Riemann Zeta Function: The Riemann Hypothesis Origin, the Factorization Error, and the Count of the Primes

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Contents

Introduction

Riemann's 1859 Zeta paper defines the Zeta function and uses its properties to approximate the count of prime numbers up to a number t, and the density of the primes at the number t

The few pages paper outlines a book that was never written by Riemann. The paper sums up Riemann's results on the Zeta function, and on the count of the prime numbers, with a few connecting words, and no proofs/explanations.

Attempts to write the book were made by Titchmarsh, and by Edwards, but none followed through Riemann's writing.

Only by staying faithful to Riemann's development of the Zeta function we can

Get to the origin and meaning of the Riemann Hypothesis,

Correct the factorization error,

Apply Riemann's Formula for the Count of the Primes,

Acquire the tools needed for the Hypothesis Proof

In 2006, I attempted to write that book, [Dan1], and this is an extensive revision of that attempt.

0.1 The Factorization Error

Riemann defines the auxiliary function

$$\xi(z) \equiv (z-1)\pi^{-z/2}\Gamma(z/2+1)\zeta(z)$$

that has the same zeros as $\zeta(z)$ in $0 < x < 1$

We show that it has the factorization

$$\xi(z) = \xi(0)\prod_{n=1}^{\infty}\left(1 - \frac{z}{1/2 - i\alpha_n}\right)\left(1 - \frac{z}{1/2 + i\alpha_n}\right)$$

$$= \xi(0)\prod_{n=1}^{\infty}\left(1 + \frac{z^2 - z}{\alpha_n^2 + 1/4}\right)$$

where the $1/2 \pm \alpha_n$ are the zeros of $\xi(z)$.

We further show that if the zeros are all on the line $x = 1/2$, this factorization for $\xi(z)$ produces the term

$$\sum_{n=1}^{\infty}\left[\mathrm{Li}(t^{\frac{1}{2}+i\alpha_n}) + \mathrm{Li}(t^{\frac{1}{2}-i\alpha_n})\right]$$

in the formula for the count of the primes, where $\mathrm{Li}(t)$ is the Logarithmic integral.

Riemann obtained the erroneous factorization

$$\xi(z) = \xi\Big|_{z=1/2}\prod_{n=1}^{\infty}\left(1 + \frac{(z-1/2)^2}{\alpha_n^2}\right)$$

that does not produce the Logarithmic integral series term.

Already in 1860, Genocchi pointed out that the formula should use

$$\xi\Big|_{z=0}.$$

But that only diverted attention from the error in the factors of the product, and the main problem seemed to be that a derivation was missing.

In 1893, Hadamard supplied the derivation and obtained

$$\xi(z) = \xi(0)\prod_{\rho}\left(1 - \frac{z}{\rho}\right),$$

where the ρ's are the zeros of $\xi(z)$.

Hadamard formula does not exhibit the connection to the α_n's, and so far as I can tell, the connection to the Logarithmic integral series in the formula for the count of the primes, was never made.

Indeed, to make that connection, one has to follow through the whole paper. Thus, producing the correct derivations and results of the Zeta paper amounts to execution of the book that is outlined in the Zeta paper.

I use common notations and terms such as

$z = x + iy$ not $s = \sigma + i\tau$

$z \to w(z)$, not $s \to t(s)$

$\Gamma(z)$ not $\Pi(s-1)$.

"zeros of ξ" not "roots of the equation $\xi(z) = 0$."

Otherwise, Riemann's notations are kept unchanged.

1

The Count of Prime Numbers

1.1 Gauss approximation

In his 1859 Zeta Paper, Riemann's aim was

...to report on a study of the frequency with which prime numbers occur.

A topic that seems worthy of such reporting, because of the interest shown in it by Gauss and Dirichlet over many years. $\Diamond$

Gauss (1849) computed

$$\int_{u=2}^{u=t} \frac{du}{\log u}$$

to approximate $\pi(t)$, the Number of Primes < t.

His results are listed in the following **Gauss table** [Ed, p.3]

t	$\pi(t)$	$\int_{u=2}^{u=t} \frac{du}{\log u}$	$Error$
$500,000$	$41,556$	$41,606.4$	50.4
$1,000,000$	$78,501$	$78,627.5$	126.5
$1,500,000$	$114,112$	$114,263.1$	151.1
$2,000,000$	$148,883$	$149,054.8$	171.8
$2,500,000$	$183,016$	$183,245.0$	229.0
$3,000,000$	$216,745$	$216,970.6$	225.6

1.2 Gauss Approximation, and the Prime Number Theorem

We have

$$\int_{u=2}^{u=t} \frac{du}{\log u} = Li(t) - Li(2),$$

where

$$\mathrm{Li}(t) = \lim_{\varepsilon \downarrow 0} \left\{ \int_{u=0}^{u=1-\varepsilon} \frac{du}{\log u} + \int_{u=1+\varepsilon}^{u=t} \frac{du}{\log u} \right\}$$

is the Logarithmic Integral.

By the Prime Number Theorem (Hadamard-1896),

$$\frac{\pi(t)}{Li(t)} \to 1,$$

as $t \to \infty$.

The Prime Number Theorem substantiates the Gauss approximation

2

Zeta Series in $\operatorname{Re} z > 1$

2.1 The Euler Product

For $\operatorname{Re} z > 1$**,**

$$\zeta(z) \equiv \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + ... + \frac{1}{n^z} + ...$$

$$= \frac{1}{1 - \frac{1}{2^z}} \frac{1}{1 - \frac{1}{3^z}} \frac{1}{1 - \frac{1}{5^z}} .... \frac{1}{1 - \frac{1}{p^z}} ....$$

Riemann wrote:

...My starting point was the observation of Euler that the product

$$\prod_{p=prime} \frac{1}{1 - 1 / p^z}$$

equals

$$\sum_{n=natural} \frac{1}{n^z},$$

where p *ranges over all the prime numbers, and* n *over all the natural numbers.*

I denote by

$$\zeta(z),$$

the function of the complex variable z, defined by these two expressions when they converge. $\Diamond$

Proof:

Euler's product formula, defined for $k = 2, 3, \ldots$ by

$$\frac{1}{1-\frac{1}{2^k}}\frac{1}{1-\frac{1}{3^k}}\frac{1}{1-\frac{1}{5^k}}\cdots\frac{1}{1-\frac{1}{p^k}}\cdots = \frac{1}{1^k}+\frac{1}{2^k}+\frac{1}{3^k}+\cdots+\frac{1}{n^k}+\cdots,$$

can be extended to complex numbers z with $\mathrm{Re}\, z > 1$, because

$$\sum_{n=natural}\left|\frac{1}{n^z}\right| \le \sum_{n=natural}\frac{1}{n^x}$$

Thus, in $\mathrm{Re}\, z > 1$.

$$\zeta(z) \equiv \sum_{n=natural}\frac{1}{n^z} = \prod_{p=prime}\frac{1}{1-1\,/\,p^z}.\square$$

For a proof that the equality between the sum and the product in Euler Product Formula holds for complex numbers, see [Karat] , or [Ed1]

3

Zeta Integral in $\operatorname{Re} z > 1$

3.1 Gamma Integral in $\operatorname{Re} z > 0$

For $x > 0$,

$$\boxed{\Gamma(z) = \int_{t=0}^{t=\infty} \frac{t^{z-1}}{e^t} dt}$$

Proof:

Euler's integral,

$$\int_{t=0+}^{t=\infty} \frac{t^{z-1}}{e^t} dt$$

can be written as

$$\int_{t=0+}^{t=1} \frac{t^{z-1}}{e^t} dt + \int_{t=1}^{t=\infty} \frac{t^{z-1}}{e^t} dt .$$

For the first integral,

$$\int_{t=0+}^{t=1} \left| \frac{t^{z-1}}{e^t} \right| dt \leq \int_{t=0+}^{t=1} t^{x-1} dt = \frac{1}{x} t^x \Big|_{t=0+}^{t=1} = \frac{1}{x} - \frac{1}{x} \lim_{t \to 0+} t^x .$$

Therefore the first integral converges uniformly for any $x \geq \delta > 0$.

For the second integral,

$$\int_{t=1}^{t=\infty} \left| \frac{t^{z-1}}{e^t} \right| dt = \int_{t=1}^{t=\infty} \frac{t^{x-1}}{e^t} dt$$

Integrating by parts,

$$= \underbrace{t^{x-1}(-e^{-t})\Big|_{t=1}^{t=\infty}}_{=1/e} + (x-1)\int_{t=1}^{t=\infty} \frac{t^{x-2}}{e^t} dt\,.$$

$$= \frac{1}{e} + (x-1)\underbrace{t^{x-2}(-e^{-t})\Big|_{t=1}^{t=\infty}}_{=1/e} + (x-1)(x-2)\int_{t=1}^{t=\infty} \frac{t^{x-3}}{e^t} dt$$

....

Eventually, after $[x]$ such integrations, the power of t in the integrand becomes $x-[x]-1$, and we have

$$(x-1)...(x-[x])\int_{t=1}^{t=\infty} \frac{1}{t^{[x]+1-x}e^t} dt < (x-1)...(x-[x])\int_{t=1}^{t=\infty} \frac{1}{e^t} dt < \infty$$

Therefore, the Gamma integral is defined, and converges uniformly for $x > 0$.$\square$

3.2 Gamma for any z

$\Gamma(z)$ is extended for any z by

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)(z+2)....(z+n)}$$

$$= \lim_{n\to\infty} \frac{n^z}{z(1+z)(1+z/2)...(1+z/n)}$$

This is analytic for any z except for the poles at $z = 0, -1, -2, ...$

3.3 Zeta Integral in $x > 1$

For $x > 1$,

$$\boxed{\zeta(z) = \frac{1}{\Gamma(z)} \int_{t=0}^{t=\infty} \frac{t^{z-1}}{e^t - 1} dt = \frac{\displaystyle\int_{t=0}^{t=\infty} \frac{t^{z-1}}{e^t - 1} dt}{\displaystyle\int_{t=0}^{t=\infty} \frac{t^{z-1}}{e^t} dt}}$$

Riemann wrote:

$$\sum_{n=natural} \frac{1}{n^z}$$

and

$$\prod_{p=prime} \frac{1}{1 - 1/p^z}$$

converge only when $\mathrm{Re}z > 1$.

However, it is easy to find for $\zeta(z)$ *an expression that holds for any* z.

First of all, by applying the equation

$$\int_{t=0}^{t=\infty} e^{-nt} t^{z-1} dt = \frac{1}{n^z} \Gamma(z)$$

we get

$$\Gamma(z)\zeta(z) = \int_{t=0}^{t=\infty} \frac{t^{z-1}}{e^t - 1} dt. \Diamond$$

Proof:

By 3.1, for $x > 0$, Euler's Gamma function is

$$\Gamma(z) = \int_{u=0}^{u=\infty} \frac{u^{z-1}}{e^u} du$$

By the change of variable $u = nt$,

$$\Gamma(z)\frac{1}{n^z} = \int_{t=0}^{t=\infty} \frac{t^{z-1}}{e^{nt}} dt$$

That is, for $n = 1, 2, 3, ...N$,

$$\Gamma(z)\frac{1}{1^z} = \int_{t=0}^{t=\infty} \frac{t^{z-1}}{e^{t}} dt,$$

$$\Gamma(z)\frac{1}{2^z} = \int_{t=0}^{t=\infty} \frac{t^{z-1}}{e^{2t}} dt,$$

.....................................,

$$\Gamma(z)\frac{1}{N^z} = \int_{t=0}^{t=\infty} \frac{t^{z-1}}{e^{Nt}} dt.$$

Therefore, summing this equalities, for $x > 0$,

$$\Gamma(z)\left(\frac{1}{1^z} + \frac{1}{2^z} + ...\frac{1}{N^z}\right) = \int_{t=0}^{t=\infty} \left(\frac{1}{e^t} + \frac{1}{e^{2t}} + ...\frac{1}{e^{Nt}}\right) t^{z-1} dt.$$

Since $\sum_{n=1}^{\infty}\frac{1}{n^z}$ converges only for $x > 1$, then, letting $N \to \infty$, we obtain for $x > 1$,

$$\Gamma(z)\left(\frac{1}{1^z}+\frac{1}{2^z}+...\right) = \lim_{N\to\infty}\int_{t=0}^{t=\infty}\left(\frac{1}{e^t}+\frac{1}{e^{2t}}+...\frac{1}{e^{Nt}}\right)t^{z-1}dt\,.$$

Uniform convergence of

$$\frac{1}{e^t}+\frac{1}{e^{2t}}+... = \frac{1}{e^t-1}$$

for $t \geq \delta > 0$, allows bringing the limit in, and summing under the integral sign.

Thus, for $x > 1$,

$$\Gamma(z)\left(\frac{1}{1^z}+\frac{1}{2^z}+...\right) = \int_{t=0}^{t=\infty}\frac{t^{z-1}}{e^t-1}dt.$$

The integral,

$$\int_{t=0+}^{t=\infty}\frac{t^{z-1}}{e^t-1}dt$$

can be written as

$$\int_{t=0+}^{t=1}\frac{t^{z-1}}{e^t-1}dt + \int_{t=1}^{t=\infty}\frac{t^{z-1}}{e^t-1}dt\,.$$

For the second integral

$$\int_{t=1}^{t=\infty} \left| \frac{t^{z-1}}{e^t - 1} \right| dt = \int_{t=1}^{t=\infty} \frac{t^{x-1}}{(e^{t/2} - 1)(e^{t/2} + 1)} dt \leq \int_{t=1}^{t=\infty} \frac{t^{x-1}}{e^{t/2}} dt$$

The last integral converges similarly to $\int_{t=1}^{t=\infty} \frac{t^{z-1}}{e^t} dt$, in 3.1.

For the first integral,

$$\int_{t=0+}^{t=1} \left| \frac{t^{z-1}}{e^t - 1} \right| dt = \int_{t=0+}^{t=1} \frac{t^{x-1}}{e^t - 1} dt$$

$$= \int_{t=0+}^{t=1} t^{x-1} \left(\frac{1}{e^t} + \frac{1}{e^{2t}} + \ldots \right) dt .$$

The integral diverges in $0 < x \leq 1$, and converges only in $x > 1$.

Therefore, for $x > 1$, $\zeta(z) = \frac{1}{\Gamma(z)} \int_{t=0}^{t=\infty} \frac{t^{z-1}}{e^t - 1} dt . \square$

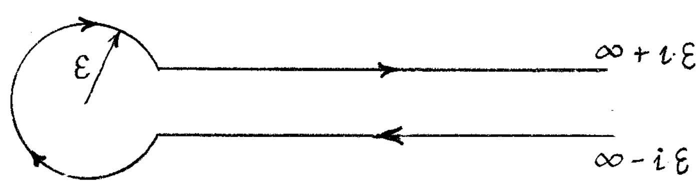
4

Zeta for any z

4.1 The Path Integral Formula for Zeta

$$\boxed{\zeta(z) = \frac{1}{2\sin \pi z \Gamma(z)} i \int\limits_{\lambda=\infty-i0}^{\lambda=\infty+i0} \frac{(-\lambda)^{z-1}}{e^{\lambda}-1} d\lambda}$$

the path starts at $\infty - i0$,
encircles $z = 0$,
and returns to $\infty + i0$.



Riemann wrote

...Consider the integral

$$\int\limits_{\infty-i0}^{\infty+i0} \frac{(-\lambda)^{z-1}}{e^{\lambda}-1} d\lambda$$

along a closed path from

$$\lambda = \infty - i0$$

to

$$\lambda = \infty + i0$$

clockwise around a domain that contains the singularity at

$$z = 0,$$

but none of the singularities at

$$z = 2\pi i n,$$

for $n = 1, 2, 3...$

To define the multi-valued function

$$(-\lambda)^{z-1} = e^{(z-1)\log(-\lambda)},$$

we choose the branch of

$$\log(-\lambda)$$

that is real for

$$\lambda < 0.$$

The integral equals

$$(e^{-i\pi z} - e^{i\pi z}) \int_{t=0}^{t=\infty} \frac{(-t)^{z-1}}{e^t - 1} dt,$$

where the integration path is along the real axis.

Thus, we obtain

$$2\sin \pi z \Gamma(z)\zeta(z) = i \int_{\infty - i0}^{\infty + i0} \frac{(-\lambda)^{z-1}}{e^{\lambda} - 1} d\lambda. \Diamond$$

PROOF: 4.2-4.6

4.2 The Integration Path

To evaluate the integral

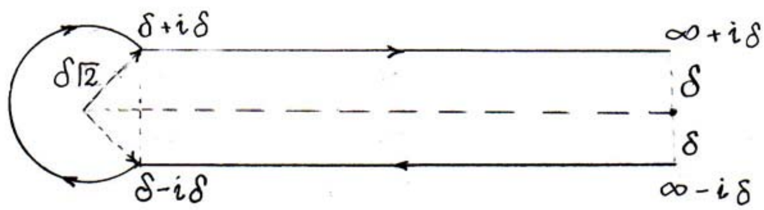
$$\int_{\lambda=\infty-i\delta}^{\lambda=\infty+i\delta} \frac{(-\lambda)^{z-1}}{e^{\lambda}-1} d\lambda,$$

we choose a path that

runs from $\lambda = \infty - i\delta$ to $\lambda = \delta - i\delta$, along $\lambda = t - i\delta$.

runs from $\lambda = \delta - i\delta$ to $\lambda = \delta + i\delta$, encircling $z = 0$ along $\lambda = \delta\sqrt{2}e^{i\theta} \equiv \varepsilon e^{i\theta}$.

runs from $\lambda = \delta + i\delta$ to $\lambda = \infty + i\delta$, along $\lambda = t + i\delta$.



Then, the integral equals

$$\int_{\lambda=\infty-i\delta}^{\lambda=\delta-i\delta} \frac{(-\lambda)^{z-1}}{e^{\lambda}-1} d\lambda + \int_{\lambda=\delta-i\delta}^{\lambda=\delta+i\delta} \frac{(-\lambda)^{z-1}}{e^{\lambda}-1} d\lambda + \int_{\lambda=\delta+i\delta}^{\lambda=\infty+i\delta} \frac{(-\lambda)^{z-1}}{e^{\lambda}-1} d\lambda.$$

To stay in the same branch of $\log(-\lambda)$, we should not cross the cut along the positive x axis.

