Riemann's Zeta Paper: The Product Formula Error

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The Product Formula Error

Riemann's 1859 Zeta paper defines the Zeta function and uses its properties to approximate the count of prime numbers up to a number t, and the density of the primes at the number t.

To that end, Riemann defines the auxiliary function

$$\xi(z) \equiv (z-1)\pi^{-z/2}\Gamma(z/2+1)\zeta(z)$$

that has the same zeros as $\zeta(z)$ in $0 < x < 1$.

We show that it has the factorization

$$\xi(z) = \xi(0)\prod_{n=1}^{\infty}\left(1 - \frac{z}{1/2 - i\alpha_n}\right)\left(1 - \frac{z}{1/2 + i\alpha_n}\right) = \xi(0)\prod_{n=1}^{\infty}\left(1 + \frac{z^2 - z}{\alpha_n^2 + 1/4}\right)$$

where the $1/2 \pm \alpha_n$ are the zeros of $\xi(z)$.

We further show that if the zeros are all on the line $x = 1/2$, this factorization for $\xi(z)$ produces the term

$$\sum_{n=1}^{\infty}\left[\mathrm{Li}(t^{1/2+i\alpha_n}) + \mathrm{Li}(t^{1/2-i\alpha_n})\right]$$

in the formula for the count of the primes, where $\mathrm{Li}(t)$ is the Logarithmic integral.

Riemann obtained the erroneous product formula

$$\xi(z) = \xi|_{z=1/2}\prod_{n=1}^{\infty}\left(1 + \frac{(z-1/2)^2}{\alpha_n^2}\right)$$

that does not produce the Logarithmic integral series term.

Already in 1860, Genocchi pointed out that the formula should use

$$\xi|_{z=0}.$$

But that only diverted attention from the error in the factor in the product, and the main problem seemed to be that a derivation was missing.

In 1893, Hadamard supplied the derivation and obtained

$$\xi(z) = \xi(0)\prod_{\rho}\left(1 - \frac{z}{\rho}\right),$$

where the ρ's are the zeros of $\xi(z)$.

Hadamard formula does not exhibit the connection to the α_n's, and so far as I can tell, the connection to the Logarithmic integral series in the formula for the count of the primes, was never made.

Indeed, to make that connection, one has to follow through the whole paper. Thus, producing the correct derivations and results of the Zeta paper amounts to execution of the book that is outlined in the Zeta paper.

I use common notations and terms such as

$z = x + iy$, not $s = \sigma + i\tau$.
$z \to w(z)$, not $s \to t(s)$.
$\Gamma(z)$, not $\Pi(s-1)$.
"zeros of ξ" not "roots of the equation $\xi(z) = 0$."

Otherwise, Riemann's notations are kept unchanged.

My quote of Riemann is based on the two translations in [1], and [2].

Part I

The Zeta Function

Chapter 1

The Count of Prime Numbers

1.1 Gauss approximation

Riemann's aim was

> *...to report on a study of the frequency with which prime numbers occur.*
>
> *A topic that seems worthy of such reporting, because of the interest shown in it by Gauss and Dirichlet over many years...*

Gauss (1849) computed that

$$\int_{u=2}^{u=t} \frac{du}{\log u}$$

approximates the

$$\text{Number of Primes} < \text{t},$$

denoted

$$\pi(t).$$

His results are listed in the following **Gauss table** [1, p.3]

t	$\pi(t)$	$\int_{u=2}^{u=t} \frac{du}{\log u}$	$Error$
$500,000$	$41,556$	$41,606.4$	50.4
$1,000,000$	$78,501$	$78,627.5$	126.5
$1,500,000$	$114,112$	$114,263.1$	151.1
$2,000,000$	$148,883$	$149,054.8$	171.8
$2,500,000$	$183,016$	$183,245.0$	229.0
$3,000,000$	$216,745$	$216,970.6$	225.6

1.2 Gauss Approximation, and the Prime Number Theorem

We have

$$\int_{u=2}^{u=t} \frac{du}{\log u} = Li(t) - Li(2),$$

where

$$Li(t) \equiv \lim_{\varepsilon \downarrow 0} \left\{ \int_{u=0}^{u=1-\varepsilon} \frac{du}{\log u} + \int_{u=1+\varepsilon}^{u=t} \frac{du}{\log u} \right\}$$

is the Logarithmic Integral.
By the Prime Number Theorem (Hadamard-1896),

$$\frac{\pi(t)}{Li(t)} \to 1, \text{ as } t \to \infty.$$

The Prime Number Theorem substantiates the Gauss approximation

Chapter 2

Definition of Zeta in $\mathrm{Re}z > 1$

2.1 For $\mathrm{Re}z > 1$, $\zeta(z) \equiv \sum_{n=natural} \frac{1}{n^z} = \prod_{p=prime} \frac{1}{1-1/p^z}$

Riemann wrote:

> *...My starting point was the observation of Euler that the product*
>
> $$\prod_{p=prime} \frac{1}{1-1/p^z}$$
>
> *equals*
>
> $$\sum_{n=natural} \frac{1}{n^z},$$
>
> *where p ranges over all the prime numbers, and n over all the natural numbers.*
>
> *I denote by*
>
> $$\zeta(z),$$
>
> *the function of the complex variable z, defined by these two expressions when they converge.*

Euler's product formula, defined for $k = 2, 3, ...$ by

$$\prod_{p=prime} \frac{1}{1 - 1/p^k} = \sum_{n=natural} \frac{1}{n^k},$$

can be extended to complex numbers z with $Rez > 1$.

Thus, in $Rez > 1$,

$$\zeta(z) \equiv \sum_{n=natural} \frac{1}{n^z} = \prod_{p=prime} \frac{1}{1 - 1/p^z}.$$

Chapter 3

Definition of Zeta in $\mathrm{Re}(z) > 0$

3.1 $\Gamma(z) = \int_{t=0}^{t=\infty} e^{-t} t^{z-1} dt$, extended to $x > 0$.

Euler's Gamma that for $x > 0$ is defined by

$$\Gamma(x) \equiv \int_{t=0}^{t=\infty} e^{-t} t^{x-1} dt,$$

coincides with the analytic function

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)...(z+n)} = \lim_{n\to\infty} \frac{n^z}{z(1+z)(1+z/2)...(1+z/n)}$$

that converges for any z except for poles that it has at $z = 0, -1, -2, ...$

Since the integral

$$\int_{t=0}^{t=\infty} e^{-t} t^{z-1} dt$$

converges for $x > 0$, we have

$$\Gamma(z) = \int_{t=0}^{t=\infty} e^{-t}t^{z-1}dt, \text{ for } x > 0.$$

3.2 For $x > 0$, $\Gamma(z)\zeta(z) = \int_{t=0}^{t=\infty} \frac{t^{z-1}}{e^t-1}dt.$

Riemann wrote:

$$\sum_{n=natural} \frac{1}{n^z}, \text{ and } \prod_{p=prime} \frac{1}{1-1/p^z},$$

converge only when

$$\text{Re}(z) > 1.$$

However, it is easy to find an expression of the function that is always valid.

By applying the equation

$$\int_{t=0}^{t=\infty} e^{-nt}t^{z-1}dt = \frac{1}{n^z}\Gamma(z),$$

first of all we get

$$\Gamma(z)\zeta(z) = \int_{t=0}^{t=\infty} \frac{t^{z-1}}{e^t - 1}dt,$$

for $x > 1$.

By Section 3.1, for $x > 0$, Euler's Gamma function is

$$\Gamma(z) = \int_{u=0}^{u=\infty} e^{-u}u^{z-1}du.$$

By the change of variable $u = nt$,

$$\Gamma(z)\frac{1}{n^z} = \int_{t=0}^{t=\infty} e^{-nt}t^{z-1}dt$$

Therefore, for $x > 0$,

$$\Gamma(z)\sum_{n=1}^{N}\frac{1}{n^z} = \int_{t=0}^{t=\infty} (\sum_{n=1}^{N} e^{-nt})t^{z-1}dt.$$

Letting $N \to \infty$, we obtain, now limited to $x > 1$,

$$\Gamma(z)\sum_{n=1}^{\infty} 1/n^z = \lim_{N\to\infty}\int_{t=0}^{t=\infty} (\sum_{n=1}^{N} e^{-nt})t^{z-1}dt.$$

Uniform convergence allows summation under the integral sign, and we have for $x > 1$,

$$\Gamma(z)\sum_{n=1}^{\infty} 1/n^z = \int_{t=0}^{t=\infty} \frac{t^{z-1}}{e^t - 1}dt.$$

That is, for $x > 1$,

$$\zeta(z) = \frac{1}{\Gamma(z)}\int_{t=0}^{t=\infty} \frac{t^{z-1}}{e^t - 1}dt.$$

However, the right hand side is well defined for $x > 0$. Thus,

$$\frac{1}{\Gamma(z)}\int_{t=0}^{t=\infty} \frac{t^{z-1}}{e^t - 1}dt.$$

defines Zeta in $\mathrm{Re}(z) > 0$.

Chapter 4

Definition of Zeta for any z

4.1 $2\sin\pi z\Gamma(z)\zeta(z) = i\int\limits_{\infty - i0}^{\infty + i0}\frac{(-\lambda)^{z-1}}{e^{\lambda}-1}d\lambda$, **on a path that starts from $\infty - i0$, encircles $z = 0$ clockwise, and returns to $\infty + i0$.**

Riemann wrote

...Consider the integral

$$\int\limits_{\infty - i0}^{\infty + i0}\frac{(-\lambda)^{z-1}}{e^{\lambda}-1}d\lambda$$

along a closed path from

$$\lambda = \infty - i0$$

to

$$\lambda = \infty + i0,$$

clockwise around a domain that contains the singularity at

$$z = 0,$$

but none of the singularities at

$$z = 2\pi i n,$$

for $n = 1, 2, 3...$

To define the multi-valued function

$$(-\lambda)^{z-1} = e^{(z-1)\log(-\lambda)},$$

we choose the branch of

$$\log(-\lambda)$$

that is real for

$$\lambda < 0.$$

The integral equals

$$(e^{-i\pi z} - e^{i\pi z}) \int_{t=0}^{t=\infty} \frac{(-t)^{z-1}}{e^t - 1} dt,$$

where the integration path is along the real axis.

Thus, we obtain

$$2 \sin \pi z \Gamma(z)\zeta(z) = i \int_{\infty - i0}^{\infty + i0} \frac{(-\lambda)^{z-1}}{e^\lambda - 1} d\lambda.$$

4.1.1 The integration path

To evaluate the integral

$$\int_{\lambda=\infty - i\delta}^{\lambda=\infty + i\delta} \frac{(-\lambda)^{z-1}}{e^\lambda - 1} d\lambda,$$

we choose a path that

- runs from $\lambda = \infty - i\delta$ to $\lambda = \delta - i\delta$, along $\lambda = t - i\delta$.
- runs from $\lambda = \delta - i\delta$ to $\lambda = \delta + i\delta$, encircling $z = 0$ along $\lambda = \delta\sqrt{2}e^{i\theta} \equiv \varepsilon e^{i\theta}$.
- runs from $\lambda = \delta + i\delta$ to $\lambda = \infty + i\delta$, along $\lambda = t + i\delta$.

Then, the integral equals

$$\int_{\lambda=\infty-i\delta}^{\lambda=\delta-i\delta} \frac{(-\lambda)^{z-1}}{e^{\lambda}-1} d\lambda + \int_{\lambda=\delta-i\delta}^{\lambda=\delta+i\delta} \frac{(-\lambda)^{z-1}}{e^{\lambda}-1} d\lambda + \int_{\lambda=\delta+i\delta}^{\lambda=\infty+i\delta} \frac{(-\lambda)^{z-1}}{e^{\lambda}-1} d\lambda.$$

To use the same branch of $\log(-\lambda)$, we should not cross the cut along the positive x axis.

4.1.2 The first Integral

For the first integral, we rotate

$$[t - i\delta]$$

clockwise, multiplying it by

$$e^{-i\pi}$$

to obtain

$$-[t - i\delta.]$$

That is,

$$-\lambda = -[t - i\delta] = [t - i\delta]e^{-i\pi}.$$

Therefore,

$$(-\lambda)^{z-1} = ([t - i\delta]e^{-i\pi})^{z-1} = -e^{-i\pi z}[t - i\delta]^{z-1},$$

and we have

$$\int_{\lambda=\infty-i\delta}^{\lambda=\delta-i\delta} \frac{(-\lambda)^{z-1}}{e^{\lambda}-1}d\lambda = e^{-i\pi z}\int_{t=\delta}^{t=\infty} \frac{(t-i\delta)^{z-1}}{e^{t-i\delta}-1}dt.$$

For $\delta \downarrow 0$,

$$\left|\frac{(t-i\delta)^{z-1}}{e^{t-i\delta}-1}\right| \to \left|\frac{t^{z-1}}{e^{t}-1}\right|.$$

Therefore, by Lebesgue Dominant Convergence, as $\delta \downarrow 0$,

$$\text{first integral} \to e^{-i\pi z}\int_{t=0+}^{t=\infty} \frac{t^{z-1}}{e^{t}-1}dt.$$

4.1.3 The Second Integral

In the second integral,

$$|\lambda| = \delta\sqrt{2} \equiv \varepsilon$$

and

$$\left|\int_{\lambda=\delta-i\delta}^{\lambda=\delta+i\delta} \frac{(-\lambda)^{z-1}}{e^{\lambda}-1}d\lambda\right| \le \int_{\theta=0}^{\theta=2\pi} \frac{\varepsilon^{x-1}}{|exp(\varepsilon e^{i\theta})-1|}\varepsilon d\theta.$$

To apply Lebesgue Dominant Convergence, we verify by L'Hospital that as $\varepsilon \downarrow 0$,

$$\frac{\varepsilon^{2x}}{|exp(\varepsilon e^{i\theta})-1|^{2}} \to 0$$

Now,

$$\left|exp(\varepsilon e^{i\theta})-1\right|^{2} = \left(e^{\varepsilon e^{i\theta}}-1\right)\left(e^{\varepsilon e^{-i\theta}}-1\right)$$

$$= e^{2\varepsilon\cos\theta} + 1 - e^{\varepsilon\cos\theta}\left(e^{i\varepsilon\sin\theta} + e^{-i\varepsilon\sin\theta}\right)$$
$$= e^{2\varepsilon\cos\theta} + 1 - 2e^{\varepsilon\cos\theta}\cos(\varepsilon\sin\theta)$$
$$\to 0, \text{ as } \varepsilon \downarrow 0.$$

$$\frac{d}{d\varepsilon}\left|exp(\varepsilon e^{i\theta}) - 1\right|^2 = \frac{d}{d\varepsilon}\left(e^{2\varepsilon\cos\theta} + 1 - 2e^{\varepsilon\cos\theta}\cos(\varepsilon\sin\theta)\right)$$
$$= 2\cos\theta e^{2\varepsilon\cos\theta} - 2e^{\varepsilon\cos\theta}\left(\cos\theta\cos(\varepsilon\sin\theta) - \sin\theta\sin(\varepsilon\sin\theta)\right)$$
$$= 2\cos\theta e^{2\varepsilon\cos\theta} - 2e^{\varepsilon\cos\theta}\left(\cos(\theta + \varepsilon\sin\theta)\right)$$
$$= 2e^{\varepsilon\cos\theta}\left(e^{\varepsilon\cos\theta}\cos\theta - \cos(\theta + \varepsilon\sin\theta)\right)$$
$$\to 0, \text{ as } \varepsilon \downarrow 0.$$

$$\frac{d^2}{d\varepsilon^2}\left|exp(\varepsilon e^{i\theta}) - 1\right|^2 = \frac{d}{d\varepsilon}2e^{\varepsilon\cos\theta}\left(e^{\varepsilon\cos\theta}\cos\theta - \cos(\theta + \varepsilon\sin\theta)\right)$$
$$= 2e^{\varepsilon\cos\theta}\cos\theta\left(e^{\varepsilon\cos\theta}\cos\theta - \cos(\theta + \varepsilon\sin\theta)\right) +$$
$$+2e^{\varepsilon\cos\theta}\left(e^{\varepsilon\cos\theta}\cos\theta\cos\theta + \sin(\theta + \varepsilon\sin\theta)\sin\theta\right)$$
$$= (\to 0) + (\to 2) \text{ as } \varepsilon \downarrow 0.$$

