Riemann's Formula for the Count of the Primes the effect of the Hypothesis Series

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Abstract

We approximate with Riemann's formula that includes the Hypothesis series, the count of the primes till 10^21.

We confirm Riemann's suspicion that the effect of the Hypothesis series on the count of the primes is unpredictable.

Consequently, given a large enough number, any approximation has to specify the error bounds due to the Hypothesis series.

Keywords Riemann Zeta, Riemann Formula, Count of Primes, Riemann Hypothesis, Riemann Hypothesis Series.

Mathematics Subject Classification 11M26

Introduction

How many prime numbers up to 10?

2, 3, 5, and 7 are the four prime numbers up to 10.

Lehmer have listed the prime numbers up to 10,006,721, and we can count with his list.

A supercomputer was used to count how many primes are up to a Billion times a Trillion.

But isn't there some formula to replace counting?

The computing of the number of the primes up to a number x, started with Gauss who approximated it by the logarithmic integral.

In his 1859 Zeta paper, Riemann obtained a formula for the count of the primes, that uses all the zeros of the Zeta function on the line $x = \frac{1}{2}$, to solve the problem completely, provided that all the zeros of the Zeta function in $0 < x < 1$, are on the line $x = \frac{1}{2}$.

The Riemann formula has four terms. But only the first and the third of these terms have non-negligible values. The first is a dominant term that can be computed precisely. The third term is smaller and depends on the provision regarding the zeros of the Zeta function.

This provision became known as the Riemann Hypothesis, but it was never hypothesized by Riemann. Not seeing an easy proof for it, Riemann used only the first term of his formula, and obtained an approximation far superior to Gauss for the count of the primes. Thus, the first term in Riemann's Formula is known as the Riemann Approximation term.

We shall refer to the third term that depends on the Hypothesis, and was neglected since Riemann, as the Riemann-Hypothesis-Series.

It is obtained provided that all the zeros of the Zeta function in the strip $0 < x < 1$, lie on the line $x = \frac{1}{2}$.

Each term of the Hypothesis Series is evaluated at a zero of the Zeta function on the line $x = \frac{1}{2}$. Since there are infinitely many such zeros, the Series has infinitely many terms

Riemann wondered about the effect of the Hypothesis series, but left it out of his approximation formula [Riem].

We have proved in [Dan, Theorem 14.1] that

Riemann's Formula for the Count of the Primes is valid with Riemann Hypothesis Series, with uncertainty under 10^{-16}.

This allows us to use Riemann's formula for the count of the primes with great certainty.

Actually, our computations indicated that if not for the limitations of the software, Riemann's Formula can be confirmed to any degree of certainty.

Here, we approximate $F(10 \wedge 7)$ by Riemann's Formula, and compare it to Lehmer's count. We confirm Riemann's suspicion that

the Hypothesis Series convergence is unpredictable.

Even with many Hypothesis Series terms, the convergence is slow. The Cesaro-Arithmetic-Means converge more smoothly but not faster.

The slow convergence of the Hypothesis series, forces us to use statistics to estimate $F(10 \wedge 21)$.

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Riemann's Formula for the Count of the Primes.

1.1 Riemann's Derivation

Riemann Denoted the count the number of primes < t, by $F(t)$.

By [Ed], [Riem], and [Dan], Riemann used the Euler product formula

$$\zeta(z) \equiv \sum_{n=natural} \frac{1}{n^z} = \prod_{p=prime} \frac{1}{1 - 1\ /\ p^z}$$

to obtain

$$\frac{\log \zeta(z)}{z} = \int_{t=1}^{t=\infty} \left(F(t) + \frac{1}{2} F(t^{1/2}) + \frac{1}{3} F(t^{1/3}) + ... \right) \frac{dt}{t^{z+1}}$$

Riemann denoted

$$f(t) = F(t) + \frac{1}{2} F(t^{1/2}) + \frac{1}{3} F(t^{1/3}) + \frac{1}{4} F(t^{1/4}) + ...$$

Inverting the equation,

$$\frac{\log \zeta(z)}{z} = \int_{t=1}^{t=\infty} f(t) \frac{dt}{t^{z+1}}$$

he obtained

$$f(t) = \frac{1}{2\pi} \int_{y=-\infty}^{y=\infty} \frac{\log \zeta(z)}{z} t^z dy.$$

The entire function

$$\xi(z) \equiv (z-1)\pi^{-\frac{z}{2}} \Gamma(\tfrac{z}{2} + 1) \zeta(z)$$

has the same zeros as $\zeta(z)$ in $0 < x < 1$, and has the factorization

$$\xi(z) = \xi(0)\prod_{n=1}^{\infty}\left(1-\frac{z}{\frac{1}{2}-i\alpha_n}\right)\left(1-\frac{z}{\frac{1}{2}+i\alpha_n}\right) = \xi(0)\prod_{n=1}^{\infty}\left(1+\frac{z^2-z}{\alpha_n^2+\frac{1}{4}}\right)$$

where $\frac{1}{2}-i\alpha_n$, and $\frac{1}{2}+i\alpha_n$ are the zeros of $\xi(z)$, and are sequenced by their size.

Replacing $\zeta(z)$ with $\xi(z)$

$$f(t) = \frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left[\frac{\log(z-1)}{z}+\frac{\log\Gamma\left(\frac{1}{2}z+1\right)}{z}-\frac{\log\xi(0)}{z}\right.$$

$$\left.-\frac{1}{z}\sum_{n=1}^{\infty}\log\left(1+\frac{z^2-z}{\alpha_n^2+\frac{1}{4}}\right)\right].$$

ξ has infinitely many zeros on the line $x = \frac{1}{2}$.