4.3 For $\delta \downarrow 0$, the First Integral $\to e^{-i\pi z} \int\limits_{t=0+}^{t=\infty} \frac{t^{z-1}}{e^t - 1} dt$

Proof:

For the first integral, we rotate

$$[t - i\delta]$$

clockwise, multiplying it by

$$e^{-i\pi}$$

to obtain

$$-[t - i\delta].$$

That is,

$$-\lambda = -[t - i\delta] = [t - i\delta]e^{-i\pi}.$$

Therefore,

$$(-\lambda)^{z-1} = ([t - i\delta]e^{-i\pi})^{z-1} = -e^{-i\pi z}[t - i\delta]^{z-1}$$

and we have

$$\int\limits_{\lambda=\infty-i\delta}^{\lambda=\delta-i\delta} \frac{(-\lambda)^{z-1}}{e^{\lambda} - 1} d\lambda = e^{-i\pi z} \int\limits_{t=\delta}^{t=\infty} \frac{(t - i\delta)^{z-1}}{e^{t-i\delta} - 1} dt.$$

Since $1 \geq \cos\delta$, we have,

$$2e^t \geq e^{t-i\delta} + e^{t+i\delta}$$

$$(e^{t-i\delta} - 1)(e^{t+i\delta} - 1) \geq (e^t - 1)^2$$

$$\frac{1}{\left|e^{t-i\delta} - 1\right|} \leq \frac{1}{e^t - 1}$$

$$\left|\frac{(t-i\delta)^{z-1}}{e^{t-i\delta}-1}\right| \le 2\left|\frac{t^{z-1}}{e^{t}-1}\right|$$

Also, for $\delta \downarrow 0$,

$$\left|\frac{(t-i\delta)^{z-1}}{e^{t-i\delta}-1}\right| \to \left|\frac{t^{z-1}}{e^{t}-1}\right|.$$

Therefore, by Lebesgue Dominant Convergence, as $\delta \downarrow 0$,

$$\text{first integral} \to e^{-i\pi z}\int_{t=0+}^{t=\infty}\frac{t^{z-1}}{e^{t}-1}dt\,.\square$$

4.4 For $\delta \downarrow 0$, and $x > 1$, the Second Integral $\to 0$

Proof:

In the second integral,

$$\left|\lambda\right| = \delta\sqrt{2} \equiv \varepsilon$$

and

$$\left|\int_{\lambda=\delta-i\delta}^{\lambda=\delta+i\delta}\frac{(-\lambda)^{z-1}}{e^{\lambda}-1}d\lambda\right| \le \int_{\theta=0}^{\theta=2\pi}\frac{\varepsilon^{x-1}}{\left|\, exp(\varepsilon e^{i\theta})-1\,\right|}\varepsilon d\theta\,.$$

To apply Lebesgue Dominant Convergence, we need to confirm that as $\varepsilon \downarrow 0$,

$$\frac{\varepsilon^{x}}{\left|\, exp(\varepsilon e^{i\theta})-1\,\right|} \to 0\,.$$

We will avoid differentiating square roots by showing that

$$\frac{\varepsilon^{2x}}{\left|\,exp(\varepsilon e^{i\theta})-1\,\right|^2} \to 0\,.$$

Now,

$$\left|exp(\varepsilon e^{i\theta})-1\right|^2 = \left(e^{\varepsilon e^{i\theta}}-1\right)\left(e^{\varepsilon e^{-i\theta}}-1\right)$$

$$= e^{2\varepsilon\cos\theta}+1-e^{\varepsilon\cos\theta}\left(e^{i\varepsilon\sin\theta}+e^{-i\varepsilon\sin\theta}\right)$$

$$= e^{2\varepsilon\cos\theta}+1-2e^{\varepsilon\cos\theta}\cos(\varepsilon\sin\theta)$$

$$\to 0\,,\ \textbf{as}\ \ \varepsilon\downarrow 0\,.$$

$$\frac{d}{d\varepsilon}\left|exp(\varepsilon e^{i\theta})-1\right|^2 =$$

$$= \frac{d}{d\varepsilon}\left(e^{2\varepsilon\cos\theta}+1-2e^{\varepsilon\cos\theta}\cos(\varepsilon\sin\theta)\right)$$

$$= 2\cos\theta e^{2\varepsilon\cos\theta}-2e^{\varepsilon\cos\theta}\left(\cos\theta\cos(\varepsilon\sin\theta)-\sin\theta\sin(\varepsilon\sin\theta)\right)$$

$$= 2\cos\theta e^{2\varepsilon\cos\theta}-2e^{\varepsilon\cos\theta}\left(\cos(\theta+\varepsilon\sin\theta)\right)$$

$$= 2e^{\varepsilon\cos\theta}\left(e^{\varepsilon\cos\theta}\cos\theta-\cos(\theta+\varepsilon\sin\theta)\right)$$

$$\to 0\,,\ \textbf{as}\ \ \varepsilon\downarrow 0\,.$$

$$\frac{d^2}{d\varepsilon^2}\left|exp(\varepsilon e^{i\theta})-1\right|^2 =$$

$$= \frac{d}{d\varepsilon}2e^{\varepsilon\cos\theta}\left(e^{\varepsilon\cos\theta}\cos\theta-\cos(\theta+\varepsilon\sin\theta)\right)$$

$$= 2e^{\varepsilon\cos\theta}\cos\theta\left(e^{\varepsilon\cos\theta}\cos\theta-\cos(\theta+\varepsilon\sin\theta)\right)+$$

$$+2e^{\varepsilon\cos\theta}\left(e^{\varepsilon\cos\theta}\cos\theta\cos\theta+\sin(\theta+\varepsilon\sin\theta)\sin\theta\right)$$

$$\to 2 \text{ as } \varepsilon \downarrow 0.$$

Therefore, by L'Hospital, for $x > 1$, and for $\varepsilon \downarrow 0$,

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{2x}}{\left|exp(\varepsilon e^{i\theta})-1\right|^2}=\lim_{\varepsilon\to 0}\frac{D_\varepsilon \varepsilon^{2x}}{D_\varepsilon\left|exp(\varepsilon e^{i\theta})-1\right|^2}$$

$$=\lim_{\varepsilon\to 0}\frac{D_\varepsilon^2 \varepsilon^{2x}}{D_\varepsilon^2\left|exp(\varepsilon e^{i\theta})-1\right|^2}$$

$$=\lim_{\varepsilon\to 0}\frac{2x(2x-1)\varepsilon^{2x-2}}{D_\varepsilon^2\left|exp(\varepsilon e^{i\theta})-1\right|^2}$$

$$=\frac{(\to 0)}{(\to 2)}, \quad \text{as } \varepsilon \downarrow 0.$$

Hence, as $\varepsilon \downarrow 0$, and for $x > 1$,

$$\frac{\varepsilon^x}{\left| exp(\varepsilon e^{i\theta})-1 \right|}\to 0.$$

Therefore, by Lebesgue Dominant Convergence, for $\delta \downarrow 0$, and for $x > 1$, Second Integral $\to 0$.□

4.5 For $\delta \downarrow 0$, the Third Integral $\to -e^{i\pi z}\int\limits_{t=0+}^{t=\infty}\frac{t^{z-1}}{e^t-1}dt$

Proof:

For the third integral we rotate

$$[t + i\delta]$$

counter-clockwise, multiplying it by

$$e^{i\pi}$$

to obtain

$$-[t + i\delta].$$

Hence,

$$-\lambda = -[t + i\delta] = [t + i\delta]e^{i\pi}$$

$$(-\lambda)^{z-1} = ([t + i\delta]e^{i\pi})^{z-1} = -e^{i\pi z}[t + i\delta]^{z-1}$$

and we have

$$\int_{\lambda=\delta+i\delta}^{\lambda=\infty+i\delta} \frac{(-\lambda)^{z-1}}{e^{\lambda} - 1} d\lambda = -e^{i\pi z} \int_{t=\delta}^{t=\infty} \frac{(t + i\delta)^{z-1}}{e^{t+i\delta} - 1} dt$$

Thus, by Lebesgue Dominant Convergence, as $\delta \downarrow 0$,

$$\text{Third Integral} \to -e^{i\pi z} \int_{t=0+}^{t=\infty} \frac{t^{z-1}}{e^{t} - 1} dt . \square$$

4.6 The Path-Integral Formula 4.1

Proof:

For $\delta \downarrow 0$, and for $x > 1$, we obtain

$$\int_{\infty-i0}^{\infty+i0} \frac{(-\lambda)^{z-1}}{e^{\lambda} - 1} d\lambda = (e^{-i\pi z} - e^{i\pi z}) \int_{t=0}^{t=\infty} \frac{t^{z-1}}{e^{t} - 1} dt$$

$$= -2i \sin(\pi z) \int_{t=0}^{t=\infty} \frac{t^{z-1}}{e^{t} - 1} dt$$

By 3.2, for $x > 1$,

$$\int_{t=0}^{t=\infty} \frac{t^{z-1}}{e^t - 1} dt = \Gamma(z)\zeta(z)$$

Hence,

$$\int_{\infty - i0}^{\infty + i0} \frac{(-\lambda)^{z-1}}{e^{\lambda} - 1} d\lambda = -2i \sin(\pi z)\Gamma(z)\zeta(z)$$

Therefore,

$$\zeta(z) = \frac{1}{2\sin \pi z \Gamma(z)} i \int_{\lambda = \infty - i0}^{\lambda = \infty + i0} \frac{(-\lambda)^{z-1}}{e^{\lambda} - 1} d\lambda .$$

The right hand side is defined for any $z \neq 1$.

Since it equals Zeta in the half plane $x > 1$, it is the analytic continuation of Zeta to the complex plane for any $z \neq 1$.$\square$

4.7 A Second path Integral Formula for Zeta

$$\zeta(z) = -\frac{\Gamma(1-z)}{2\pi i} \int_{\infty - i0}^{\infty + i0} \frac{(-\lambda)^{z-1}}{e^{\lambda} - 1} d\lambda.$$

Proof:

By 4.1,

$$\zeta(z) = -\frac{1}{2i \sin(\pi z)\Gamma(z)} \int_{\infty - i0}^{\infty + i0} \frac{(-\lambda)^{z-1}}{e^{\lambda} - 1} d\lambda .$$

Substituting

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)},$$

we obtain

$$\zeta(z) = -\frac{\Gamma(1-z)}{2\pi i}\int_{\infty-i0}^{\infty+i0} \frac{(-\lambda)^{z-1}}{e^{\lambda}-1}d\lambda. \square$$

4.8 Zeta is finite for all $z \neq 1$

Riemann wrote

For any $z \neq 1$*,*

$\zeta(z)$ *is a single-valued, and finite function.* $\lozenge$

Proof:

$\Gamma(1-z)$ has simple poles at $z = 1,2,3...$, but only $z = 1$ is a pole

of $\frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + ...$, or $\frac{1}{2\sin(\pi z)\Gamma(z)} i \int_{\infty-i0}^{\infty+i0} \frac{(-\lambda)^{z-1}}{e^{\lambda}-1}d\lambda$.

It follows that the integral

$$\int_{\infty-i0}^{\infty+i0} \frac{(-\lambda)^{z-1}}{e^{\lambda}-1}d\lambda$$

has zeros that cancel the poles at $z = 2,3...$,

Zeta has only one pole at $z = 1$, and is finite for all $z \neq 1$.

4.9 Zeta has zeros at $z = -2n$ for $n = 1, 2, 3...$

Proof:

The Bernoulli numbers

$$B_n$$

are defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{1}{n!} B_n t^n$$

that converges in $|t| < 2\pi$.

We have

$$\zeta(-2n) = \frac{(-1)^n}{n+1} B_{2n+1},$$

and

$$B_{2n+1} = 0$$

Thus, Zeta has zeros at

$$z = -2n$$

for $n = 1, 2, 3... \square$

5

The Zeta Functional Equation

5.1 $2\sin \pi z \Gamma(z)\zeta(z) = (2\pi)^z[(-i)^{z-1} + i^{z-1}]\left\{\frac{1}{1^{1-z}} + \frac{1}{2^{1-z}} + \frac{1}{3^{1-z}} + ...\right\}$

Riemann wrote

> *If* $x < 0$, *we integrate along a clockwise-oriented path around the complementary domain in the complex plane.*
>
> *Then, the integral is infinitesimal, because it is over values that have infinitely large modulus.*
>
> *In the complementary domain, the integrand has singularities only at*
>
> $$\lambda = \pm 2\pi i n\,,$$
>
> *for* $n = 1, 2, 3...$
>
> *Therefore, the integral equals the sum of the clockwise-oriented path- integrals around these singularities.*
>
> *Since the clockwise-oriented path-integral around a singularity* $z = 2\pi i n$ *is*
>
> $$(-2\pi i)(-2\pi i n)^{z-1}\,,$$
>
> *we have*
>
> $$2\sin \pi z \Gamma(z)\zeta(z) = (2\pi)^z \sum_n n^{z-1}[(-i)^{z-1} + i^{z-1}].\Diamond$$

Proof:

By 4.1, we have

$$2\sin \pi z \Gamma(z)\zeta(z) = i\int_{\infty - i0}^{\infty + i0} \frac{(-\lambda)^{z-1}}{e^{\lambda} - 1} d\lambda$$

We will evaluate the integral along a clockwise-oriented path around the complementary domain.

By the Residue Theorem, the clockwise-oriented integral

$$\int_{|\lambda - 2\pi in| = \varepsilon} \frac{(-\lambda)^{z-1}}{e^{\lambda} - 1} d\lambda$$

equals

$$(-2\pi i)\mathrm{Res}\left[\frac{(-\lambda)^{z-1}}{e^{\lambda} - 1}\right]_{\lambda = 2\pi in} = -2\pi i \lim_{\lambda \to 2\pi in}\left[(\lambda - 2\pi in)\frac{(-\lambda)^{z-1}}{e^{\lambda} - 1}\right]$$

By L'Hospital,

$$\lim_{\lambda \to 2\pi in} \frac{(\lambda - 2\pi in)}{e^{\lambda} - 1} = \lim_{\lambda \to 2\pi in} \frac{1}{e^{\lambda}} = 1.$$

Hence, around the singularity

$$\lambda = 2\pi in,$$

the integral equals

$$(-2\pi i)\lim_{\lambda \to 2\pi in}(-\lambda)^{z-1} = (-2\pi i)(-2\pi in)^{z-1}.$$

Similarly, around the singularity

$$\lambda = -2\pi in\,,$$

the integral equals

$$(-2\pi i)(2\pi in)^{z-1}.$$

The integral around the complementary domain is the sum of the integrals around all the singularities and it equals

$$= (-2\pi i)\sum_{n=1}^{\infty}[(-2\pi in)^{z-1} + (2\pi in)^{z-1}]$$

$$= -i(2\pi)^{z}[(-i)^{z-1} + (i)^{z-1}]\sum_{n=1}^{\infty} n^{z-1}$$

Thus,

$$2\sin \pi z \Gamma(z)\zeta(z) = (2\pi)^{z}[(-i)^{z-1} + i^{z-1}]\Big\{\tfrac{1}{1^{1-z}} + \tfrac{1}{2^{1-z}} + \tfrac{1}{3^{1-z}} + ...\Big\}.\square$$

5.2 Zeta Functional Equation

Define $\boxed{\eta(z) \equiv \pi^{-z/2}\Gamma(z\,/\,2)\zeta(z)}$**.**

Then $\boxed{\eta(z) = \eta(1-z)}$ **for any** z**.**

The equation $\eta(z) = \eta(1-z)$ is solved also by $z(1-z)$.

Riemann wrote

By using known properties of $\Gamma(z)$ *we obtain the following relation between* $\zeta(z)$ *and* $\zeta(1-z)$

$\pi^{-z/2}\Gamma(z/2)\zeta(z)$ *is unchanged if* $1-z$ *replaces* z. $\Diamond$

Proof:

By section 5.1,

$$2\sin \pi z\Gamma(z)\zeta(z) = (2\pi)^z[(-i)^{z-1} + i^{z-1}]\left\{\tfrac{1}{1^{1-z}} + \tfrac{1}{2^{1-z}} + \tfrac{1}{3^{1-z}} + ...\right\}$$

Put

$$\tfrac{1}{1^{1-z}} + \tfrac{1}{2^{1-z}} + \tfrac{1}{3^{1-z}} + ... = \zeta(1-z),$$

and

$$[(-i)^{z-1} + i^{z-1}] = i(-i)^z - i(i)^z$$

$$= i[e^{z\log(-i)} - e^{z\log i}]$$

$$= i[e^{z(-i\pi/2)} - e^{z(i\pi/2)}] = 2\sin\frac{\pi z}{2}$$

Then,

$$2\sin \pi z\Gamma(z)\zeta(z) = 2^z\pi^z 2\sin(\pi z/2)\zeta(1-z).$$

Substitute

$$\sin(\pi z/2) = \frac{\pi}{\Gamma(z/2)\Gamma(1-z/2)},$$

and

$$\sin(\pi z)\Gamma(z) = \frac{\pi}{\Gamma(1-z)}.$$

Then,

$$\frac{1}{\Gamma(1-z)}\zeta(z) = 2^z \pi^z \frac{1}{\Gamma(z\ /\ 2)\Gamma(1-z\ /\ 2)}\zeta(1-z).$$

That is,

$$\pi^{-z/2}\Gamma(z\ /\ 2)\zeta(z) = \pi^{-(1-z)/2}\pi^{1/2}2^z\Gamma(1-z)\frac{1}{\Gamma(1-z\ /\ 2)}\zeta(1-z).$$

By [Mag, p. 3]

$$\Gamma(z) = \pi^{-1/2}2^{z-1}\Gamma(\frac{z}{2})\Gamma(\frac{1+z}{2}).$$

Hence,

$$\Gamma(1-z) = \pi^{-1/2}2^{-z}\Gamma([1-z]\ /\ 2)\Gamma(1-z\ /\ 2).$$

Namely,

$$\pi^{1/2}2^z\Gamma(1-z)\frac{1}{\Gamma(1-z\ /\ 2)} = \Gamma([1-z]\ /\ 2).$$

Therefore,

$$\pi^{-z/2}\Gamma(z\ /\ 2)\zeta(z) = \pi^{-(1-z)/2}\Gamma([1-z]\ /\ 2)\zeta(1-z).$$

That is,

$$\eta(z) = \eta(1-z).\square$$

5.3 Functional equation with respect to $z = \frac{1}{2}$.

For $\boxed{x = \frac{1}{2} - \alpha}$,

$$\boxed{\eta(\frac{1}{2} - \alpha + iy) = \eta(\frac{1}{2} + \alpha - iy)}$$

Proof:

Make the x-translation

$$x = \tfrac{1}{2} - \alpha.$$

Then,

$$z = x + iy = \tfrac{1}{2} - \alpha + iy,$$

$$1 - z = 1 - x - iy = \tfrac{1}{2} + \alpha - iy.$$

The functional equation for $\zeta(z)$ is

$$\eta(\tfrac{1}{2} - \alpha + iy) = \eta(\tfrac{1}{2} + \alpha - iy).$$

The points

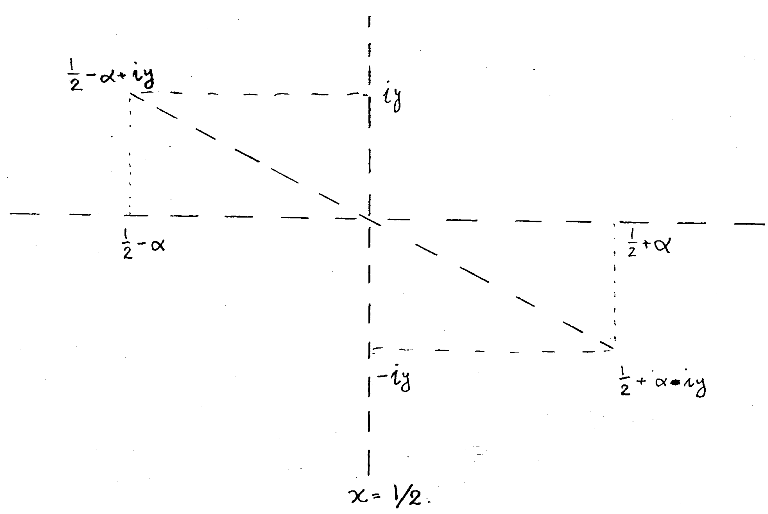
$$\tfrac{1}{2} - \alpha + iy,$$

and

$$\tfrac{1}{2} + \alpha - iy,$$

are symmetric with respect to

$$z = \tfrac{1}{2}.$$



6

Poisson Elliptic Function $\psi(t)$.

