\delta(\underbrace{\partial_x\sigma_{xx} + \partial_y\sigma_{xy} + \partial_z\sigma_{xz}}_{f_x}) = 0,$$

$$\delta f_x = 0, \text{ in } G$$

Similarly,

$$\delta f_y = 0, \text{ in } G$$

$$\delta f_z = 0, \text{ in } G.$$

Then, the partial derivatives

$$\frac{\partial U^*}{\partial P_1}, \frac{\partial U^*}{\partial P_2}, \ldots$$

are not defined.

15.

Castigliano Theorems and the Principle of Virtual Work

15.1 Proof of Castigliano Theorem for Forces

If under several concentrated body forces applied at several points, the strain energy U depends on the displacements,

$$U = U(u_1,...u_n),$$

the infinitesimal work along $\delta u_1,....\delta u_n$ is

$$\delta W = P_1\delta u_1 + ... + P_n\delta u_n.$$

By the Principle of Virtual Work,

$$\delta U = \delta W = P_1\delta u_1 + ... + P_n\delta u_n$$

Therefore,

$$\frac{\partial U}{\partial u_1} = P_1, ... \frac{\partial U}{\partial u_n} = P_n. \square$$

15.2 Proof of Castigliano Theorem for Displacements

If under several concentrated body forces applied at several points, the Complementary Strain energy U^* depends on the Forces,

$$U^* = U^*(P_1, ...P_n),$$

the infinitesimal work by $\delta P_1, ... \delta P_n$ is

$$\delta W^* = u_1 \delta P_1 + ... + u_n \delta P_n.$$

By the Principle of Virtual Work,

$$\delta U^* = \delta W^* = u_1 \delta P_1 + ... + u_n \delta P_n$$

Therefore,

$$\frac{\partial U^*}{\partial P_1} = u_1, ... \frac{\partial U^*}{\partial P_n} = u_n. \square$$

16.

Castigliano Theorems and the Hamiltonian

16.1 The Hamiltonian of Concentrated Forces

is the potential

$$\mathrm{H}(P_1,...P_n,u_1,...u_n) = U - P_1u_1 - .... - P_nu_n$$

where

$$U = U(u_1,...u_n) = U^*(P_1,...P_n)$$

16.2 *Castigliano Theorem for the Forces is Hamilton's Equations*

$$\frac{\partial U}{\partial u_i} = P_i$$

Proof: By the Principle of Minimal Potential, $\frac{\partial \mathrm{H}}{\partial u_i} = 0$. $\square$

16.3 *Castigliano Theorem for the displacements is Hamilton's Equations*

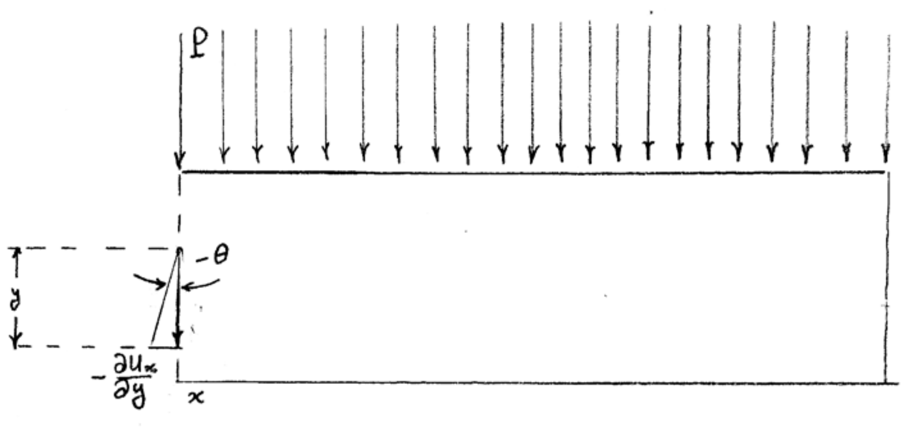
$$\frac{\partial U}{\partial u_i} = P_i$$

Proof: By the Principle of Minimal Potential, $\frac{\partial \mathrm{H}}{\partial P_i} = 0$.$\square$

17.

