Riemann Hypothesis Proof

H. Vic Dannon
vic0@comcast.net
March, 2009
Revised December, 2009

Abstract We show that in $0 < x < 1$ any factor of Riemann's function $\xi(z)$ must have its zero on the line $x = 1/2$. Since $\xi(z)$ has in $0 < x < 1$ the same zeros as $\zeta(z)$, this proves Riemann's Hypothesis that all the zeros of $\zeta(z)$ in $0 < x < 1$, are on $x = 1/2$.

Keywords: Riemann Hypothesis, Zeta function, Gamma function, Analytic functions.

2000 Mathematics Subject Classification: 11M26, 33E20, 30A10, 26D15, 30D20.

Contents

Introduction

In his 1859 Zeta paper, Riemann obtained a formula for the count of the primes up to a given number. Riemann's formula has four terms. But only the first and the third terms have non-negligible values. The first is the dominant term, and can be computed precisely. The third is smaller and depends on the provision that all the zeros of the Zeta function in $0 < x < 1$ are on the line $x = 1/2$.

This provision became known as the Riemann Hypothesis, but it was never hypothesized by Riemann, nor was it used by him. Not seeing an easy proof for it, Riemann used only the first term of his formula and obtained an approximation far superior to Gauss for the count of the primes.

In 1935, and 1936, Titchmarsh and Comrie [Hasel, p. xii] confirmed by computations that the first 1042 zeros of Zeta in $0 < x < 1$ with

$$0 < \operatorname{Im} z < 1468$$

lie exactly on the line $x = 1/2$. Following that, Touring (1953) extended the upper limit to $\operatorname{Im} z = 1540$, and Lehmer (1956) confirmed that the first 25,000 zeros of Zeta in $0 < x < 1$ with

$$0 < \operatorname{Im} z < 21{,}944$$

lie on the line $x = 1/2$.

By 2002, the first 50 billion Zeta zeros have been located on the line $x = 1/2$, and in a 2008 meeting a far greater number was mentioned.

The fact that the first 50 billion zeros are on $x = 1/2$, does not constitute a proof for the infinitely many zeros in the infinite area of the strip $0 < x < 1$.

But expecting massive endless computations to disprove the Riemann Hypothesis by finding a zero off the line $x = 1/2$, is statistically implausible.

In fact, statistical tests indicate that the Riemann Hypothesis holds with extremely high statistical certainty. In 2008, we applied a Chi-Squared Goodness-of-Fit-Test to the Riemann formula for the count of the primes, and confirmed that the Riemann Hypothesis holds with certainty that is limited only by the software [Dan2].

In sections 1 to 4, we present Riemann's function $\xi(z)$, and known facts about its zeros. In section 5, we recall that $\left|\xi(x-iy)\right| = \left|\xi(x+iy)\right|$.

This fact serves as a key result for the proof.

In 8.2, we present a key result that follows from the Weierstrass factorization of $\xi(z)$, and in 9.1 a key result due to Hadamard.

The last key result due to Titchmarsh, is presented in section 10.

There, we prove that in $0 < x < 1$, any factor of $\xi(z)$ must have its zero on the line $x = 1/2$.

Since $\xi(z)$ has in $0 < x < 1$ the same zeros as $\zeta(z)$, this proves that all the zeros of $\zeta(z)$ in $0 < x < 1$, are on $x = 1/2$.

1. $\xi(z)$, and $\zeta(z)$ have the same zeros in $0 < x < 1$

For $x + iy = z \in \mathbb{C}$, Riemann gave the <u>Definition</u>

1.1
$$\xi(z) \equiv \tfrac{1}{2} z(z-1)\frac{1}{\pi^{\frac{1}{2}z}}\Gamma(\tfrac{1}{2}z)\zeta(z)$$

Since $\Gamma(z)$ has no zeros in $0 < x < 1$, [Saks], we have the <u>Proposition</u>

1.2 $|\xi(z)|$ *has in* $0 < x < 1$ *the same zeros as* $\zeta(z)$

2. $\xi(z)$ has in $0 < x < 1$, infinitely many zeros that are sequenced by size and increase to ∞.