6.1 Poisson Elliptic Function

For $t \geq \delta > 0$, **the functions**

$$\psi_N(t) \equiv \frac{1}{e^{\pi t}} + \frac{1}{e^{2^2 \pi t}} + \frac{1}{e^{3^2 \pi t}} + ... + \frac{1}{e^{N^2 \pi t}},$$

converge uniformly to the Poisson Elliptic Function

$$\boxed{\psi(t) \equiv \frac{1}{e^{\pi t}} + \frac{1}{e^{2^2 \pi t}} + \frac{1}{e^{3^2 \pi t}} + ...}\ .$$

Riemann cites Jacobi's treatise on elliptic functions [5, p. 184], as his source for the functional equation for $\psi(t)$. But in [Jaco, p. 260], Jacobi attributes the formula to Poisson.

6.2 Poisson Functional Equation

$$\boxed{\psi(t) = \psi(1\ /\ t)t^{-1/2} + \tfrac{1}{2}t^{-1/2} - \tfrac{1}{2}}$$

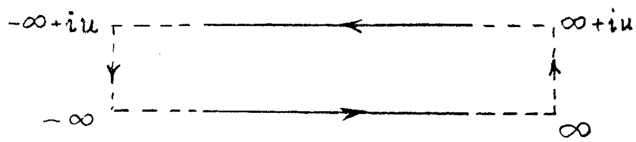
Proof:

Fix u, and integrate the function

$$e^{-\pi(\theta-iu)^2} = e^{-\pi\theta^2}e^{2\pi i\theta u}e^{\pi u^2}$$

in the anti-clockwise direction, along the rectangle with vertices

$(\infty,0)$, (∞,iu), $(-\infty,iu)$, $(-\infty,0)$.



Two of these integrals vanish:

$$\left|\int_{\theta=\infty}^{\theta=\infty+iu} e^{-\pi\theta^2}e^{2\pi i\theta u}e^{\pi u^2}d\theta\right| \le \int_{\theta=\infty}^{\theta=\infty+iu}\left|e^{-\pi\theta^2}e^{2\pi i\theta u}e^{\pi u^2}\right|d\theta$$

$$= \underbrace{e^{-\pi\infty^2}}_{0}\underbrace{\left|e^{2\pi i\theta u}\right|}_{1}e^{\pi u^2}\,|u| = 0.$$

and

$$\left|\int_{\theta=-\infty+iu}^{\theta=-\infty} e^{-\pi\theta^2}e^{2\pi i\theta u}e^{\pi u^2}d\theta\right| \le \int_{\theta=-\infty+iu}^{\theta=-\infty}\left|e^{-\pi\theta^2}e^{2\pi i\theta u}e^{\pi u^2}\right|d\theta$$

$$= \underbrace{e^{-\pi(-\infty)^2}}_{0}\underbrace{\left|e^{2\pi i\theta u}\right|}_{1}e^{\pi u^2}\,|u| = 0.$$

By Cauchy's Theorem,

$$0 = \oint e^{-\pi\theta^2}e^{2\pi i\theta u}e^{\pi u^2}d\theta$$

$$= e^{\pi u^2} \int_{\theta=\infty+iu}^{\theta=-\infty+iu} e^{-\pi\theta^2} e^{2\pi i\theta u} d\theta + \underbrace{e^{\pi(0)^2}}_{1} \int_{\theta=-\infty}^{\theta=\infty} e^{-\pi\theta^2} \underbrace{e^{2\pi i\theta(0)}}_{1} d\theta$$

Therefore,

$$e^{\pi u^2} \int_{\theta=-\infty+iu}^{\theta=\infty+iu} e^{-\pi\theta^2} e^{2\pi i\theta u} d\theta = \int_{\theta=-\infty}^{\theta=\infty} e^{-\pi\theta^2} d\theta = 1.$$

That is,

$$e^{-\pi u^2} = \int_{\theta=-\infty+iu}^{\theta=\infty+iu} e^{-\pi\theta^2} e^{2\pi i\theta u} d\theta$$

$$= \int_{\theta=-\infty}^{\theta=\infty} e^{-\pi\theta^2} e^{2\pi i\theta u} d\theta.$$

For $m = ..., -3, -2, -1, 0, 1, 2, 3, ...,$ substitute

$$u = m\sqrt{t},$$

and obtain the equations

$$e^{-m^2\pi t} = \int_{\theta=-\infty}^{\theta=\infty} e^{-\pi\theta^2} e^{2\pi i m\sqrt{t}\theta} d\theta.$$

The change of variable

$$\omega = \sqrt{t}\theta$$

gives

$$e^{-m^2\pi t} = \frac{1}{\sqrt{t}} \int\limits_{\omega=-\infty}^{\omega=\infty} e^{-\omega^2\pi/t} e^{2\pi i m\omega} d\omega .$$

Thus, for $m = ..., -3, -2, -1, \ 0, \ 1, \ 2, \ 3, ...,$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{1}{e^{(-3)^2\pi t}} = \frac{1}{\sqrt{t}} \int\limits_{\omega=-\infty}^{\omega=\infty} \frac{1}{e^{\omega^2\pi/t}} e^{2\pi i(-3)\omega} d\omega$$

$$\frac{1}{e^{(-2)^2\pi t}} = \frac{1}{\sqrt{t}} \int\limits_{\omega=-\infty}^{\omega=\infty} \frac{1}{e^{\omega^2\pi/t}} e^{2\pi i(-2)\omega} d\omega$$

$$\frac{1}{e^{(-1)^2\pi t}} = \frac{1}{\sqrt{t}} \int\limits_{\omega=-\infty}^{\omega=\infty} \frac{1}{e^{\omega^2\pi/t}} e^{2\pi i(-1)\omega} d\omega$$

$$\frac{1}{e^{(0)^2\pi t}} = \frac{1}{\sqrt{t}} \int\limits_{\omega=-\infty}^{\omega=\infty} \frac{1}{e^{\omega^2\pi/t}} e^{2\pi i(0)\omega} d\omega$$

$$\frac{1}{e^{1^2\pi t}} = \frac{1}{\sqrt{t}} \int\limits_{\omega=-\infty}^{\omega=\infty} \frac{1}{e^{\omega^2\pi/t}} e^{2\pi i(1)\omega} d\omega$$

$$\frac{1}{e^{2^2\pi t}} = \frac{1}{\sqrt{t}} \int\limits_{\omega=-\infty}^{\omega=\infty} \frac{1}{e^{\omega^2\pi/t}} e^{2\pi i(2)\omega} d\omega$$

$$\frac{1}{e^{3^2\pi t}} = \frac{1}{\sqrt{t}} \int\limits_{\omega=-\infty}^{\omega=\infty} \frac{1}{e^{\omega^2\pi/t}} e^{2\pi i(3)\omega} d\omega .$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Summation of both sides gives

$$1+2\left(\frac{1}{e^{\pi t}}+\frac{1}{e^{2^2\pi t}}+\frac{1}{e^{3^2\pi t}}+....\right)=$$

$$=\frac{1}{\sqrt{t}}\left(...+\int_{\omega=-\infty}^{\omega=\infty}\frac{1}{e^{\omega^2\pi/t}}e^{2\pi i(-3)\omega}d\omega+\int_{\omega=-\infty}^{\omega=\infty}\frac{1}{e^{\omega^2\pi/t}}e^{2\pi i(-2)\omega}d\omega\right.$$

$$+\int_{\omega=-\infty}^{\omega=\infty}\frac{1}{e^{\omega^2\pi/t}}e^{2\pi i(-1)\omega}d\omega++\int_{\omega=-\infty}^{\omega=\infty}\frac{1}{e^{\omega^2\pi/t}}e^{2\pi i(0)\omega}d\omega$$

$$\left.+\int_{\omega=-\infty}^{\omega=\infty}\frac{1}{e^{\omega^2\pi/t}}e^{2\pi i(1)\omega}d\omega+\int_{\omega=-\infty}^{\omega=\infty}\frac{1}{e^{\omega^2\pi/t}}e^{2\pi i(2)\omega}d\omega+....\right).$$

By Poisson Summation Formula [Spieg, p.109],

$$...+\int_{\omega=-\infty}^{\omega=\infty}F(\omega)e^{2\pi i(-3)\omega}d\omega+\int_{\omega=-\infty}^{\omega=\infty}F(\omega)e^{2\pi i(-2)\omega}d\omega$$

$$+\int_{\omega=-\infty}^{\omega=\infty}F(\omega)e^{2\pi i(-1)\omega}d\omega+\int_{\omega=-\infty}^{\omega=\infty}F(\omega)e^{2\pi i(0)\omega}d\omega$$

$$+\int_{\omega=-\infty}^{\omega=\infty}F(\omega)e^{2\pi i(1)\omega}d\omega+\int_{\omega=-\infty}^{\omega=\infty}F(\omega)e^{2\pi i(2)\omega}d\omega+...$$

$$=...+F(-3)+F(-2)+F(-1)+F(0)+F(1)+F(2)+F(3)+...$$

Applying to

$$F(\omega) = \frac{1}{e^{\omega^2 \pi / t}},$$

we have

$$1 + 2\left(\frac{1}{e^{\pi t}} + \frac{1}{e^{2^2 \pi t}} + \frac{1}{e^{3^2 \pi t}} + \ldots.\right)$$

$$= \frac{1}{\sqrt{t}}\left(\ldots + \frac{1}{e^{(-3)^2 \pi / t}} + \frac{1}{e^{(-2)^2 \pi / t}} + \frac{1}{e^{(-1)^2 \pi / t}}\right.$$

$$\left. + \frac{1}{e^{(0)^2 \pi / t}} + \frac{1}{e^{(1)^2 \pi / t}} + \frac{1}{e^{(2)^2 \pi / t}} + \frac{1}{e^{(3)^2 \pi / t}} + \ldots\right)$$

$$= \frac{1}{\sqrt{t}}\left\{1 + 2\left(\frac{1}{e^{\pi / t}} + \frac{1}{e^{2^2 \pi / t}} + \frac{1}{e^{3^2 \pi / t}} + \ldots.\right)\right\}.$$

That is,

$$1 + 2\psi(t) = \frac{1}{\sqrt{t}}(1 + 2\psi(1/t)).\square$$

7

Riemann's Formula for Zeta

7.1 Riemann's Formula for Zeta

For any $z \neq 1$,

$$\boxed{\eta(z) = \frac{1}{z(z-1)} + \int_{t=1}^{t=\infty} \psi(t)(t^{-(z+1)/2} + t^{z/2-1})dt}$$

where $\psi(t)$ **is Poisson Elliptic Function,**

and $\eta(z) = \pi^{-z/2}\Gamma(z / 2)\zeta(z)$**.**

Riemann wrote

By using known properties of $\Gamma(z)$ *we obtain the following relation between* $\zeta(z)$*,and* $\zeta(1-z)$

$\pi^{-z/2}\Gamma(z / 2)\zeta(z)$ *is unchanged if* $1-z$ *replaces* z*.*

This property of Zeta let me substitute $\Gamma(z / 2)$ *instead of* $\Gamma(z)$*, into the general term of* $\sum 1 / n^z$*.*

This substitution gives a very convenient formula for $\zeta(z)$

We have

$$\pi^{-z/2}\Gamma(z\,/\,2)\frac{1}{n^z} = \int\limits_{t=0}^{t=\infty} e^{-n^2\pi t} t^{z/2-1} dt$$

If we set

$$\sum_{n=1}^{\infty} e^{-n^2\pi t} = \psi(t),$$

we get

$$\pi^{-z/2}\Gamma(z\,/\,2)\zeta(z) = \int\limits_{t=0}^{t=\infty} \psi(t) t^{z/2-1} dt\,.$$

Since

$$2\psi(t) + 1 = t^{-1/2}[2\psi(1\,/\,t) + 1]$$ ***(Jacobi, Fund., p.184),***

it follows that

$$\pi^{-z/2}\Gamma(z\,/\,2)\zeta(z) = \int\limits_{t=1}^{t=\infty} \psi(t) t^{z/2-1} dt$$

$$+\int\limits_{t=0}^{t=1} \psi(1\,/\,t) t^{(z-3)/2} dt$$

$$+\frac{1}{2}\int\limits_{t=0}^{t=1} (t^{(z-3)/2} - t^{z/2-1}) dt$$

$$= \frac{1}{z(z-1)} + \int\limits_{t=1}^{t=\infty} \psi(t)(t^{z/2-1} + t^{-(1+z)/2}) dt\,.\Diamond$$

Proof:

Riemann generated the partial sums of the Poisson Elliptic Function from Euler's Gamma function written for $\frac{1}{2}z$

$$\Gamma(\tfrac{1}{2}z) = \int_{u=0}^{u=\infty} \frac{1}{e^u} u^{\frac{1}{2}z-1} du, \text{ for } x > 0.$$

The change of variable

$$u = n^2 \pi t,$$

gives for $x > 0$,

$$\Gamma(\tfrac{1}{2}z) = \pi^{\frac{1}{2}z} n^z \int_{t=0}^{t=\infty} \frac{1}{e^{n^2 \pi t}} t^{\frac{1}{2}z-1} dt.$$

Thus, for $n = 1, 2, 3, \ldots N$, we write the N equalities

$$\frac{1}{\pi^{\frac{1}{2}z}} \Gamma(\tfrac{1}{2}z) \frac{1}{1^z} = \int_{t=0}^{t=\infty} \frac{1}{e^{\pi t}} t^{\frac{1}{2}z-1} dt,$$

$$\frac{1}{\pi^{\frac{1}{2}z}} \Gamma(\tfrac{1}{2}z) \frac{1}{2^z} = \int_{t=0}^{t=\infty} \frac{1}{e^{2^2 \pi t}} t^{\frac{1}{2}z-1} dt,$$

$$\frac{1}{\pi^{\frac{1}{2}z}} \Gamma(\tfrac{1}{2}z) \frac{1}{3^z} = \int_{t=0}^{t=\infty} \frac{1}{e^{3^2 \pi t}} t^{\frac{1}{2}z-1} dt,$$

.................................,

$$\frac{1}{\pi^{\frac{1}{2}z}} \Gamma(\tfrac{1}{2}z) \frac{1}{N^z} = \int_{t=0}^{t=\infty} \frac{1}{e^{N^2 \pi t}} t^{\frac{1}{2}z-1} dt.$$

Summing the N equalities, we have for $x > 0$,

$$\frac{1}{\pi^{\frac{1}{2}z}}\Gamma(\tfrac{1}{2}z)\left(\frac{1}{1^z}+\frac{1}{2^z}+\frac{1}{3^z}+...\frac{1}{N^z}\right)$$

$$=\int_{t=0}^{t=\infty}\left(\frac{1}{e^{\pi t}}+\frac{1}{e^{2^2\pi t}}+\frac{1}{e^{3^2\pi t}}+.......+\frac{1}{e^{N^2\pi t}}\right)t^{\frac{1}{2}z-1}dt\,.$$

Letting $N \to \infty$, the series

$$\frac{1}{1^z}+\frac{1}{2^z}+\frac{1}{3^z}+...$$

converges only in the half plane $x > 1$. Thus, only for $x > 1$,

$$\underbrace{\frac{1}{\pi^{\frac{1}{2}z}}\Gamma(\tfrac{1}{2}z)\underbrace{\left(\frac{1}{1^z}+\frac{1}{2^z}+\frac{1}{3^z}+...\right)}_{\zeta(z)}}_{\eta(z)}=\lim_{N\to\infty}\int_{t=0}^{t=\infty}\left(\frac{1}{e^{\pi t}}+.......+\frac{1}{e^{N^2\pi t}}\right)t^{\frac{1}{2}z-1}dt\,.$$

Now, the uniform convergence of the Poisson Elliptic for $t \geq \delta > 0$, allows us to take the limit under the integral sign,

$$\lim_{N\to\infty}\int_{t=0+}^{t=\infty}\left(\frac{1}{e^{\pi t}}+\frac{1}{e^{2^2\pi t}}+...+\frac{1}{e^{N^2\pi t}}\right)t^{\frac{1}{2}z-1}dt$$

$$=\int_{t=0+}^{t=\infty}\lim_{N\to\infty}\left(\frac{1}{e^{\pi t}}+\frac{1}{e^{2^2\pi t}}+...+\frac{1}{e^{N^2\pi t}}\right)t^{\frac{1}{2}z-1}dt$$

$$= \int_{t=0+}^{t=\infty} \left(\frac{1}{e^{\pi t}} + \frac{1}{e^{2^2 \pi t}} + \frac{1}{e^{3^2 \pi t}} + \ldots.. \right) t^{\frac{1}{2}z-1} dt$$

$$= \int_{t=0}^{t=\infty} \psi(t) t^{\frac{1}{2}z-1} dt,$$

where $\psi(t)$ is the Poisson Elliptic Function.

We aim to remove the singularity at $t = 0$, from under the integral sign. To that end, we write the last integral as the sum of two integrals

$$\int_{t=0}^{t=\infty} \psi(t) t^{\frac{1}{2}z-1} dt = \int_{t=0}^{t=1} \psi(t) t^{\frac{1}{2}z-1} dt + \int_{t=1}^{t=\infty} \psi(t) t^{\frac{1}{2}z-1} dt.$$

In $\int_{t=0}^{t=1}$, we replace $\psi(t)$ with $\psi(1/t) t^{-1/2} + \frac{1}{2} t^{-1/2} - \frac{1}{2}$.

$$= \int_{t=0}^{t=1} \left(\psi(1/t) t^{-1/2} + \tfrac{1}{2}[t^{-1/2} - 1] \right) t^{\frac{1}{2}z-1} dt + \int_{t=1}^{t=\infty} \psi(t) t^{\frac{1}{2}z-1} dt$$

$$= \int_{t=0}^{t=1} \psi(1/t) t^{\frac{1}{2}z-\frac{3}{2}} dt + \frac{1}{2} \int_{t=0}^{t=1} \left(t^{\frac{1}{2}z-\frac{3}{2}} - t^{\frac{1}{2}z-1} \right) dt + \int_{t=1}^{t=\infty} \psi(t) t^{\frac{1}{2}z-1} dt$$

Replacing $\frac{1}{2} \int_{t=0}^{t=1} \left(t^{\frac{1}{2}z-\frac{3}{2}} - t^{\frac{1}{2}z-1} \right) dt$ with $\dfrac{1}{z-1} - \dfrac{1}{z} = \dfrac{1}{z(z-1)}$,

$$= \frac{1}{z(z-1)} + \int_{t=0}^{t=1} \psi(1/t) t^{\frac{1}{2}z-\frac{3}{2}} dt + \int_{t=1}^{t=\infty} \psi(t) t^{\frac{1}{2}z-1} dt .$$

The change of variable $\tau = 1/t$, in $\int_{t=0}^{t=1} \psi(1/t) t^{\frac{1}{2}z-\frac{3}{2}} dt$, gives

$$\int_{\tau=\infty}^{\tau=1} \psi(\tau) \tau^{-\frac{1}{2}z+\frac{3}{2}} (-d\tau / \tau^2) = \int_{\tau=1}^{\tau=\infty} \psi(\tau) \tau^{-\frac{1}{2}(z+1)} d\tau .$$

Therefore,

$$\int_{t=0}^{t=\infty} \psi(t) t^{\frac{1}{2}z-1} dt = \frac{1}{z(z-1)} + \int_{t=1}^{t=\infty} \psi(t) \left(t^{-\frac{1}{2}(z+1)} + t^{\frac{1}{2}z-1} \right) dt .$$

If $x > 0$,

$$\psi(t) \left(t^{-\frac{1}{2}(z+1)} + t^{\frac{1}{2}z-1} \right) = (e^{-\pi t} + e^{-2^2 \pi t} + ...) t^{-\frac{1}{2}} \left(t^{-\frac{1}{2}z} + t^{\frac{1}{2}z} \right).$$

If $x < 0$, the change of notation

$$x = -u, \quad y = -v, \quad s = u + iv,$$

gives the same integrand

$$\psi(t) \left(t^{-\frac{1}{2}(z+1)} + t^{\frac{1}{2}z-1} \right) = (e^{-\pi t} + e^{-2^2 \pi t} + ...) t^{-\frac{1}{2}} \left(t^{-\frac{1}{2}s} + t^{\frac{1}{2}s} \right).$$

Therefore, the integral

$$\int_{t=1}^{t=\infty} \psi(t)\Big(t^{-\frac{1}{2}(z+1)} + t^{\frac{1}{2}z-1}\Big)dt$$

is an entire function, and the function

$$\mu(z) \equiv \frac{1}{z(z-1)} + \int_{t=1}^{t=\infty} \psi(t)\Big(t^{-\frac{1}{2}(z+1)} + t^{\frac{1}{2}z-1}\Big)dt$$

is analytic in $\mathbb{C}$ less a pole at $z = 1$,

satisfies the functional equation

$$\mu(1-z) = \frac{1}{(1-z)(-z)} + \int_{t=1}^{t=\infty} \psi(t)\Big(t^{\frac{1}{2}z-1} + t^{-\frac{1}{2}(1+z)}\Big)dt$$

$$= \mu(z).$$

equals $\eta(z)$ in $x > 1$.