Therefore, by L'Hospital, for $x > 1$, and for $\varepsilon \downarrow 0$

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{2x}}{\left|exp(\varepsilon e^{i\theta}) - 1\right|^2} = \lim_{\varepsilon\to 0}\frac{D_\varepsilon \varepsilon^{2x}}{D_\varepsilon\left|exp(\varepsilon e^{i\theta}) - 1\right|^2}$$
$$= \lim_{\varepsilon\to 0}\frac{D_\varepsilon^2 \varepsilon^{2x}}{D_\varepsilon^2\left|exp(\varepsilon e^{i\theta}) - 1\right|^2} = \lim_{\varepsilon\to 0}\frac{2x(2x-1)\varepsilon^{2x-2}}{D_\varepsilon^2\left|exp(\varepsilon e^{i\theta}) - 1\right|^2}$$

$$= \frac{(\to 0)}{(\to 2)} \text{ as } \varepsilon \downarrow 0.$$

Hence, as $\varepsilon \downarrow 0$,

$$\frac{\varepsilon^x}{|exp(\varepsilon e^{i\theta}) - 1|} \to 0$$

Therefore, by Lebesgue Dominant Convergence, as $\delta \downarrow 0$

$$\text{Second Integral} \to 0$$

4.1.4 The Third Integral

For the third integral we rotate

$$[t + i\delta]$$

counter-clockwise, multiplying it by

$$e^{i\pi}$$

to obtain

$$-[t + i\delta].$$

Hence,

$$-\lambda = -[t + i\delta] = [t + i\delta]e^{i\pi}.$$

$$(-\lambda)^{z-1} = ([t + i\delta]e^{i\pi})^{z-1} = -e^{i\pi z}[t + i\delta]^{z-1},$$

and we have

$$\int_{\lambda=\delta+i\delta}^{\lambda=\infty+i\delta} \frac{(-\lambda)^{z-1}}{e^{\lambda} - 1} d\lambda = -e^{i\pi z} \int_{t=\delta}^{t=\infty} \frac{(t + i\delta)^{z-1}}{e^{t+i\delta} - 1} dt.$$

Thus, by Lebesgue Dominant Convergence, as $\delta \downarrow 0$,

$$\text{Third Integral} \to -e^{i\pi z} \int_{t=0+}^{t=\infty} \frac{t^{z-1}}{e^t - 1} dt.$$

4.1.5 The Path-Integral Formula

For $\delta \downarrow 0$, we obtain

$$\int_{\infty - i0}^{\infty + i0} \frac{(-\lambda)^{z-1}}{e^{\lambda} - 1} d\lambda = (e^{-i\pi z} - e^{i\pi z}) \int_{t=0}^{t=\infty} \frac{t^{z-1}}{e^t - 1} dt.$$

$$= -2i \sin(\pi z) \int_{t=0}^{t=\infty} \frac{t^{z-1}}{e^t - 1} dt.$$

By Section 3.2, for $x > 0$,

$$\int_{t=0}^{t=\infty} \frac{t^{z-1}}{e^t - 1} dt = \Gamma(z)\zeta(z)$$

Hence,

$$\int_{\infty - i0}^{\infty + i0} \frac{(-\lambda)^{z-1}}{e^{\lambda} - 1} d\lambda = -2i \sin(\pi z)\Gamma(z)\zeta(z).$$

4.2 Definition of Zeta for any z

Riemann wrote

> *For any* $z \neq 1$,
>
> $$\zeta(z)$$
>
> *is a single-valued, and finite function.*

4.2.1 Zeta defined for any z

We have,

$$\zeta(z) = -\frac{1}{2i\sin(\pi z)\Gamma(z)} \int_{\infty-i0}^{\infty+i0} \frac{(-\lambda)^{z-1}}{e^{\lambda}-1} d\lambda.$$

Substituting

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)},$$

we obtain

$$\zeta(z) = -\frac{\Gamma(1-z)}{2\pi i} \int_{\infty-i0}^{\infty+i0} \frac{(-\lambda)^{z-1}}{e^{\lambda}-1} d\lambda.$$

The right side defines Zeta for any z.

4.2.2 Zeta is finite for all $z \neq 1$

$\Gamma(1-z)$ has simple poles at $z = 1, 2, 3...$, but only $z = 1$ is a pole of $\sum\limits_{n=1}^{\infty} \frac{1}{n^z}$.

It follows that the integral

$$\int_{\infty-i0}^{\infty+i0} \frac{(-\lambda)^{z-1}}{e^{\lambda}-1} d\lambda$$

has zeros that cancel the poles at $z = 2, 3...$, Zeta has only one pole at $z = 1$, and is finite for all $z \neq 1$.

4.3 Zeta has zeros at $z = -2n$, for $n = 1, 2, 3...$

The Bernoulli numbers

$$B_n$$

are defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{1}{n!} B_n t^n$$

that converges in $|t| < 2\pi$.

We have

$$\zeta(-2n) = \frac{(-1)^n}{n+1} B_{2n+1},$$

and

$$B_{2n+1} = 0.$$

Thus, Zeta has zeros at

$$z = -2n,$$

for $n = 1, 2, 3...$

Chapter 5

The Zeta Functional Equation

5.1 $2\sin \pi z \Gamma(z)\zeta(z) = (2\pi)^z[(-i)^{z-1} + i^{z-1}] \sum_n n^{z-1}$.

Riemann wrote

> *If $x < 0$, we integrate along a clockwise-oriented path around the complementary domain in the complex plane.*
>
> *Then, the integral is infinitesimal, because it is over values that have infinitely large modulus.*
>
> *In the complementary domain, the integrand has singularities only at*
>
> $$\lambda = \pm 2\pi i n,$$
>
> *for $n = 1, 2, 3...$*
>
> *Therefore, the integral is equal to the sum of the clockwise-oriented path- integrals around these singularities.*
>
> *Since the clockwise-oriented path-integral around a singularity $z = 2\pi i n$ is*
>
> $$(-2\pi i)(-2\pi i n)^{z-1},$$

we have

$$2\sin\pi z\Gamma(z)\zeta(z) = (2\pi)^z \sum_n n^{z-1}[(-i)^{z-1} + i^{z-1}].$$

By Section 4.1,we have

$$2\sin\pi z\Gamma(z)\zeta(z) = i\int_{\infty - i0}^{\infty + i0} \frac{(-\lambda)^{z-1}}{e^\lambda - 1}d\lambda$$

We will evaluate the integral along a clockwise-oriented path around the complementary domain.
By the Residue Theorem, the clockwise-oriented integral

$$\int_{|\lambda - 2\pi in| = \varepsilon} \frac{(-\lambda)^{z-1}}{e^\lambda - 1}d\lambda,$$

equals

$$(-2\pi i)\mathrm{Res}\left[\frac{(-\lambda)^{z-1}}{e^\lambda - 1}\right]_{\lambda = 2\pi in} = -2\pi i \lim_{\lambda \to 2\pi in}\left[(\lambda - 2\pi in)\frac{(-\lambda)^{z-1}}{e^\lambda - 1}\right].$$

By L'Hospital,

$$\lim_{\lambda \to 2\pi in}\frac{(\lambda - 2\pi in)}{e^\lambda - 1} = \lim_{\lambda \to 2\pi in}\frac{1}{e^\lambda} = 1.$$

Hence, around the singularity

$$\lambda = 2\pi in,$$

the integral equals

$$(-2\pi i)\lim_{\lambda \to 2\pi in}(-\lambda)^{z-1} = (-2\pi i)(-2\pi in)^{z-1}.$$

Similarly, around the singularity

$$\lambda = -2\pi i n,$$

the integral equals

$$(-2\pi i)(2\pi i n)^{z-1}.$$

The integral around the complementary domain is the sum of the integrals around all the singularities and it equals

$$= (-2\pi i)\sum_{n=1}^{\infty}[(-2\pi i n)^{z-1} + (2\pi i n)^{z-1}]$$

$$= -i(2\pi)^z[(-i)^{z-1} + (i)^{z-1}]\sum_{n=1}^{\infty} n^{z-1}.$$

Thus,

$$2\sin \pi z \Gamma(z)\zeta(z) = (2\pi)^z[(-i)^{z-1} + i^{z-1}]\sum_{n=1}^{\infty} n^{z-1}.$$

5.2 Functional Equation: If $\eta(z) \equiv \pi^{-z/2}\Gamma(z/2)\zeta(z)$, then $\eta(z) = \eta(1-z)$, for any z.

The equation

$$\eta(z) = \eta(1-z)$$

is solved also by

$$z(1-z).$$

Riemann wrote

By using known properties of $\Gamma(z)$ *we obtain the following relation between* $\zeta(z)$, *and* $\zeta(1-z)$

$\pi^{-z/2}\Gamma(z/2)\zeta(z)$ *is unchanged if* $1-z$ *replaces* z.

By section 5.1,

$$2\sin\pi z\Gamma(z)\zeta(z) = (2\pi)^z[(-i)^{z-1} + i^{z-1}]\sum_{n=1}^{\infty} n^{z-1}.$$

Put

$$\sum_{n=1}^{\infty} n^{z-1} = \sum_{n=1}^{\infty} \frac{1}{n^{1-z}} = \zeta(1-z),$$

and

$$[(-i)^{z-1} + i^{z-1}] = i(-i)^z - i(i)^z = i[e^{z\log(-i)} - e^{z\log i}]$$

$$= i[e^{z(-i\pi/2)} - e^{z(i\pi/2)}] = 2\sin\tfrac{\pi z}{2}.$$

Then,

$$2\sin\pi z\Gamma(z)\zeta(z) = 2^z\pi^z 2\sin(\pi z/2)\zeta(1-z).$$

Substitute

$$\sin(\pi z/2) = \frac{\pi}{\Gamma(z/2)\Gamma(1-z/2)},$$

and

$$\sin(\pi z)\Gamma(z) = \frac{\pi}{\Gamma(1-z)}.$$

Then,

$$\frac{1}{\Gamma(1-z)}\zeta(z) = 2^z\pi^z\frac{1}{\Gamma(z/2)\Gamma(1-z/2)}\zeta(1-z).$$

That is,

$$\pi^{-z/2}\Gamma(z/2)\zeta(z) = \pi^{-(1-z)/2}\pi^{1/2}2^{z}\Gamma(1-z)\frac{1}{\Gamma(1-z/2)}\zeta(1-z).$$

By [3, p. 3]

$$\Gamma(z) = \pi^{-1/2}2^{z-1}\Gamma(\tfrac{z}{2})\Gamma(\tfrac{1+z}{2}).$$

Hence,

$$\Gamma(1-z) = \pi^{-1/2}2^{-z}\Gamma([1-z]/2)\Gamma(1-z/2).$$

Namely,

$$\pi^{1/2}2^{z}\Gamma(1-z)\frac{1}{\Gamma(1-z/2)} = \Gamma([1-z]/2).$$

Therefore,

$$\pi^{-z/2}\Gamma(z/2)\zeta(z) = \pi^{-(1-z)/2}\Gamma([1-z]/2)\zeta(1-z).$$

That is,

$$\eta(z) = \eta(1-z).$$

Chapter 6

Riemann's "Very Convenient Formula" for Zeta.

6.1 $\eta(z) = \frac{1}{z(z-1)} + \int\limits_{t=1}^{t=\infty} \psi(t)(t^{-(z+1)/2} + t^{z/2-1})dt$, where $\psi(t) \equiv \sum\limits_{n=1}^{\infty} e^{-n^2\pi t}$ for $t > 0$.

Riemann wrote

> *By using known properties of $\Gamma(z)$ we obtain the following relation between $\zeta(z)$, and $\zeta(1-z)$*
>
> $\pi^{-z/2}\Gamma(z/2)\zeta(z)$ *is unchanged if* $1-z$ *replaces* z.
>
> *This property of Zeta let me substitute $\Gamma(z/2)$ instead of $\Gamma(z)$, into the general term of $\sum 1/n^z$.*
>
> *This substitution gives a very convenient formula for $\zeta(z)$.*

Riemann's formula for $\zeta(z)$ that follows from the functional equation, was misunderstood by Titchmarsh [4] to be a proof

of the functional equation.
Every book except for Whittaker and Watson [5] follows Titchmarsh error that the derivation of this formula for $\zeta(z)$ is a second proof of the functional equation.
Anyone who read just a few lines of Riemann's 1859 paper knows that Riemann would never bother to give a second proof of anything.
It would take more than a few lines to realize that had this been even a *remotely possible* second proof of the functional equation, Riemann would have stated that.

6.2 Motivation for Riemann's Zeta Formula

Riemann wrote

$$\pi^{-z/2}\Gamma(z/2)\frac{1}{n^z} = \int_{t=0}^{t=\infty} e^{-n^2\pi t} t^{z/2-1} dt.$$

If we set

$$\sum_{n=1}^{\infty} e^{-n^2\pi t} = \psi(t),$$

we get

$$\pi^{-z/2}\Gamma(z/2)\zeta(z) = \int_{t=0}^{t=\infty} \psi(t) t^{z/2-1} dt.$$

For $x > 0$, consider Euler's Gamma function

$$\Gamma(z/2) = \int_{u=0}^{u=\infty} e^{-u} u^{z/2-1} du,$$

By the change of variable

$$u = n^2 \pi t,$$

we have

$$\pi^{-z/2}\Gamma(z/2)\frac{1}{n^z} = \int_{t=0}^{t=\infty} e^{-n^2\pi t} t^{z/2-1} dt,$$

Therefore, for $x > 0$,

$$\pi^{-z/2}\Gamma(z/2)\sum_{n=1}^{N}\frac{1}{n^z} = \int_{t=0}^{t=\infty} (\sum_{n=1}^{N} e^{-n^2\pi t}) t^{z/2-1} dt,$$

Letting $N \to \infty$, we obtain for $x > 1$

$$\pi^{-z/2}\Gamma(z/2)\sum_{n=1}^{\infty} 1/n^z = \lim_{N\to\infty}\int_{t=0}^{t=\infty} (\sum_{n=1}^{N} e^{-n^2\pi t}) t^{z/2-1} dt.$$

That is, in $x > 1$,

$$\lim_{N\to\infty}\int_{t=0}^{t=\infty} (\sum_{n=1}^{N} e^{-n^2\pi t}) t^{z/2-1} dt \equiv \nu(z)$$

equals

$$\pi^{-z/2}\Gamma(z/2)\zeta(z) \equiv \eta(z).$$

Since $\nu(z)$ is defined for $x > 0$, $\nu(z)$ may equal $\eta(z)$ in $x > 0$.