Since $\zeta(z)$ has infinitely many zeros in $0 < x < 1$ [Titch], we have the Proposition

2.1 $\xi(z)$ *has infinitely many zeros* $z_1, z_2, z_3, \ldots$ *in* $0 < x < 1$.

Also, $\xi(z)$ is an entire function, not identically zero. Thus, by [Saks, p. 296] we have the Proposition

2.2 *The zeros* $z_1, z_2, z_3, \ldots$, *are sequenced by size, and increase to* ∞.

$$|z_1| \leq |z_2| \leq |z_3| \leq \ldots \uparrow \infty.$$

3. the multiplicity of each zero of $\xi(z)$ is finite

Since an analytic function that vanishes on a converging sequence is identically zero, there is no infinite sequence of identical $\xi(z)$ zeros. That is, we have the Proposition

3.1 *The multiplicity of each zero of* $\xi(z)$ *is finite.*

4. Diagonally symmetric zeros of $\xi(z)$

For $\zeta(z)$, Riemann derived the functional equation $\xi(z) = \xi(1-z)$.

Therefore, we have the <u>Proposition</u>

4.1 *If* $z_n = x_n + iy_n$ *is a zero of* $\xi(z)$, *then so is* $1 - z_n = 1 - x_n - iy_n$.

These two zeros are diagonally symmetric with respect to $z = \frac{1}{2}$, because denoting $x_n = \frac{1}{2} + \alpha_n$, we have $1 - x_n = \frac{1}{2} - \alpha_n$.

5. Observing $\left|\xi(x - iy)\right| = \left|\xi(x + iy)\right|$, for $0 < x < 1$,

Proof: Riemann obtained an integral formula for $\xi(z)$,

$$\xi(z) = \tfrac{1}{2} + z(z-1)\int_{t=1}^{t=\infty} \psi(t)\left(t^{-\frac{1}{2}(z+1)} + t^{\frac{1}{2}z-1}\right)dt,$$

where the infinite series $\psi(t) \equiv \dfrac{1}{e^{\pi t}} + \dfrac{1}{e^{2^2\pi t}} + \dfrac{1}{e^{3^2\pi t}} + \dots..,$

converges uniformly for $t \geq 1$. Therefore,

$$\left|\xi(\overline{z})\right| = \left|\tfrac{1}{2} + \overline{z}(\overline{z}-1)\int_{t=1}^{t=\infty} \psi(t)\left(t^{-\frac{1}{2}(\overline{z}+1)} + t^{\frac{1}{2}\overline{z}-1}\right)dt\right|$$

$$= \left|\overline{\tfrac{1}{2} + z(z-1)\int_{t=1}^{t=\infty} \psi(t)\left(t^{-\frac{1}{2}(z+1)} + t^{\frac{1}{2}z-1}\right)dt}\right|$$

$$= \left|\overline{\xi(z)}\right| = \left|\xi(z)\right|.$$

That is, we have

$$\left|\xi(x - iy)\right| = \left|\xi(x + iy)\right|. \square$$

6. A convergent infinite product is zero if and only if at least one of its factors is zero

Analytic functions are effectively represented as infinite products [Saks, VII].

By the definition in [Saks, p. 286],

The Infinite Product $a_1 a_2 a_3 \ldots$ *converges to* p *if and only if there is an index* n_0, *so that all the terms after it* $a_{n_0+1}, a_{n_0+2}, a_{n_0+3}, \ldots$ *are non-zero, and*

$$a_{n_0+1} a_{n_0+2} \ldots a_{n_0+m} \to q \neq 0.$$

Consequently,

$$p = a_1 a_2 \ldots a_{n_0} q.$$

Therefore, [Saks, p. 287], has the Proposition

6.1 T*he value of a convergent infinite product is equal to zero if and only if at least one of its factors is zero*.

The product of $a_n = 1 - \frac{1}{n}$ that has no such q, *diverges* to zero, although it satisfies the convergence necessary condition $a_n \to 1$ [Saks, p. 287].

If a convergent infinite product equals zero, this is due to only finitely many vanishing factors, without which, the remaining infinite product is non-zero.