If $\mathrm{Im}(\alpha_n) = 0$ for all n, then all the zeros of ξ are on the line $x = \frac{1}{2}$, and the integration of $-\frac{1}{z}\sum_{n=1}^{\infty}\log\left(1+\frac{z^2-z}{\alpha_n^2+\frac{1}{4}}\right)$ yields the sum

$$\sum_{n=1}^{\infty}\left[\mathrm{Li}(t^{\frac{1}{2}+i\alpha_n})+\mathrm{Li}(t^{\frac{1}{2}-i\alpha_n})\right]$$

where

$$\mathrm{Li}(t) \equiv \int_{u=0}^{u=t}\frac{du}{\log u} \equiv \lim_{\varepsilon\downarrow 0}\left\{\int_{u=0}^{u=1-\varepsilon}\frac{du}{\log u}+\int_{u=1+\varepsilon}^{u=t}\frac{du}{\log u}\right\}$$

is the logarithmic integral.

Then, integrating the rest of the terms for $f(t)$, Riemann derived,

$$
\boxed{
\begin{aligned}
f(t) = {} & \mathrm{LogIntegral}(t) \\
& - \log 2 \\
& - \sum_{n=1}^{\infty}\Big[\mathrm{L}\,ogIntegral(t^{\frac{1}{2}+i\alpha_n}) + \mathrm{L}\,ogIntegral(t^{\frac{1}{2}-i\alpha_n})\Big] \\
& + \int_{u=t}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}\,du
\end{aligned}
}
$$

$F(t)$ is obtained by Mobius Inversion of the definition of $f(t)$

The Mobius Inversion of $f(t) = F(t) + \frac{1}{2}F(t^{1/2}) + \frac{1}{3}F(t^{1/3}) + \ldots$gives

$$
\boxed{
\begin{aligned}
F(t) = {} & f(t) \\
& - \frac{1}{2} f(t^{1/2}) \\
& + \ldots \\
& + \frac{(-1)^{\mu(m)}}{m} f(t^{1/m}) \\
& + \ldots \\
& + \frac{(-1)^{\mu(m_{j_0})}}{m_{j_0}} f(t^{1/m_{j_0}})
\end{aligned}
}
$$

1.2 Formula Cut-Off, and the Mobius Inversion Table

While $f(t)$ is an infinite series, the $F(t)$ summation terminates at m_{j_0} so that

$$t^{1/m_{j_0}} > 2,$$

and

$$t^{1/[m_{j_0+1}]} < 2.$$

For instance, if $t = 10$, we find that

$$10^{1/3} = 2.15 > 2,$$

and

$$10^{1/5} = 1.58 < 2$$

Therefore, the formula cut-off is at $m_{j_0} = 3$

m is a natural number that has no prime factors squared.

Any number that has a prime factor squared is skipped.

$\mu(m)$ is the number of prime factors of m.

Mobius Inversion Formula is constructed with the aid of the following table

m	p_1	p_2	p_3	p_4.....	$\mu(m)$	$\frac{(-1)^{\mu(m)}}{m}$
1	–				0	1
2	2				1	$-1/2$
3	3				1	$-1/3$
$4 = 2^2$						
5	5				1	$-1/5$
6	2	3			2	$1/6$
7	7				1	$-1/7$
$8 = 2^3$						
$9 = 3^2$						
10	2	5			2	$1/10$
11	11				1	$-1/11$
$12 = 2^2 3$						
13	13				1	$-1/13$

Appendix A has the values of m, and μ, up to $m = 113$.

For $t = 10$, we find the admissible $m's$ from the following Mobius Inversion Table

m	p_1	p_2	p_3	$p_4 \ldots\ldots$	$\mu(m)$	$\frac{(-1)^{\mu(m)}}{m}$
1	$-$				0	1
2	2				1	$-1/2$
3	3				1	$-1/3$
$4 = 2^2$						

Therefore, the number of primes up to 10 is

$$F(10) = f(10) - \frac{1}{2} f(10^{1/2}) - \frac{1}{3} f(10^{1/3})$$

Note that $F(10) = 4$.

We proceed to write this formula for $F(10)$ in detail.

1.3 Riemann's Formula for F(10)

If the Riemann Hypothesis holds, then for $t = 10$,

$$f(10) = \text{LogIntegral}(10) - \log 2 - \sum_{\alpha} \Big[\text{LogIntegral}(10^{\frac{1}{2}+i\alpha}) + \text{LogIntegral}(10^{\frac{1}{2}-i\alpha}) \Big] + \int_{u=10}^{u=\infty} \frac{1}{u \log u} \frac{1}{u^2 - 1} du$$

$$-\tfrac{1}{2}f(\sqrt{10}) = -\tfrac{1}{2}\text{LogIntegral}(\sqrt{10})$$
$$+\tfrac{1}{2}\log 2$$
$$+\tfrac{1}{2}\sum_{\alpha}\left[\text{LogIntegral}(\sqrt{10}^{\frac{1}{2}+i\alpha}) + \text{LogIntegral}(\sqrt{10}^{\frac{1}{2}-i\alpha})\right]$$
$$-\tfrac{1}{2}\int_{u=\sqrt{10}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{3}f(\sqrt[3]{10}) = -\tfrac{1}{3}\text{LogIntegral}(\sqrt[3]{10})$$
$$+\tfrac{1}{3}\log 2$$
$$+\tfrac{1}{3}\sum_{\alpha}\left[\text{LogIntegral}(\sqrt[3]{10}^{\frac{1}{2}+i\alpha}) + \text{LogIntegral}(\sqrt[3]{10}^{\frac{1}{2}-i\alpha})\right]$$
$$-\tfrac{1}{3}\int_{u=\sqrt[3]{10}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