6.3 Riemann's Zeta Formula Proof.

Riemann wrote

Since

$$2\psi(t) + 1 = t^{-1/2}[2\psi(1/t) + 1] \text{ (Jacobi, Fund., p.184),}$$

it follows that

$$\pi^{-z/2}\Gamma(z/2)\zeta(z) = \int_{t=1}^{t=\infty} \psi(t)t^{z/2-1}dt$$

$$+\int_{t=0}^{t=1} \psi(1/t)t^{(z-3)/2}dt + \tfrac{1}{2}\int_{t=0}^{t=1} (t^{(z-3)/2} - t^{z/2-1})dt$$

$$= \frac{1}{z(z-1)} + \int_{t=1}^{t=\infty} \psi(t)(t^{z/2-1} + t^{-(1+z)/2})dt.$$

We have seen that in $x > 1$

$$\lim_{N\to\infty} \int_{t=0}^{t=\infty} (\sum_{n=1}^{N} e^{-n^2\pi t})t^{z/2-1}dt$$

equals

$$\eta(z)$$

However, the integral is well-defined in $x > 0$, and $\eta(z)$ is well-defined for any z.

So we want to extend the equality between the integral and $\eta(z)$ up to the maximal possible domain.

To that end, we find a different expression for the integral, that converges for any z.
We first use uniform convergence to take the limit under the integral sign, and we obtain

$$\int_{t=0}^{t=\infty} (\sum_{n=1}^{\infty} e^{-n^2\pi t}) t^{z/2-1} dt = \int_{t=0}^{t=\infty} \psi(t) t^{z/2-1} dt,$$

where for $t > 0$,

$$\psi(t) \equiv \sum_{n=1}^{\infty} e^{-n^2\pi t}$$

is Jacobi's elliptic function.
By [6]

$$\psi(t) = \psi(1/t)t^{-1/2} + \tfrac{1}{2}t^{-1/2} - \tfrac{1}{2}.$$

Substituting this for $\psi(t)$,

$$\int_{t=0}^{t=\infty} \psi(t) t^{z/2-1} dt =$$

$$= \int_{t=0}^{t=1} \left[\psi(1/t)t^{-1/2} + \tfrac{1}{2}\left(t^{-1/2} - 1\right)\right] t^{z/2-1} dt + \int_{t=1}^{t=\infty} \psi(t) t^{z/2-1} dt$$

$$= \int_{t=0}^{t=1} \psi(1/t) t^{z/2-3/2} dt + \tfrac{1}{2} \int_{t=0}^{t=1} \left(t^{z/2-3/2} - t^{z/2-1}\right) dt + \int_{t=1}^{t=\infty} \psi(t) t^{z/2-1} dt$$

Since

$$\tfrac{1}{2} \int_{t=0}^{t=1} \left(t^{z/2-3/2} - t^{z/2-1}\right) dt = \frac{1}{z-1} - \frac{1}{z} = \frac{1}{z(z-1)},$$

we have

$$= \frac{1}{z(z-1)} + \int_{t=0}^{t=1} \psi(1/t) t^{z/2-3/2} dt + \int_{t=1}^{t=\infty} \psi(t) t^{z/2-1} dt.$$

By the change of variable $\tau = 1/t$,

$$\int_{t=0}^{t=1} \psi(1/t) t^{z/2-3/2} dt = \int_{\tau=\infty}^{\tau=1} \psi(\tau) \tau^{-z/2+3/2} (-d\tau/\tau^2) = \int_{\tau=1}^{\tau=\infty} \psi(\tau) \tau^{-(z+1)/2} d\tau.$$

Therefore,

$$= \frac{1}{z(z-1)} + \int_{t=1}^{t=\infty} \psi(t) \left(t^{-(z+1)/2} + t^{z/2-1} \right) dt$$

to be denoted by

$$\equiv \mu(z).$$

Then, $Re(z) < 0$, has the effect of transforming each of

$$\int_{t=1}^{t=\infty} \psi(t) t^{-(z+1)/2} dt$$

and

$$\int_{t=1}^{t=\infty} \psi(t) t^{z/2-1} dt$$

into a converging integral on $0 < t < 1$

Hence,

$$\int_{t=1}^{t=\infty} \psi(t) \left(t^{-(z+1)/2} + t^{z/2-1} \right) dt$$

is defined also for $Re(z) < 0$, and for any z.
If we can show that

$$\mu(z) = \mu(1-z),$$

then the equality between $\mu(z)$ and $\eta(z)$ in $x > 1$, will allow to conclude equality for any z.
Indeed,

$$\mu(1-z) = \frac{1}{(1-z)(-z)} + \int_{t=1}^{t=\infty} \psi(t) \left(t^{z/2-1} + t^{-(z+1)/2}\right) dt = \mu(z).$$

Thus,

$$\eta(z) = \frac{1}{z(z-1)} + \int_{t=1}^{t=\infty} \psi(t) \left(t^{-(z+1)/2} + t^{z/2-1}\right) dt$$

for any z.

Chapter 7

Whittaker's Formula for Zeta in $x > 0$.

7.1 For $x > 0$, $\eta(z) = \lim_{N\to\infty} \int_{t=0}^{t=\infty} (\sum_{n=1}^{N} e^{-n^2\pi t}) t^{z/2-1} dt$.

Appears in [5].

By 6.1, for any z,

$$\eta(z) = \pi^{-z/2}\Gamma(z/2)\zeta(z) = \frac{1}{z(z-1)} + \int_{t=1}^{t=\infty} \psi(t) \left(t^{-(z+1)/2} + t^{z/2-1} \right) dt.$$

For $x > 0$,

$$= \lim_{N\to\infty} \int_{t=0}^{t=\infty} \left(\sum_{n=1}^{N} e^{-n^2\pi t} \right) t^{z/2-1} dt.$$

Therefore, for $x > 0$,

$$\eta(z) = \lim_{N\to\infty} \int_{t=0}^{t=\infty} \left(\sum_{n=1}^{N} e^{-n^2\pi t} \right) t^{z/2-1} dt$$

Chapter 8

The function $\xi(z)$

8.1 $\xi(z) \equiv \frac{1}{2}z(z-1)\eta(z)$.

Riemann wrote

> *Set*
> $$\tfrac{1}{2}z(z-1)\pi^{-z/2}\Gamma(z/2)\zeta(z) = \xi(z).$$

8.2 Formulas for $\xi(z)$

By 6.1, we have

$$\xi(z) = \tfrac{1}{2}z(z-1)\left(\frac{1}{z(z-1)} + \int_{t=1}^{t=\infty} \psi(t)(t^{-(z+1)/2} + t^{z/2-1})dt\right)$$

$$= \frac{1}{2} - \frac{z(1-z)}{2}\int_{t=1}^{t=\infty} \psi(t)\left(t^{-(z+1)/2} + t^{z/2-1}\right)dt$$

Riemann wrote,

$$\xi(z) = \tfrac{1}{2} - z(1-z) \int_{t=1}^{t=\infty} \psi(t) t^{-3/4} \cos\left(\tfrac{1}{2}[i(\tfrac{1}{2}-z)]\log t\right) dt$$

or

$$\xi(z) = 4 \int_{t=1}^{t=\infty} \left(t^{3/2}\psi'(t)\right)' t^{-1/4} \cos\left(\tfrac{1}{2}[i(\tfrac{1}{2}-z)]\log t\right) dt$$

8.2.1 $\xi(z) = \tfrac{1}{2} - z(1-z) \int_{t=1}^{t=\infty} \psi(t) t^{-3/4} \cos\left(\tfrac{1}{2}[i(\tfrac{1}{2}-z)]\log t\right) dt$

$$\xi(z) = \frac{1}{2} - \frac{z(1-z)}{2} \int_{t=1}^{t=\infty} \psi(t) \left(t^{-(z+1)/2} + t^{z/2-1}\right) dt$$

$$= \frac{1}{2} - \frac{z(1-z)}{2} \int_{t=1}^{t=\infty} \psi(t) t^{-3/4} \left(t^{-i(1/2)(i[(1/2)-z])} + t^{i(1/2)(i[(1/2)-z])}\right) dt$$

$$= \frac{1}{2} - \frac{z(1-z)}{2} \int_{t=1}^{t=\infty} \psi(t) t^{-3/4} \left(e^{-i(1/2)(i[(1/2)-z])\log t} + e^{i(1/2)(i[(1/2)-z])\log t}\right) dt$$

$$= \frac{1}{2} - z(1-z) \int_{t=1}^{t=\infty} \psi(t) t^{-3/4} \cos\left(\tfrac{1}{2} i[(1/2)-z]\log t\right) dt$$

8.2.2 $\xi(z) = 4 \int\limits_{t=1}^{t=\infty} (t^{3/2}\psi'(t))' t^{-1/4} \cos\left(\frac{1}{2}[i(\frac{1}{2} - z)] \log t\right) dt$

$$\xi(z) = \frac{1}{2} - \frac{z(1-z)}{2} \int\limits_{t=1}^{t=\infty} \psi(t) \left(t^{-(z+1)/2} + t^{z/2-1}\right) dt$$

$$= \frac{1}{2} - \frac{z(1-z)}{2} \int\limits_{t=1}^{t=\infty} \psi(t) d_t \left(\frac{t^{(1-z)/2}}{(1-z)/2} + \frac{t^{z/2}}{z/2}\right)$$

$$= \frac{1}{2} - \left[\frac{z(1-z)}{2}\left(\frac{t^{(1-z)/2}}{(1-z)/2} + \frac{t^{z/2}}{z/2}\right)\psi(t)\right]_{t=1}^{t=\infty} +$$

$$+ \frac{z(1-z)}{2} \int\limits_{t=1}^{t=\infty} \left(\frac{t^{(1-z)/2}}{(1-z)/2} + \frac{t^{z/2}}{z/2}\right) d\psi(t)$$

$$= \frac{1}{2} + \left(zt^{(1-z)/2} + (1-z)t^{z/2}\right)\psi(t)\Big|_{t=1} +$$

$$+ \int\limits_{t=1}^{t=\infty} \left(zt^{(1-z)/2} + (1-z)t^{z/2}\right)\psi'(t) dt$$

$$= \frac{1}{2} + \psi(1) + \int\limits_{t=1}^{t=\infty} \left(zt^{-z/2-1} + (1-z)t^{(z-1)/2-1}\right) t^{3/2}\psi'(t) dt$$

$$= \frac{1}{2} + \psi(1) + \int\limits_{t=1}^{t=\infty} t^{3/2}\psi'(t) d_t \left(-2t^{-z/2} - 2t^{(z-1)/2}\right)$$

$$= \frac{1}{2} + \psi(1) + \left[t^{3/2}\psi'(t)(-2)\left(t^{-z/2} + t^{(z-1)/2}\right)\right]_{t=1}^{t=\infty} +$$

$$-\int_{t=1}^{t=\infty}(-2)\left(t^{-z/2}+t^{(z-1)/2}\right)d\left(t^{3/2}\psi'(t)\right)$$

$$=\frac{1}{2}+\psi(1)+4\psi'(1)+$$

$$+2\int_{t=1}^{t=\infty}t^{-1/4}\left(t^{-i(1/2)i[(1/2)-z]}+t^{i(1/2)i[(1/2)-z]}\right)d\left(t^{3/2}\psi'(t)\right)$$

$$=\frac{1}{2}+\psi(1)+4\psi'(1)+$$

$$+2\int_{t=1}^{t=\infty}t^{-1/4}\left(e^{-i(1/2)(i[(1/2)-z])\log t}+e^{i(1/2)(i[(1/2)-z])\log t}\right)d\left(t^{3/2}\psi'(t)\right)$$

$$=\frac{1}{2}+\psi(1)+4\psi'(1)+$$

$$+4\int_{t=1}^{t=\infty}t^{-1/4}\cos\left(\tfrac{1}{2}i[(1/2)-z]\log t\right)d\left(t^{3/2}\psi'(t)\right)$$

To evaluate $\psi'(1)$, differentiate Jacobi's formula

$$\psi(t)=\psi(1/t)t^{-1/2}+\tfrac{1}{2}t^{-1/2}-\tfrac{1}{2}.$$

$$\psi'(t)=\psi'(1/t)(-1/t^2)t^{-1/2}+\psi(1/t)(-\tfrac{1}{2}t^{-3/2})-\tfrac{1}{4}t^{-3/2}$$

$$\psi'(1)=\psi'(1)(-1)+\psi(1)(-\tfrac{1}{2})-\tfrac{1}{4}$$

$$4\psi'(1)+\psi(1)+\tfrac{1}{2}=0$$

Thus,

$$\xi(z)=4\int_{t=1}^{t=\infty}t^{-1/4}\cos\left(\tfrac{1}{2}i[(1/2)-z]\log t\right)d\left(t^{3/2}\psi'(t)\right)$$

Chapter 9

Taylor series for the function $\xi(\frac{1}{2} + iw)$ in w

We use $\xi(1/2 + iw)$ to get a correct product formula for ξ.

9.1 Formulas for $\xi(\frac{1}{2} + iw)$

Riemann wrote

> *Set*
>
> $$z = 1/2 + iw,$$
>
> *and*
>
> $$\tfrac{1}{2}z(z-1)\pi^{-z/2}\Gamma(z/2)\zeta(z) = \xi(w).$$
>
> *Then,*
>
> $$\xi(w) = \tfrac{1}{2} - \left(\tfrac{1}{4} + w^2\right) \int\limits_{t=1}^{t=\infty} \psi(t) t^{-3/4} \cos\left(\tfrac{1}{2} w \log t\right) dt.$$

or also

$$\xi(w) = 4 \int_{t=1}^{t=\infty} \frac{d(t^{3/2}\psi'(t))}{dt} t^{-1/4} \cos(\tfrac{1}{2} w \log t) dt.$$

Substituting

$$w = i(\tfrac{1}{2} - z) = y + i(\tfrac{1}{2} - x).$$

into the 8.2.1 formula

$$\xi(z) = \tfrac{1}{2} - z(1-z) \int_{t=1}^{t=\infty} \psi(t) t^{-3/4} \cos\left(\tfrac{1}{2}[i(\tfrac{1}{2} - z)] \log t\right) dt,$$

we have

$$\xi(\tfrac{1}{2} + iw) = \tfrac{1}{2} - \left(\tfrac{1}{4} + w^2\right) \int_{t=1}^{t=\infty} \psi(t) t^{-3/4} \cos\left(\tfrac{1}{2} w \log t\right) dt.$$

Substituting w into the 8.2.2 formula

$$\xi(z) = 4 \int_{t=1}^{t=\infty} \left(t^{3/2}\psi'(t)\right)' t^{-1/4} \cos\left(\tfrac{1}{2}[i(\tfrac{1}{2} - z)] \log t\right) dt,$$

we have

$$\xi(\tfrac{1}{2} + iw) = 4 \int_{t=1}^{t=\infty} \left(t^{3/2}\psi'(t)\right)' t^{-1/4} \cos(\tfrac{1}{2} w \log t) dt.$$

9.2 $\xi(\frac{1}{2}+iw) = \sum_{n=0}^{\infty} A_n (w^2)^n$ that converges very rapidly

Riemann wrote

> *$\xi(w)$ is finite for all finite values of w, and can be expanded into a very rapidly convergent series in powers of w^2.*

By the second formula,

$$\xi(\tfrac{1}{2}+iw) = 4\int_{t=1}^{t=\infty} \left(t^{3/2}\psi'(t)\right)' t^{-1/4}\cos(\tfrac{1}{2}w\log t)dt$$

$$= 4\int_{t=1}^{t=\infty} \left(t^{3/2}\psi'(t)\right)' t^{-1/4}\left[1-\tfrac{1}{2!}(\tfrac{1}{2}\log t)^2 w^2 + \tfrac{1}{4!}(\tfrac{1}{2}\log t)^4 w^4 - ...\right] dt$$

$$= A_0 - A_1 w^2 + A_2 w^4 - ...$$

where,

$$A_n = 4\frac{1}{(2n)!}\int_{t=1}^{t=\infty} \left(t^{3/2}\psi'(t)\right)' t^{-1/4}\left[\tfrac{1}{2}\log t\right]^{2n} dt.$$

Hadamard [7] proved that the rapid convergence is equivalent to the product formula

$$\xi(z) = \xi(0)\prod_{\rho=zero\ of\ \xi}\left(1-\frac{z}{\rho}\right)$$

Chapter 10

The zeros of $\xi(z)$.