7. Absolutely Convergent Infinite Product

We'll need Absolute Convergence, so that the infinite products will converge independently of the order of their factors.

By the definition in [Saks, p. 288],

$$(1+u_1)(1+u_2)\ldots \textit{ converges absolutely} \Leftrightarrow (1+|u_1|)(1+|u_2|)\ldots \textit{converges.}$$

Since

$$(1+u_1)(1+u_2)\ldots \text{ converges if and only if } u_1 + u_2 + \ldots \text{converges,}$$

we have in [Saks, p. 289] the Proposition

7.1 $(1+u_1)(1+u_2)\ldots$ *converges absolutely* $\Leftrightarrow$ $|u_1| + |u_2| + \ldots$*converges*.

Consequently, since the value of an absolutely convergence series does not depend on the order of the summation, we have Proposition

7.2 *The value of an absolutely convergent infinite product does not depend on the order of its factors.*

8. The Weierstrass Factorization of $\xi(z)$

By **2.2**, the zeros of $\xi(z)$ in $0 < x < 1$,

$$z_1, z_2, z_3, \ldots,$$

are sequenced by size, and increase to ∞.

$$|z_1| \le |z_2| \le |z_3| \le \ldots \uparrow \infty.$$

Since $\xi(z)$ is an analytic function in the complex plane so that

$$\xi(0) \ne 0,$$

and since the z_n 's are sequenced by their size and increase to ∞,

Then, by Weierstrass [Saks,VII, 2.13], we have

$$\xi(z) = e^{h(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{Q_n(z)},$$

where

the polynomials $Q_n(z)$ guarantee the uniform convergence of the product in the open plane, and

$h(z)$ is analytic in the complex plane.

The zeros $1 - z_n$ of **4.1** appear in the factorization, and we have

Proposition

8.1
$$\xi(z) = e^{h(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right)\left(1 - \frac{z}{1 - z_n}\right) e^{Q_n(z)}.$$

While the $1 - z_n$'s appear in the factorization, $\overline{z}_1, \overline{z}_2, \overline{z}_3 \ldots.$ DO NOT.

We stress this crucial point with Proposition

8.2 $\overline{z}_1, \overline{z}_2, \overline{z}_3 \ldots$ **do not appear in the Weierstrass Factorization of** $\xi(z)$.

Proof: The appearance of the zero $\overline{z}_1$ of **5**, will require the inclusion of the factor $1 - \dfrac{\overline{z}}{\overline{z}_1}$, that vanishes whenever $\overline{z} = \overline{z}_1$.

It is well known that the function

$$g(z) = \overline{z},$$

is not differentiable with respect to z.

Therefore, the factor $1 - \dfrac{\overline{z}}{\overline{z}_1}$ is not differentiable with respect to z.

Consequently, multiplying by $1 - \dfrac{\overline{z}}{\overline{z}_1}$ will produce a function that is not differentiable with respect to z.

In particular, including any of the factors

$$1 - \frac{\overline{z}}{\overline{z}_1},\ 1 - \frac{\overline{z}}{\overline{z}_2},\ 1 - \frac{\overline{z}}{\overline{z}_3}, \ldots$$

in the factorization will make $\xi(z)$ not differentiable.

But since $\xi(z)$ is an analytic function in the complex plane, it is differentiable of any order, at any point z in the complex plane.

A similar argument prevents the appearance of any of the factors

$$1 - \frac{\overline{z}}{1 - \overline{z}_1},\ 1 - \frac{\overline{z}}{1 - \overline{z}_2},\ 1 - \frac{\overline{z}}{1 - \overline{z}_3}, \ldots . \square$$

9. The Hadamard factorization of $\xi(z)$

Hadamard showed that for $\xi(z)$, we have

$$e^{Q_n(z)} = 1,$$

and

$$e^{h(z)} = \tfrac{1}{2}.$$

Thus, simplifying the Weierstrass product representation of $\xi(z)$.