Substituting in the formula for $F(10)$ at the end of section **1.3**,

$$F(10) = \text{LogIntegral}(10) - \tfrac{1}{2}\text{LogIntegral}(\sqrt{10}) - \tfrac{1}{3}\text{LogIntegral}(\sqrt[3]{10})$$

$$+(-1+\tfrac{1}{2}+\tfrac{1}{3})\log 2$$

$$-\sum_{\alpha}\Big[\Big(\text{LogIntegral}(10^{\frac{1}{2}+i\alpha}) + \text{LogIntegral}(10^{\frac{1}{2}-i\alpha})\Big)$$
$$+\tfrac{1}{2}\sum_{\alpha}\Big(\text{LogIntegral}(\sqrt{10}^{\frac{1}{2}+i\alpha}) + \text{LogIntegral}(\sqrt{10}^{\frac{1}{2}-i\alpha})\Big)$$
$$+\tfrac{1}{3}\sum_{\alpha}\Big(\text{LogIntegral}(\sqrt[3]{10}^{\frac{1}{2}+i\alpha}) + \text{LogIntegral}(\sqrt[3]{10}^{\frac{1}{2}-i\alpha})\Big)$$

$$+\int_{u=10}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du - \tfrac{1}{2}\int_{u=\sqrt{10}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du - \tfrac{1}{3}\int_{u=\sqrt[3]{10}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

If we aim to compute by hand, this is our final formula.

But in order to ready the sum of the three infinite series for computer run, it is crucial that we write them as one infinite series, each of its terms is the sum of three pairs of logarithmic integrals.

Each term of the series is evaluated at a distinct zero of the Zeta function on the line $x = \frac{1}{2}$, and at its conjugate zero.

Consequently, we have one infinite series that depends on the Hypothesis.

This is how the Hypothesis Term becomes the Hypothesis Series.

$$
\begin{aligned}
F(10) = \quad & \text{LogIntegral}(10) - \tfrac{1}{2}\text{LogIntegral}(\sqrt{10}) - \tfrac{1}{3}\text{LogIntegral}(\sqrt[3]{10}) \\
& +(-1 + \tfrac{1}{2} + \tfrac{1}{3})\log 2 \\
& +\sum_{\alpha}\Big[-\Big(\text{LogIntegral}(10^{\frac{1}{2}+i\alpha}) + \text{LogIntegral}(10^{\frac{1}{2}-i\alpha})\Big) \\
& \quad +\tfrac{1}{2}\Big(\text{LogIntegral}(\sqrt{10}^{\frac{1}{2}+i\alpha}) + \text{LogIntegral}(\sqrt{10}^{\frac{1}{2}-i\alpha})\Big) \\
& \quad +\tfrac{1}{3}\Big(\text{LogIntegral}(\sqrt[3]{10}^{\frac{1}{2}+i\alpha}) + \text{LogIntegral}(\sqrt[3]{10}^{\frac{1}{2}-i\alpha})\Big)\Big] \\
& +\int_{u=10}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du - \tfrac{1}{2}\int_{u=\sqrt{10}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du - \tfrac{1}{3}\int_{u=\sqrt[3]{10}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du
\end{aligned}
$$

If the Hypothesis holds, this expression holds true.

The first term is the Riemann Approximation Term.

In the example above, it is the sum of 3 LogIntegrals.

It is the dominant term, and it is superior to any approximation attempt before Riemann.

The second term is called the Log Term.

In the above example it is made of three summands.

For large numbers, it is negligible.

The third term is the Hypothesis Series.

It is the term that Riemann, not having a proof for the Hypothesis, did not use.

Here, we will establish the validity of this term, and hence, the validity of the Riemann Hypothesis, up to a high statistical certainty.

The fourth term is called the Integral Term.

In the above example, it is the sum of three integrals.

It is negligible for large numbers.

1.4 Riemann's Approximation

Riemann did not use the Hypothesis Series. Riemann wrote

> "…*It is very likely that all of the zeros are real. One would like to have a rigorous proof of this, but after several fleeting attempts to no avail, I have temporarily set aside the search for this proof because it appeared to be unnecessary for the immediate purpose of my investigation*…"

Riemann used

$$f(t) \approx \mathrm{L}\,ogIntegral(t)$$

Then,

$$F(t) \approx \mathrm{Li}(t) - \frac{1}{2}\mathrm{Li}(t^{\frac{1}{2}}) - \frac{1}{3}\mathrm{Li}(t^{\frac{1}{3}}) - \frac{1}{5}\mathrm{Li}(t^{\frac{1}{5}}) + ... + \frac{(-1)^{\mu_{m_{j_0}}}}{m_{j_0}}\mathrm{Li}(t^{m_{j_0}}).$$

That is, only the LogIntegral term, cutoff at m_{j_0}, is used.

Riemann's approximation is better than the Gauss approximation

$$F(t) \approx \mathrm{Li}(t),$$

as the Lehmer table [Ed, p. 35] shows.

t	Riemann's Error	Gauss's Error
$1,000,000$	30	130
$2,000,000$	-9	122
$3,000,000$	0	155
$4,000,000$	33	206
$5,000,000$	-64	125
$6,000,000$	24	228
$7,000,000$	-38	179
$8,000,000$	-6	223
$9,000,000$	-53	187
$10,000,000$	88	339

Riemann was interested in the effect of the Hypothesis Series on the count of the Primes. He wrote

The finite sum of oscillatory terms over zeros of Zeta that are less than t,

$$-2\frac{t^{-\frac{1}{2}}}{\log t}\sum_{\alpha=\text{zeros}<t}\cos(\alpha\log t),$$

causes irregular fluctuations in the density of the primes. In a future count, it would be interesting to trace the fluctuations of the density of the primes $F'(t)$ *to the particular oscillatory terms* $f'(t)$.