Thus, $\mu(z)$ is the analytic continuation of $\eta(z)$ in $\mathbb{C}$ without $z = 1$.$\square$

8

The function $\xi(z)$

8.1 Definition of $\xi(z)$

$$\boxed{\begin{aligned}\xi(z) &\equiv \tfrac{1}{2}z(z-1)\eta(z)\\ &= \tfrac{1}{2}z(z-1)\pi^{-\frac{z}{2}}\Gamma(\tfrac{z}{2})\zeta(z)\end{aligned}}$$

8.2 Formulas for $\xi(z)$

$$\boxed{\xi(z) = \tfrac{1}{2} + z(z-1)\int_{t=1}^{t=\infty} \psi(t)\Big(t^{-\frac{1}{2}(z+1)} + t^{\frac{1}{2}z-1}\Big)dt}$$

$$\boxed{\xi(0) = \tfrac{1}{2}}$$

8.3 Riemann's First Formula for $\xi(z)$.

$$\xi(z) = \tfrac{1}{2} - z(1-z)\int_{t=1}^{t=\infty} \psi(t)t^{-\frac{3}{4}}\cos\Big(\tfrac{1}{2}[i(\tfrac{1}{2}-z)]\log t\Big)dt$$

Proof: By 7.1, we have

$$\xi(z) = \frac{1}{2} z(z-1)\left(\frac{1}{z(z-1)} + \int_{t=1}^{t=\infty} \psi(t)(t^{-(z+1)/2} + t^{z/2-1})dt \right)$$

$$\xi(z) = \frac{1}{2} - \frac{z(1-z)}{2} \int_{t=1}^{t=\infty} \psi(t)\left(t^{-\frac{z+1}{2}} + t^{\frac{z}{2}-1}\right)dt$$

$$= \frac{1}{2} - \frac{z(1-z)}{2} \int_{t=1}^{t=\infty} \psi(t)t^{-\frac{3}{4}}\left(t^{-i\frac{1}{2}\left(i[\frac{1}{2}-z]\right)} + t^{i\frac{1}{2}\left(i[\frac{1}{2}-z]\right)}\right)dt$$

$$= \frac{1}{2} - \frac{z(1-z)}{2} \int_{t=1}^{t=\infty} \psi(t)t^{-\frac{3}{4}}\left(e^{-i\frac{1}{2}\left(i[(\frac{1}{2}-z]\right)\log t} + e^{i\frac{1}{2}\left(i[\frac{1}{2}-z]\right)\log t}\right)dt$$

$$= \tfrac{1}{2} - z(1-z) \int_{t=1}^{t=\infty} \psi(t)t^{-\frac{3}{4}} \cos\left(\tfrac{1}{2} i[\tfrac{1}{2}-z]\log t\right)dt\,.\square$$

8.4 Riemann's Second Formula for $\xi(z)$.

$$\boxed{\xi(z) = 4 \int_{t=1}^{t=\infty} (t^{\frac{3}{2}}\psi'(t))' t^{-\frac{1}{4}} \cos\left(\tfrac{1}{2}[i(\tfrac{1}{2}-z)]\log t\right)dt}$$

Proof:

$$\xi(z) = \frac{1}{2} - \frac{z(1-z)}{2} \int_{t=1}^{t=\infty} \psi(t)\left(t^{-\frac{z+1}{2}} + t^{\frac{z}{2}-1}\right)dt$$

$$= \frac{1}{2} - \frac{z(1-z)}{2} \int_{t=1}^{t=\infty} \psi(t) d_t \left(\frac{t^{\frac{1-z}{2}}}{\frac{1-z}{2}} + \frac{t^{\frac{z}{2}}}{\frac{z}{2}} \right)$$

$$= \frac{1}{2} - \left[\frac{z(1-z)}{2} \left(\frac{t^{\frac{1-z}{2}}}{\frac{1-z}{2}} + \frac{t^{\frac{z}{2}}}{\frac{z}{2}} \right) \psi(t) \right]_{t=1}^{t=\infty} + \frac{z(1-z)}{2} \int_{t=1}^{t=\infty} \left(\frac{t^{\frac{1-z}{2}}}{\frac{1-z}{2}} + \frac{t^{\frac{z}{2}}}{\frac{z}{2}} \right) d\psi(t)$$

$$= \frac{1}{2} + \left(zt^{\frac{1-z}{2}} + (1-z)t^{\frac{z}{2}} \right) \psi(t) \Big|_{t=1} + \int_{t=1}^{t=\infty} \left(zt^{\frac{1-z}{2}} + (1-z)t^{\frac{z}{2}} \right) \psi'(t) dt$$

$$= \tfrac{1}{2} + \psi(1) + \int_{t=1}^{t=\infty} \left(zt^{-\frac{z}{2}-1} + (1-z)t^{\frac{z-1}{2}-1} \right) t^{\frac{3}{2}} \psi'(t) dt$$

$$= \tfrac{1}{2} + \psi(1) + \int_{t=1}^{t=\infty} t^{\frac{3}{2}} \psi'(t) d_t \left(-2t^{-\frac{z}{2}} - 2t^{\frac{z-1}{2}} \right)$$

$$= \frac{1}{2} + \psi(1) + \left[t^{\frac{3}{2}} \psi'(t)(-2) \left(t^{-\frac{z}{2}} + t^{\frac{z-1}{2}} \right) \right]_{t=1}^{t=\infty} - \int_{t=1}^{t=\infty} (-2) \left(t^{-\frac{z}{2}} + t^{\frac{z-1}{2}} \right) d\left(t^{\frac{3}{2}} \psi'(t) \right)$$

$$= \frac{1}{2} + \psi(1) + 4\psi'(1) + 2\int_{t=1}^{t=\infty} t^{-\frac{1}{4}} \left(t^{-i\frac{1}{2}i[\frac{1}{2}-z]} + t^{i\frac{1}{2}i[\frac{1}{2}-z]} \right) d\left(t^{\frac{3}{2}} \psi'(t) \right)$$

$$= \tfrac{1}{2} + \psi(1) + 4\psi'(1) + 2\int_{t=1}^{t=\infty} t^{-\frac{1}{4}} \left(e^{-i\frac{1}{2}\left(i[\frac{1}{2}-z]\right)\log t} + e^{i\frac{1}{2}\left(i[\frac{1}{2}-z]\right)\log t} \right) d\left(t^{\frac{3}{2}} \psi'(t) \right)$$

$$= \tfrac{1}{2} + \psi(1) + 4\psi'(1) + 4\int_{t=1}^{t=\infty} t^{-\frac{1}{4}} \cos\left(\tfrac{1}{2} i[\tfrac{1}{2} - z] \log t \right) d\left(t^{\frac{3}{2}} \psi'(t) \right)$$

To evaluate $\psi'(1)$, differentiate Poisson Functional Equation

$$\psi(t) = \psi(1 / t)t^{-1/2} + \frac{1}{2}t^{-1/2} - \frac{1}{2}.$$

$$\psi'(t) = \psi'(1 / t)(-1 / t^2)t^{-1/2} + \psi(1 / t)(-\frac{1}{2}t^{-3/2}) - \frac{1}{4}t^{-3/2}$$

$$\psi'(1) = \psi'(1)(-1) + \psi(1)(-\frac{1}{2}) - \frac{1}{4}$$

$$4\psi'(1) + \psi(1) + \frac{1}{2} = 0\,.$$

Thus,

$$\xi(z) = 4\int_{t=1}^{t=\infty} t^{-\frac{1}{4}} \cos\left(\frac{1}{2}i[\tfrac{1}{2} - z]\log t\right) d\left(t^{\frac{3}{2}}\psi'(t)\right)$$

$$= 4\int_{t=1}^{t=\infty} t^{-\frac{1}{4}} \cos\left(\tfrac{1}{2}i[\tfrac{1}{2} - z]\log t\right)(t^{\frac{3}{2}}\psi'(t))'\,dt\,.\square$$

9

The Function $\xi(\frac{1}{2}+iw)$

We use $\xi(\frac{1}{2}+iw) \equiv \xi(z)$ to get a correct factorization for ξ.

9.1 Formula for $\xi(\frac{1}{2}+iw)$

$$\boxed{\xi(\frac{1}{2}+iw) = 4\int_{t=1}^{t=\infty}\left(t^{\frac{3}{2}}\psi'(t)\right)' t^{-\frac{1}{4}}\cos(\frac{1}{2}w\log t)dt}$$

Riemann wrote

Set

$$z = \tfrac{1}{2}+iw$$

and

$$\tfrac{1}{2}z(z-1)\pi^{-\frac{z}{2}}\Gamma(\tfrac{z}{2})\zeta(z) = \xi(w)$$

Then,

$$\xi(w) = \tfrac{1}{2} - \left(\tfrac{1}{4}+w^2\right)\int_{t=1}^{t=\infty}\psi(t)t^{-\frac{3}{4}}\cos\left(\tfrac{1}{2}w\log t\right)dt$$

or also

$$\xi(w) = 4\int_{t=1}^{t=\infty}\frac{d(t^{\frac{3}{2}}\psi'(t))}{dt}t^{-\frac{1}{4}}\cos(\tfrac{1}{2}w\log t)dt\,.\Diamond$$

Proof:

Substituting

$$w = i(\tfrac{1}{2} - z) = y + i(\tfrac{1}{2} - x)$$

into Riemann's First Formula of 8.3

$$\xi(z) = \tfrac{1}{2} - z(1-z) \int_{t=1}^{t=\infty} \psi(t) t^{-\frac{3}{4}} \cos\left(\tfrac{1}{2}[i(\tfrac{1}{2} - z)] \log t\right) dt\,,$$

we obtain Riemann's First Formula here

$$\xi(\tfrac{1}{2} + iw) = \tfrac{1}{2} - \left(\tfrac{1}{4} + w^2\right) \int_{t=1}^{t=\infty} \psi(t) t^{-\frac{3}{4}} \cos\left(\tfrac{1}{2} w \log t\right) dt\,. \Box$$

Substituting w into Riemann's Second Formula of 8.4

$$\xi(z) = 4 \int_{t=1}^{t=\infty} \left(t^{\frac{3}{2}} \psi'(t)\right)' t^{-\frac{1}{4}} \cos\left(\tfrac{1}{2}[i(\tfrac{1}{2} - z)] \log t\right) dt\,,$$

we obtain

$$\xi(\tfrac{1}{2} + iw) = 4 \int_{t=1}^{t=\infty} \left(t^{\frac{3}{2}} \psi'(t)\right)' t^{-\frac{1}{4}} \cos(\tfrac{1}{2} w \log t) dt. \Box$$

9.2 Taylor Series for $\xi(\frac{1}{2} + iw)$

$$\boxed{\begin{aligned} \xi(\tfrac{1}{2} + iw) &= A_0 - A_1 w^2 + A_2 w^4 - \ldots \\ &= A_0 + A_1 (z - \tfrac{1}{2})^2 + A_2 (z - \tfrac{1}{2})^4 + \ldots \end{aligned}}$$

where $A_n > 0$.

Riemann wrote

> $\xi(w)$ *is finite for all finite values of* w*, and can be expanded into a very rapidly convergent series in powers of* w^2. $\Diamond$

Proof:

By the second formula for $\xi(w)$,

$$\xi(\tfrac{1}{2}+iw) = 4\int_{t=1}^{t=\infty}\left(t^{\frac{3}{2}}\psi'(t)\right)' t^{-\frac{1}{4}}\cos(\tfrac{1}{2}w\log t)dt$$

$$= 4\int_{t=1}^{t=\infty}\left(t^{\frac{3}{2}}\psi'(t)\right)' t^{-\frac{1}{4}}\left[1-\tfrac{1}{2!}(\tfrac{1}{2}\log t)^2 w^2+\tfrac{1}{4!}(\tfrac{1}{2}\log t)^4 w^4-...\right]dt$$

$$= A_0 - A_1 w^2 + A_2 w^4 - ...$$

where,

$$A_n = 4\frac{1}{(2n)!}\int_{t=1}^{t=\infty}\left(t^{\frac{3}{2}}\psi'(t)\right)' t^{-\frac{1}{4}}\left[\tfrac{1}{2}\log t\right]^{2n} dt. \square$$

To obtain $\xi(z)$, Substitute $w = i(\frac{1}{2}-z)$. Then,

$$\xi(z) = A_0 - A_1(i[z-\tfrac{1}{2}])^2 + A_2(i[z-\tfrac{1}{2}])^4 - A_3(i[z-\tfrac{1}{2}])^6 + ...$$

$$= A_0 + A_1(z-\tfrac{1}{2})^2 + A_2(z-\tfrac{1}{2})^4 + A_3(z-\tfrac{1}{2})^6 + ... \square$$

The $A_n > 0$ because

$$(t^{\frac{3}{2}}\psi'(t))' = \left(t^{\frac{3}{2}}\left(e^{-1^2\pi t} + e^{-2^2\pi t} + e^{-3^2\pi t} + ...\right)'\right)'$$

$$= \left(t^{\frac{3}{2}}(-\pi)\left(1^2 e^{-1^2\pi t} + 2^2 e^{-2^2\pi t} + 3^2 e^{-3^2\pi t} + ...\right)\right)'$$

$$= \tfrac{3}{2}t^{\frac{1}{2}}(-\pi)\left(1^2 e^{-1^2\pi t} + ...\right) + t^{\frac{3}{2}}\pi^2\left(1^4 e^{-1^2\pi t} + ...\right)$$

$$= \pi t^{\frac{1}{2}}\left(1^2 e^{-1^2\pi t}[1^2\pi t - \tfrac{3}{2}] + 2^2 e^{-2^2\pi t}[2^2\pi t - \tfrac{3}{2}] + ...\right) > 0\,.\square$$

Hadamard [Hada] proved that the rapid convergence is equivalent to the factorization

$$\xi(z) = \xi(0)\prod_{\rho = zero\ of\ \xi}\left(1 - \frac{z}{\rho}\right).$$

10

The Zeros of $\xi(z)$

10.1 The zeros of $\xi(z)$

Hadamard denoted the zeros of

$$\xi(z)$$

by

$$\rho = x_\rho + iy_\rho,$$

and

$$1 - \rho = 1 - x_\rho - iy_\rho.$$

Since $z = \frac{1}{2} + iw$, the corresponding zeros of

$$\xi(\tfrac{1}{2} + iw)$$

are

$$\alpha, \qquad \text{and} \qquad -\alpha.$$

We have

$$\boxed{\rho = \tfrac{1}{2} + i\alpha}$$

and

$$\boxed{1 - \rho = \tfrac{1}{2} - i\alpha}$$

We shall use

$$\boxed{-z(1-z) = z^2 - z = (z - \tfrac{1}{2})^2 - \tfrac{1}{4} = -(w^2 + \tfrac{1}{4})}$$

$$\boxed{\rho(1-\rho) = \alpha^2 + \tfrac{1}{4}}$$

10.2 All the zeros of $\xi(z)$ are in $0 < x < 1$

Riemann wrote

Since for $z = x + iy$, *with* $x > 1$

$$\log\zeta(z) = -\sum\log(1 - 1 / p^z)$$

is finite,

and since the same is true for the logarithms of the other factors of

$$\xi(w),$$

the function

$$\xi(w)$$

will vanish only if

$$-\tfrac{1}{2} < \mathrm{Im}(w) < \tfrac{1}{2}.\Diamond$$

Proof:

For $x > 1$, the Euler product

$$\prod_{p=prime}\frac{1}{1-\frac{1}{p^z}} = \zeta(z)$$

has no vanishing factor, and is non-zero.

Namely, for $x > 1$,

$$\zeta(z) \neq 0.$$

and

$$\xi(z) = \frac{1}{2}z(z-1)\pi^{-z/2}\Gamma(z / 2)\zeta(z) \neq 0$$

Thus, if ρ is a zero of $\xi(z)$

$$Re(\rho) < 1.$$

Since

$$\xi(z) = \xi(1-z),$$

we have

$$\xi(\rho) = 0 \Leftrightarrow \xi(1-\rho) = 0.$$

That is,

if ρ is a zero of $\xi(z)$ so is $1-\rho$

Hence,

$$1 > Re(1-\rho) = 1 - Re(\rho)$$

Therefore,

$$0 < Re(\rho)$$

Thus,

All the zeros of $\xi(z)$ are in $0 < x < 1$.

Namely, in

$$-\tfrac{1}{2} < x - \tfrac{1}{2} < \tfrac{1}{2}$$

or, since $x + iy = \frac{1}{2} + iw$,

$$-\tfrac{1}{2} < Im(w) < \tfrac{1}{2}.\square$$

10.3 For R large enough, $\log\left|\xi(z)\right| \leq R\log R$ in $\left|w\right| = \left|z - \frac{1}{2}\right| \leq 2R$

Proof:

By 9.2,

$$\xi(z) = A_0 + A_1(z - \tfrac{1}{2})^2 + A_2(z - \tfrac{1}{2})^4 + A_3(z - \tfrac{1}{2})^6 + ...$$

where $A_n > 0$.

Therefore, if $\left|z - \tfrac{1}{2}\right| \le 2R$, and R is large,

Then,

$$\left|z\right| \le 2R + \tfrac{1}{2} = 2n$$

and

$$\left|\xi(z)\right| \le \left|\xi(2n)\right|$$

$$= (2n-1)\pi^{-n}\Gamma(n+1)\zeta(2n)$$

$$\le 2n\pi^{-n}n!\zeta(2)$$

$$\le n^{n+1}$$

Hence, for large enough R

$$\left|\xi(z)\right| \le R^R$$

and

$$\log\left|\xi(z)\right| \le R\log R \,\square.$$

This enabled Hadamard to obtain his estimate for the number of zeros of $\xi(z)$.