Hadamard factorization for $\xi(z)$ is the 3rd key result for the Hypothesis proof. This is Proposition

9.1 $$\xi(z) = \tfrac{1}{2}\left(1 - \frac{z}{z_1}\right)\left(1 - \frac{z}{1 - z_1}\right)\left(1 - \frac{z}{z_2}\right)\left(1 - \frac{z}{1 - z_2}\right)\ldots,$$

where the product converges absolutely, and uniformly with respect to the $z's$, its value does not depend on the order of its factors, and it vanishes only at the zeros of $\xi(z)$.

The conjugate roots, $\overline{z}_1, 1 - \overline{z}_1, \overline{z}_2, 1 - \overline{z}_2, \overline{z}_3, 1 - \overline{z}_3, \ldots$. do not appear in the factorization of $\xi(z)$.

A proof of this established result, is in the Appendix. We note that the $\overline{z}_n$'s do not appear in the factorization of $\xi(z)$, by **8.2**,

10. $x_n = \frac{1}{2}$, for any zero z_n

Proof: To keep it readable, we'll take $n = 7$, and assume that z_7 has multiplicity 2. Say, $z_7 = z_8$, and $|z_6| < |z_7| = |z_8| < |z_9|$

Since $z_7 = x_7 + iy_7$ is a zero of ξ,

$$\left|\xi(x_7 + iy_7)\right| = 0.$$

By **5**,

$$\left|\xi(x_7 - iy_7)\right| = 0.$$

Thus,

$$\xi(x_7 - iy_7) = 0.$$

Applying **9.1** to $\xi(x_7 - iy_7)$,

$$0=\left(1-\frac{x_7-iy_7}{z_1}\right)\left(1-\frac{x_7-iy_7}{1-z_1}\right)\left(1-\frac{x_7-iy_7}{z_2}\right)\left(1-\frac{x_7-iy_7}{1-z_2}\right)...$$

Since we assumed $z_7=z_8$, we have

$$0=\left(1-\frac{x_7-iy_7}{z_1}\right)\left(1-\frac{x_7-iy_7}{1-z_1}\right)...\left(1-\frac{x_7-iy_7}{z_6}\right)\left(1-\frac{x_7-iy_7}{1-z_6}\right)\times$$

$$\times\left(1-\frac{x_7-iy_7}{z_7}\right)^2\left(1-\frac{x_7-iy_7}{1-z_7}\right)^2\left(1-\frac{x_7-iy_7}{z_9}\right)\left(1-\frac{x_7-iy_7}{1-z_9}\right)...$$

Clearly, for $m\neq 7,8$, the factors with $z_m\neq z_7$ are all non-zero.

Indeed,

$1-\dfrac{x_7-iy_7}{z_m}=0\Rightarrow z_m=\overline{z}_7$, contradicting **8.2**.

And,

$1-\dfrac{x_7-iy_7}{1-z_m}=0\Rightarrow 1-z_m=x_7-iy_7\Rightarrow z_m=1-\overline{z}_7$, contradicting **8.2**.

Since all the factors with $z_m\neq z_7$ are non-zero, by **6.1**, we must have

$$\left(1-\frac{x_7-iy_7}{z_7}\right)^2\left(1-\frac{x_7-iy_7}{1-z_7}\right)^2=0.$$

Thus,

$$\left|1-\frac{x_7-iy_7}{z_7}\right|^2\left|1-\frac{x_7-iy_7}{1-z_7}\right|^2=0.$$

Hence,

$$\frac{4\left|y_7\right|^2}{\left|x_7+iy_7\right|^2}\frac{\left|1-2x_7\right|^2}{\left|1-x_7-iy_7\right|^2}=0.$$

That is,

$$\frac{4y_7^2}{x_7^2+y_7^2}\frac{\left|1-2x_7\right|^2}{(1-x_7)^2+y_7^2}=0$$

Now, the three terms that include y_7^2 are non-vanishing.

To see that, note the 4th key result for the Hypothesis Proof, due to Titchmarsh, [Titch, p.329-332],

in $0 < x < 1$, *and* $0 \le y \le y_1$, $\zeta(z)$ *has only one zero on the line* $x = 1/2$, *at* $y_1 = 14.134725132$,

Thus, y_7 must be out of the Titchmarsh rectangle.