1.5 Riemann's Formula for F(t)

Assuming that the Riemann Hypothesis holds, we have

$$f(t) = \mathrm{Li}(t) - \log 2 - \sum_{\alpha}\Big[\mathrm{Li}(t^{\frac{1}{2}+i\alpha}) + \mathrm{Li}(t^{\frac{1}{2}-i\alpha})\Big] + \int_{u=t}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{2}f(t^{1/2}) = -\tfrac{1}{2}\mathrm{Li}(t^{1/2}) + \tfrac{1}{2}\log 2 + \tfrac{1}{2}\sum_{\alpha}\Big[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/2}) + \mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/2})\Big]$$

$$-\tfrac{1}{2}\int_{u=t^{1/2}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{3}f(t^{1/3}) = -\tfrac{1}{3}\mathrm{Li}(t^{1/3}) + \tfrac{1}{3}\log 2 + \tfrac{1}{3}\sum_{\alpha}\Big[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/3}) + \mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/3})\Big]$$

$$-\tfrac{1}{3}\int_{u=t^{1/3}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{5}f(t^{1/5}) = -\tfrac{1}{5}\mathrm{Li}(t^{1/5}) + \tfrac{1}{5}\log 2 + \tfrac{1}{5}\sum_{\alpha}\Big[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/5}) + \mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/5})\Big]$$

$$-\tfrac{1}{5}\int_{u=t^{1/5}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{6}f(t^{1/6}) = \tfrac{1}{6}\mathrm{Li}(t^{1/6}) - \tfrac{1}{6}\log 2 - \tfrac{1}{6}\sum_{\alpha}\Big[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/6}) + \mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/6})\Big]$$

$$+\tfrac{1}{6}\int_{u=t^{1/6}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{7}f(t^{1/7}) = -\tfrac{1}{7}\mathrm{Li}(t^{1/7}) + \tfrac{1}{7}\log 2 + \tfrac{1}{7}\sum_{\alpha}\Big[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/7}) + \mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/7})\Big]$$

$$-\tfrac{1}{7}\int_{u=t^{1/7}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{10}f(t^{1/10}) = \tfrac{1}{10}\mathrm{Li}(t^{1/10}) - \tfrac{1}{10}\log 2 - \tfrac{1}{10}\sum_{\alpha}\Big[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/10}) + \mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/10})\Big]$$

$$+\tfrac{1}{10}\int_{u=t^{1/10}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{11}f(t^{\frac{1}{11}}) = -\tfrac{1}{11}\mathrm{Li}(t^{\frac{1}{11}}) + \tfrac{1}{11}\log 2 + \tfrac{1}{11}\sum_{\alpha}\Big[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/11}) + \mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/11})\Big]$$

$$-\tfrac{1}{11}\int_{u=t^{1/11}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{13}f(t^{\frac{1}{13}}) = -\tfrac{1}{13}\mathrm{Li}(t^{\frac{1}{13}}) + \tfrac{1}{13}\log 2 + \tfrac{1}{13}\sum_{\alpha}\Big[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/13}) + \mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/13})\Big]$$

$$-\tfrac{1}{13}\int_{u=t^{1/13}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

....

Then, the count of the primes that are smaller than t is

$$F(t) = \mathrm{Li}(t) - \log 2 - \sum_{\alpha}\Big[\mathrm{Li}(t^{\frac{1}{2}+i\alpha}) + \mathrm{Li}(t^{\frac{1}{2}-i\alpha})\Big] + \int_{u=t}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{2}\mathrm{Li}(t^{1/2}) + \tfrac{1}{2}\log 2 + \tfrac{1}{2}\sum_{\alpha}\Big[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/2}) + \mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/2})\Big]$$

$$-\tfrac{1}{2}\int_{u=t^{1/2}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{3}\mathrm{Li}(t^{1/3}) + \tfrac{1}{3}\log 2 + \tfrac{1}{3}\sum_{\alpha}\Big[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/3}) + \mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/3})\Big]$$

$$-\tfrac{1}{3}\int_{u=t^{1/3}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

+..

$$+\frac{(-1)^{\mu}}{m_{j_0}}\left(\mathrm{Li}(t^{1/m_{j_0}}) - \log 2 - \sum_{\alpha}\left[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/m_{j_0}}) + \mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/m_{j_0}}) \right] + \int_{u=t^{1/m_{j_0}}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du \right).$$

Grouping like-terms, Riemann's Formula is,

$$\boxed{\begin{aligned} F(t) = &\left\{ \mathrm{Li}(t) - \tfrac{1}{2}\mathrm{Li}(t^{1/2}) - \tfrac{1}{3}\mathrm{Li}(t^{1/3}) + ... + \frac{(-1)^{\mu}}{m_{j_0}}\mathrm{Li}(t^{1/m_{j_0}}) \right\} \\ &+ \left\{ -1 + \tfrac{1}{2} + \tfrac{1}{3} + ... + \frac{(-1)^{\mu}}{m_{j_0}} \right\} \mathrm{Log}\, 2 \\ &+ \sum_{\alpha}\Bigg(-\left[\mathrm{Li}(t^{\frac{1}{2}+i\alpha}) + \mathrm{Li}(t^{\frac{1}{2}-i\alpha}) \right] \\ &\qquad + \tfrac{1}{2}\left[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/2}) + \mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/2}) \right] \\ &\qquad + \tfrac{1}{3}\left[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/3}) + \mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/3}) \right] + ... \\ &\quad - \frac{(-1)^{\mu}}{m_{j_0}}\left[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/m_{j_0}}) + \mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/m_{j_0}}) \right] \\ &+ \Bigg\{ \int_{u=t}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du - \tfrac{1}{2}\int_{u=t^{1/2}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du \\ &\quad - \tfrac{1}{3}\int_{u=t^{1/3}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du .... + \frac{(-1)^{\mu}}{m_{j_0}}\int_{u=t^{1/m_{j_0}}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du \Bigg\} \end{aligned}}$$

where the summation terminates at m_{j_0} so that

$$t^{1/m_{j_0}} > 2, \quad \text{and } t^{1/[m_{j_0+1}]} < 2.$$

The infinite sum over the zeta zeros is the Hypothesis Series.