10.4 Hadamard Estimate

$n(R)$= number of zeros of $\xi(z)$ in $\left|z - \tfrac{1}{2}\right| \le R$

If R is large enough, then $n(R) \le 2R\log R$

Proof:

By Jensen's Theorem [Ed, p. 40], applied to $\left|w\right| = \left|z - \frac{1}{2}\right| \le 2R$

$$\log\left|\xi(1/2)\frac{(2R)^{n(2R)}}{(\rho_1 - 1/2)\ldots(\rho_{n(2R)} - 1/2)}\right| = \frac{1}{2\pi}\int_{\theta=0}^{\theta=2\pi}\log\left|f(Re^{i\theta})\right|d\theta$$

Substitute

$$\log\left|\xi(Re^{i\theta})\right| \le R\log R$$

$$\left|\rho_k - 1/2\right| = \left|\alpha_k\right|$$

$$\log\left|\frac{(2R)^{n(2R)}}{\alpha_1\ldots\alpha_{n(2R)}}\right| = n(R)\log 2 + \log\left|\frac{R^{n(R)}}{\alpha_1\ldots\alpha_{n(R)}}\right| + \log\left|\frac{(2R)^{n(2R)-n(R)}}{\alpha_{n(R)+1}\ldots\alpha_{n(2R)}}\right|$$

Since for $k = 1\ldots n(R)$,

$$|\ \alpha_k\ | \le R$$

and

$$|\ \alpha_{n(R)+k}\ | \le 2R$$

we have

$$n(R)\log 2 \le \log\left|\frac{(2R)^{n(2R)}}{\alpha_1\ldots\alpha_{n(2R)}}\right|.$$

Therefore,

$$\log\left|\xi(1/2)\right| + n(R)\log 2 \le R\log R.$$

Hence, for R large enough,

$$n(R) \le 2R\log R.\square$$

Hadamard estimate is sufficient for the derivation of the factorization for $\xi(z)$.

10.5 Riemann Estimate:

$n(Y)$ = number of zeros of $\xi(z)$ in $0 < y < Y$, $0 < x < 1$.

$$n(Y) \leq \frac{Y}{2\pi}\log\frac{Y}{2\pi} - \frac{Y}{2\pi}.$$

Riemann wrote

Consider the counter-clockwise closed-path integral

$$\oint_{around\ (0,1)\times(0,Y)} d\log\xi(\frac{1}{2} + iw)$$

around the domain with

$$0 < x < 1, \quad \textit{and} \quad 0 < y < Y.$$

With relative error of the order of $1/Y$, *the integral is equal to*

$$iY\left(\log\frac{Y}{2\pi} - 1\right)$$

On the other hand, the integral equals

$$2\pi i \times n(Y). \Diamond$$

Proof:

In 1914, Backlund [Back] gave a proof.

10.6 The Hypothesis

All the zeros of $\xi(z)$ **are on** $x = \frac{1}{2}$

Equivalently, All the zeros of $\xi(\frac{1}{2} + iw)$ **have** $Im(w) = 0$**.**

Riemann wrote

It is very likely that all of the zeros of $\xi(w)$ *are real.*

One would like to have a rigorous proof of this, but after several fleeting attempts to no avail, I have temporarily set aside the search for this proof because it appeared to be unnecessary for the immediate purpose of my investigation. $\Diamond$

The Hypothesis is required in the application of the factorization to the count of the primes, and Riemann's erroneous factorization did not indicate the need for the Hypothesis.

While the correct factorization is needed for a derivation, the formula for the count of the primes may be arrived at empirically, and Riemann realized later that he needed the Hypothesis for a derivation, and tried to prove it.

After the publication of the 1859 paper, Riemann wrote

...The Theorem which I merely cited, that between 0*, and* Y*, there are around*

$$\frac{Y}{2\pi}\left(\log\frac{Y}{2\pi}-1\right)$$

real zeros of the function ξ*, follows from a new development of* ξ*, which I had not simplified enough to report it...* $\Diamond$

Apparently, Riemann had no proof for the Hypothesis.

11

Factorization of ξ

11.1 Weierstrass Factorization of $\xi(z)$

$$\xi(z) = e^{h(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\rho_n}\right)\left(1 - \frac{z}{1-\rho_n}\right) e^{Q_n(z)}$$

where

- the zeros of $\xi(z)$, the ρ_n's are sequenced by their size and increase to ∞
- the polynomials $Q_n(z)$ guarantee the uniform convergence of the product in the open plane
- $h(z)$ is an entire function

Proof:

Since $\xi(z)$ is an entire function so that $\xi(0) \neq 0$,

by Weierstrass [Sak, Chapter 7, 2.13]

$$\xi(z) = e^{h(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\rho_n}\right) e^{Q_n(z)}$$

Since the ρ_n's are sequenced by size, and since $1 - \rho_n$ is a zero too, the product representation is

$$\xi(z) = e^{h(z)} \prod_{n=1}^{\infty} \Big(1 - \frac{z}{\rho_n}\Big)\Big(1 - \frac{z}{1-\rho_n}\Big) e^{Q_n(z)}. \square$$

11.2 $$\xi(z) = e^{h(z)} \prod_{n=1}^{\infty} \Big(1 - \frac{z}{\rho_n}\Big)\Big(1 - \frac{z}{1-\rho_n}\Big).$$

Proof:

To obtain $e^{Q_n(z)} = 1$, we show that

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{\rho_n}\right)\left(1 - \frac{z}{1-\rho_n}\right) = \prod_n \left(1 - \frac{z(1-z)}{\rho_n(1-\rho_n)}\right)$$

converges.

Since the convergence of

$$\prod_m \left(1 - \frac{z(1-z)}{\rho_m(1-\rho_m)}\right)$$

is equivalent to the convergence of

$$\sum_m \frac{1}{\rho_m(1-\rho_m)},$$

and since

$$\left|\rho_m(1-\rho_m)\right| = \left|(\rho_m - 1/2)^2 - 1/4\right| > \left|\rho_m - 1/2\right|^2,$$

it is sufficient to show that

$$\sum_m \frac{1}{\left|\rho_m - 1/2\right|^2} = \sum_m \frac{1}{\left|\alpha_m^2\right|} < \infty.$$

Hence, we need to show that $\sum_{m>N} \frac{1}{\left|\alpha_m^2\right|} < \infty$.

The α_m are arranged so that

$$\frac{1}{\left|\alpha_m\right|^2}$$

is decreasing.

For m large enough,

$$m = N, N+1, N+2, \ldots$$

define positive numbers $R_m > 1$ so that

$$\log R_m > 1$$

and

$$m = 4R_m \log R_m .$$

Then,

$$\log m > \log R_m .$$

By Hadamard Estimate of 10.3, the number of zeros of $\xi(z)$ in $\left|w\right| \leq R_m$ is bounded by $2R_m \log R_m$.

Hence,

$$\left|\alpha_m\right| > R_m$$

and we have

$$\sum_{m>N} \frac{1}{\left|\alpha_m\right|^2} \leq \sum_{m>N} \frac{1}{R_m^2}$$

$$= 4^2 \sum_{m>N} \frac{1}{m^2} (\log R_m)^2$$

$$\leq 4^2 \sum_{m>N} \frac{\left(\log m\right)^2}{m^2}$$

$$= 4^2 \sum_{m>N} \frac{1}{m^{3/2}} \frac{\left(\log m\right)^2}{m^{1/2}}$$

Since $\dfrac{\left(\log m\right)^2}{m^{1/2}} \to 0$, as $m \to \infty$, we have

$$\frac{\left(\log m\right)^2}{m^{1/2}} < 1, \text{ for } m > N$$

and

$$\sum_{m>N} \frac{1}{\left|\alpha_m\right|^2} < \infty .$$

Therefore,

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{\rho_n}\right)\left(1 - \frac{z}{1-\rho_n}\right) \text{ converges,}$$

$$e^{Q_n(z)} = 1,$$

and

$$\xi(z) = e^{h(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\rho_n}\right)\left(1 - \frac{z}{1-\rho_n}\right) . \square$$

11.3 Factorization of $\xi(\frac{1}{2} + iw)$, and $\xi(z)$

$$\xi(\tfrac{1}{2} + iw) = \xi(0) \prod_{n=1}^{\infty} \left(1 - \frac{w^2 + \frac{1}{4}}{\alpha_n^2 + \frac{1}{4}}\right)$$

$$\xi(z) = \xi(0)\prod_n \left(1 - \frac{z - z^2}{\alpha_n^2 + \frac{1}{4}}\right)$$

Proof:

First, the factorization factors are

$$\left(1 - \frac{z}{\rho_n}\right)\left(1 - \frac{z}{1-\rho_n}\right) = 1 - \frac{z(1-z)}{\rho_n(1-\rho_n)} = 1 - \frac{w^2 + \frac{1}{4}}{\alpha_n^2 + \frac{1}{4}}.\Box$$

Second, we show that $e^{h(z)} = \xi(0)$.

The Hadamard Factorization Theorem [Holl, p. 68] applies to a function $f(z)$ for which

$$\limsup_{R\to\infty} \frac{1}{\log R} \log\log \max_{|z|=R} \left|f(z)\right| \equiv \varrho < \infty$$

ϱ is called **the order of** $f(z)$.

Hadamard replaced Weierstrass

$$h(z)$$

with a polynomial

$$Q(z)$$

so that

$$\deg Q(z) \le \varrho$$

By 10.2, if R is large enough, $\log\left|\xi(z)\right| \le R\log R$ in $\left|w\right| \le 2R$.

Hence,

$$\frac{1}{\log R}\log\log \max_{|w|=R}\left|\xi(z)\right| \le \frac{1}{\log R}\log\left(R\log R\right)$$

$$= 1 + \frac{\log \log R}{\log R}$$

Since by L'Hospital $\lim_{R \to \infty} \frac{\log \log R}{\log R} = \lim_{R \to \infty} \frac{\frac{1}{\log R}\frac{1}{R}}{\frac{1}{R}} = 0$,

$$\limsup_{R \to \infty} \frac{1}{\log R} \log \log \max_{|w|=R} \left| \xi(z) \right| = 1.$$

That is,

$$\xi(z) \text{ is of order } \varrho = 1$$

Hence,

$$\deg Q(z) \leq 1.$$

and

$$Q(w) = A + Bw$$

Therefore,

$$\xi(1 / 2 + iw) = e^{A+Bw} \prod_n \left(1 - \frac{w^2 + 1/4}{\alpha_n^2 + 1/4} \right),$$

where the product is an even function of w.

Indeed, by 9.2,

$$\xi(\tfrac{1}{2} + iw) = A_0 - A_1 w^2 + A_2 w^4 + \ldots.$$

is an even function of w.

Consequently,

$$B = 0,$$

and

$$\xi(\tfrac{1}{2} + iw) = e^A \prod_n \left(1 - \frac{w^2 + 1/4}{\alpha_n^2 + 1/4} \right).$$

Setting $w = -\frac{1}{2}i$

$$\xi(0) = e^{A},$$

and

$$\xi(\tfrac{1}{2} + iw) = \xi(0)\prod_{n}\left(1 - \frac{w^2 + 1/4}{\alpha_n^2 + 1/4}\right)$$

$$\xi(z) = \xi(0)\prod_{n}\left(1 - \frac{z}{\rho_n}\right)\left(1 - \frac{z}{1-\rho_n}\right)$$

$$= \xi(0)\prod_{n}\left(1 + \frac{z^2 - z}{\alpha_n^2 + \frac{1}{4}}\right).\square$$

11.4 Riemann's Factorization Error

Riemann wrote

Since the density of the zeros of size w increases like

$$\log\frac{w}{2\pi},$$

the series

$$\sum_{\alpha = zero \text{ of } \xi(w)} \log(1 - \tfrac{w^2}{\alpha^2})$$

converges, and grows like

$$|w|\log|w|.$$

$$\log\xi(w) - \sum_{\alpha = zero \text{ of } \xi(w)} \log(1 - \tfrac{w^2}{\alpha^2})$$

is a function that is continuous, and finite for finite w.

For $w \to \infty$,

$$\frac{1}{w^2}\left[\log\xi(w) - \sum_{\alpha = zero \text{ of } \xi(w)} \log(1 - \tfrac{w^2}{\alpha^2})\right] \to 0.$$

Therefore,

$$\log\xi(w) - \sum_{\alpha = zero \text{ of } \xi(w)} \log(1 - \tfrac{w^2}{\alpha^2}) = const.$$

Setting $w = 0$*, gives*

$$\log\xi(0) = const\,.\Diamond$$

Evidently, $w = 0 \Leftrightarrow z = \frac{1}{2}$, and Riemann should have

$$\log\xi(\tfrac{1}{2}) = const\,,$$

as observed by Genocchi in 1860. Riemann's derivation does not obtain the correct coefficient of the factorization which is $\xi(0)$.

Now, exponentiating both sides, Riemann's factorization is

$$\xi(w) = \xi(0)\prod_{\alpha = zero \text{ of } \xi(w)}\left(1 - \frac{w^2}{\alpha^2}\right)$$

Riemann's factor, $1 - \dfrac{w^2}{\alpha^2}$, is irreconcilable with the correct factor

$$1 - \frac{w^2 + 1/4}{\alpha^2 + 1/4}.\Box$$

12.

The Number of Primes < t

12.1 $\pi(t)$, and $F(t)$.

Riemann wrote

...We can now determine $\pi(t)$, the number of primes less than t.

Let

$$F(t)$$

be equal to

$$\pi(t) \textit{ if } t \textit{ is not a prime,}$$

and to

$$\pi(t) + \frac{1}{2}, \textit{ if } t \textit{ is a prime.}$$

so that if $F(t)$ jumps at t,

$$F(t) = \frac{1}{2}[F(t+0) + F(t-0].\Diamond$$

We define

$$\pi(t) \equiv \text{number of prime numbers } p \text{ so that } p < t.$$

Thus,

$$\pi(1) = 0$$

$$\pi(\sqrt{2}) = 0$$

$$\pi(2) = 0$$

$$\pi(e) = 1$$

$$\pi(3) = 1$$

$$\pi(\pi) = 2$$

To apply Fourier Integral Theorem, we need the auxiliary function

$$F(t) \equiv \begin{cases} \pi(t) & if \quad t \neq prime \\ \pi(t) + 1/2 & if \quad t = prime \end{cases}$$

that satisfies the Dirichlet condition

$$\frac{1}{2}[F(t+0) + F(t-0] = F(t)$$

Thus,

t	$\pi(t)$	$F(t)$
$t < 2$	0	0
$t = 2$	0	$1/2$
$2 < t < 3$	1	1
$t = 3$	1	$1 + 1/2$
$3 < t < 5$	2	2
$t = 5$	2	$2 + 1/2$

13

Zeta in terms of $f(t)$

13.1 Definition of $f(t)$

$$\boxed{f(t) \equiv F(t) + \frac{1}{2}F(t^{1/2}) + \frac{1}{3}F(t^{1/3}) + \ldots}$$

13.2 $\zeta(z)$ in terms of $f(t)$

$$\boxed{\frac{\log\zeta(z)}{z} = \int_{t=1}^{t=\infty} \frac{1}{t^{z+1}} f(t)dt}$$

Riemann wrote

If $x > 1$,

$$\log\zeta(z) = -\sum_{p=prime} \log\left(1 - \frac{1}{p^z}\right)$$

$$= \sum_{p=prime} \frac{1}{p^z} + \frac{1}{2}\sum_{p=prime} \frac{1}{p^{2z}} + \frac{1}{3}\sum_{p=prime} \frac{1}{p^{3z}} + \ldots$$

Substitute

$$\frac{1}{p^z} = z\int_{t=p}^{t=\infty}\frac{dt}{t^{z+1}}$$

$$\frac{1}{p^{2z}} = z\int_{t=p^2}^{t=\infty}\frac{dt}{t^{z+1}}$$

……………………….

Then,

$$\frac{\log\zeta(z)}{z} = \int_{t=1}^{t=\infty}\frac{1}{t^{z+1}}f(t)dt$$

where

$$f(t) = F(t) + \frac{1}{2}F(t^{1/2}) + \frac{1}{3}F(t^{1/3}) + \ldots \Diamond$$

Proof:

For $x > 1$,

$$\zeta(z) = \frac{1}{1-\frac{1}{2^z}}\frac{1}{1-\frac{1}{3^z}}\frac{1}{1-\frac{1}{5^z}}\ldots.$$

$$\log\zeta(z) = -\log\left(1-\frac{1}{2^z}\right) - \log\left(1-\frac{1}{3^z}\right) - \log\left(1-\frac{1}{5^z}\right) - \ldots.$$

$$= \frac{1}{2^z} + \frac{1}{2}\frac{1}{2^{2z}} + \frac{1}{3}\frac{1}{2^{3z}} + \ldots.$$

$$+\frac{1}{3^z} + \frac{1}{2}\frac{1}{3^{2z}} + \frac{1}{3}\frac{1}{3^{3z}} + \ldots.$$

$$+\frac{1}{5^z} + \frac{1}{2}\frac{1}{5^{2z}} + \frac{1}{3}\frac{1}{5^{3z}} + \ldots.$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{5^z} + ...$$

$$+\frac{1}{2}\left(\frac{1}{2^{2z}} + \frac{1}{3^{2z}} + \frac{1}{5^{2z}} + ...\right)$$

$$+\frac{1}{3}\left(\frac{1}{2^{3z}} + \frac{1}{3^{3z}} + \frac{1}{5^{3z}} + ...\right)$$

$$+ ....$$

Since

$$dF(t) = F(t+) - F(t-) \equiv \begin{cases} 0, & if \quad t \neq prime \\ 1, & if \quad t = prime \end{cases}$$

The Series

$$\frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{5^z} + ...$$

is the Riemann-Stieltjes integral $\int\limits_{t=1}^{t=\infty} \frac{1}{t^z} dF(t)$

and we have

$$\frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{5^z} + ... = \int\limits_{t=1}^{t=\infty} \frac{1}{t^z} dF(t).$$

Similarly,

$$\frac{1}{2^{2z}} + \frac{1}{3^{2z}} + \frac{1}{5^{2z}} + ... = \int\limits_{u=1}^{u=\infty} \frac{1}{u^{2z}} dF(u)$$

$$= \int\limits_{t=1}^{t=\infty} \frac{1}{t^z} dF(t^{\frac{1}{2}})$$

and

$$\frac{1}{2^{3z}}+\frac{1}{3^{3z}}+\frac{1}{5^{3z}}+....=\int_{t=1}^{t=\infty}\frac{1}{t^z}dF(t^{\frac{1}{3}})$$

……………………………………….