The zeros of

$$\xi(z)$$

will be denoted by

$$\rho,$$

and

$$1 - \rho.$$

The zeros of

$$\xi(1/2 + iw)$$

will be denoted by

$$\alpha.$$

We see that

$$\rho = 1/2 + i\alpha,$$

and

$$1 - \rho = 1/2 - i\alpha.$$

We shall use

$$-z(1-z) = z^2 - z = (z - 1/2)^2 - 1/4 = -(w^2 + 1/4)$$

$$\rho(1-\rho) = \alpha^2 + 1/4$$

10.1 All the zeros of $\xi(z)$ are in $0 < x < 1$.

Riemann wrote

> *Since for $z = x + iy$, with $x > 1$,*
>
> $$\log \zeta(z) = -\sum \log(1 - 1/p^z)$$
>
> *is finite,*
>
> *and since the same is true for the logarithms of the other factors of*
>
> $$\xi(w),$$
>
> *the function*
>
> $$\xi(w)$$
>
> *will vanish only if*
>
> $$-\tfrac{1}{2} < \mathrm{Im}(w) < \tfrac{1}{2}.$$

For $x > 1$, the Euler product

$$\prod_{p=prime} \frac{1}{1 - 1/p^z} = \zeta(z)$$

has no vanishing factor, and is non-zero.
Namely, for $x > 1$,

$$\zeta(z) \neq 0.$$

and

$$\xi(z) = \tfrac{1}{2} z(z-1)\pi^{-z/2}\Gamma(z/2)\zeta(z) \neq 0.$$

Thus, if ρ is a zero of $\xi(z)$,

$$Re(\rho) < 1.$$

Since $\xi(z) = \xi(1-z)$, we have

$$\xi(\rho) = 0 \Leftrightarrow \xi(1-\rho) = 0.$$

That is,

$$\text{if } \rho \text{ is a zero of } \xi(z), \text{ so is } 1-\rho.$$

Hence,

$$1 > Re(1-\rho) = 1 - Re(\rho).$$

Therefore,

$$0 < Re(\rho).$$

Thus,

$$\text{All the zeros of } \xi(z) \text{ are in } 0 < x < 1.$$

Namely, in

$$-\tfrac{1}{2} < x - \tfrac{1}{2} < \tfrac{1}{2}.$$

That is,

$$\tfrac{1}{2} > -x + \tfrac{1}{2} > -\tfrac{1}{2}.$$

or

$$\tfrac{1}{2} > Im(w) > -\tfrac{1}{2}$$

Chapter 11

The Number of Zeros of $\xi(z)$

11.1 If R is large enough, $\log|\xi(z)|$ is bounded in $|w| = |z - 1/2| \leq 2R$, by $R \log R$

By 9.2,

$$\xi(z) = A_0 - A_1(i[z-1/2])^2 + A_2(i[z-1/2])^4 - A_3(i[z-1/2])^6 + ...$$

$$= A_0 + A_1(z-1/2)^2 + A_2(z-1/2)^4 + A_3(z-1/2)^6 + ...$$

where

$$A_n = 4\frac{1}{(2n)!} \int_{t=1}^{t=\infty} (t^{3/2}\psi'(t))' t^{-1/4} \left(\frac{\log t}{2}\right)^{2n} dt.$$

The $A_n > 0$ because

$$(t^{3/2}\psi'(t))' = \left(t^{3/2}\left(\sum_{n=1}^{\infty} \exp(-n^2\pi t)\right)'\right)'$$

$$= \left(t^{3/2}\left(-\pi \sum_{n=1}^{\infty} n^2 \exp(-n^2\pi t)\right)\right)'$$

$$= \frac{3}{2}t^{1/2}(-\pi)\sum_{n=1}^{\infty} n^2 \exp(-n^2\pi t) + t^{3/2}\pi^2 \sum_{n=1}^{\infty} n^4 \exp(-n^2\pi t)$$

$$= \pi \sum_{n=1}^{\infty} n^2 t^{1/2} \mathrm{e}^{-n^2\pi t}(n^2\pi t - 3/2).$$

Therefore, in $|z - 1/2| \leq 2R$,

$$|\xi(z)| \leq |\xi(2R + 1/2)|,$$

and if $2R + 1/2 < 2n < 2R + 2 + 1/2$,

$$|\xi(z)| \leq |\xi(2n)| = (2n-1)\pi^{-n}\Gamma(n+1)\zeta(2n)$$

$$\leq 2n\pi^{-n} n!\zeta(2) \leq n^{n+1} \leq (R + 5/4)^{R+9/4}.$$

Hence, for large enough R,

$$|\xi(z)| \leq R^R,$$

and $\log|\xi(z)|$ in $|z - 1/2| \leq 2R$ is bounded by $R \log R$.

This enabled Hadamard to obtain his estimate for the number of zeros of $\xi(z)$.

11.2 Hadamard Estimate: If R is large enough, then $n(R)$=the number of zeros of $\xi(z)$ in $|w| = |z - 1/2| \leq R$ is bounded by $2R \log R$

By Jensen's Theorem [1, p. 40], applied to $|w| = |z - 1/2| \leq 2R$

$$\log\left|\xi(1/2)\frac{(2R)^{n(2R)}}{(\rho_1 - 1/2)...(\rho_{n(2R)} - 1/2)}\right| = \frac{1}{2\pi}\int_{\theta=0}^{\theta=2\pi} \log\left|f(R\mathrm{e}^{\mathrm{i}\theta})\right| d\theta.$$

Substitute

$$\log\left|\xi(Re^{i\theta})\right| \leq R\log R$$

$$|\rho_k - 1/2| = |\alpha_k|$$

$$\log\left|\frac{(2R)^{n(2R)}}{\alpha_1 ... \alpha_{n(2R)}}\right| = n(R)\log 2 + \log\left|\frac{R^{n(R)}}{\alpha_1 ... \alpha_{n(R)}}\right| + \log\left|\frac{(2R)^{n(2R)-n(R)}}{\alpha_{n(R)+1} ... \alpha_{n(2R)}}\right|$$

Since for $k = 1...n(R)$,

$$|\alpha_k| \leq R,$$

and

$$|\alpha_{n(R)+k}| \leq 2R,$$

we have

$$n(R)\log 2 \leq \log\left|\frac{(2R)^{n(2R)}}{\alpha_1 ... \alpha_{n(2R)}}\right|.$$

Therefore,

$$\log|\xi(1/2)| + n(R)\log 2 \leq R\log R.$$

Hence, for R large enough,

$$n(R) \leq 2R\log R$$

Hadamard estimate is sufficient for the derivation of the product formula for $\xi(z)$.

11.3 Riemann Estimate: The number of zeros of $\xi(z)$, in $0 < y < Y$, and $0 < x < 1$, is bounded by $\frac{Y}{2\pi}\log\frac{Y}{2\pi} - \frac{Y}{2\pi}$.

Riemann wrote

Consider the counter-clockwise closed-path integral

$$\int_{\text{counter clockwise}} d \log \xi(\tfrac{1}{2} + iw)$$

around the domain with

$$0 < x < 1, \text{ and } 0 < y < Y.$$

With relative error of the order of $1/Y$, *the integral is equal to*

$$iY \left(\log \frac{Y}{2\pi} - 1 \right).$$

On the other hand, the integral equals

$$2\pi i \,(\text{Number of zeros of } \xi \text{ in the domain }).$$

In 1914, Backlund [8] gave a proof.

Chapter 12

The Hypothesis

12.1 The Hypothesis: All the zeros of $\xi(z)$ are on $x = 1/2$.

In terms of $\xi(w)$, having

$$x = 1/2$$

means

$$Im(w) = 0.$$

Riemann wrote

> *It is very likely that all of the zeros of $\xi(w)$ are real. One would like to have a rigorous proof of this, but after several fleeting attempts to no avail, I have temporarily set aside the search for this proof because it appeared to be unnecessary for the immediate purpose of my investigation.*

The Hypothesis is required in the application of the product formula to the count of the primes, and Riemann's erroneous product formula did not indicate the need for the Hypothesis.

While the correct product formula is needed for a derivation, the formula for the count of the primes may be arrived at empirically, and Riemann realized later that he needed the Hypothesis for a derivation, and tried to prove it. After the publication of the 1859 paper, Riemann wrote

> *…The Theorem which I merely cited that between* 0, *and* Y *there are around*
>
> $$\frac{Y}{2\pi}\left(\log\frac{Y}{2\pi}-1\right)$$
>
> *real zeros of the function* ξ *follows from a new development of* ξ, *which I had not simplified enough to report it…*

Apparently, Riemann had no proof for the Hypothesis.

In January 2006, I reported my proof of the Hypothesis in the San Antonio Mathematics meeting [9].
The proof was submitted to a journal.

Chapter 13

The Product Formula

13.1 Weierstrass Factorization Theorem

Riemann wrote

> *Since the density of the zeros of size w increases like*
>
> $$\log \frac{w}{2\pi},$$
>
> *the series*
>
> $$\sum_{\alpha = zero \text{ of } \xi(w)} \log(1 - w^2/\alpha^2)$$
>
> *converges, and grows like*
>
> $$|w| \log |w| .$$
>
> $$\log \xi(w) - \sum_{\alpha = zero \text{ of } \xi(w)} \log(1 - w^2/\alpha^2)$$
>
> *is a function that is continuous, and finite for finite w.*

For $w \to \infty$,

$$\frac{1}{w^2}\left[\log\xi(w) - \sum_{\alpha = zero \text{ of } \xi(w)} \log(1 - w^2/\alpha^2)\right] \to 0$$

Therefore,

$$\log\xi(w) - \sum_{\alpha = zero \text{ of } \xi(w)} \log(1 - w^2/\alpha^2) = const.$$

Setting $w = 0$, *gives*

$$\log\xi(0) = const.$$

Since $\xi(z)$ is an entire function so that $\xi(0) \neq 0$, by Weierstrass [10, Chapter 7, 2.13]

$$\xi(z) = e^{h(z)}\prod_{n=1}^{\infty}(1 - \frac{z}{\rho_n})e^{Q_n(z)},$$

where

- ♣ the zeros of $\xi(z)$, the ρ_n's are sequenced by their size and increase to ∞
- ♣ the polynomials $Q_n(z)$ guarantee the uniform convergence of the product in the open plane
- ♣ $h(z)$ is an entire function

Since the ρ_n's are sequenced by size, and since $1 - \rho_n$ is a zero too, the product representation is

$$\xi(z) = e^{h(z)}\prod_{n=1}^{\infty}(1 - \frac{z}{\rho_n})(1 - \frac{z}{1-\rho_n})e^{Q_n(z)},$$

where

$$(1 - \frac{z}{\rho_n})(1 - \frac{z}{1-\rho_n}) = 1 - \frac{z(1-z)}{\rho_n(1-\rho_n)} = 1 - \frac{w^2 + 1/4}{\alpha_n^2 + 1/4}$$

Following Riemann outline, this should lead to

$$\log \xi(1/2 + iw) = \log \xi|_{w=i/2} + \sum_n \log \left(1 - \frac{w^2 + 1/4}{\alpha_n^2 + 1/4}\right)$$

Since Riemann had the wrong result, we do not follow his outline here. We use Hadamard Theorem to obtain the product formula.

We first show that

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{\rho_n}\right)\left(1 - \frac{z}{1-\rho_n}\right)$$

converges. That is

$$e^{Q_n(z)} = 1.$$

13.2 $\xi(z) = e^{h(z)} \prod_{n=1}^{\infty} (1 - \frac{z}{\rho_n})(1 - \frac{z}{1-\rho_n})$

The convergence of

$$\prod_m \left(1 - \frac{z(1-z)}{\rho_m(1-\rho_m)}\right)$$

is equivalent to the convergence of

$$\sum_m \frac{1}{\rho_m(1-\rho_m)}$$

Since

$$|\rho_m(1-\rho_m)| = \left|(\rho_m - 1/2)^2 - 1/4\right| > |\rho_m - 1/2|^2 ,$$

it is sufficient to show that

$$\sum_m \frac{1}{|\rho_m - 1/2|^2} = \sum_m \frac{1}{|\alpha_m^2|} < \infty,$$

or, that the tail is bounded. That is,

$$\sum_{m>N} \frac{1}{|\alpha_m^2|} < \infty.$$

The α_m are arranged so that

$$\frac{1}{|\alpha_m|^2}$$

is decreasing.

For m large enough,

$$m = N, N+1, N+2, ...$$

define positive numbers $R_m > 1$ so that

$$\log R_m > 1$$

$$m = 4R_m \log R_m.$$

Then,

$$\log m > \log R_m.$$

By 11.2, the number of zeros of $\xi(z)$ in $|w| \le R_m$ is bounded by

$$2R_m \log R_m$$

Hence,

$$|\alpha_m| > R_m.$$

and we have

$$\sum_{m>N} \frac{1}{|\alpha_m|^2} \le \sum_{m>N} \frac{1}{R_m^2} = 4^2 \sum_{m>N} \frac{1}{m^2} (\log R_m)^2$$

$$\le 4^2 \sum_{m>N} \frac{(\log m)^2}{m^2} = 4^2 \sum_{m>N} \frac{1}{m^{3/2}} \frac{(\log m)^2}{m^{1/2}}.$$

Since

$$\frac{(\log m)^2}{m^{1/2}} \to 0, \ as\ m \to \infty,$$

we have

$$\tfrac{(\log m)^2}{m^{1/2}} < 1, \text{ for } m > N,$$

and

$$\sum_{m>N} \frac{1}{|\alpha_m|^2} < \infty.$$

13.3 $\xi(z) = \xi(0) \prod_{n=1}^{\infty} (1 - \frac{z}{\rho_n})(1 - \frac{z}{1-\rho_n})$ in $0 < x < 1$

The Hadamard Factorization Theorem [11, p. 68] applies to a function $f(z)$ for which

$$\limsup_{R\to\infty} \frac{1}{\log R} \log \log \max_{|z|=R} |f(z)| \equiv \varrho < \infty$$

Then,

$$\varrho = \text{the } \textbf{order of } f(z).$$

Hadamard replaces Weierstrass

$$h(z)$$

with a polynomial

$$Q(z)$$

so that

$$\deg Q(z) \leq \text{order of the factorized function}$$

By 11.1, if R is large enough, $\log \xi(z)$ in $|w| \leq 2R$, is bounded by $R \log R$. Hence,

$$\frac{1}{\log R} \log \log \max_{|w|=R} |\xi(z)| \leq \frac{1}{\log R} \log (R \log R) = 1 + \frac{\log \log R}{\log R}$$

Thus, by L'Hospital,

$$\limsup_{R \to \infty} \frac{1}{\log R} \log \log \max_{|w|=R} |\xi(z)| = 1.$$

That is,

$$\xi(z) \text{ is of order } \varrho = 1$$

Hence,

$$\deg Q(z) \leq 1.$$

and

$$Q(w) = A + Bw$$

Therefore,

$$\xi(1/2 + iw) = e^{A+Bw} \prod_n \left(1 - \frac{w^2 + 1/4}{\alpha_n^2 + 1/4}\right),$$

where the product is an even function of w.