Appendix B lists the formulas for the $f(t)$ term up to

$$m = 113$$

2

Approximating the Hypothesis Series

2.1 The Hypothesis Series

The Hypothesis Series for a number t is the infinite series with the general component

$$\tau(\alpha_n) = -\Big[\mathrm{Li}(t^{\frac{1}{2}+i\alpha_n}) + \mathrm{Li}(t^{\frac{1}{2}-i\alpha_n})\Big] + \tfrac{1}{2}\Big[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha_n)/2}) + \mathrm{Li}(t^{(\frac{1}{2}-i\alpha_n)/2})\Big] + \tfrac{1}{3}\Big[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha_n)/3}) + \mathrm{Li}(t^{(\frac{1}{2}-i\alpha_n)/3})\Big] + ... \cdots - \frac{(-1)^{\mu}}{m_{j_0}}\Big[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha_n)/m_{j_0}}) + \mathrm{Li}(t^{(\frac{1}{2}-i\alpha_n)/m_{j_0}})\Big]$$

where

$$\alpha_1 < \alpha_2 < \alpha_3 < ....$$

and $i\alpha_1, i\alpha_2, i\alpha_3 ....$ are the zeros of the Zeta function on the line $x = \frac{1}{2}$.

The infinite series may be approximated by its partial sums

$$y_n = \tau(\alpha_1) + \tau(\alpha_2) + \tau(\alpha_3) + ... + \tau(\alpha_n).$$

However, no method is available to accelerate the convergence.

2.2 The incompatibility of the Hypothesis Series with summation methods

All summation methods apply to partial sums of the form

$$f(1) + f(2) + f(3) + ... + f(n).$$

That is, we need to identify the series term a_n with a function f so that

$$a_n = f(n)$$

For instance, the Euler Summation formula [Abram, p.16]

$f(1) + f(2) + f(3) + ... + f(n)$

$$= \int_{x=1}^{x=n} f(x)dx + \frac{f(1) + f(n)}{2}$$

$$+[\tfrac{1}{12} f'(x) - \tfrac{1}{720} f'''(x) + \tfrac{1}{30,240} f^{(5)}(x) - ...]_{x=1}^{x=n}$$

requires that we know the function f.

Consequently, no summation methods is available to sum up the partial sums y_n.

2.3 The unpredictable convergence of the Hypothesis Series

Since the Hypothesis Series converges,

$$\tau(n) \to 0.$$

But the convergence is unpredictable to the extent that we cannot be certain that 100,000 partial sums will yield a better approximation of the Hypothesis series, than 5000 partial sums.

We observe this unpredictable convergence in the approximation of the Hypothesis series for $F(10\,\hat{}\,7)$. The precise value of $F(10\,\hat{}\,7)$ was given by Lehmer.

In the case of $F(10\,\hat{}\,21)$, the precise value is not known, and our approximation must specify the error that is mandated by the chaotic convergence of the Hypothesis series.

We employ statistics to obtain error bounds.

2.4 Approximating the Hypothesis Series by the Cesaro-Arithmetic Means

The Cesaro-Arithmetic-Means fluctuate less than the partial sums.

In the following example, they also converge faster.

For instance, the series

$$0.69314718 = \log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + ...$$

	partial sum	*error*	Arithmetic mean	*error*
one term	1	0.30685	1	0.30685
two terms	1-1/2=0.5	0.19314	(1+0.5)/2= 0.75	0.05685
three terms	1-1/2+1/3=0.8333...	0.140186	(1+0.5+0.8333...)/3 0.777...	0.08463
four terms	0.58333...	0.1098	0.7291	0.03595
five terms	0.78333...	0.09018	0.74	0.04685
six terms	0.61666...	0.07648	0.7194	0.02625
seven terms	0.75952	0.06037	0.72513	0.03198
eight terms	0.63452	0.05867	0.71380	0.02065

We will observe that the Cesaro-Arithmetic Means of the partial sums of the Hypothesis Series converge smoother than the partial sums, but the error is not necessarily smaller than the error in the partial sums.

2.4 Non-parametric Distribution of the Partial Sums of the Hypothesis Series

The unpredictable convergence of the Hypothesis Series leads to statistical analysis of its values. Given N partial sums of the series, we can find the mean μ and standard deviation σ of the partial sums distribution.

If we assume that these mean, and standard deviation come from a normal distribution, then with 95% confidence, we can expect to find the partial sums of the Hypothesis Series for $F(10\,\hat{}\,21)$ in the interval

$$(\mu - 2\sigma, \mu + 2\sigma).$$

While our computations indicate that these bounds are possible, the error bounds are too large, and may not be improved further.

To obtain smaller bounds, it is common to assume that by the Central Limit Theorem, the means of the partial sums are normally distributed about the mean μ, with standard deviation $\dfrac{\sigma}{\sqrt{N}}$.

But our computations indicate that for the Hypothesis Series for $F(10\,\hat{}\,21)$, the intervals

$$(\mu_{5000} - 2\frac{\sigma_{5000}}{\sqrt{5000}}, \mu_{5000} + 2\frac{\sigma_{5000}}{\sqrt{5000}})$$

$$(\mu_{20,000} - 2\frac{\sigma_{20,000}}{\sqrt{20,000}}, \mu_{20,000} + 2\frac{\sigma_{20,000}}{\sqrt{20,000}})$$

$$(\mu_{100,000} - 2\frac{\sigma_{100,000}}{\sqrt{100,000}}, \mu_{100,000} + 2\frac{\sigma_{100,000}}{\sqrt{100,000}})$$

are disjoint.

Thus, the error bounds that are based on the Central Limit Theorem are too small.