Therefore,

$$\log\zeta(z)=\int_{t=1}^{t=\infty}\frac{1}{t^z}dF(t)+\frac{1}{2}\int_{t=1}^{t=\infty}\frac{1}{t^z}dF(t^{1/2})+\frac{1}{3}\int_{t=1}^{t=\infty}\frac{1}{t^z}dF(t^{1/3})+...$$

Integrating by parts,

$$\int_{t=1}^{t=\infty}\frac{1}{t^z}d_tF(t^{1/n})=\left[\frac{1}{t^z}F(t^{1/n})\right]_{t=1}^{t=\infty}-\int_{t=1}^{t=\infty}F(t^{1/n})(-z)t^{-z-1}dt$$

Since $F(1)=0$, and $\frac{1}{t^z}\Big|_{t=\infty}=0$,

$$=z\int_{t=1}^{t=\infty}F(t^{1/n})\frac{1}{t^{z+1}}dt.$$

Consequently,

$$\log\zeta(z)=z\int_{t=1}^{t=\infty}F(t)\frac{1}{t^{z+1}}dt+\frac{1}{2}z\int_{t=1}^{t=\infty}F(t^{1/2})\frac{1}{t^{z+1}}dt+...$$

That is,

$$\frac{\log\zeta(z)}{z}=\int_{t=1}^{t=\infty}\left(F(t)+\frac{1}{2}F(t^{1/2})+\frac{1}{3}F(t^{1/3})+...\right)\frac{dt}{t^{z+1}}.\Box$$

14.

$f(t)$ in terms of Zeta

14.1 $f(t)$ in terms of $\zeta(z)$

$$\boxed{f(t) = \frac{1}{2\pi i} \int_{y=-\infty}^{y=\infty} \frac{\log \zeta(z)}{z} t^z dz},$$

Riemann wrote,

If for $x > 1$*,*

$$g(z) = \int_{u=0}^{u=\infty} h(u) u^{-z} d(\log t),$$

then by the Fourier Theorem

$$h$$

can be written in terms of

$$g(z).$$

If

$h(u)$ *is real*

$g(x + iy) = g_1(y) + g_2(y),$

then the equation splits into

$$g_1(y) = \int_{u=0}^{u=\infty} h(u)u^{-x}\cos(y\log u)d(\log u),$$

and

$$g_2(y) = -i\int_{u=0}^{u=\infty} h(u)u^{-x}\sin(y\log u)d(\log u).$$

Multiply both equations by

$$[\cos(y\log t) + i\sin(y\log t)]dy,$$

and integrate from $y = -\infty$ *to* $y = \infty$.

Then, the right hand side of either equation is

$$\pi h(t)t^{-x}.$$

Adding the equations, and multiplying by it^x,

$$2\pi i h(t) = \int_{x-i\infty}^{x+i\infty} g(z)t^z dz,$$

where x *is fixed through the integration.*

Thus, if $h(t)$ *has a jump at* t,

then,

$$h(t) = \frac{1}{2}[h(t+0) + h(t-0)].$$

Since $f(t)$ *has the same property, we get with complete generality*

$$f(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} \frac{\log\zeta(z)}{z}t^z dz. \Diamond$$

Proof:

By 13.1,

$$\frac{\log \zeta(z)}{z} = \int_{u=0}^{u=\infty} \frac{1}{u^{z+1}} f(u) du\,.$$

For $0 \le u \le 1$ we have

$$F(u) = 0\,,$$

and

$$f(u) = 0\,.$$

Hence,

$$\frac{\log \zeta(z)}{z} = \int_{u=1}^{u=\infty} \frac{1}{u^{z+1}} f(u) du$$

$$= \int_{u=0}^{u=\infty} f(u) u^{-z} d(\log u)\,.$$

$$= \int_{u=0}^{u=\infty} f(u) u^{-x} e^{-iy\log u} d(\log u)$$

We apply Fourier Inversion to write f in terms of $\dfrac{\log \zeta(z)}{z}$.

To that end, fix x, multiply both sides of the equation above by

$$t^z dz = t^x t^{iy} d(x + iy) = it^x e^{iy\log t} dy$$

and integrate from $y = -\infty$ to $y = \infty$.

Then,

$$\int_{y=-\infty}^{y=\infty} \frac{\log\zeta(z)}{z} t^z dz = it^x \int_{y=-\infty}^{y=\infty} e^{iy\log t}\left(\int_{u=0}^{u=\infty} f(u)u^{-x}e^{-iy\log u}d(\log u)\right)dy\,.$$

Since

$$F(t) = \frac{1}{2}[F(t+0)+F(t-0)]$$

we have

$$f(t) = \frac{1}{2}[f(t+0)+f(t-0)]$$

and by Fourier integral Theorem, we can change the order of integration.

$$= it^x \int_{u=0}^{u=\infty} f(u)u^{-x}\left(\int_{y=-\infty}^{y=\infty} e^{iy(\log t-\log u)}dy\right)d(\log u)$$

$$= it^x \int_{u=0}^{u=\infty} f(u)u^{-x}\left[2\pi\delta(\log t-\log u)\right]d(\log u)$$

$$= 2\pi i f(t)$$

Therefore,

$$f(t) = \frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty} \frac{\log\zeta(z)}{z} t^z dz$$

$$= \frac{1}{2\pi}\int_{y=-\infty}^{y=\infty} \frac{\log\zeta(z)}{z} t^z dy\,,$$

for fixed x.□

15

$f(t)$ in terms of $\xi(z)$

15.1 $f(t)$ in terms of $\xi(z)$

$$\boxed{\begin{aligned} f(t) = \frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty} \frac{t^z}{\log t}\, d\Bigg\{ &\frac{\log(z-1)}{z} + \frac{\log\Gamma\big(z/2+1\big)}{z} \\ &- \frac{\log\xi(0)}{z} - \frac{1}{z}\sum_{\alpha}\log\left(1+\frac{z^2-z}{\alpha^2+1/4}\right)\Bigg\} \end{aligned}}$$

Riemann wrote

$$f(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} \frac{\log\zeta(z)}{z} t^z dz.$$

For

$$\log\zeta(z)$$

we may substitute

$$\frac{1}{2} z\log\pi - \log(z-1) - \log\Gamma\left(\frac{1}{2}z+1\right)$$

$$+\sum_{\alpha=zero\ of\ \xi}\log\left(1+\frac{\big(z-1/2\big)^2}{\alpha^2}\right) + \log\xi(0)$$

But the integrals of these terms are divergent at infinity.

So we have to integrate the equation for f by parts

$$f(t) = -\frac{1}{2\pi i}\frac{1}{\log t}\int_{x-i\infty}^{x+i\infty}\frac{d\left[\frac{1}{z}\log\zeta(z)\right]}{dz}t^z dz\,.\Diamond$$

Proof:

Since

$$\xi(z) = \frac{1}{2}z(z-1)\pi^{-z/2}\Gamma(z\,/\,2)\zeta(z)\,,$$

we have

$$\log\zeta(z) = \frac{1}{2}z\log\pi - \log(z-1) - \log\Gamma\left(\frac{1}{2}z+1\right) + \log\xi(z).$$

Replacing $\xi(z)$ by its factorization of 11.3, we have

$$\frac{\log\zeta(z)}{z} = \tfrac{1}{2}\log\pi - \frac{\log(z-1)}{z} - \frac{\log\Gamma\left(\frac{1}{2}z+1\right)}{z} +$$

$$+\frac{1}{z}\sum_{\alpha}\log\left(1+\frac{z^2-z}{\alpha^2+1\,/\,4}\right) + \frac{\log\xi(0)}{z}$$

If we substitute this expression for $\dfrac{\log\zeta(z)}{z}$ into 14.1 to obtain $f(t)$, the first term,

$$\tfrac{1}{2}\log\pi\,,$$

gives

$$\frac{\frac{1}{2}\log\pi}{2\pi i}\int_{x-i\infty}^{x+i\infty}t^z dz = \frac{\log\pi}{4\pi i\log t}t^x\left[t^{iy}\right]_{y=-\infty}^{y=\infty}$$

that diverges.

On the other hand, integrating by parts gives

$$f(t) = \frac{1}{2\pi i}\frac{1}{\log t}\lim_{Y\to\infty}\left[\frac{\log\zeta(z)}{z}t^z\right]_{y=-Y}^{y=Y} - \frac{1}{2\pi i}\frac{1}{\log t}\int_{x-i\infty}^{x+i\infty} t^z d\left(\frac{\log\zeta(z)}{z}\right)$$

Now,

$$\left|\log\zeta(z)\right| = \left|\log\prod_{p=prime}\frac{1}{1-\frac{1}{p^z}}\right|$$

$$= \left|\sum_{p=prime}\log\left(1-\frac{1}{p^z}\right)\right|$$

$$\leq \sum_{p=prime}\left|\log\left(1-\frac{1}{p^z}\right)\right|$$

$$= \sum_{p=prime}\left|\frac{1}{p^z}+\frac{1}{2}\frac{1}{p^{2z}}+\frac{1}{3}\frac{1}{p^{3z}}+...\right|$$

$$\leq \sum_{p=prime}\left(\frac{1}{\left|p^z\right|}+\frac{1}{2}\frac{1}{\left|p^{2z}\right|}+\frac{1}{3}\frac{1}{\left|p^{3z}\right|}+...\right)$$

$$= \sum_{p=prime}\left(\frac{1}{p^x}+\frac{1}{2}\frac{1}{p^{2x}}+\frac{1}{3}\frac{1}{p^{3x}}+...\right)$$

$$= \sum_{p=prime}-\log\left(1-\frac{1}{p^x}\right)$$

$$= -\sum_{p=prime}\log\left(1-\frac{1}{p^x}\right)$$

$$= -\log\prod_{p=prime}\frac{1}{1-\frac{1}{p^x}}$$

$$= -\log \zeta(x)$$

$$\leq \left| \log \zeta(x) \right|$$

Thus, $\left| \log \zeta(z) \right|$ is bounded independent of y, and the boundary term vanishes as $Y \to \infty$. We have

$$\left| \left[\frac{\log \zeta(z)}{z} t^z \right]_{y=-Y}^{y=Y} \right| \leq 2 \frac{t^x \left| \log \zeta(x) \right|}{\sqrt{x^2 + Y^2}} \to 0, \text{ as } \quad Y \to \infty.$$

Therefore,

$$f(t) = -\frac{1}{2\pi i} \frac{1}{\log t} \int_{x-i\infty}^{x+i\infty} t^z d\left(\frac{\log \zeta(z)}{z} \right)$$

$$= \frac{1}{2\pi i} \int_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} \times$$

$$\times d\left(\frac{\log(z-1)}{z} + \frac{\log \Gamma\left(\frac{1}{2} z + 1\right)}{z} - \frac{\log \xi(0)}{z} - \frac{1}{z} \sum_{\alpha} \log\left(1 + \frac{z^2 - z}{\alpha^2 + 1/4} \right) \right). \square$$

16

The $-\dfrac{\log \xi(0)}{z}$ term

16.1 $$\boxed{-\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d\left(\frac{1}{z}\right)=\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{1}{z}t^z dz=1}$$

Proof:

Integration by parts gives

$$-\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d\left(\frac{1}{z}\right)=-\frac{1}{2\pi i}\frac{1}{\log t}\lim_{Y\to\infty}\left[\frac{t^z}{z}\right]_{y=-Y}^{y=Y}+\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{1}{z}t^z dz$$

The boundary term vanishes because

$$\left|\left[\frac{1}{z}t^z\right]_{y=-Y}^{y=Y}\right|\leq 2\frac{t^x}{\sqrt{x^2+Y^2}}\to 0, \quad \text{as } Y\to\infty.$$

Substituting in the integral

$$dz = idy,$$

$$t^z = t^x e^{iy\log t},$$

$$\frac{1}{z}=\int_{u=1}^{u=\infty}u^{-z-1}du,$$

The integral becomes

$$= t^x \frac{1}{2\pi} \int_{y=-\infty}^{y=\infty} \left[\int_{u=1}^{u=\infty} u^{-z-1} du \right] e^{iy\log t} dy$$

By the change of variable

$$u = e^{\omega}$$

$$du = e^{\omega} d\omega$$

we have

$$= t^x \frac{1}{2\pi} \int_{y=-\infty}^{y=\infty} \left[\int_{\omega=0}^{\omega=\infty} e^{-\omega z} d\omega \right] e^{iy\log t} dy$$

$$= t^x \frac{1}{2\pi} \int_{y=-\infty}^{y=\infty} \left[\int_{\omega=0}^{\omega=\infty} e^{-\omega x} e^{-i\omega y} d\omega \right] e^{iy\log t} dy$$

By Fourier Integral Theorem for $\begin{cases} e^{-\omega x}, & \omega > 0 \\ 0, & \omega < 0 \end{cases}$, we change integration order, and obtain

$$= t^x \int_{\omega=0}^{\omega=\infty} e^{-\omega x} \left[\frac{1}{2\pi} \int_{y=-\infty}^{y=\infty} e^{iy(\log t - \omega)} dy \right] d\omega$$

$$= t^x \int_{\omega=0}^{\omega=\infty} e^{-\omega x} \delta(\omega - \log t) d\omega$$

$$= t^x e^{-x\log t}$$

$$= 1. \square$$

16.2 $$\boxed{\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d\left(-\frac{\log\xi(0)}{z}\right)=\log\xi(0)=-\log 2}$$

Proof:

$$\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d\left(-\frac{\log\xi(0)}{z}\right)=\log\xi(0)\underbrace{\left(-\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d\left(\frac{1}{z}\right)\right)}_{=1,\ by\ 16.1}$$

$$=\log\xi(0)$$

By 8.2, $\xi(0)=\frac{1}{2}$,

$$=\log\tfrac{1}{2}$$

$$=-\log 2\,.\square$$

17

Terms with $\log\left(1-\dfrac{z}{\beta}\right)$

17.1 $$-d\left(\tfrac{1}{z}\log\Gamma\left(\tfrac{1}{2}z+1\right)\right)=\sum_{n=1}^{\infty}d\left(\tfrac{1}{z}\log\left(1+\tfrac{1}{2n}z\right)\right)$$

Proof:

For the Gamma function

$$\Gamma\left(\tfrac{1}{2}z+1\right)=\lim_{N\to\infty}\frac{N^{\frac{1}{2}z}}{\prod_{n=1}^{\infty}\left(1+\tfrac{1}{2n}z\right)}$$

Hence,

$$-\log\Gamma\left(\tfrac{1}{2}z+1\right)=\lim_{N\to\infty}\left(\sum_{n=1}^{N}\log\left(1+\tfrac{1}{2n}z\right)-\tfrac{1}{2}z\log N\right).$$

Dividing both sides by z, and differentiating,

$$-\frac{d}{dz}\left(\frac{1}{z}\log\Gamma\left(\tfrac{1}{2}z+1\right)\right)=\frac{d}{dz}\lim_{N\to\infty}\left(\sum_{n=1}^{N}\tfrac{1}{z}\log\left(1+\tfrac{1}{2n}z\right)-\tfrac{1}{2}\log N\right)$$

By the uniform convergence of the series,

$$= \lim_{N\to\infty} \frac{d}{dz}\left(\sum_{n=1}^{N} \tfrac{1}{z}\log\left(1+\tfrac{1}{2n}z\right) - \tfrac{1}{2}\log N \right)$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{d}{dz}\left(\tfrac{1}{z}\log\left(1+\tfrac{z}{2n}\right)\right)$$

$$= \sum_{n=1}^{\infty} \frac{d}{dz}\left(\tfrac{1}{z}\log\left(1+\tfrac{z}{2n}\right)\right).$$

Therefore,

$$-d\left(\frac{1}{z}\log\Gamma\left(\frac{1}{2}z+1\right)\right) = \sum_{n=1}^{\infty} d\left(\frac{1}{z}\log\left(1+\frac{z}{2n}\right)\right).\Box$$

17.2

Let $\boxed{\beta = \sigma + i\tau}$ **be a constant. Then**

$$\boxed{\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right] = \begin{cases} \displaystyle\int_{u=\infty}^{u=t} \frac{u^{\beta-1}}{\log u} du + const, \text{ if } \sigma < 0 \\ \displaystyle\int_{u=0}^{u=t} \frac{u^{\beta-1}}{\log u} du\$ + const, \text{ if } \sigma > 0 \end{cases}}$$

Riemann wrote

consider

$$\pm\frac{1}{2\pi i}\frac{1}{\log t}\int_{x-i\infty}^{x+i\infty}\frac{d}{dz}\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right]t^z dz$$

Now,

$$\frac{d}{d\beta}\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right]=\frac{1}{(\beta-z)\beta}$$

If $x>\sigma$*,*

$$-\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}\frac{1}{(\beta-z)\beta}t^z dz=\frac{t^\beta}{\beta}=\begin{cases}\displaystyle\int_{u=\infty}^{u=t}u^{\beta-1}du, & \sigma<0\\ \\ \displaystyle\int_{u=0}^{u=t}u^{\beta-1}du, & \sigma>0\end{cases}$$

Therefore,

$$\frac{1}{2\pi i}\frac{1}{\log t}\int_{x-i\infty}^{x+i\infty}\frac{d}{dz}\left(\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right)t^z dz=$$

$$=-\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)t^z dz$$

$$=\begin{cases}\displaystyle\int_{u=\infty}^{u=t}\frac{u^{\beta-1}}{\log u}du+const, & \sigma<0\\ \\ \displaystyle\int_{u=0}^{u=t}\frac{u^{\beta-1}}{\log u}du+const, & \sigma>0\end{cases}.\Diamond$$

Proof:

Integration by parts gives

$$\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right]=$$

$$=\frac{1}{2\pi i\log t}\lim_{Y\to\infty}\left[\frac{t^z}{z}\log\left(1-\frac{z}{\beta}\right)\right]_{y=-Y}^{y=Y}-\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{z}\log\left(1-\frac{z}{\beta}\right)dz$$

We first show that the boundary term vanishes. Since

$$\left|\log\left(1-\frac{z}{\beta}\right)\right|=\left|\frac{z}{\beta}+\frac{1}{2}\left(\frac{z}{\beta}\right)^2+\frac{1}{3}\left(\frac{z}{\beta}\right)^3+\ldots\right|$$

$$\leq\left|\frac{z}{\beta}\right|+\frac{1}{2}\left|\frac{z}{\beta}\right|^2+\frac{1}{3}\left|\frac{z}{\beta}\right|^3+\ldots$$

$$=\left|\log\left(1-\left|\frac{z}{\beta}\right|\right)\right|,$$

we have

$$\left|\left[\frac{t^z}{z}\log\left(1-\frac{z}{\beta}\right)\right]_{y=-Y}^{y=Y}\right|\leq\frac{2t^x}{\sqrt{x^2+Y^2}}\left|\log\left(1-\frac{\sqrt{x^2+Y^2}}{|\beta|}\right)\right|$$

and by L'Hospital,

$$\to 0\quad\text{as}\quad Y\to\infty.$$

Therefore, the boundary term vanishes, and

$$\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right]=-\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)t^z dz$$

Substituting

$$dz=idy$$

$$t^z=t^x e^{iy\log t}$$

$$\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)=\int\frac{1}{\beta(\beta-z)}d\beta\,,$$

we have

$$=-t^{x}\frac{1}{2\pi}\int_{y=-\infty}^{y=\infty}\left(\int_{\beta}\frac{1}{\beta(\beta-z)}d\beta\right)e^{iy\log t}dy\,.$$

$$=t^{x}\int_{\beta}\left(\frac{1}{2\pi}\int_{y=-\infty}^{y=\infty}\frac{1}{z-\beta}e^{iy\log t}dy\right)\frac{1}{\beta}d\beta$$

$$=t^{x}\int_{\beta}\left[\frac{1}{2\pi}\int_{y=-\infty}^{y=\infty}\left(\int_{s=1}^{s=\infty}s^{-(z-\beta)-1}ds\right)e^{iy\log t}dy\right]\frac{1}{\beta}d\beta$$