On the other hand, by 9.2,

$$\xi(1/2 + iw) = A_0 - A_1 w^2 + A_2 w^4 + ...$$

is an even function of w.
Consequently,

$$B = 0$$

and

$$\xi(1/2 + iw) = e^A \prod_n \left(1 - \frac{w^2 + 1/4}{\alpha_n^2 + 1/4}\right),$$

Setting $w = -i/2$,

$$\xi(0) = e^A$$

and

$$\xi(z) = \xi(0) \prod_n \left(1 - \frac{z}{\rho_n}\right) \left(1 - \frac{z}{1 - \rho_n}\right)$$

Part II

The Number of Primes $< t$

Chapter 14

$\pi(t)$ and $F(t)$

Riemann wrote

> *...We can now determine $\pi(t)$, the number of primes less than t.*
>
> *Let*
>
> $$F(t)$$
>
> *be*
>
> $$\textit{equal to } \pi(t) \textit{ if } t \textit{ is not a prime,}$$
>
> *and*
>
> $$\textit{equal to } \pi(t) + \tfrac{1}{2}, \textit{ if } t \textit{ is a prime,}$$
>
> *so that if $F(t)$ jumps at t,*
>
> $$F(t) = \tfrac{1}{2}[F(t+0) + F(t-0].$$

We define

$$\pi(t) \equiv \text{number of prime numbers } p \text{ so that } p < t.$$

Thus,

$$\begin{aligned} \pi(1) &= 0 \\ \pi(\sqrt{2}) &= 0 \\ \pi(2) &= 0 \\ \pi(e) &= 1 \\ \pi(3) &= 1 \\ \pi(\pi) &= 2 \end{aligned}$$

To apply Fourier Integral Theorem, we'll need the auxiliary function

$$F(t) \equiv \begin{cases} \pi(t) & if \quad t \neq prime \\ \pi(t) + 1/2 & if \quad t = prime \end{cases}$$

that satisfies the Dirichlet condition

$$\tfrac{1}{2}[F(t+0) + F(t-0] = F(t).$$

Thus,

t	$\pi(t)$	$F(t)$
$t < 2$	0	0
$t = 2$	0	$1/2$
$2 < t < 3$	1	1
$t = 3$	1	$1 + 1/2$
$3 < t < 5$	2	2
$t = 5$	2	$2 + 1/2$

Chapter 15

Zeta and $F(t)$

15.1 $\frac{\log \zeta(z)}{z} = \int\limits_{t=1}^{t=\infty} \frac{1}{t^{z+1}} \left(F(t) + \frac{1}{2}F(t^{1/2}) + \frac{1}{3}F(t^{1/3}) + ... \right) dt$

Riemann wrote

> *If* $x > 1$,
>
> $$\log \zeta(z) = - \sum_{p=prime} \log \left(1 - \frac{1}{p^z} \right)$$
>
> $$= \sum_{p=prime} \frac{1}{p^z} + \frac{1}{2} \sum_{p=prime} \frac{1}{p^{2z}} + \frac{1}{3} \sum_{p=prime} \frac{1}{p^{3z}} + ...$$
>
> *Substitute*
>
> $$\frac{1}{p^z} = z \int\limits_{t=p}^{t=\infty} \frac{dt}{t^{z+1}}$$
>
> $$\frac{1}{p^{2z}} = z \int\limits_{t=p^2}^{t=\infty} \frac{dt}{t^{z+1}}$$

....

Then,

$$\frac{\log\zeta(z)}{z} = \int_{t=1}^{t=\infty} \frac{1}{t^{z+1}} f(t)dt,$$

where

$$f(t) = F(t) + \tfrac{1}{2}F(t^{1/2}) + \tfrac{1}{3}F(t^{1/3}) + ...$$

Since

$$F(t+0) - F(t-0) \equiv \begin{cases} 0, & if \quad t \neq prime \\ 1, & if \quad t = prime \end{cases}$$

we have

$$\sum_{p=prime} \frac{1}{p^z} = \int_{t=1}^{t=\infty} \frac{1}{t^z}\left[F(t+0) - F(t-0)\right] = \int_{t=1}^{t=\infty} \frac{1}{t^z} dF(t)$$

$$\sum_{p=prime} \frac{1}{p^{2z}} = \int_{t=1}^{t=\infty} \frac{1}{t^z}\left[F(t^{1/2}+0) - F(t^{1/2}-0)\right] = \int_{t=1}^{t=\infty} \frac{1}{t^z} dF(t^{1/2})$$

$$\sum_{p=prime} \frac{1}{p^{3z}} = \int_{t=1}^{t=\infty} \frac{1}{t^z}\left[F(t^{1/3}+0) - F(t^{1/3}-0)\right] = \int_{t=1}^{t=\infty} \frac{1}{t^z} dF(t^{1/3})$$

....

Therefore,

$$\log \zeta(z) = \sum_{p=prime} \frac{1}{p^z} + \frac{1}{2} \sum_{p=prime} \frac{1}{p^{2z}} + \frac{1}{3} \sum_{p=prime} \frac{1}{p^{3z}} + ...$$

$$= \int_{t=1}^{t=\infty} \frac{1}{t^z} dF(t) + \frac{1}{2} \int_{t=1}^{t=\infty} \frac{1}{t^z} dF(t^{1/2}) + \frac{1}{3} \int_{t=1}^{t=\infty} \frac{1}{t^z} dF(t^{1/3}) + ...$$

Integrating by parts,

$$\int_{t=1}^{t=\infty} \frac{1}{t^z} d_t F(t^{1/n}) = \left[\frac{1}{t^z} F(t^{1/n}) \right]_{t=1}^{t=\infty} - \int_{t=1}^{t=\infty} F(t^{1/n})(-z) t^{-z-1} dt$$

Since $F(1) = 0$, and $\frac{1}{t^z}|_{t=\infty} = 0$, we have

$$= z \int_{t=1}^{t=\infty} F(t^{1/n}) \frac{1}{t^{z+1}} dt.$$

Consequently,

$$\log \zeta(z) = z \int_{t=1}^{t=\infty} F(t) \frac{1}{t^{z+1}} dt + \frac{1}{2} z \int_{t=1}^{t=\infty} F(t^{1/2}) \frac{1}{t^{z+1}} dt + ...$$

and

$$\frac{\log \zeta(z)}{z} = \int_{t=1}^{t=\infty} \left(F(t) + \frac{1}{2} F(t^{1/2}) + \frac{1}{3} F(t^{1/3}) + ... \right) \frac{dt}{t^{z+1}}.$$

Chapter 16

$f(t)$ and Zeta

16.1 $f(t) = \frac{1}{2\pi i} \int\limits_{y=-\infty}^{y=\infty} \frac{\log\zeta(z)}{z} t^z dz = \frac{1}{2\pi} \int\limits_{y=-\infty}^{y=\infty} \frac{\log\zeta(z)}{z} t^z dy$, for fixed x, and for any z

Riemann wrote,

> *If for $x > 1$,*
>
> $$g(z) = \int\limits_{u=0}^{u=\infty} h(u) u^{-z} d(\log t),$$
>
> *then by the Fourier Theorem*
>
> $$h$$
>
> *can be written in terms of*
>
> $$g(z)$$
>
> *If*
>
> - $h(u)$ *is real*

- $g(x+iy) = g_1(y) + g_2(y)$

then the equation splits into

$$g_1(y) = \int_{u=0}^{u=\infty} h(u)u^{-x}\cos(y\log u)d(\log u)$$

$$g_2(y) = -i\int_{u=0}^{u=\infty} h(u)u^{-x}\sin(y\log u)d(\log u)$$

Multiply both equations by

$$[\cos(y\log t) + i\sin(y\log t)]dy,$$

and integrate from $y = -\infty$ to $y = \infty$.

Then, the right hand side of either equation is

$$\pi h(t)t^{-x}.$$

Adding the equations, and multiplying by it^x,

$$2\pi i h(t) = \int_{x-i\infty}^{x+i\infty} g(z)t^z dz,$$

where x is fixed through the integration.

Thus, if $h(t)$ has a jump at t,

then,

$$h(t) = \tfrac{1}{2}[h(t+0) + h(t-0)].$$

Since $f(t)$ has the same property, we get with complete generality

$$f(t) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} \frac{\log \zeta(z)}{z} t^z dz$$

For $0 \leq u \leq 1$, we have

$$F(u) = 0$$

and

$$f(u) = 0$$

Hence,

$$\frac{\log \zeta(z)}{z} = \int_{u=1}^{u=\infty} \frac{1}{u^{z+1}} f(u) du = \int_{u=0}^{u=\infty} f(u) u^{-z} d(\log u).$$

Fourier Inversion applies to write f in terms of

$$\frac{log\zeta(z)}{z}$$

To that end, fix x, multiply by

$$t^z dz = t^x t^{iy} d(x+iy) = it^x e^{iy \log t} dy,$$

and integrate from $y = -\infty$ to $y = \infty$.

Then,

$$\int_{y=-\infty}^{y=\infty} \frac{\log \zeta(z)}{z} t^z dz =$$

$$= it^x \int_{y=-\infty}^{y=\infty} e^{iy\log t} \left(\int_{u=0}^{u=\infty} f(u)u^{-x}e^{-iy\log u} d(\log u) \right) dy$$

Since

$$F(t) = \tfrac{1}{2}[F(t+0) + F(t-0)],$$

we have

$$f(t) = \tfrac{1}{2}[f(t+0) + f(t-0)]$$

and by Fourier integral Theorem, we can change the order of integration.

$$= it^x \int_{u=0}^{u=\infty} f(u)u^{-x} \left(\int_{y=-\infty}^{y=\infty} e^{iy(\log t - \log u)} dy \right) d(\log u)$$

$$= it^x \int_{u=0}^{u=\infty} f(u)u^{-x} \left[2\pi\delta(\log t - \log u)\right] d(\log u)$$

$$= 2\pi i f(t).$$

Therefore,

$$f(t) = \frac{1}{2\pi i} \int_{y=-\infty}^{y=\infty} \frac{\log\zeta(z)}{z} t^z dz = \frac{1}{2\pi} \int_{y=-\infty}^{y=\infty} \frac{\log\zeta(z)}{z} t^z dy,$$

for fixed x, and for any z

Chapter 17

$f(t)$ and $\xi(z)$

17.1 $f(t) = \frac{1}{2\pi i}\int\limits_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d_z\Phi$, where $\Phi = \frac{\log(z-1)}{z} + \frac{\log\Gamma(z/2+1)}{z} - \frac{\log\xi(0)}{z} - \frac{1}{z}\sum\limits_{\alpha}\log\left(1+\frac{z^2-z}{\alpha^2+1/4}\right)$

Riemann wrote

$$f(t) = \frac{1}{2\pi i}\int\limits_{x-i\infty}^{x+i\infty} \frac{\log\zeta(z)}{z} t^z dz.$$

For

$$\log\zeta(z)$$

we may substitute

$\frac{1}{2}z\log\pi - \log(z-1) - \log\Gamma\left(\frac{1}{2}z+1\right)$

$$+\sum_{\alpha = zero\ of\ \xi}\log\left(1+\frac{(z-1/2)^2}{\alpha^2}\right) + \log\xi(0)$$

But the integrals of these terms are divergent at infinity. So we have to integrate the equation for f by parts

$$f(t) = -\frac{1}{2\pi i}\frac{1}{\log t}\int_{x-i\infty}^{x+i\infty} \frac{d\left[\frac{1}{z}\log\zeta(z)\right]}{dz} t^z dz$$

Since

$$\xi(z) = \tfrac{1}{2}z(z-1)\pi^{-z/2}\Gamma(z/2)\zeta(z),$$

we have

$$\log\zeta(z) = \tfrac{1}{2}z\log\pi - \log(z-1) - \log\Gamma\left(\tfrac{1}{2}z+1\right) + \log\xi(z).$$

Replacing $\xi(z)$ by the product formula of Section 13.3

$$\frac{\log\zeta(z)}{z} = \tfrac{1}{2}\log\pi - \frac{\log(z-1)}{z} - \frac{\log\Gamma\left(\frac{1}{2}z+1\right)}{z} + \\ + \frac{1}{z}\sum_{\alpha}\log\left(1+\frac{z^2-z}{\alpha^2+1/4}\right) + \frac{\log\xi(0)}{z}.$$

If this formula is used to obtain $f(t)$, the first term

$$\tfrac{1}{2}\log\pi$$

gives

$$\frac{\frac{1}{2}\log\pi}{2\pi i}\int_{x-i\infty}^{x+i\infty} t^z dz = \frac{\log\pi}{4\pi i\log t} t^x \left[t^{iy}\right]_{y=-\infty}^{y=\infty}$$

that diverges.