Consequently, it is safer to not assume that the partial sums of the Hypothesis Series are distributed normally, or otherwise parametrically.

The partial sums of the Hypothesis Series seem to converge unpredictably, and seem to be distributed non-parametrically.

3

F(10^7), and its exact terms

3.1 F(10^7)

Lehmer [Lehm] listed all the primes up to 10^7. Each page of the Lehmer listing has 50 columns, and each column has 100 primes. The last prime preceding 10^7 is 9,999,991. That prime is on the 80th row of column 46 in page 133. Therefore, its place in the Lehmer table is at

$$132 \times 5000 + 45 \times 100 + 80 = 664,580.$$

Since the Lehmer tables start at 1, the number of primes up to 10^7 is

$$F(10 \wedge 7) = 664,580 - 1 = 664,579.$$

3.2 m(10^7)

We sum up to $m(10^7) \quad 23$, because

$$\sqrt[23]{10^7} \quad 2.015 \quad 2$$

and

$$\sqrt[26]{10^7} \quad 1.858 \quad 2.$$

By Appendix A, we use the 16 numbers

1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23.

That is, each of the four terms in Riemann's Formula is the sum of 16 terms.

3.3 The Riemann approximation term for F(10^7)

This is the main term.

s1= $N[LogIntegral[10 \wedge 7]$

$$-\tfrac{1}{2}\,LogIntegral[\sqrt{10\,\hat{}\,7}]$$
$$-\tfrac{1}{3}\,LogIntegral[\sqrt[3]{10\,\hat{}\,7}]$$
$$-\tfrac{1}{5}\,LogIntegral[\sqrt[5]{10\,\hat{}\,7}]$$
$$+\tfrac{1}{6}\,LogIntegral[\sqrt[6]{10\,\hat{}\,7}]$$
$$-\tfrac{1}{7}\,LogIntegral[\sqrt[7]{10\,\hat{}\,7}]$$
$$+\tfrac{1}{10}\,LogIntegral[\sqrt[10]{10\,\hat{}\,7}]$$
$$-\tfrac{1}{11}\,LogIntegral[\sqrt[11]{10\,\hat{}\,7}]$$
$$-\tfrac{1}{13}\,LogIntegral[\sqrt[13]{10\,\hat{}\,7}]$$
$$+\tfrac{1}{14}\,LogIntegral[\sqrt[14]{10\,\hat{}\,7}]$$
$$+\tfrac{1}{15}\,LogIntegral[\sqrt[15]{10\,\hat{}\,7}]$$
$$-\tfrac{1}{17}\,LogIntegral[\sqrt[17]{10\,\hat{}\,7}]$$
$$-\tfrac{1}{19}\,LogIntegral[\sqrt[19]{10\,\hat{}\,7}]$$
$$+\tfrac{1}{21}\,LogIntegral[\sqrt[21]{10\,\hat{}\,7}]$$
$$+\tfrac{1}{22}\,LogIntegral[\sqrt[22]{10\,\hat{}\,7}]$$
$$-\tfrac{1}{23}\,LogIntegral[\sqrt[23]{10\,\hat{}\,7}]$$

= **664667.4443892473`**

3.4 The Log2 term for F(10^7)

The log2 term is

s2=

N[(-1+1/2+1/3+1/5-1/6+1/7-1/10+1/11+1/13-1/14-1/15+1/17+1/19-1/21-1/22+1/23)*Log[2]]

= 0.0637901

This turns out to be negligible compared with the first term.

3.5 The Integral term for F(10^7)

$$
\begin{aligned}
\textbf{s4=}\,\mathrm{N}[&\int_{10^{\wedge}7}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx \\
&-\frac{1}{2}\int_{\sqrt{10^{\wedge}7}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx \\
&-\frac{1}{3}\int_{\sqrt[3]{10^{\wedge}7}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx \\
&-\frac{1}{5}\int_{\sqrt[5]{10^{\wedge}7}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx \\
&+\frac{1}{6}\int_{\sqrt[6]{10^{\wedge}7}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx \\
&-\frac{1}{7}\int_{\sqrt[7]{10^{\wedge}7}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx \\
&+\frac{1}{10}\int_{\sqrt[10]{10^{\wedge}7}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx \\
&-\frac{1}{11}\int_{\sqrt[11]{10^{\wedge}7}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx \\
&-\frac{1}{13}\int_{\sqrt[13]{10^{\wedge}7}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx \\
&+\frac{1}{14}\int_{\sqrt[14]{10^{\wedge}7}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx \\
&+\frac{1}{15}\int_{\sqrt[15]{10^{\wedge}7}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx \\
&-\frac{1}{17}\int_{\sqrt[17]{10^{\wedge}7}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx \\
&-\frac{1}{19}\int_{\sqrt[19]{10^{\wedge}7}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx \\
&+\frac{1}{21}\int_{\sqrt[21]{10^{\wedge}7}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx \\
&+\frac{1}{22}\int_{\sqrt[22]{10^{\wedge}7}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx
\end{aligned}
$$

$$-\frac{1}{23}\int_{\sqrt[23]{10\hat{}7}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx$$

$$= \mathbf{-0.000519957}$$

This too, turns out to be negligible compared with the first term.