That is, $y_7 \ge y_1$, and any one of the terms with y_7^2 is greater than $y_1^2 > 196$.

Therefore, we have

$$\left|1 - 2x_7\right| = 0.$$

That is,

$$x_7 = \tfrac{1}{2}.$$

Replacing 7 with n, and 2 with k, we conclude that if z_n is a zero of multiplicity k, then we'll apply the arguments above to obtain that $x_n = \frac{1}{2}$.$\square$

Discussion

The proof ties together key results due to Riemann, Weierstrass, Hadamard, and Titchmarsh.

Each of these results is crucial to the proof:

Hadamard Factorization, 9.1, lists all the zeros of zeta, and enables us to deal with them all.

Riemann's result, 5, applied to Hadamard's factorization yields an infinite product with mainly non-zero factors.

Weierstrass result 8.2, enables us to toss away the non-zero factors, and focus on the few that vanish.

Titchmarsh result in 10, allows us to eliminate the rest of the non-zero terms, and conclude that all the zeros are on the line $x = \frac{1}{2}$.

The crucial role of these components of the proof, suggests that this may be the only path to a comprehensible, direct proof.

APPENDIX: Proof of Hadamard Factorization 9.1

A. $e^{Q_n(z)} = 1$

To establish $e^{Q_n(z)} = 1$, we aim to show that for all z in $0 < x < 1$,

$$\prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right)\left(1-\frac{z}{1-z_n}\right) = \prod_{n}\left(1-\frac{z(1-z)}{z_n(1-z_n)}\right) \quad \text{converges}$$

absolutely.

By **7.1**, we need to show that $\sum_m \frac{1}{\left|z_m(1-z_m)\right|}$ converges.

For instance, $\cos(\pi z) = \prod_{n=1}^{\infty}\left(1-\left(\frac{2z}{2n-1}\right)^2\right)$ converges absolutely, because

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} \text{ converges.}$$

Note that,

$$\left|z_m(1-z_m)\right|^2 = \left|(z_m-\tfrac{1}{2})^2-\tfrac{1}{4}\right|^2$$

$$= \left|(x_m-\tfrac{1}{2}+iy_m)^2-\tfrac{1}{4}\right|^2$$

$$= \left|(x_m-\tfrac{1}{2})^2-y_m^2+2i(x_m-\tfrac{1}{2})y_m-\tfrac{1}{4}\right|^2$$

$$= \left((x_m-\tfrac{1}{2})^2-y_m^2-\tfrac{1}{4}\right)^2+4(x_m-\tfrac{1}{2})^2y_m^2$$

$$= (x_m-\tfrac{1}{2})^4+y_m^4+\tfrac{1}{16}-\tfrac{1}{2}(x_m-\tfrac{1}{2})^2+\tfrac{1}{2}y_m^2+2(x_m-\tfrac{1}{2})^2y_m^2$$

$$= \left((x_m-\tfrac{1}{2})^2+y_m^2\right)^2+\tfrac{1}{16}-\tfrac{1}{2}(x_m-\tfrac{1}{2})^2+\tfrac{1}{2}y_m^2$$

$$= \left|z_m - \tfrac{1}{2}\right|^4 + \tfrac{1}{16} - \tfrac{1}{2}(x_m - \tfrac{1}{2})^2 + \tfrac{1}{2}y_m^2$$

Now, $\frac{1}{16} - \frac{1}{2}(x_m - \frac{1}{2})^2 + \frac{1}{2}y_m^2 > 0$, because $\left|x_m - \frac{1}{2}\right| \le \frac{1}{2}$, and by a result of Titchmarsh (that we state in section 10), $y_m \ge y_1 > 14$.

Therefore,

$$\left|z_m(1 - z_m)\right| > \left|z_m - \tfrac{1}{2}\right|^2.$$

Hence,

$$\frac{1}{\left|z_m(1 - z_m)\right|} < \frac{1}{\left|z_m - \frac{1}{2}\right|^2},$$

and it is sufficient to show that $\displaystyle\sum_{m>N} \frac{1}{\left|z_m - \frac{1}{2}\right|^2} < \infty$.