On the other hand, integrating by parts gives

$$f(t) = \frac{1}{2\pi i}\frac{1}{\log t}\lim_{Y\to\infty}\left[\frac{\log\zeta(z)}{z}t^z\right]_{y=-Y}^{y=Y} - \frac{1}{2\pi i}\frac{1}{\log t}\int_{x-i\infty}^{x+i\infty} t^z d_z\left[\frac{\log\zeta(z)}{z}\right].$$

Now,

$$|\log\zeta(z)| = \sum_{p=prime}\left|\log\left(1-\tfrac{1}{p^z}\right)\right|$$

$$\leq \sum_{p=prime}\left[\frac{1}{|p^z|}+\frac{1}{2}\frac{1}{|p^{2z}|}+\frac{1}{3}\frac{1}{|p^{3z}|}+...\right]$$

$$= \sum_{p=prime}\left[\frac{1}{p^x}+\frac{1}{2}\frac{1}{p^{2x}}+\frac{1}{3}\frac{1}{p^{3x}}+...\right]$$

$$= \sum_{p=prime}\left|\log\left(1-\frac{1}{p^x}\right)\right| = |\log\zeta(x)|$$

This ensures the vanishing of the boundary terms

$$\left|\left[\frac{\log\zeta(z)}{z}t^z\right]_{y=-Y}^{y=Y}\right| \leq 2\frac{t^x|\log\zeta(x)|}{\sqrt{x^2+Y^2}} \to 0,\ as\ Y\to\infty.$$

Therefore,

$$f(t) = -\frac{1}{2\pi i}\frac{1}{\log t}\int_{x-i\infty}^{x+i\infty} t^z d_z\left[\frac{\log\zeta(z)}{z}\right]$$

$$= \frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}\times$$

$$d_z\left[\tfrac{\log(z-1)}{z}+\tfrac{\log\Gamma\left(\frac{1}{2}z+1\right)}{z}-\tfrac{\log\xi(0)}{z}-\tfrac{1}{z}\sum_{\alpha}\log\left(1+\tfrac{z^2-z}{\alpha^2+1/4}\right)\right]$$

Chapter 18

The $-\dfrac{\log \xi(0)}{z}$ term

18.1 $-\frac{1}{2\pi i}\int\limits_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d_z\left[\frac{1}{z}\right] = \frac{1}{2\pi i}\int\limits_{y=-\infty}^{y=\infty} \frac{1}{z} t^z dz = 1$

Integration by parts gives

$$-\frac{1}{2\pi i}\int\limits_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d_z\left[\frac{1}{z}\right] =$$

$$= -\frac{1}{2\pi i}\,\frac{1}{\log t}\lim_{Y\to\infty}\left[\frac{t^z}{z}\right]_{y=-Y}^{y=Y} + \frac{1}{2\pi i}\int\limits_{y=-\infty}^{y=\infty} \frac{1}{z} t^z dz$$

The boundary terms vanish because

$$\left|\left[\frac{1}{z}t^z\right]_{y=-Y}^{y=Y}\right| \le 2\frac{t^x}{\sqrt{x^2+Y^2}} \to 0,\ as\ Y \to \infty.$$

Therefore, we have

$$= \frac{1}{2\pi i}\int\limits_{y=-\infty}^{y=\infty} \frac{1}{z} t^z dz$$

Substituting

$$dz = idy,$$

$$t^z = t^x e^{iy \log t}$$

$$\tfrac{1}{z} = \int_{u=1}^{u=\infty} u^{-z-1} du,$$

$$= t^x \frac{1}{2\pi} \int_{y=-\infty}^{y=\infty} \left[\int_{u=1}^{u=\infty} u^{-z-1} du \right] e^{iy \log t} dy$$

By the change of variable

$$u = e^{\omega},$$

$$du = e^{\omega} d\omega,$$

$$= t^x \frac{1}{2\pi} \int_{y=-\infty}^{y=\infty} \left[\int_{\omega=0}^{\omega=\infty} e^{-\omega z} d\omega \right] e^{iy \log t} dy$$

$$= t^x \frac{1}{2\pi} \int_{y=-\infty}^{y=\infty} \left[\int_{\omega=0}^{\omega=\infty} e^{-\omega x} e^{-i\omega y} d\omega \right] e^{iy \log t} dy$$

By Fourier Integral Theorem for $\left\{ \begin{array}{ll} e^{-\omega x}, & \omega > 0 \\ 0, & \omega < 0 \end{array} \right.$, we change integration order

$$= t^x \int_{\omega=0}^{\omega=\infty} e^{-\omega x} \left[\frac{1}{2\pi} \int_{y=-\infty}^{y=\infty} e^{iy(\log t - \omega)} dy \right] d\omega$$

$$= t^x \int\limits_{\omega=0}^{\omega=\infty} e^{-\omega x}\delta(\omega - \log t)d\omega$$

$$= t^x e^{-x\log t} = 1$$

18.2 $-\frac{1}{2\pi i}\int\limits_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d_z \left[\frac{\log\xi(0)}{z}\right] = \log\xi(0) = -\log 2$

Riemann wrote

$$\frac{1}{2\pi i}\frac{1}{\log t}\int\limits_{x-i\infty}^{x+i\infty} \frac{1}{z^2}\,(\log\xi(0))t^z dz = \log\xi(0)$$

$\log\xi(0)$ factors out of

$$-\frac{1}{2\pi i}\int\limits_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d_z \left[\frac{\log\xi(0)}{z}\right]$$

and by section 18.1, we obtain just $\log\xi(0)$.

Since,

$$\xi(0) = \pi^{-0/2}\Gamma(1)(0-1)\zeta(0) = -\zeta(0) = \tfrac{1}{2}.$$

we have

$$\log\xi(0) = \log\tfrac{1}{2} = -\log 2.$$

Chapter 19

Terms with $\log\left(1-\frac{z}{\beta}\right)$

19.1 $-d_z\left[\frac{1}{z}\log\Gamma\left(\frac{1}{2}z+1\right)\right]=\sum_{n=1}^{\infty}d_z\left[\frac{1}{z}\log\left(1+\frac{z}{2n}\right)\right]$

Riemann wrote

$$-\log\Gamma\left(\tfrac{1}{2}z+1\right)=\lim_{N\to\infty}\left[\sum_{n=1}^{N}\log\left(1+\frac{z}{2n}\right)-\frac{z}{2}\log N\right].$$

Hence,

$$\frac{d\left(\frac{1}{z}\log\Gamma\left(\frac{1}{2}z+1\right)\right)}{dz}=\sum_{n=1}^{\infty}\frac{d\left(\frac{1}{z}\log\left(1+\frac{z}{2n}\right)\right)}{dz}$$

For the Gamma function

$$\Gamma\left(\tfrac{1}{2}z+1\right)=\lim_{N\to\infty}\frac{N^{z/2}}{\prod_{n=1}^{\infty}\left(1+\frac{z}{2n}\right)}$$

$$-\log\Gamma\left(\tfrac{1}{2}z+1\right) = \lim_{N\to\infty}\left[\sum_{n=1}^{N}\log\left(1+\frac{z}{2n}\right) - \tfrac{1}{2}z\log N\right]$$

$$-\frac{\log\Gamma\left(\tfrac{1}{2}z+1\right)}{z} = \lim_{N\to\infty}\left[\sum_{n=1}^{N}\frac{1}{z}\log\left(1+\frac{z}{2n}\right) - \tfrac{1}{2}\log N\right]$$

$$-d_z\left[\frac{\log\Gamma\left(\tfrac{1}{2}z+1\right)}{z}\right] = \sum_{n=1}^{\infty} d_z\left[\frac{1}{z}\log\left(1+\frac{z}{2n}\right)\right]$$

19.2 $\frac{1}{2\pi i}\int\limits_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right] = \int\limits_{u=\infty}^{u=t}\frac{u^{\beta-1}}{\log u}du +$ *const* **if** $\sigma < 0$**, or** $= \int\limits_{u=0}^{u=t}\frac{u^{\beta-1}}{\log u}du +$ *const* **if** $\sigma > 0$

Riemann wrote

For

$$\beta = \sigma + i\tau,$$

consider

$$\pm\frac{1}{2\pi i}\frac{1}{\log t}\int\limits_{x-i\infty}^{x+i\infty}\frac{d}{dz}\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right]t^z dz.$$

Now,

$$\frac{d}{d\beta}\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right] = \frac{1}{(\beta-z)\beta}.$$

If $x > \sigma$,

$$-\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}\frac{1}{(\beta - z)\beta}t^z dz = \frac{t^\beta}{\beta} = \begin{cases}\int\limits_{u=\infty}^{u=t} u^{\beta-1}du, & \sigma < 0 \\ \int\limits_{u=0}^{u=t} u^{\beta-1}du, & \sigma > 0\end{cases}$$

Therefore,

$$\frac{1}{2\pi i}\frac{1}{\log t}\int_{x-i\infty}^{x+i\infty}\frac{d}{dz}\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right]t^z dz$$

$$= -\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)t^z dz$$

$$= \begin{cases}\int\limits_{u=\infty}^{u=t} \frac{u^{\beta-1}}{\log u}du + const, & \sigma < 0 \\ \int\limits_{u=0}^{u=t} \frac{u^{\beta-1}}{\log u}du + const, & \sigma > 0\end{cases}$$

Integration by parts gives

$$\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right] =$$

$$= \frac{1}{2\pi i\log t}\left[\lim_{Y\to\infty}\frac{t^z}{z}\log\left(1-\frac{z}{\beta}\right)\right]_{y=-Y}^{y=Y} - \frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{z}\log\left(1-\frac{z}{\beta}\right)dz$$

Now,

$$\left|\log\left(1-\frac{z}{\beta}\right)\right| = \left|\frac{z}{\beta}+\frac{1}{2}\left(\frac{z}{\beta}\right)^2+\frac{1}{3}\left(\frac{z}{\beta}\right)^3+...\right|$$

$$\leq \left|\frac{z}{\beta}\right|+\frac{1}{2}\left|\frac{z}{\beta}\right|^2+\frac{1}{3}\left|\frac{z}{\beta}\right|^3+...$$

$$= \left|\log\left(1-\left|\frac{z}{\beta}\right|\right)\right|$$

This ensures that the boundary terms vanish

$$\left|\left[\frac{t^z}{z}\log\left(1-\frac{z}{\beta}\right)\right]_{y=-Y}^{y=Y}\right| \leq \frac{2t^x}{\sqrt{x^2+Y^2}}\left|\log\left(1-\frac{\sqrt{x^2+Y^2}}{|\beta|}\right)\right|$$

$$\to 0, \text{ as } Y \to \infty.$$

Therefore,

$$= -\frac{1}{2\pi i}\int\limits_{y=-\infty}^{y=\infty}\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)t^z dz.$$

Substituting

$$dz = idy,$$

$$t^z = t^x e^{iy\log t}$$

$$\tfrac{1}{z}\log\left(1-\tfrac{z}{\beta}\right) = \int \tfrac{1}{\beta(\beta-z)}d\beta$$

$$= -t^x\frac{1}{2\pi}\int\limits_{y=-\infty}^{y=\infty}\left(\int\frac{1}{\beta(\beta-z)}d\beta\right)e^{iy\log t}dy$$

$$= t^x \int \left[\frac{1}{2\pi} \int_{y=-\infty}^{y=\infty} \frac{1}{z-\beta} e^{iy\log t} dy \right] \frac{1}{\beta} d\beta$$

$$= t^x \int \left[\frac{1}{2\pi} \int_{y=-\infty}^{y=\infty} \left(\int_{s=1}^{s=\infty} s^{-(z-\beta)-1} ds \right) e^{iy\log t} dy \right] \frac{1}{\beta} d\beta$$

By the change of variable

$$s = e^{\omega}$$

$$ds = e^{\omega} d\omega$$

$$= t^x \int \left[\frac{1}{2\pi} \int_{y=-\infty}^{y=\infty} \left(\int_{\omega=0}^{\omega=\infty} e^{-\omega(z-\beta)} d\omega \right) e^{iy\log t} dy \right] \frac{1}{\beta} d\beta$$

$$= t^x \int \left[\frac{1}{2\pi} \int_{y=-\infty}^{y=\infty} \left(\int_{\omega=0}^{\omega=\infty} e^{-\omega(x-\sigma)} e^{-i\omega(y-\tau)} d\omega \right) e^{iy\log t} dy \right] \frac{1}{\beta} d\beta$$

If $x > \sigma$, by Fourier Integral Theorem for $\begin{cases} e^{-\omega(x-\sigma)}, & \omega > 0 \\ 0, & \omega < 0 \end{cases}$,
we change integration order

$$= t^x \int \left[\int_{\omega=0}^{\omega=\infty} e^{-\omega(x-\sigma)} e^{i\omega\tau} \left(\frac{1}{2\pi} \int_{y=-\infty}^{y=\infty} e^{iy(\log t-\omega)} dy \right) d\omega \right] \frac{1}{\beta} d\beta$$

$$= t^x \int \left[\int_{\omega=0}^{\omega=\infty} e^{-\omega(x-\sigma)} e^{i\omega\tau} \delta(\omega - \log t) d\omega \right] \frac{1}{\beta} d\beta$$

$$= t^x \int e^{-(x-\sigma)\log t} e^{i\tau \log t} \frac{1}{\beta} d\beta$$

$$= t^x e^{-x \log t} \int e^{(\sigma+i\tau)\log t} \frac{1}{\beta} d\beta$$

$$= \int \frac{1}{\beta} t^\beta d\beta$$

$$= \begin{cases} \int \left(\int\limits_{u=\infty}^{u=t} u^{\beta-1} du \right) d\beta, & \sigma < 0 \\ \\ \int \left(\int\limits_{u=0}^{u=t} u^{\beta-1} du \right) d\beta, & \sigma > 0 \end{cases}$$

$$= \begin{cases} \int\limits_{u=\infty}^{u=t} \left[\int u^{\beta-1} d\beta \right] du, & \sigma < 0 \\ \\ \int\limits_{u=0}^{u=t} \left[\int u^{\beta-1} d\beta \right] du, & \sigma > 0 \end{cases}$$

$$= \begin{cases} \int\limits_{u=\infty}^{u=t} \frac{u^{\beta-1}}{\log u} du + const, & \sigma < 0 \\ \\ \int\limits_{u=0}^{u=t} \frac{u^{\beta-1}}{\log u} du + const, & \sigma > 0 \end{cases}$$

19.3 $\frac{1}{2\pi i} \int\limits_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d_z \left[\frac{1}{z} \log \left(1 - \frac{z}{\beta} \right) \right] = \int\limits_{u=\infty}^{u=t} \frac{u^{\beta-1}}{\log u} du$, if $\sigma < 0$

Riemann wrote

If $\sigma < 0$,

$$\frac{1}{2\pi i}\frac{1}{\log t}\int_{x-i\infty}^{x+i\infty}\frac{d}{dz}\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right]t^z dz$$

$$=\int_{u=\infty}^{u=t}\frac{u^{\beta-1}}{\log u}du + const.$$

The constant of integration drops out by letting

$$\beta \to -\infty.$$

19.3.1 The left side integral

As in 19.2, integration by parts gives

$$\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right] = -\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)t^z dz.$$

Now,

$$\left|-\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)t^z dz\right| \le \frac{t^x}{2\pi}\int_{y=-\infty}^{y=\infty}\left|\frac{1}{z}\log\left(1-\left|\frac{z}{\beta}\right|\right)\right|dy$$

if we let $\sigma \to -\infty$,

$$\left|\frac{1}{z}\log\left(1-\left|\frac{z}{\beta}\right|\right)\right| \to \left|\frac{1}{z}\log 1\right| = 0$$

Therefore, by Lebesgue Dominant Convergence,

$$\frac{1}{2\pi i}\frac{1}{\log t}\int_{y=-\infty}^{y=\infty}t^z d_z\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right] \to 0.$$

19.3.2 The right side Integral

$$\left| \int_{u=\infty}^{u=t} \frac{u^{\beta-1}}{\log u} du \right| \leq \int_{u=\infty}^{u=t} \left| \frac{u^{\beta-1}}{\log u} \right| du$$

If we let $\sigma \to -\infty$,

$$\left| \frac{u^{\beta-1}}{\log u} \right| = \frac{|u|^{\sigma-1}}{|\log u|} \to \frac{|u|^{-\infty}}{|\log u|} = 0,$$

since $\log u \neq 0$ in $t < u < \infty$, for $t > 1$.

Therefore, by Lebesgue Dominant Convergence,

$$\int_{u=\infty}^{u=t} \frac{u^{\beta-1}}{\log u} du \to 0.$$

Consequently,

$$const = 0,$$

and

$$\frac{1}{2\pi i} \int_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d_z \left[\frac{1}{z} \log \left(1 - \frac{z}{\beta} \right) \right] = \int_{u=\infty}^{u=t} \frac{u^{\beta-1}}{\log u} du.$$

19.4 $\frac{1}{2\pi i} \int_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d_z \left[\frac{1}{z} \log \left(1 - \frac{z}{\beta} \right) \right] = \int_{u=0}^{u=t} \frac{u^{\beta-1}}{\log u} du$, if $\sigma > 0$.