By the change of variable

$$s=e^{\omega}$$

$$ds=e^{\omega}d\omega\,,$$

$$=t^{x}\int_{\beta}\left[\frac{1}{2\pi}\int_{y=-\infty}^{y=\infty}\left(\int_{\omega=0}^{\omega=\infty}e^{-\omega(z-\beta)}d\omega\right)e^{iy\log t}dy\right]\frac{1}{\beta}d\beta$$

$$=t^{x}\int_{\beta}\left[\frac{1}{2\pi}\int_{y=-\infty}^{y=\infty}\left(\int_{\omega=0}^{\omega=\infty}e^{-\omega(x-\sigma)}e^{-i\omega(y-\tau)}d\omega\right)e^{iy\log t}dy\right]\frac{1}{\beta}d\beta$$

If $x>\sigma$, by Fourier Integral Theorem for $\begin{cases}e^{-\omega(x-\sigma)}, & \omega>0\\ 0, & \omega<0\end{cases}$,

we change integration order

$$= t^x \int_\beta \left[\int_{\omega=0}^{\omega=\infty} e^{-\omega(x-\sigma)} e^{i\omega\tau} \underbrace{\left(\frac{1}{2\pi} \int_{y=-\infty}^{y=\infty} e^{iy(\log t - \omega)} dy \right)}_{\delta(\omega - \log t)} d\omega \right] \frac{1}{\beta} d\beta$$

$$= t^x \int_\beta \left[\int_{\omega=0}^{\omega=\infty} e^{-\omega(x-\sigma)} e^{i\omega\tau} \delta(\omega - \log t) d\omega \right] \frac{1}{\beta} d\beta$$

$$= t^x \int_\beta e^{-(x-\sigma)\log t} e^{i\tau \log t} \frac{1}{\beta} d\beta$$

$$= t^x \underbrace{e^{-x\log t}}_{t^{-x}} \int_\beta \underbrace{e^{(\sigma + i\tau)\log t}}_{t^\beta} \frac{1}{\beta} d\beta$$

$$= \int_\beta \frac{1}{\beta} t^\beta d\beta$$

$$= \begin{cases} \displaystyle\int_\beta \left(\int_{u=\infty}^{u=t} u^{\beta-1} du \right) d\beta & \sigma < 0 \\ \displaystyle\int_\beta \left(\int_{u=0}^{u=t} u^{\beta-1} du \right) d\beta & \sigma > 0 \end{cases} = \begin{cases} \displaystyle\int_{u=\infty}^{u=t} \left[\int_\beta u^{\beta-1} d\beta \right] du, & \sigma < 0 \\ \displaystyle\int_{u=0}^{u=t} \left[\int_\beta u^{\beta-1} d\beta \right] du, & \sigma > 0 \end{cases}$$

$$= \begin{cases} \displaystyle\int_{u=\infty}^{u=t} \frac{u^{\beta-1}}{\log u} du + const, & \sigma < 0 \\ \displaystyle\int_{u=0}^{u=t} \frac{u^{\beta-1}}{\log u} du + const, & \sigma > 0 \end{cases}$$

17.3 If $\sigma < 0$,

$$\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right]=\int_{u=\infty}^{u=t}\frac{u^{\beta-1}}{\log u}du$$

Riemann wrote

If $\sigma < 0$*,*

$$\frac{1}{2\pi i}\frac{1}{\log t}\int_{x-i\infty}^{x+i\infty}\frac{d}{dz}\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right]t^z dz=\int_{u=\infty}^{u=t}\frac{u^{\beta-1}}{\log u}du+const$$

The constant of integration drops out by letting

$$\beta\to-\infty.\Diamond$$

Proof:

By 17.2, if $\sigma < 0$,

$$\frac{1}{2\pi i}\frac{1}{\log t}\int_{x-i\infty}^{x+i\infty}\frac{d}{dz}\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right]t^z dz=\int_{u=\infty}^{u=t}\frac{u^{\beta-1}}{\log u}du+const$$

To establish that the constant vanishes, we'll show that both integrals vanish as $\sigma\to-\infty$.

As in 17.2, integration by parts of the left-side-integral gives

$$\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right]=-\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)t^z dz$$

The integral is bounded by

$$\left| -\frac{1}{2\pi i} \int_{y=-\infty}^{y=\infty} \frac{1}{z} \log\left(1 - \frac{z}{\beta}\right) t^z dz \right| \leq \frac{t^x}{2\pi} \int_{y=-\infty}^{y=\infty} \left| \frac{1}{z} \log\left(1 - \left|\frac{z}{\beta}\right|\right) \right| dy\,.$$

If we let $\sigma \to -\infty$,

$$\left| \frac{1}{z} \log\left(1 - \left|\frac{z}{\beta}\right|\right) \right| \to \left| \frac{1}{z} \log 1 \right| = 0\,.$$

Therefore, by Lebesgue Dominant Convergence,

$$\frac{1}{2\pi i} \frac{1}{\log t} \int_{y=-\infty}^{y=\infty} t^z d_z \left[\frac{1}{z} \log\left(1 - \frac{z}{\beta}\right) \right] \to 0\,, \quad \text{as } \sigma \to -\infty\,.$$

For the right-side-integral, $\log u \neq 0$, in $t < u < \infty$, for $t > 1$. And

$$\left| \int_{u=\infty}^{u=t} \frac{u^{\beta-1}}{\log u} du \right| \leq \int_{u=\infty}^{u=t} \left| \frac{u^{\beta-1}}{\log u} \right| du\,.$$

If we let $\sigma \to -\infty$,

$$\left| \frac{u^{\beta-1}}{\log u} \right| = \frac{|u|^{\sigma-1}}{|\log u|} \to \frac{|u|^{-\infty}}{|\log u|} = 0\,,$$

Therefore, by Lebesgue Dominant Convergence,

$$\int_{u=\infty}^{u=t} \frac{u^{\beta-1}}{\log u} du \to 0\,, \text{ as } \quad \sigma \to -\infty\,.$$

Consequently,

$$const = 0\,.$$

and

$$\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right]=\int_{u=\infty}^{u=t}\frac{u^{\beta-1}}{\log u}du\,.\square$$

17.4 If $\sigma > 0$,

$$\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right]=\int_{u=0}^{u=t}\frac{u^{\beta-1}}{\log u}du$$

Riemann wrote

If $\sigma > 0$*,*

$$\frac{1}{2\pi i}\frac{1}{\log t}\int_{x-i\infty}^{x+i\infty}\frac{d}{dz}\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right]t^z dz=\int_{u=0}^{u=t}\frac{u^{\beta-1}}{\log u}du+const.$$

The integral from $u=0$ *to* $u=t$ *will be infinitesimal, if the path of integration is in the upper half-plane, and we let*

$$\tau\to\infty,$$

or if the path of integration is in the lower half-plane, and we let

$$\tau\to-\infty.$$

Then, we evaluate

$$\log\left(1-\frac{z}{\beta}\right)$$

on the left side so that the integration constant drops out. $\Diamond$

Proof:

By 17.2, if $\sigma > 0$,

$$\frac{1}{2\pi i}\frac{1}{\log t}\int_{x-i\infty}^{x+i\infty}\frac{d}{dz}\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right]t^z dz = \int_{u=0}^{u=t}\frac{u^{\beta-1}}{\log u}du + const.$$

To establish that the constant vanishes, we'll show that both integrals vanish as $\tau \to \infty$, or $\tau \to -\infty$.

The left-side-Integral

As in 17.3, integration by parts of the left-side-integral leads to

$$\left|\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right]\right| \le \frac{t^x}{2\pi}\int_{y=-\infty}^{y=\infty}\left|\frac{1}{z}\log\left(1-\left|\frac{z}{\beta}\right|\right)\right|dy.$$

If we let $\left|\tau\right| \to \infty$,

$$\left|\frac{1}{z}\log\left(1-\left|\frac{z}{\beta}\right|\right)\right| \to \left|\frac{1}{z}\log 1\right| = 0.$$

Therefore, by Lebesgue Dominant Convergence,

$$\frac{1}{2\pi i}\frac{1}{\log t}\int_{y=-\infty}^{y=\infty} t^z d_z\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right] \to 0\,, \text{ as } \left|\tau\right| \to \infty\,.$$

The Right-Side-Integral

To show that the right-side-integral vanishes too, we make the change of variable

$$u = e^{\omega}$$

$$du = e^{\omega}d\omega$$

Then,

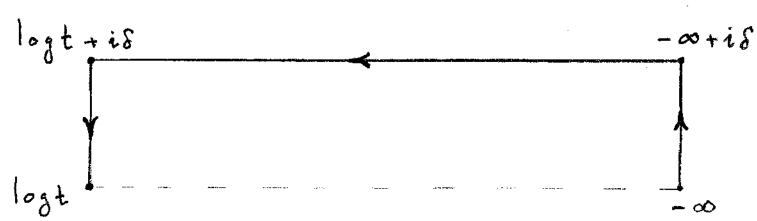
$$\int_{u=0}^{u=t}\frac{u^{\beta-1}}{\log u}du = \int_{\omega=-\infty}^{\omega=\log t}\frac{e^{\omega\beta}}{\omega}d\omega$$

And we take a path in the upper half-plane that

runs from $\omega = -\infty$ to $\omega = -\infty + i\delta$, along $\omega = -\infty + i\varepsilon$

runs from $\omega = -\infty + i\delta$ to $\omega = \log t + i\delta$, along $\omega = v + i\delta$

runs from $\omega = \log t + i\delta$ to $\omega = \log t$, along $\omega = \log t + i\varepsilon$



Then,

$$\int_{\omega=-\infty}^{\omega=\log t}\frac{e^{\omega\beta}}{\omega}d\omega = \int_{\omega=-\infty}^{\omega=-\infty+i\delta}\frac{e^{\omega\beta}}{\omega}d\omega + \int_{\omega=-\infty+i\delta}^{\omega=\log t+i\delta}\frac{e^{\omega\beta}}{\omega}d\omega + \int_{\omega=\log t+i\delta}^{\omega=\log t}\frac{e^{\omega\beta}}{\omega}d\omega$$

The First Integral

$$\left| \int_{\omega=-\infty}^{\omega=-\infty+i\delta} \frac{e^{\omega\beta}}{\omega} d\omega \right| \leq \int_{\varepsilon=0}^{\varepsilon=\delta} \frac{1}{\left|-\infty + i\varepsilon\right|} \left| e^{(-\infty+i\varepsilon)(\sigma+i\tau)} \right| d\varepsilon$$

$$= \int_{\varepsilon=0}^{\varepsilon=\delta} \frac{1}{\infty} e^{-(\infty\sigma+\varepsilon\tau)} d\varepsilon$$

Since $\sigma > 0$, and $\varepsilon > 0$, then for $\tau \to \infty$,

$$\frac{1}{\infty} e^{-\infty\sigma-\varepsilon\tau} \to \frac{1}{\infty} e^{-\infty(\sigma+\varepsilon)} = 0\,.$$

Hence, by Lebesgue Dominant Convergence

$$\int_{\omega=-\infty}^{\omega=-\infty+i\delta} \frac{e^{\omega\beta}}{\omega} d\omega \to 0, \quad \text{as} \quad \tau \to \infty.$$

The Second Integral

$$\left| \int_{\omega=-\infty+i\delta}^{\omega=\log t+i\delta} \frac{e^{\omega\beta}}{\omega} d\omega \right| \leq \int_{\upsilon=-\infty}^{\upsilon=\log t} \frac{1}{\left|\upsilon + i\delta\right|} \left| e^{(\upsilon+i\delta)(\sigma+i\tau)} \right| d\upsilon$$

$$= \int_{\upsilon=-\infty}^{\upsilon=\log t} \frac{1}{\left|\upsilon + i\delta\right|} e^{\upsilon\sigma-\delta\tau} d\upsilon$$

Since $\sigma > 0$, and $\delta > 0$, then for $\tau \to \infty$,

$$\frac{1}{\left|\upsilon + i\delta\right|}e^{\upsilon\sigma - \delta\tau} \to \frac{1}{\left|\upsilon + i\delta\right|}e^{\upsilon\sigma - \delta\infty} = \frac{1}{\left|\upsilon + i\delta\right|}e^{-\infty} = 0.$$

Hence, by Lebesgue Dominant convergence,

$$\int_{\omega=-\infty+i\delta}^{\omega=\log t+i\delta} \frac{e^{\omega\beta}}{\omega}d\omega \to 0, \quad \text{as} \quad \tau \to \infty.$$

The Third Integral

$$\left|\int_{\omega=\log t+i\delta}^{\omega=\log t} \frac{e^{\omega\beta}}{\omega}d\omega\right| \le \int_{\varepsilon=0}^{\varepsilon=\delta} \frac{1}{\left|\log t + i\varepsilon\right|}\left|e^{(\log t + i\varepsilon)(\sigma + i\tau)}\right|d\varepsilon$$

$$= \int_{\varepsilon=0}^{\varepsilon=\delta} \frac{1}{\left|\log t + i\varepsilon\right|}e^{\sigma\log t - \varepsilon\tau}d\varepsilon$$

Since $\sigma > 0$, and $\varepsilon > 0$, then for $\tau \to \infty$,

$$\frac{\exp(\sigma\log t - \varepsilon\tau)}{\left|\log t + i\varepsilon\right|} \to \frac{\exp(-\infty)}{\left|\log t + i\varepsilon\right|} = 0.$$

Hence, by Lebesgue Dominant convergence,

$$\int_{\omega=\log t+i\delta}^{\omega=\log t} \frac{e^{\omega\beta}}{\omega}d\omega \to 0, \quad \text{as} \quad \tau \to \infty$$

Thus, the right-side-integral vanishes.

Similarly, if we take a path in the lower half plane and let

$\tau \to -\infty$, the right side integral vanishes.

Consequently, if $\sigma > 0$, then

$$const = 0,$$

and

$$\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d_z\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right] = \int_{u=0}^{u=t} \frac{u^{\beta-1}}{\log u}du \,.\square$$

17.5 If $\sigma > 0$, then

$$\int_{u=0}^{u=t} \frac{u^{\beta-1}}{\log u}du = \begin{cases} \mathrm{Li}(t^\beta) - \pi i, & \text{for upper-half-plane path} \\ \mathrm{Li}(t^\beta) + \pi i, & \text{for lower-half-plane path} \end{cases}$$

Riemann wrote

If $\sigma > 0$, the integral

$$\int_{u=0}^{u=t} \frac{u^{\beta-1}}{\log u}du$$

takes on two values which differ by

$$2\pi i$$

depending on whether the path of integration is in the upper-half-plane or in the lower-half-plane. $\lozenge$

Proof:

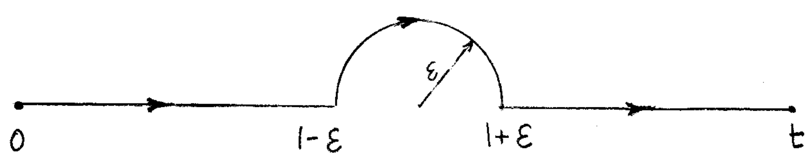
The integrand $\dfrac{u^{\beta-1}}{\log u}$ is singular at $u = 1$, and the path of integration has to bypass the singularity.

Thus, an upper-half-plane path will

run from $u = 0$ to $u = 1 - \varepsilon$

encircle the singularity clockwise from $u = 1 - \varepsilon$ to $u = 1 + \varepsilon$

run from $u = 1 + \varepsilon$ to $u = t$.



Then,

$$\int_{u=0}^{u=t} \frac{u^{\beta-1}}{\log u} du = \int_{u=0}^{u=1-\varepsilon} \frac{u^{\beta-1}}{\log u} du + \int_{\omega=\log(1-\varepsilon)}^{\omega=\log(1+\varepsilon)} \frac{e^{\omega\beta}}{\omega} d\omega + \int_{u=1+\varepsilon}^{u=t} \frac{u^{\beta-1}}{\log u} du$$

By the Residue Theorem for the clockwise semi-circle

$$\int_{\omega=\log(1-\varepsilon)}^{\omega=\log(1+\varepsilon)} \frac{e^{\omega\beta}}{\omega} d\omega = -2\pi i \left(\frac{\pi}{2\pi}\right) \operatorname{Res}\left[\frac{e^{\omega\beta}}{\omega}\right]_{\omega=0}$$

$$= -\pi i \lim_{\omega \to 0} \left[\omega \frac{e^{\omega\beta}}{\omega}\right]$$

$$= -\pi i$$

Hence,

$$\int_{u=0}^{u=t} \frac{u^{\beta-1}}{\log u} du = \int_{u=0}^{u=1-\varepsilon} \frac{u^{\beta-1}}{\log u} du + \int_{u=1+\varepsilon}^{u=t} \frac{u^{\beta-1}}{\log u} du + \left(-\pi i\right)$$

By the change of variable

$$\upsilon = u^{\beta}$$

$$d\upsilon = \beta u^{\beta-1} du$$

$$\log \upsilon = \beta \log u$$

$$\int_{\upsilon=0}^{\upsilon=t^{\beta}} \frac{d\upsilon}{\log \upsilon} = \underbrace{\left\{ \int_{\upsilon=0}^{\upsilon=(1-\varepsilon)^{\beta}} \frac{d\upsilon}{\log \upsilon} + \int_{\upsilon=(1+\varepsilon)^{\beta}}^{\upsilon=t^{\beta}} \frac{d\upsilon}{\log \upsilon} \right\}}_{\to \text{Li}(t^{\beta}), \text{ as } \varepsilon \downarrow 0} - \pi i$$

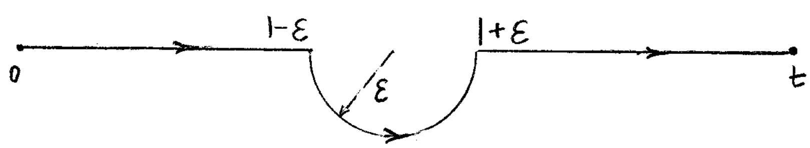
Letting $\varepsilon \downarrow 0$,

$$= Li(t^{\beta}) - \pi i$$

Hence,

$$\int_{u=0}^{u=t} \frac{u^{\beta-1}}{\log u} du = Li(t^{\beta}) - \pi i\,.$$

Similarly, with a lower half-plane path that encircles the singularity counter-clockwise,



$$\int_{u=0}^{u=t} \frac{u^{\beta-1}}{\log u} du = Li(t^{\beta}) + \pi i\,.$$

17.6 If $\sigma > 0$,

$$\boxed{\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)\right]=\begin{cases}\mathrm{Li}(t^\beta)-\pi i, & \text{for upper-half-plane path}\\ \mathrm{Li}(t^\beta)+\pi i, & \text{for lower-half-plane path}\end{cases}}$$

Proof:

By 17.4, and 17.5.□

18

The $\dfrac{\log\left(z-1\right)}{z}$ term

18.1 $$\boxed{\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d\left(\tfrac{1}{z}\log(z-1)\right)=\mathrm{Li}(t)}$$

Proof:

$\log(1-z)$ is defined with a cut along the positive real numbers.