4

The Hypothesis Series for F(10^7)

4.1 The Hypothesis Series for F(10^7)

Each term of the infinite series is the sum of 16 pairs of logarithmic integrals computed at the Zeta-zeros $i\alpha$, and $-i\alpha$.

$$\tau(\alpha) = \mathrm{Li}(10^{7(\frac{1}{2}-\alpha)}) + \mathrm{Li}(10^{7(\frac{1}{2}+\alpha)})$$
$$+\tfrac{1}{2}[\mathrm{Li}(10^{7(\frac{1}{2}-\alpha)/2}) + \mathrm{Li}(10^{7(\frac{1}{2}+\alpha)/2})]$$
$$+\tfrac{1}{3}[\mathrm{Li}(10^{7(\frac{1}{2}-\alpha)/3}) + \mathrm{Li}(10^{7(\frac{1}{2}+\alpha)/3})]$$
$$+\tfrac{1}{5}[\mathrm{Li}(10^{7(\frac{1}{2}-\alpha)/5}) + \mathrm{Li}(10^{7(\frac{1}{2}+\alpha)/5})]$$
$$-\tfrac{1}{6}[\mathrm{Li}(10^{7(\frac{1}{2}-\alpha)/6}) + \mathrm{Li}(10^{7(\frac{1}{2}+\alpha)/6})]$$
$$+\tfrac{1}{7}[\mathrm{Li}(10^{7(\frac{1}{2}-\alpha)/7}) + \mathrm{Li}(10^{7(\frac{1}{2}+\alpha)/7})]$$
$$-\tfrac{1}{10}[\mathrm{Li}(10^{7(\frac{1}{2}-\alpha)/10}) + \mathrm{Li}(10^{7(\frac{1}{2}+\alpha)/10})]$$
$$+\tfrac{1}{11}[\mathrm{Li}(10^{7(\frac{1}{2}-\alpha)/11}) + \mathrm{Li}(10^{7(\frac{1}{2}+\alpha)/11})]$$
$$+\tfrac{1}{13}[\mathrm{Li}(10^{7(\frac{1}{2}-\alpha)/13}) + \mathrm{Li}(10^{7(\frac{1}{2}+\alpha)/13})]$$
$$-\tfrac{1}{14}[\mathrm{Li}(10^{7(\frac{1}{2}-\alpha)/14}) + \mathrm{Li}(10^{7(\frac{1}{2}+\alpha)/14})]$$
$$-\tfrac{1}{15}[\mathrm{Li}(10^{7(\frac{1}{2}-\alpha)/15}) + \mathrm{Li}(10^{7(\frac{1}{2}+\alpha)/15})]$$
$$+\tfrac{1}{17}[\mathrm{Li}(10^{7(\frac{1}{2}-\alpha)/17}) + \mathrm{Li}(10^{7(\frac{1}{2}+\alpha)/17})]$$
$$+\tfrac{1}{19}[\mathrm{Li}(10^{7(\frac{1}{2}-\alpha)/19}) + \mathrm{Li}(10^{7(\frac{1}{2}+\alpha)/19})]$$

$$-\frac{1}{21}[\mathrm{Li}(10^{7(\frac{1}{2}-\alpha)/21}) + \mathrm{Li}(10^{7(\frac{1}{2}+\alpha)/21})]$$

$$-\frac{1}{22}[\mathrm{Li}(10^{7(\frac{1}{2}-\alpha)/22}) + \mathrm{Li}(10^{7(\frac{1}{2}+\alpha)/22})]$$

$$+\frac{1}{23}[\mathrm{Li}(10^{7(\frac{1}{2}-\alpha)/23}) + \mathrm{Li}(10^{7(\frac{1}{2}+\alpha)/23})]$$

By [Derb, p. 390, endpoint 128], for complex numbers we have to use in MATHEMATICA

$$\mathrm{Li}\Big((10 \hat{\ } 7)^{1/2+i\alpha}\Big) = \mathrm{ExpIntegralEi}[(1\ /\ 2 + i\alpha)\mathrm{Log}[10 \hat{\ } 7]].$$

4.2 Approximating the Hypothesis Series for F(10^7) with the first 50 partial sums

To demonstrate the method we approximate the Hypothesis series with its first 50 partial sums.

We apply the following procedure

1. Input the first 50 zeros of the Zeta function on the line $x = \frac{1}{2}$, multiplied by i

$\delta = \{i\alpha_1, i\alpha_2, i\alpha_3, ..., i\alpha_{50}\}$

= **{14.1347 ⅈ,21.022 ⅈ,25.0109 ⅈ,30.4249 ⅈ,32.9351 ⅈ,37.5862 ⅈ,40.9187 ⅈ,43.3271 ⅈ,48.0052 ⅈ,49.7738 ⅈ,52.9703 ⅈ,56.4462 ⅈ,59.347 ⅈ,60.8318 ⅈ,65.1125 ⅈ,67.0798 ⅈ,69.5464 ⅈ,72.0672 ⅈ,75.7047 ⅈ,77.1448 ⅈ,79.3374 ⅈ,82.9104 ⅈ,84.7355 ⅈ,87.4253 ⅈ,88.8091 ⅈ,92.4919 ⅈ,94.6513 ⅈ,95.8706 ⅈ,98.8312 ⅈ,101.318 ⅈ,103.726 ⅈ,105.447 ⅈ,107.169 ⅈ,111.03 ⅈ,111.875 ⅈ,114.32 ⅈ,116.227 ⅈ,118.791 ⅈ,121.37 ⅈ,122.947 ⅈ,124.257 ⅈ,127.517 ⅈ,129.579 ⅈ,131.088 ⅈ,133.498 ⅈ,134.757 ⅈ,138.116 ⅈ,139.736 ⅈ,141.124 ⅈ,143.112ⅈ}**