Riemann wrote

If $\sigma > 0$,

$$\frac{1}{2\pi i}\frac{1}{\log t}\int_{x-i\infty}^{x+i\infty}\frac{d}{dz}\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right]t^z dz$$

$$=\int_{u=0}^{u=t}\frac{u^{\beta-1}}{\log u}du + const$$

The integral from $u = 0$ *to* $u = t$ *will be infinitesimal, if the path of integration is in the upper half-plane, and we let*

$$\tau \to \infty,$$

or if the path of integration is in the lower half-plane, and we let

$$\tau \to -\infty.$$

Then, we evaluate

$$\log\left(1-\frac{z}{\beta}\right)$$

on the left side so that the integration constant drops out.

We want to show that both integrals vanish if $\sigma > 0$, and if $\tau \to \infty$, or $\tau \to -\infty$.

19.4.1 The left side Integral

Similarly to Section 19.3.1, integration by parts leads to

$$\left|\frac{1}{2\pi i}\int\limits_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right]\right| \leq \frac{t^x}{2\pi}\int\limits_{y=-\infty}^{y=\infty}\left|\frac{1}{z}\log\left(1-\left|\frac{z}{\beta}\right|\right)\right|dy$$

if we let $\tau \to -\infty$, or $\tau \to \infty$,

$$\left|\frac{1}{z}\log\left(1-\left|\frac{z}{\beta}\right|\right)\right| \to \left|\frac{1}{z}\log 1\right| = 0$$

Therefore, by Lebesgue Dominant Convergence,

$$\frac{1}{2\pi i}\frac{1}{\log t}\int\limits_{y=-\infty}^{y=\infty} t^z d_z\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right] \to 0.$$

19.4.2 The right side integral

We want to show that if we take a path in the upper half plane and let $\tau \to \infty$, the right side integral vanishes.

By the change of variable

$$u = e^{\omega},$$
$$du = e^{\omega}d\omega.$$

we have

$$\int\limits_{u=0}^{u=t}\frac{u^{\beta-1}}{\log u}du = \int\limits_{\omega=-\infty}^{\omega=\log t}\frac{e^{\omega\beta}}{\omega}d\omega$$

We take a path in the upper half-plane that

- runs from $\omega = -\infty$ to $\omega = -\infty + i\delta$, along $\omega = -\infty + i\varepsilon$
- runs from $\omega = -\infty + i\delta$ to $\omega = \log t + i\delta$, along $\omega = \upsilon + i\delta$
- runs from $\omega = \log t + i\delta$ to $\omega = \log t$, along $\omega = \log t + i\varepsilon$

Then,

$$\int_{\omega=-\infty}^{\omega=\log t} \frac{e^{\omega\beta}}{\omega} d\omega = \int_{\omega=-\infty}^{\omega=-\infty+i\delta} \frac{e^{\omega\beta}}{\omega} d\omega + \int_{\omega=-\infty+i\delta}^{\omega=\log t+i\delta} \frac{e^{\omega\beta}}{\omega} d\omega + \int_{\omega=\log t+i\delta}^{\omega=\log t} \frac{e^{\omega\beta}}{\omega} d\omega$$

The First Integral

$$\left| \int_{\omega=-\infty}^{\omega=-\infty+i\delta} \frac{e^{\omega\beta}}{\omega} d\omega \right| \leq \int_{\varepsilon=0}^{\varepsilon=\delta} \frac{|\exp(-\infty + i\varepsilon)(\sigma + i\tau)|}{|-\infty + i\varepsilon|} d\varepsilon$$

$$= \int_{\varepsilon=0}^{\varepsilon=\delta} \frac{\exp(-(\infty\sigma + \varepsilon\tau))}{\infty} d\varepsilon$$

Since $\sigma > 0$, and $\varepsilon > 0$, then for $\tau \to \infty$,

$$\frac{\exp(-\infty\sigma - \varepsilon\tau)}{\infty} \to \frac{\exp(-\infty(\sigma + \varepsilon))}{\infty} = 0$$

Hence, by Lebesgue Dominant Convergence

$$\int_{\omega=-\infty}^{\omega=-\infty+i\delta} \frac{e^{\omega\beta}}{\omega} d\omega \to 0.$$

The Second Integral

$$\left| \int_{\omega=-\infty+i\delta}^{\omega=\log t+i\delta} \frac{e^{\omega\beta}}{\omega} d\omega \right| \leq \int_{\upsilon=-\infty}^{\upsilon=\log t} \frac{|\exp(\upsilon+i\delta)(\sigma+i\tau)|}{|\upsilon+i\delta|} d\upsilon$$

$$= \int_{\upsilon=-\infty}^{\upsilon=\log t} \frac{\exp(\upsilon\sigma-\delta\tau)}{|\upsilon+i\delta|} d\upsilon$$

Since $\sigma > 0$, and $\delta > 0$, then for $\tau \to \infty$,

$$\frac{\exp(\upsilon\sigma-\delta\tau)}{|\upsilon+i\delta|} \to \frac{\exp(\upsilon\sigma-\delta\infty)}{|\upsilon+i\delta|} = \frac{\exp(-\infty)}{|\upsilon+i\delta|} = 0.$$

Hence, by Lebesgue Dominant convergence,

$$\int_{\omega=-\infty+i\delta}^{\omega=\log t+i\delta} \frac{e^{\omega\beta}}{\omega} d\omega \to 0.$$

The Third Integral

$$\left| \int_{\omega=\log t+i\delta}^{\omega=\log t} \frac{e^{\omega\beta}}{\omega} d\omega \right| \leq \int_{\varepsilon=0}^{\varepsilon=\delta} \frac{|\exp(\log t+i\varepsilon)(\sigma+i\tau)|}{|\log t+i\varepsilon|} d\varepsilon$$

$$= \int_{\varepsilon=0}^{\varepsilon=\delta} \frac{\exp(\sigma\log t-\varepsilon\tau)}{|\log t+i\varepsilon|} d\varepsilon$$

Since $\sigma > 0$, and $\varepsilon > 0$, then for $\tau \to \infty$,

$$\frac{\exp(\sigma \log t - \varepsilon\tau)}{|\log t + i\varepsilon|} \to \frac{\exp(-\infty)}{|\log t + i\varepsilon|} = 0.$$

Hence, by Lebesgue Dominant convergence,

$$\int_{\omega=\log t+i\delta}^{\omega=\log t} \frac{e^{\omega\beta}}{\omega} d\omega \to 0.$$

Thus, the right side integral vanishes.

Similarly, if we take a path in the lower half plane and let $\tau \to -\infty$, the right side integral vanishes.

Consequently, if $\sigma > 0$, then

$$const = 0$$

and

$$\frac{1}{2\pi i} \int_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d_z \left[\frac{1}{z} \log\left(1 - \frac{z}{\beta}\right)\right] = \int_{u=0}^{u=t} \frac{u^{\beta-1}}{\log u} du.$$

19.5 If $\sigma > 0$, then $\int_{u=0}^{u=t} \frac{u^{\beta-1}}{\log u} du = Li(t^\beta) - \pi i$, for upper half plane path, and $= Li(t^\beta) + \pi i$, for lower half plane path

Riemann wrote

If $\sigma > 0$, the integral

$$\int\limits_{u=0}^{u=t} \frac{u^{\beta-1}}{\log u} du$$

takes on two values which differ by

$$2\pi i$$

depending on whether the path of integration is in the upper half-plane or in the lower half-plane.

The integrand is singular at $u = 1$, and the path of integration has to bypass the singularity.
Thus, an upper half-plane path will

- run from $u = 0$ to $u = 1 - \varepsilon$
- encircle the singularity clockwise from $u = 1-\varepsilon$ to $u = 1+\varepsilon$
- run from $u = 1 + \varepsilon$ to $u = t$

Then,

$$\int\limits_{u=0}^{u=t} \frac{u^{\beta-1}}{\log u} du = \int\limits_{u=0}^{u=1-\varepsilon} \frac{u^{\beta-1}}{\log u} du + \int\limits_{\omega=\log(1-\varepsilon)}^{\omega=\log(1+\varepsilon)} \frac{e^{\omega\beta}}{\omega} d\omega + \int\limits_{u=1+\varepsilon}^{u=t} \frac{u^{\beta-1}}{\log u} du.$$

By the Residue Theorem for the clockwise semi-circle

$$\int\limits_{\omega=\log(1-\varepsilon)}^{\omega=\log(1+\varepsilon)} \frac{e^{\omega\beta}}{\omega} d\omega = -2\pi i \left(\frac{\pi}{2\pi}\right) \left[\text{Res} \frac{e^{\omega\beta}}{\omega}\right]_{\omega=0}$$

$$= -\pi i \lim_{\omega \to 0} \left[\omega \frac{e^{\omega\beta}}{\omega} \right]$$

$$= -\pi i$$

Hence,

$$\int_{u=0}^{u=t} \frac{u^{\beta-1}}{\log u} du = \int_{u=0}^{u=1-\varepsilon} \frac{u^{\beta-1}}{\log u} du + \int_{u=1+\varepsilon}^{u=t} \frac{u^{\beta-1}}{\log u} du + (-\pi i)$$

By the change of variable

$$\upsilon = u^{\beta},$$

$$d\upsilon = \beta u^{\beta-1} du,$$

$$\log \upsilon = \beta \log u$$

$$\int_{\upsilon=0}^{\upsilon=t^{\beta}} \frac{d\upsilon}{\log \upsilon} = \left\{ \int_{\upsilon=0}^{\upsilon=(1-\varepsilon)^{\beta}} \frac{d\upsilon}{\log \upsilon} + \int_{\upsilon=(1+\varepsilon)^{\beta}}^{\upsilon=t^{\beta}} \frac{d\upsilon}{\log \upsilon} \right\} - \pi i$$

Letting $\varepsilon \downarrow 0$,

$$= Li(t^{\beta}) - \pi i$$

Hence,

$$\int_{u=0}^{u=t} \frac{u^{\beta-1}}{\log u} du = Li(t^{\beta}) - \pi i$$

Similarly, with a lower half-plane path that encircles the singularity counter-clockwise,

$$\int_{u=0}^{u=t} \frac{u^{\beta-1}}{\log u} du = Li(t^\beta) + \pi i$$

19.6 If $\sigma > 0$, $\frac{1}{2\pi i} \int_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d_z \left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right] = Li(t^\beta) - \pi i$, for upper half plane path, and $= Li(t^\beta) + \pi i$, for lower half plane path

By section 19.4,

$$\frac{1}{2\pi i} \int_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d_z \left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right] = \int_{u=0}^{u=t} \frac{u^{\beta-1}}{\log u} du$$

By section 19.5,

$$= \begin{cases} Li(t^\beta) - \pi i, & for\ upper\ half\ plane\ path \\ Li(t^\beta) + \pi i, & for\ lower\ half\ plane\ path \end{cases}$$

where

$$Li(t^\beta) = \lim_{\varepsilon \downarrow 0} \left\{ \int_{\upsilon=0}^{\upsilon=(1-\varepsilon)^\beta} \frac{d\upsilon}{\log \upsilon} + \int_{\upsilon=(1+\varepsilon)^\beta}^{\upsilon=t^\beta} \frac{d\upsilon}{\log \upsilon} \right\}.$$

Chapter 20

The $\frac{\log(z-1)}{z}$ term

20.1 $\frac{1}{2\pi i}\int\limits_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d_z \left[\frac{\log(z-1)}{z}\right] = Li(t)$

$\log(1-z)$ is defined with a cut along the positive real numbers. Therefore, to obtain

$$z-1$$

in the main branch of $\log(1-z)$, we rotate

$$1-z$$

clockwise, by multiplying it by

$$e^{-i\pi}$$

That is,

$$(1-z)e^{-i\pi} = z-1,$$

and

$$\frac{1}{2\pi i}\int\limits_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d_z \left[\frac{\log(z-1)}{z}\right] = \frac{1}{2\pi i}\int\limits_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d_z \left[\frac{\log(e^{-i\pi})(1-z)}{z}\right]$$

$$= \frac{1}{2\pi i} \int_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d_z \left[\frac{\log(1-z)}{z} \right] + \frac{1}{2\pi i} \int_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d_z \left[\frac{(-i\pi)}{z} \right]$$

20.1.1 The First Integral

The integral

$$\frac{1}{2\pi i} \int_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d_z \left[\frac{\log(1-z)}{z} \right]$$

has a term of the form

$$\frac{1}{z} \log \left(1 - \frac{z}{\beta} \right)$$

where

$$\sigma = 1 > 0.$$

Therefore, by section 19.6, if we take an upper half-plane path with clockwise oriented semicircle around the singularity of the logarithmic integral at $u = 1$

$$\frac{1}{2\pi i} \int_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d_z \left[\frac{\log(1-z)}{z} \right] = Li(t) - \pi i$$

20.1.2 The Second Integral

$$\frac{1}{2\pi i} \int_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d_z \left[\frac{-\pi i}{z} \right] = (-\pi i) \frac{1}{2\pi i} \int_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d_z \left[\frac{1}{z} \right]$$

By section 18.1,

$$\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left[\frac{1}{z}\right]=-1.$$

Hence,

$$\text{Second Integral}=\pi i.$$

Consequently,

$$\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left[\frac{\log(z-1)}{z}\right]=(Li(t)-\pi i)+\pi i$$

$$=Li(t).$$

Chapter 21

The $-\frac{1}{z}\sum_{\alpha}\log\left(1+\frac{z^2-z}{\alpha^2+1/4}\right)$ term

Riemann wrote,

$$-\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left[\frac{1}{z}\sum_{\alpha}\log\left(1+\frac{\left(z-\frac{1}{2}\right)^2}{\alpha^2}\right)\right]$$
$$=-\sum_{\alpha}\left[Li(t^{1/2+i\alpha})+Li(t^{1/2-i\alpha})\right]$$

The summation is over all positive zeros of ξ (or all zeros with positive real part), ordered by their size.

♣ *With a more precise discussion of the function ξ, it is easy to show that the sum*

$$\sum_{\alpha}\left[Li(t^{1/2+i\alpha})+Li(t^{1/2-i\alpha})\right]$$

equals

$$\lim_{Y\to\infty}\frac{1}{2\pi i}\int_{x-iY}^{x+iY}\frac{d}{dz}\left[\frac{1}{z}\sum_{\alpha}\log\left(1+\frac{(z-1/2)^2}{\alpha^2}\right)\right]t^z dz$$

♣ *If the zeros are not sequenced by their size, the sum may have any arbitrary real value.*

21.1 The assumption of the Hypothesis

The claim that the series

$$\sum_{\alpha}\left[Li(t^{1/2+i\alpha})+Li(t^{1/2-i\alpha})\right]$$

equals the integral

$$\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left[\frac{1}{z}\sum_{\alpha}\log\left(1+\frac{z^2-z}{\alpha^2+1/4}\right)\right]$$

includes the assumption of the Hypothesis that

$$0=Im(\alpha)=x_0-\tfrac{1}{2}$$

That is, the zeros are assumed to be on the line $x=1/2$.

21.2 Implicit Claim

The following claim was not made by Riemann, but is implicit in the derivation of his formula for the count of the prime numbers

$$\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left[\frac{1}{z}\sum_{\alpha}\log\left(1+\frac{z^2-z}{\alpha^2+1/4}\right)\right]$$

$$=\sum_{\alpha}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left[\frac{1}{z}\log\left(1+\frac{z^2-z}{\alpha^2+1/4}\right)\right]$$

In 1908, Landau [12] proved that the summation over the zeros, and the integration can be interchanged.