Note that the necessary condition, $\dfrac{1}{\left|z_m - \frac{1}{2}\right|^2} \downarrow 0$, holds, since $z_m \uparrow \infty$.

To show the convergence, for m large enough,

$$m = N, N + 1, N + 2, \ldots,$$

define positive numbers $R_m > 1$ so that

$$\log R_m > 1,$$

and

$$m = 4R_m \log R_m.$$

Then,

$$\log m > \log R_m.$$

Hadamard showed [Dan3, 10.3], that the number of zeros of $\xi(z)$ in $\left|z - \frac{1}{2}\right| \le R_m$ is bounded by $2R_m \log R_m$.

Since we took $m = 4R_m \log R_m$, we have

$$\left|z_m - \tfrac{1}{2}\right| > R_m .$$

Therefore,

$$\sum_{m>N} \frac{1}{\left|z_m - \frac{1}{2}\right|^2} \leq \sum_{m>N} \frac{1}{R_m^2}$$

$$= 4^2 \sum_{m>N} \frac{1}{m^2}(\log R_m)^2$$

$$\leq 4^2 \sum_{m>N} \frac{\left(\log m\right)^2}{m^2}$$

$$= 4^2 \sum_{m>N} \frac{1}{m^{3/2}} \frac{\left(\log m\right)^2}{m^{1/2}} .$$

Since $\dfrac{\left(\log m\right)^2}{m^{1/2}} \to 0$, as $m \to \infty$, we have

$$\frac{\left(\log m\right)^2}{m^{1/2}} < 1, \text{ for } m > N ,$$

and

$$\sum_{m>N} \frac{1}{\left|z_m - \frac{1}{2}\right|^2} < \infty .$$

Therefore,

$$\prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right)\left(1 - \frac{z}{1 - z_n}\right) \text{ converges absolutely,}$$

and

$$e^{Q_n(z)} = 1.$$

Since the Weierstrass product converges uniformly, we have

$$\xi(z) = e^{h(z)} \prod_{n=1}^{\infty} \Big(1 - \frac{z}{z_n}\Big)\Big(1 - \frac{z}{1 - z_n}\Big),$$

where the infinite product converges absolutely, and uniformly with respect to the $z's$.□

B. The order of an entire function

By [Holl, p. 68], the Hadamard Factorization Theorem applies to an entire function $f(z)$ for which

$$\limsup_{R\to\infty} \frac{1}{\log R} \log\log \max_{|z|=R} \big|f(z)\big| \equiv \varrho < \infty .$$

ϱ is called **the order of** $f(z)$.

For instance, if $f(z) = \cos(\pi z)$,

$$\begin{aligned} \big|\cos(\pi z)\big| &= \tfrac{1}{2}\big|e^{i\pi z} + e^{-i\pi z}\big| \\ &\le \tfrac{1}{2}\Big(\big|e^{i\pi z}\big| + \big|e^{-i\pi z}\big|\Big) \\ &= \tfrac{1}{2}\Big(\big|e^{i\pi(x+iy)}\big| + \big|e^{-i\pi(x+iy)}\big|\Big) \\ &= \tfrac{1}{2}\Big(e^{-\pi y} + e^{\pi y}\Big) \\ &\le e^{\pi y} \\ &\le e^{\pi |z|}. \end{aligned}$$

Thus,

$$\frac{1}{\log R} \log\log \max_{|z|=R} \big|\cos(\pi z)\big| \le \frac{1}{\log R} \log\log \max_{|z|=R} e^{\pi |z|}$$

$$= \frac{1}{\log R} \log \log e^{\pi R}$$

$$= \frac{1}{\log R} \log(\pi R)$$

$$= \frac{1}{\log R} (\log \pi + \log R)$$

$$= 1 + \frac{\log \pi}{\log R}$$

$$\to 1, \text{ as } R \to \infty .$$

Hence, $f(z) = \cos(\pi z)$ has order $\varrho = 1$.