Castigliano and Beam Bending without Shear

A beam of length l parallel to the ground, is attached to a wall at one of the beam ends. A uniform force P pointing in the ground direction, applies at each point along the beam, and bends it down by y.



17.1 The x-Deflection due to the y-length Bending

$$u_x = y(-\theta) = y\frac{\partial u_y}{\partial x}$$

Proof: $u_x = y(-\theta) = y\left(-\dfrac{\partial u_x}{\partial y}\right)$

Zero shear means that $\dfrac{\partial u_y}{\partial x} + \dfrac{\partial u_x}{\partial y} = 0$, Therefore,

$$u_x = y\frac{\partial u_y}{\partial x}$$

17.2 The x-Strain due to the y-length Bending

$$\varepsilon_{xx} = y\left(\left.\frac{\partial^2 u_y}{\partial x^2}\right|_{y=0}\right).$$

17.3 The x-Stress due to the y-length Bending

$$\sigma_{xx} = Ey\left(\left.\frac{\partial^2 u_y}{\partial x^2}\right|_{y=0}\right).$$

17.4 The Bending Moment at x,

$$M(x) = Px = \left.\frac{\partial^2 u_y}{\partial x^2}\right|_{y=0} E\underbrace{\iint_A y^2 dA}_{I}$$

where I is the Inertia Moment.

Proof:

$$M(x) = \iint_A \sigma_{xx} y dA$$

$$= \iint_A \left. \frac{\partial^2 u_y}{\partial x^2} \right|_{y=0} E y^2 dA$$

$$= \left. \frac{\partial^2 u_y}{\partial x^2} \right|_{y=0} E \underbrace{\iint_A y^2 dA}_{I}.$$

17.5 The x-Stress due to the y-length Bending

$$\sigma_{xx} = \frac{M(x)}{I} y.$$

Proof: $Ey\left(\left. \frac{\partial^2 u_y}{\partial x^2} \right|_{y=0} \right) = Ey \frac{M(x)}{EI} = y \frac{M(x)}{I}.\square$

17.6 The Strain Energy of the Bending at $x = l$,

$$U(l) = \frac{1}{2EI} \int_{x=0}^{x=l} M^2(x) dx$$

Proof: $U = \frac{1}{2E} \int_{x=0}^{x=l} \iint_A \sigma_{xx}^2 dA dx$

$$= \frac{1}{2E} \int_{x=0}^{x=l} \iint_A \left(\frac{M(x)}{I} y \right)^2 dAdx$$

$$= \frac{1}{2EI^2} \int_{x=0}^{x=l} M^2(x) \underbrace{\iint_A y^2 dAdx}_{I}$$

$$= \frac{1}{2EI} \int_{x=0}^{x=l} M^2(x) dx . \square$$

17.7 The vertical deflection at $x = l$ is $\dfrac{Pl^3}{3EI}$

Proof: By Castigliano, the vertical deflection at $x = l$ is

$$\frac{\partial U(l)}{\partial P} = \partial_P \frac{1}{2EI} \int_{x=0}^{x=l} M^2(x) dx$$

$$= \frac{1}{2EI} \int_{x=0}^{x=l} \partial_P M^2(x) dx$$

$$= \frac{1}{2EI} \int_{x=0}^{x=l} 2 \underbrace{M}_{Px} \underbrace{\partial_P M(x)}_{x} dx$$

$$= \frac{P}{EI} \int_{x=0}^{x=l} x^2 dx = \frac{Pl^3}{3EI} . \square$$

17.8 The deflection angel is $\dfrac{Pl^2}{2EI}$

Proof: By Castigliano, the deflection angel is

$$\frac{\partial U(l)}{\partial M} = \partial_M \frac{1}{2EI} \int_{x=0}^{x=l} M^2(x)dx$$

$$= \frac{1}{2EI} \int_{x=0}^{x=l} \partial_M M^2(x)dx$$

$$= \frac{1}{2EI} \int_{x=0}^{x=l} 2 \underbrace{M}_{Px} dx$$

$$= \frac{P}{EI} \int_{x=0}^{x=l} xdx = \frac{Pl^2}{2EI}. \square$$

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