21.3 $-\frac{1}{2\pi i}\int\limits_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left[\frac{1}{z}\sum_\alpha\log\left(1+\frac{z^2-z}{\alpha^2+1/4}\right)\right]$ **equals** $-\sum_\alpha\left[Li(t^{1/2+i\alpha})+Li(t^{1/2-i\alpha})\right]$ **if the zeros of** ξ **are on** $x=1/2$ **and sequenced by size**

Integrating term by term by Section 21.2

$$-\frac{1}{2\pi i}\int\limits_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left[\frac{1}{z}\sum_\alpha\log\left(1+\frac{z^2-z}{\alpha^2+1/4}\right)\right]=$$

$$=\sum_\alpha\frac{1}{2\pi i}\int\limits_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left[\frac{1}{z}\log\left(1+\frac{z^2-z}{\alpha^2+1/4}\right)\right]$$

$$=\sum_\alpha\frac{1}{2\pi i}\int\limits_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left\{\frac{1}{z}\left[\log\left(1-\frac{z}{\frac{1}{2}+i\alpha}\right)+\log\left(1-\frac{z}{\frac{1}{2}-i\alpha}\right)\right]\right\}$$

The integral

$$\frac{1}{2\pi i}\int\limits_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left[\frac{1}{z}\log\left(1-\frac{z}{\frac{1}{2}+i\alpha}\right)\right]$$

has a term of the form

$$\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)=\frac{1}{z}\log\left(1-\frac{z}{\sigma+i\tau}\right)$$

where

$$\sigma=\tfrac{1}{2}>0$$

and

$$\tau=\alpha>0$$

By 19.6 on an upper half-plane path, the integral equals

$$Li(t^{1/2+i\alpha})-i\pi$$

Similarly, using a path in the lower half plane, the integral

$$\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left[\frac{1}{z}\log\left(1-\frac{z}{\frac{1}{2}-i\alpha}\right)\right]$$

equals

$$Li(t^{1/2-i\alpha})+i\pi.$$

Therefore,

$$\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left\{\frac{1}{z}\left[\log\left(1-\frac{z}{\frac{1}{2}+i\alpha}\right)+\log\left(1-\frac{z}{\frac{1}{2}-i\alpha}\right)\right]\right\}$$

$$=Li(t^{1/2+i\alpha})+Li(t^{1/2-i\alpha})$$

Consequently,

$$-\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left[\frac{1}{z}\sum_{\alpha}\log\left(1+\frac{z^2-z}{\alpha^2+1/4}\right)\right]$$

$$=-\sum_{\alpha}\left[Li(t^{1/2+i\alpha})+Li(t^{1/2-i\alpha})\right]$$

21.4 The Hypothesis Source

- ♠ α, and $-\alpha$ are zeros of ξ
- ♠ $|\alpha_n|$ is increasing

combined with

$$Im(\alpha) \neq 0,$$

have the effect that

$$\sigma > 0,$$

is not guaranteed in any way, and the equality to the series

$$= -\sum_{\alpha} \left[Li(t^{1/2+i\alpha}) + Li(t^{1/2-i\alpha}) \right]$$

can-not be deduced.

This might have been the argument that led Riemann to make the Hypothesis.

Chapter 22

The $\frac{1}{z}\log\Gamma\left(\frac{1}{2}z+1\right)$ term

22.1 $\frac{1}{2\pi i}\int\limits_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left[\frac{1}{z}\log\Gamma\left(\frac{1}{2}z+1\right)\right]=\int\limits_{u=t}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$

For $t>1$,

$$\int\limits_{u=t}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du=$$

$$=\int\limits_{u=t}^{u=\infty}\frac{1}{u\log u}\frac{1/u^2}{1-1/u^2}du$$

$$=\int\limits_{u=t}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2}\left(1+\frac{1}{u^2}+\frac{1}{u^4}+...\right)du$$

$$=\int\limits_{u=t}^{u=\infty}\frac{1}{u\log u}\sum_{n=1}^{\infty}\frac{1}{u^{2n}}du$$

The uniform convergence of

$$\sum_{n=1}^{\infty} \frac{1}{u^{2n}}$$

allows term-wise integration

$$= \sum_{n=1}^{\infty} \int_{u=t}^{u=\infty} \frac{1}{u \log u} \frac{1}{u^{2n}} du$$

$$= \sum_{n=1}^{\infty} - \int_{u=\infty}^{u=t} \frac{u^{-2n-1}}{\log u} du$$

The integral

$$\int_{u=\infty}^{u=t} \frac{u^{-2n-1}}{\log u} du$$

has

$$\beta = \sigma = -2n,$$

and by Section 19.3, it equals

$$= \frac{1}{2\pi i} \int_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d_z \left[\frac{1}{z} \log \left(1 + \frac{z}{2n} \right) \right]$$

Therefore, we have

$$= \sum_{n=1}^{\infty} - \frac{1}{2\pi i} \int_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d_z \left[\frac{1}{z} \log \left(1 + \frac{z}{2n} \right) \right]$$

The uniform convergence allows interchanging summation and integration

$$= -\frac{1}{2\pi i} \int_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} \sum_{n=1}^{\infty} d_z \left[\frac{1}{z} \log \left(1 + \frac{z}{2n} \right) \right]$$

By section 19.1

$$-d_z \left[\frac{1}{z} \log \Gamma \left(\tfrac{1}{2} z + 1 \right) \right] = \sum_{n=1}^{\infty} d_z \left[\frac{1}{z} \log \left(1 + \frac{z}{2n} \right) \right].$$

Therefore, we conclude

$$= \frac{1}{2\pi i} \int_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d_z \left[\frac{1}{z} \log \Gamma \left(\tfrac{1}{2} z + 1 \right) \right]$$

Chapter 23

The Count and Density of the Primes

23.1 If the α's(= zeros of ξ) are positive and sequenced by size, $f(t) = Li(t) - \log 2 - \sum_{\alpha} \left[Li(t^{1/2+i\alpha}) + Li(t^{1/2-i\alpha})\right] + \int_{u=t}^{u=\infty} \frac{1}{u \log u} \frac{1}{u^2-1} du$

Riemann wrote

> *If the zeros of* ξ, α, *are positive and sequenced by size,*

$$f(t) = Li(t) - \sum_{\alpha} \left[Li(t^{1/2+i\alpha}) + Li(t^{1/2-i\alpha})\right] + \int_{u=t}^{u=\infty} \frac{1}{u \log u} \frac{1}{u^2-1} du + \log \xi(0)$$

Riemann computed

$$\xi(0) = \xi|_{w=0} = \xi|_{z=1/2}\,.$$

In 1860, Genocchi [13] observed that the correct term is

$$\xi|_{z=0} = \pi^{-0/2}\Gamma(1)(0-1)\zeta(0) = -\zeta(0) = 1/2.$$

Thus,

$$f(t) = Li(t) - \log 2 - \sum_{\alpha}\left[Li(t^{1/2+i\alpha}) + Li(t^{1/2-i\alpha})\right]$$

$$+ \int_{u=t}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

23.2 Count of the Primes is $F(t) = f(t) - \frac{1}{2}f(t^{1/2}) - \frac{1}{3}f(t^{1/3}) - \frac{1}{5}f(t^{1/5}) + \frac{1}{6}f(t^{1/6}) + \ldots\frac{(-1)^{\mu}}{m}f(t^{1/m}) + \ldots$ where $m = 1, 2, 3, \ldots$ has no prime factors squared, and μ is the number of prime factors of m

Riemann wrote

We invert

$$f(t) = \sum_{n\in N}\frac{1}{n}F(t^{1/n})$$

to obtain

$$F(t) = \sum_{m}\frac{(-1)^{\mu}}{m}f(t^{1/m}),$$

where m ranges over all the natural numbers that have no prime factors squared, and where μ is the number of prime factors of m.

This inversion formula is due to Mobius.
The formula is constructed with the aid of the following table

m	p_1	p_2	p_3	p_4	$\dots$	μ
1	$-$					0
2	2					1
3	3					1
$4 = 2^2$						
5	5					1
6	2	3				2
7	7					1
$8 = 2^3$						
$9 = 3^2$						
$10 = 2 \cdot 5$	2	5				2
11	11					1
$12 = 3 \cdot 2^2$						
13	13					1
14	2	7				2
15	3	5				2
$16 = 2^4$						

23.3 $\frac{1}{\log t} - 2\frac{t^{-1/2}}{\log t} \sum_{\alpha} \cos(\alpha \log t)$ is the approximate density of $f(t)$

Riemann wrote

To approximate $f(t)$, we take a finite sum of

$$\sum_{\alpha} \left[Li(t^{1/2+i\alpha}) + Li(t^{1/2-i\alpha}) \right],$$

where the zeros of ξ, α, are positive and sequenced by size.

The derivative of the approximated $f(t)$ is the finite sum

$$\frac{1}{\log t} - 2\frac{t^{-1/2}}{\log t}\sum_{\alpha}\cos(\alpha \log t)$$

$$+\{\to 0,\ \textit{very rapidly}\ \text{ for t} \to 0\},$$

where the zeros of ξ, α, are positive and sequenced by size.

This expression approximates the density of the primes at t

$+\frac{1}{2}$ *the density of the squared primes at t*

$+\frac{1}{3}$ *the density of the cubed primes at t*

+.......

We have

$$\frac{d}{dt}\left[Li(t^{1/2+i\alpha}) + Li(t^{1/2-i\alpha})\right] =$$

$$= \frac{d}{dt}\left\{\int_{u=0}^{u=t}\frac{u^{1/2+i\alpha-1}}{\log u}du + \int_{u=0}^{u=t}\frac{u^{1/2-i\alpha-1}}{\log u}du\right\}$$

$$= \frac{t^{-1/2}}{\log t}\left(t^{i\alpha} + t^{-i\alpha}\right)$$

$$= \frac{t^{-1/2}}{\log t}\left(e^{i\alpha \log t} + e^{-i\alpha \log t}\right)$$

$$= \frac{t^{-1/2}}{\log t} 2\cos(\alpha \log t)$$

Thus, the derivative of the approximated $f(t)$ is

$$\frac{1}{\log t} - 2\frac{t^{-1/2}}{\log t}\sum_{\alpha}\cos(\alpha \log t) + \frac{1}{t(t^2-1)\log t}$$

$$\approx \frac{1}{\log t} - 2\frac{t^{-1/2}}{\log t}\sum_{\alpha}\cos(\alpha \log t)$$

This approximates

$$F'(t) + \frac{1}{2}F'(t^{1/2}) + \frac{1}{3}F'(t^{1/3}) + \ldots$$

23.4 The Approximate Count of the Primes by non-oscillatory, unbounded terms is $Li(t) - \frac{1}{2}Li(t^{1/2}) - \frac{1}{3}Li(t^{1/3}) - \frac{1}{5}Li(t^{1/5}) + \frac{1}{6}Li(t^{1/6}) - \frac{1}{7}Li(t^{1/7}) + \ldots$

Riemann wrote

Since

$$f'(t) \approx \frac{1}{\log t} - 2\frac{t^{-1/2}}{\log t}\sum_{\alpha}\cos(\alpha \log t),$$

the well-known approximation formula

$$F(t) = Li(t)$$

is correct only to order of magnitude of

$$t^{1/2}$$

and gives a value that is somewhat too large.

Except for quantities that are bounded as t increases, the non-oscillatory terms in $F(t)$ are

$Li(t) - \frac{1}{2}Li(t^{1/2}) - \frac{1}{3}Li(t^{1/3}) - \frac{1}{5}Li(t^{1/5}) + \frac{1}{6}Li(t^{1/6}) - \frac{1}{7}Li(t^{1/7}) + ...$

If oscillatory and bounded terms are eliminated from $f(t)$ then,

$$f(t) \approx Li(t).$$

Therefore, by Section 23.2,

$$F(t) = f(t) - \frac{1}{2}f(t^{1/2}) - \frac{1}{3}f(t^{1/3}) - \frac{1}{5}f(t^{1/5}) + \frac{1}{6}f(t^{1/6}) - \frac{1}{7}Li(t^{1/7}) + ...$$

$$\approx Li(t) - \frac{1}{2}Li(t^{1/2}) - \frac{1}{3}Li(t^{1/3}) - \frac{1}{5}Li(t^{1/5}) + \frac{1}{6}Li(t^{1/6}) - \frac{1}{7}Li(t^{1/7}) + ...$$

23.5 The Error in the Approximate Count

Riemann wrote

Gauss and Goldschmidt compared

$$Li(t)$$

with the

$$\textit{Number of primes} < t$$

up to $t = 3,000,000$.

Then,

$$\textit{Number of primes} < 100,000$$

was already smaller than

$$Li(100,000).$$

That difference increases gradually, with minor fluctuations, as t increases.

Riemann's approximation

$$\pi(t) \approx \int_{u=2}^{u=t} \frac{du}{\log u} - \tfrac{1}{2} \int_{u=2}^{u=t^{1/2}} \frac{du}{\log u} - \tfrac{1}{3} \int_{u=2}^{u=t^{1/3}} \frac{du}{\log u}$$

$$-\tfrac{1}{5} \int_{u=2}^{u=t^{1/5}} \frac{du}{\log u} + \tfrac{1}{6} \int_{u=2}^{u=t^{1/6}} \frac{du}{\log u} - \tfrac{1}{7} \int_{u=2}^{u=t^{1/7}} \frac{du}{\log u} + ...$$

is compared with Gauss' approximation

$$\pi(t) \approx \int_{u=2}^{u=t} \frac{du}{\log u}$$

in the following **Lehmer table** [1, p.35]

t	Riemann Error	Gauss Error
1,000,000	30	130
2,000,000	−9	122
3,000,000	0	155
4,000,000	33	206
5,000,000	−64	125
6,000,000	24	228
7,000,000	−38	179
8,000,000	−6	223
9,000,000	−53	187
10,000,000	88	339

23.6 Fluctuations of the Density of $f(t)$

Riemann wrote

The finite sum of oscillatory terms

$$-2\frac{t^{-1/2}}{\log t}\sum_{\alpha}\cos(\alpha\log t)$$

cause irregular fluctuations in the density of the primes.

In a future count, it would be interesting to trace the fluctuations of the density of the primes

$$F'(t)$$

to the particular oscillatory terms in

$$f'(t).$$

We take the finite sum

$$\delta f(t) \equiv -2\frac{t^{-1/2}}{\log t}\sum_{\alpha<t}\cos(\alpha\log t)$$

over the zeros of ξ that are less than t.
Since the Mobius inversion applies linearly, we have

$$\delta F(t) \approx \delta f(t) - \frac{1}{2}\delta f(t^{1/2}) - \frac{1}{3}\delta f(t^{1/3}) - \frac{1}{5}\delta f(t^{1/5}) + \frac{1}{6}\delta f(t^{1/6}) - \frac{1}{7}\delta f(t^{1/7}) + \ldots$$

According to [1, p. 37],

> *So far as I know, no such investigation has ever been carried out.*

Perhaps, even now, this is still an open problem.

23.7 Fluctuations of $f(t)$

Riemann wrote

> *The behavior of $f(t)$ is more regular.*
>
> *Already for $t \leq 100$,*
>
> $$f(t) \approx Li(t) + \log \xi(0).$$

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