C. $e^{h(z)} = \frac{1}{2}$.

Hadamard showed that for $\xi(z)$, $e^{h(z)} = \frac{1}{2}$.

Hadamard replaced Weierstrass entire function

$$h(z),$$

with a polynomial

$$Q(z),$$

so that

$$\deg Q(z) \leq \varrho .$$

It is well-known ([Ed] or [Dan3, 10.2]), that if R is large enough, $\log\left|\xi(z)\right| \leq R \log R$ in $\left|z - \frac{1}{2}\right| \leq 2R$. Hence,

$$\frac{1}{\log R} \log \log \max_{\left|z-\frac{1}{2}\right|=R} \left|\xi(z)\right| \leq \frac{1}{\log R} \log\left(R \log R\right)$$

$$= 1 + \frac{\log \log R}{\log R}.$$

Since by L'Hospital $\lim\limits_{R\to\infty}\dfrac{\log\log R}{\log R}=\lim\limits_{R\to\infty}\dfrac{\frac{1}{\log R}\frac{1}{R}}{\frac{1}{R}}=0$,

we conclude

$$\limsup_{R\to\infty}\frac{1}{\log R}\log\log\max_{\left|z-\frac{1}{2}\right|=R}\left|\xi(z)\right|=1.$$

That is,

$$\xi(z) \text{ is of order } \varrho = 1.$$

Hence,

$$\deg Q(z)\le 1.$$

That is

$$Q(z)=A+B(z-\tfrac{1}{2}),$$

for some constants A, and B.

Since the factorization factors are

$$\Big(1-\frac{z}{z_n}\Big)\Big(1-\frac{z}{1-z_n}\Big)=1-\frac{z(1-z)}{z_n(1-z_n)}=1-\frac{(z-\frac{1}{2})^2-\frac{1}{4}}{(z_n-\frac{1}{2})^2-\frac{1}{4}},$$

we have,

$$\xi(z)=e^{A+B(z-\frac{1}{2})}\prod_n\left(1-\frac{(z-\frac{1}{2})^2-\frac{1}{4}}{(z_n-\frac{1}{2})^2-\frac{1}{4}}\right),$$

where the product is an even function of $z-\dfrac{1}{2}$.

But Riemann showed [Dan3, 9.2] that

$$\xi(z)=A_0+A_1(z-\tfrac{1}{2})^2+A_2(z-\tfrac{1}{2})^4+\ldots.$$

is an even function of $(z-\frac{1}{2})$.

Consequently,

$$B=0,$$

and

$$\xi(z) = e^A \prod_n \left(1 - \frac{(z - \frac{1}{2})^2 - \frac{1}{4}}{(z_n - \frac{1}{2})^2 - \frac{1}{4}} \right).$$

Setting $z = 0$,

$$e^A = \xi(0).$$

By Riemann's Integral formula for $\xi(z)$, that we used in **5**,

$$\xi(z) = \tfrac{1}{2} + z(z-1) \int_{t=1}^{t=\infty} \psi(t) \left(t^{-\frac{1}{2}(z+1)} + t^{\frac{1}{2}z - 1} \right) dt.$$

Hence, we have

$$\xi(0) = \tfrac{1}{2}.$$

Therefore,

$$\xi(z) = \frac{1}{2} \prod_n \left(1 - \frac{(z - \frac{1}{2})^2 - \frac{1}{4}}{(z_n - \frac{1}{2})^2 - \frac{1}{4}} \right).$$

$$= \frac{1}{2} \prod_n \left(1 - \frac{z}{z_n} \right) \left(1 - \frac{z}{1 - z_n} \right)$$

where the infinite product converges absolutely, and uniformly with respect to the $z's$. $\square$

Acknowledgements

This proof owes its existence to the serious critique of many since 2005. I am most obliged to them.

I thank the Canadian Number Theory Association IX Meeting participants who let me present my attempts on July 2006, in Vancouver, British Columbia.

I am very grateful to the Pacific Coast Number Theory Conference participants who let me present my attempts on December 2008, in Fort Collins, Colorado,

and let me present a preliminary version of this proof on December 2009 in Asilomar, California.

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