Therefore, to obtain

$$z-1$$

in the main branch of $\log(1-z)$, we rotate

$$1-z$$

clockwise, by multiplying it by

$$e^{-i\pi}$$

That is,

$$(1-z)e^{-i\pi}=z-1$$

and

$$\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d\left[\tfrac{1}{z}\log(z-1)\right]=\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d\left[\tfrac{1}{z}\log(e^{-i\pi})(1-z)\right]$$

$$=\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d\left[\tfrac{1}{z}\log(1-z)\right]+\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d\left[\tfrac{1}{z}(-i\pi)\right]$$

The First Integral

$$\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d\left[\tfrac{1}{z}\log(1-z)\right]$$

has a term of the form

$$\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)$$

where

$$\sigma = 1 > 0\,.$$

Therefore, if we take an upper half-plane path with clockwise oriented semicircle around the singularity of the logarithmic integral at $u = 1$, by 17.6 we have

$$\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d\left[\tfrac{1}{z}\log(1-z)\right] = \mathrm{Li}(t) - \pi i$$

The Second Integral

$$\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d\left[\tfrac{1}{z}(-\pi i)\right] = (-\pi i)\underbrace{\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d\left[\tfrac{1}{z}\right]}_{=-1,\text{ by }16.1}$$

$$= \pi i$$

Consequently,

$$\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d\left[\tfrac{1}{z}\log(z-1)\right] = \big(Li(t) - \pi i\big) + \pi i$$

$$= \mathrm{Li}(t).\square$$

19

The $-\frac{1}{z}\sum_{\alpha}\log\left(1+\frac{z^2-z}{\alpha^2+\frac{1}{4}}\right)$ term

Riemann wrote,

$$\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty} t^z \frac{d\left[\frac{1}{z}\sum_{\alpha}\log\left(1+\frac{\left(z-\frac{1}{2}\right)^2}{\alpha^2}\right)\right]}{dz}dz =$$

$$= \log t\sum_{\alpha}\left[\mathrm{Li}(t^{\frac{1}{2}+i\alpha}) + \mathrm{Li}(t^{\frac{1}{2}-i\alpha})\right]$$

The summation is over all positive zeros of ξ *(or all zeros with positive real part), ordered by their size.*

♦*With a more precise discussion of the function* ξ *it is easy to show that the sum*

$$\sum_{\alpha}\left[Li(t^{1/2+i\alpha}) + Li(t^{1/2-i\alpha})\right]$$

equals

$$\lim_{Y\to\infty}\frac{1}{2\pi i}\int_{x-iY}^{x+iY}\frac{d}{dz}\left[\frac{1}{z}\sum_{\alpha}\log\left(1+\frac{(z-\frac{1}{2})^2}{\alpha^2}\right)\right]t^z dz$$

♦*If the zeros are not sequenced by their size, the sum may have any arbitrary real value.* ♢

19.1 The assumption of the Hypothesis

The claim that

the series

$$\sum_{\alpha}\left[\mathrm{Li}(t^{\frac{1}{2}+i\alpha})+\mathrm{Li}(t^{\frac{1}{2}-i\alpha})\right]$$

equals the integral

$$\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d\left[\frac{1}{z}\sum_{\alpha}\log\left(1+\frac{z^2-z}{\alpha^2+\frac{1}{4}}\right)\right]$$

includes the assumption of the Hypothesis that

$$0=Im(\alpha)=x_0-\tfrac{1}{2}$$

That is, the zeros are assumed to be on the line

$$x=\tfrac{1}{2}.$$

19.2 Implicit Claim

The following claim is implicit in Riemann's derivation of his formula for the count of the prime numbers

$$\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d\left[\frac{1}{z}\sum_{\alpha}\log\left(1+\frac{z^2-z}{\alpha^2+\frac{1}{4}}\right)\right]=\sum_{\alpha}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d\left[\frac{1}{z}\log\left(1+\frac{z^2-z}{\alpha^2+\frac{1}{4}}\right)\right]$$

In 1908, Landau [Land] proved that the summation over the zeros,

and the integration can be interchanged.

19.3 If the zeros of ξ are on $x = \frac{1}{2}$ and are sequenced by size

$$\boxed{-\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d\left[\frac{1}{z}\sum_{\alpha}\log\left(1+\frac{z^2-z}{\alpha^2+\frac{1}{4}}\right)\right] = -\sum_{\alpha}\left[Li(t^{\frac{1}{2}+i\alpha})+Li(t^{\frac{1}{2}-i\alpha})\right]}$$

Proof:

Integrating term by term by 19.2

$$-\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d\left[\frac{1}{z}\sum_{\alpha}\log\left(1+\frac{z^2-z}{\alpha^2+\frac{1}{4}}\right)\right] =$$

$$= \sum_{\alpha}\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d\left[\frac{1}{z}\log\left(1+\frac{z^2-z}{\alpha^2+\frac{1}{4}}\right)\right]$$

$$= \sum_{\alpha}\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d\left\{\frac{1}{z}\left[\log\left(1-\frac{z}{\frac{1}{2}+i\alpha}\right)+\log\left(1-\frac{z}{\frac{1}{2}-i\alpha}\right)\right]\right\}$$

The integral

$$\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d\left[\frac{1}{z}\log\left(1-\frac{z}{\frac{1}{2}+i\alpha}\right)\right]$$

has a term of the form

$$\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)=\frac{1}{z}\log\left(1-\frac{z}{\sigma+i\tau}\right),$$

where

$$\sigma=\tfrac{1}{2}>0\,,$$

and

$$\tau=\alpha>0\,.$$

By 17.6, on an upper half-plane path, the integral equals

$$\mathrm{Li}(t^{\frac{1}{2}+i\alpha})-i\pi\,.$$

Similarly, using a path in the lower half plane, the integral

$$\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d\left[\frac{1}{z}\log\left(1-\frac{z}{\frac{1}{2}-i\alpha}\right)\right]+\mathrm{Li}(t^{\frac{1}{2}-i\alpha})+i\pi$$

Therefore,

$$\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d\left\{\frac{1}{z}\left[\log\left(1-\frac{z}{\frac{1}{2}+i\alpha}\right)+\log\left(1-\frac{z}{\frac{1}{2}-i\alpha}\right)\right]\right\}=\mathrm{Li}(t^{\frac{1}{2}+i\alpha})+\mathrm{Li}(t^{\frac{1}{2}-i\alpha})$$

Consequently,

$$-\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}\,d\left[\frac{1}{z}\sum_{\alpha}\log\left(1+\frac{z^2-z}{\alpha^2+\frac{1}{4}}\right)\right]=-\sum_{\alpha}\left[Li(t^{\frac{1}{2}+i\alpha})+Li(t^{\frac{1}{2}-i\alpha})\right]$$

19.4 The Hypothesis Origin

- α, and $-\alpha$ are zeros of ξ
- $\left|\alpha_n\right|$ is increasing

combined with

$$Im(\alpha) \neq 0,$$

have the effect that

$$\sigma > 0,$$

is not guaranteed in any way, and the equality to the series

$$-\sum_{\alpha}\Big[\mathrm{Li}(t^{\frac{1}{2}+i\alpha}) + \mathrm{Li}(t^{\frac{1}{2}-i\alpha})\Big]$$

can-not be deduced.

This might have been the argument that led Riemann to make the Hypothesis.

20

The $\frac{1}{z}\log\Gamma\left(\frac{1}{2}z+1\right)$ term

20.1

$$\boxed{\frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d\left[\tfrac{1}{z}\log\Gamma\left(\tfrac{1}{2}z+1\right)\right]=\int_{u=t}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du}$$

Proof:

For $t>1$,

$$\int_{u=t}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du=\int_{u=t}^{u=\infty}\frac{1}{u\log u}\frac{\frac{1}{u^2}}{1-\frac{1}{u^2}}du$$

$$=\int_{u=t}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2}\left(1+\frac{1}{u^2}+\frac{1}{u^4}+...\right)du$$

The uniform convergence of the sum allows term-wise integration

$$=\sum_{n=1}^{\infty}\int_{u=t}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^{2n}}du$$

$$=\sum_{n=1}^{\infty}-\int_{u=\infty}^{u=t}\frac{u^{-2n-1}}{\log u}du$$

The integral

$$\int_{u=\infty}^{u=t} \frac{u^{-2n-1}}{\log u} du$$

has

$$\beta = \sigma = -2n\,,$$

and by Section 19.3, it equals

$$= \frac{1}{2\pi i} \int_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d\Big[\tfrac{1}{z}\log\Big(1 + \tfrac{1}{2n}z\Big)\Big]$$

Therefore, we have

$$= \sum_{n=1}^{\infty} -\frac{1}{2\pi i} \int_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d\Big[\tfrac{1}{z}\log\Big(1 + \tfrac{1}{2n}z\Big)\Big]$$

The uniform convergence allows interchanging summation and integration

$$= -\frac{1}{2\pi i} \int_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} \underbrace{\sum_{n=1}^{\infty} d\Big[\tfrac{1}{z}\log\Big(1 + \tfrac{1}{2n}z\Big)\Big]}_{-d\left[\frac{1}{z}\log\Gamma\left(\frac{1}{2}z+1\right)\right],\text{ by 17.1}}$$

$$= \frac{1}{2\pi i} \int_{y=-\infty}^{y=\infty} \frac{t^z}{\log t} d\Big[\tfrac{1}{z}\log\Gamma\Big(\tfrac{1}{2}z + 1\Big)\Big] \square$$

21

Count and Density of the Primes

21.1 If the zeros of ξ, $\alpha 's$, are positive and sequenced by size,

$$\boxed{f(t) = \mathrm{Li}(t) + \log\xi(0) - \sum_{\alpha}\Big[\mathrm{Li}(t^{\frac{1}{2}+i\alpha}) + \mathrm{Li}(t^{\frac{1}{2}-i\alpha})\Big] + \int_{u=t}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du}$$

Proof: Substitute in 15.1, the results of 16.2, 18.1, 19.3, and 20.1.□

By 8.2,

$$\xi(0) = \xi\Big|_{z=0} = \tfrac{1}{2},$$

so that

$$\log\xi(0) = -\log 2.$$

But Riemann erred, and substituted for $\xi(0)$,

$$\xi\Big|_{w=0} = \xi\Big|_{z=1/2}.$$

Riemann's error was observed by Genocchi [Geno], in 1860.

21.2 The Formula for the Count of the Primes

Riemann wrote

We invert

$$f(t) = \sum_n \frac{1}{n} F(t^{1/n})$$

to obtain

$$F(t) = \sum_m \frac{(-1)^\mu}{m} f(t^{1/m}),$$

where m *ranges over all the natural numbers that have no prime factors squared, and* μ *is the number of prime factors of* m.$\Diamond$

But although

$$f(t) = F(t) + \tfrac{1}{2}F(t^{\frac{1}{2}}) + \tfrac{1}{3}F(t^{\frac{1}{3}}) + \ldots$$

is an infinite series, it's Mobius inversion

$$\boxed{\begin{aligned} F(t) = f(t) & \\ & -\tfrac{1}{2} f(t^{\frac{1}{2}}) \\ & + \ldots \\ & + \frac{(-1)^{\mu(m)}}{m} f(t^{\frac{1}{m}}) \\ & + \ldots \\ & + - \frac{(-1)^{\mu(m_{j_0})}}{m_{j_0}} f(t^{\frac{1}{m_{j_0}}}) \end{aligned}}$$

terminates at m_{j_0} so that

$$t^{\frac{1}{m_{j_0}}} > 2,$$

and

$$t^{\frac{1}{m_{j_0+1}}} < 2.$$

For instance, if $t = 10$,

$$10^{\frac{1}{3}} = 2.15 > 2,$$

and

$$10^{\frac{1}{5}} = 1.58 < 2$$

Therefore, the summation is cut-off at $m_{j_0} = 3$

The Inversion Formula is constructed under the rules

m is a natural number that has no prime factors squared.

Any number that has a prime factor squared is skipped.

$\mu(m)$ is the number of prime factors of m.

with the aid of the following table

m	p_1	p_2	p_3	p_4	$\cdots$	μ
1	$-$					0
2	2					1
3	3					1
$4 = 2^2$						
5	5					1
6	2	3				2
7	7					1
$8 = 2^3$						
$9 = 3^2$						
$10 = 2 \cdot 5$	2	5				2
11	11					1
$12 = 3 \cdot 2^2$						
13	13					1
14	2	7				2
15	3	5				2
$16 = 2^4$						

21.3 The approximate Density of $f(t)$

$$\boxed{f'(t) \approx \frac{1}{\log t} - 2\frac{t^{-1/2}}{\log t}\sum_{\alpha}\cos(\alpha \log t)}$$

Riemann wrote

To approximate $f(t)$ *we take a finite sum of*

$$\sum_{\alpha}\left[Li(t^{\frac{1}{2}+i\alpha}) + Li(t^{\frac{1}{2}-i\alpha})\right]$$

where the zeros α, *are positive and sequenced by size.*

The derivative of the approximated $f(t)$ *is the sum*

$$\frac{1}{\log t} - 2\frac{t^{-1/2}}{\log t}\sum_{\alpha}\cos(\alpha \log t) + \left\{\to 0, \textit{very rapidly} \text{ for t} \to 0\right\}$$

where the zeros α, *are positive and sequenced by size.*

This sum approximates

the density of the primes at t,

$+\frac{1}{2}$ *the density of the squared primes at* t

$+\frac{1}{3}$ *the density of the cubed primes at* t

$+$...$\Diamond$

Proof:

The density of the primes at t is

$$f'(t) = F'(t) + \tfrac{1}{2}F'(t^{\frac{1}{2}}) + \tfrac{1}{3}F'(t^{\frac{1}{3}}) + ...$$

Approximating $f(t)$ by a partial sum $\sum_{\alpha}\left[Li(t^{\frac{1}{2}+i\alpha})+Li(t^{\frac{1}{2}-i\alpha})\right]$,

$$f(t) \approx \sum_{\alpha}\left[Li(t^{\frac{1}{2}+i\alpha})+Li(t^{\frac{1}{2}-i\alpha})\right]$$

Then,

$$f'(t) \approx \frac{d}{dt}\sum_{\alpha}\left[Li(t^{\frac{1}{2}+i\alpha})+Li(t^{\frac{1}{2}-i\alpha})\right]$$

Now,

$$\frac{d}{dt}\left[Li(t^{1/2+i\alpha})+Li(t^{1/2-i\alpha})\right] = \frac{d}{dt}\left\{\int_{u=0}^{u=t}\frac{u^{1/2+i\alpha-1}}{\log u}du + \int_{u=0}^{u=t}\frac{u^{1/2-i\alpha-1}}{\log u}du\right\}$$

$$= \frac{t^{-1/2}}{\log t}\left(t^{i\alpha}+t^{-i\alpha}\right)$$

$$= \frac{t^{-1/2}}{\log t}\left(e^{i\alpha\log t}+e^{-i\alpha\log t}\right)$$

$$= \frac{t^{-1/2}}{\log t}2\cos(\alpha\log t)$$

Therefore,

$$\frac{d}{dt}\sum_{\alpha}\left[Li(t^{\frac{1}{2}+i\alpha})+Li(t^{\frac{1}{2}-i\alpha})\right] = \frac{1}{\log t} - 2\frac{t^{-1/2}}{\log t}\sum_{\alpha}\cos(\alpha\log t) + \frac{1}{t(t^2-1)\log t}$$

$$\approx \frac{1}{\log t} - 2\frac{t^{-1/2}}{\log t}\sum_{\alpha}\cos(\alpha\log t)$$

That is,

$$f'(t) \approx \frac{1}{\log t} - 2\frac{t^{-1/2}}{\log t}\sum_{\alpha}\cos(\alpha\log t)\,\square$$

21.4 Riemann's Approximation for $F(t)$

$$\boxed{F(t) \approx \mathrm{Li}(t) - \tfrac{1}{2}\mathrm{Li}(t^{\frac{1}{2}}) - \tfrac{1}{3}\mathrm{Li}(t^{\frac{1}{3}}) - \tfrac{1}{5}\mathrm{Li}(t^{\frac{1}{5}}) + ... + \frac{(-1)^{\mu_{m_{j_0}}}}{m_{j_0}}\mathrm{Li}(t^{\frac{1}{m_{j_0}}})}$$

where m_{j_0} is the cut-off for the Mobius Inversion

Riemann wrote

Since

$$f'(t) \approx \frac{1}{\log t} - 2\frac{t^{-1/2}}{\log t}\sum_{\alpha}\cos(\alpha \log t)$$

the well-known approximation formula $F(t) = \mathrm{Li}(t)$, is correct only to order of magnitude of $t^{\frac{1}{2}}$, and gives a value that is somewhat too large.

Except for quantities that are bounded as t increases, the non-oscillatory terms in $F(t)$ are

$$\mathrm{Li}(t) - \frac{1}{2}\mathrm{Li}(t^{1/2}) - \frac{1}{3}\mathrm{Li}(t^{1/3}) - \frac{1}{5}\mathrm{Li}(t^{1/5}) + \frac{1}{6}\mathrm{Li}(t^{1/6}) - \frac{1}{7}\mathrm{Li}(t^{1/7}) + ... \Diamond$$

Proof:

Substitute $f(t) \approx \mathrm{Li}(t)$ in 21.2$\Box$

21.5 Comparison with Gauss Approximation

Riemann wrote

When Gauss and Goldschmidt compared $\mathrm{Li}(t)$ *with* $F(t)$ *up to* $t =$ *three million, they found that the Count of the Primes up to* $100,000$ *was smaller than* $\mathrm{Li}(100,000)$*, and the difference increased gradually, with many fluctuations, as* t *increased.* $\diamondsuit$

Riemann's approximation of 21.4 is compared with Gauss' approximation by $\mathrm{Li}(t)$ in the Lehmer table [Ed, p.35]

t	Riemann Error	Gauss Error
$1,000,000$	30	130
$2,000,000$	-9	122
$3,000,000$	0	155
$4,000,000$	33	206
$5,000,000$	-64	125
$6,000,000$	24	228
$7,000,000$	-38	179
$8,000,000$	-6	223
$9,000,000$	-53	187
$10,000,000$	88	339

21.6 The Effect of the Hypothesis Series

Riemann wrote

The finite sum of oscillatory terms

$$-2\frac{t^{-1/2}}{\log t}\sum_{\alpha}\cos(\alpha\log t)$$

cause irregular fluctuations in the density of the primes. It would be interesting to trace the fluctuations of the density of the primes $F'(t)$ *to the particular oscillatory terms in* $f'(t)$ $\Diamond$

Under the Hypothesis that all the zeros of ξ are on $x = \frac{1}{2}$, the count of the primes includes the Hypothesis Series

$$\sum_{\alpha}\left[Li(t^{\frac{1}{2}+i\alpha}) + Li(t^{\frac{1}{2}-i\alpha})\right].$$

Being unable to compute even a partial sum of this series, Riemann considered in 21.3 the derivative of a partial sum

$$-2\frac{t^{-1/2}}{\log t}\sum_{\alpha<t}\cos(\alpha\log t),$$

and his computations indicated to him the chaotic nature of the Hypothesis Series.

He remained intrigued by the effect of the Hypothesis Series on the count of the primes,

Not believing that the partial sums of the Series itself could be computed, he states his interest in terms of the derivatives, that he could compute.

With the aid of computing software, the partial sums of the Hypothesis Series can be computed. In 2008, we studied the effect of the Hypothesis Series, in [Dan2], and in [Dan3].

21.7 Fluctuations of $f(t)$

Riemann wrote

The behavior of $f(t)$ *is more regular.*

Already for $t \leq 100$, *we have* $f(t) \approx Li(t) + \log \xi(0)$. $\Diamond$

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