2. Compute the term of the series at each of the zeros.

$\{\tau(\alpha_1),\tau(\alpha_2),\tau(\alpha_3),....\tau(\alpha_{50})\}=$

= **{-27.1113+0. i,7.77163 +0. i,-13.2144+0. i,-3.95944+0. i,-0.543427+0. i,-4.76162+0. i,1.84387 +0. i,-7.30523+0. i,-6.6019+0. i,7.08715 +0. i,4.92681 +0. i,6.68971 +0. i,-6.58084+0. i,-2.0042+0. i,-1.25439+0. i,-2.74293+0. i,-3.01527+0. i,3.91712 +0. i,-4.95623+0. i,2.99272 +0. i,0.597821 +0. i,4.44149 +0. i,-3.44063+0. i,-4.41609+0. i,3.9359 +0. i,-4.2541+0. i,3.89795 +0. i,1.61425 +0. i,0.660348 +0. i,2.06254 +0. i,-1.91944+0. i,0.0780606 +0. i,1.82515 +0. i,3.19721 +0. i,0.223178 +0. i,-3.48716+0. i,-2.74483+0. i,3.31879 +0. i,-2.69062+0. i,-2.02854+0. i,3.19817 +0. i,-2.05684+0. i,-1.6568+0. i,-2.90486+0. i,-0.689767+0. i,2.67602 +0. i,-2.62194+0. i,-0.606985+0. i,-0.39143+0. i,-1.90414+0. i}**

3. Compute the first 50 partial sums of the series

$$y_1 = \tau\left(\alpha_1\right)$$

$$y_2 = \tau\left(\alpha_1\right) + \tau\left(\alpha_2\right)$$

$$y_3 = \tau\left(\alpha_1\right) + \tau\left(\alpha_2\right) + \tau\left(\alpha_3\right)$$

....

$$y_{50} = \tau\left(\alpha_1\right) + \tau\left(\alpha_2\right) + \tau\left(\alpha_3\right) + ... + \tau\left(\alpha_{50}\right)$$

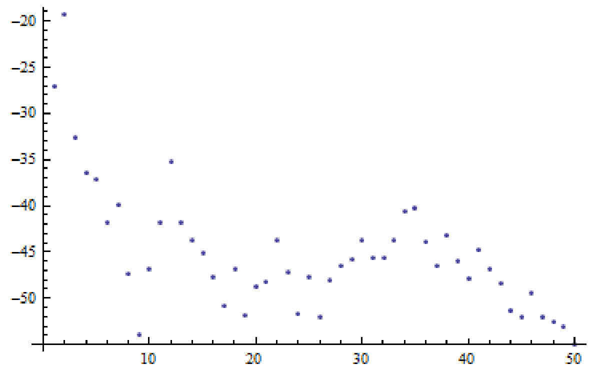
The following program inputs **t,** and prints the first 50 partial sums of the infinite series **y**,

```
y=Range[50]
y[[1]]=t[[1]]
Do [y[[j]] = y[[j-1]]+t[[j]],{j, 2,50}]
Print[y]
```

{-27.1113+0. i,-19.3397+0. i,-32.554+0. i,-36.5135+0. i,-37.0569+0. i,-41.8185+0. i,-39.9746+0. i,-47.2799+0. i,-53.8818+0. i,-46.7946+0. i,-41.8678+0. i,-35.1781+0. i,-41.7589+0. i,-43.7631+0. i,-45.0175+0. i,-

47.7605+0. i,-50.7757+0. i,-46.8586+0. i,-51.8148+0. i,-48.8221+0. i,-48.2243+0. i,-43.7828+0. i,-47.2234+0. i,-51.6395+0. i,-47.7036+0. i,-51.9577+0. i,-48.0598+0. i,-46.4455+0. i,-45.7852+0. i,-43.7226+0. i,-45.6421+0. i,-45.564+0. i,-43.7389+0. i,-40.5417+0. i,-40.3185+0. i,-43.8056+0. i,-46.5505+0. i,-43.2317+0. i,-45.9223+0. i,-47.9508+0. i,-44.7527+0. i,-46.8095+0. i,-48.4663+0. i,-51.3712+0. i,-52.0609+0. i,-49.3849+0. i,-52.0069+0. i,-52.6138+0. i,-53.0053+0. i,-54.9094+0. i}

The Plot of **y**, obtained with **ListPlot[y],** is



According to Lehmer's computation, the observed Hypothesis series is

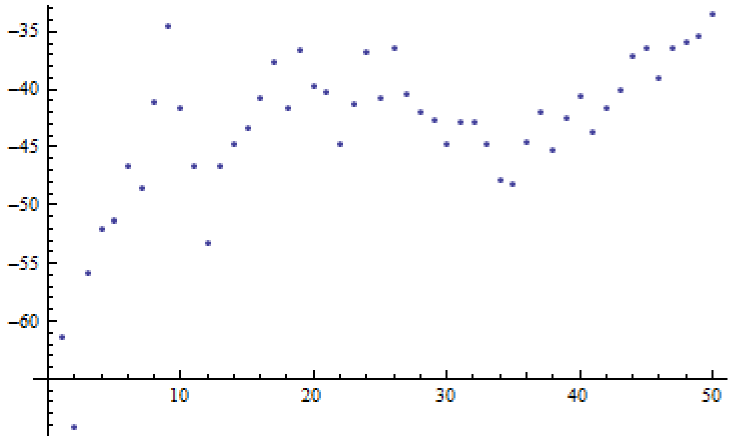
664579-664667.4443892473`= -88.4443892473355`

The errors between the observed Hypothesis series and the partial sums of the infinite series are

yy=Re[-88.4443892473355`-y]

=-61.3331,-69.1047,-55.8904,-51.9309,-51.3875,-46.6259,-48.4697,-41.1645,-34.5626,-41.6498,-46.5766,-53.2663,-46.6854,-44.6812,-43.4269,-40.6839,-37.6686,-41.5858,-36.6295,-39.6223,-40.2201,-44.6616,-41.2209,-36.8049,-40.7408,-36.4867,-40.3846,-41.9989,-42.6592,-44.7217,-42.8023,-42.8804,-44.7055,-47.9027,-48.1259,-44.6387,-41.8939,-45.2127,-42.5221,-40.4935,-43.6917,-41.6349,-39.9781,-37.0732,-36.3834,-39.0595,-36.4375,-35.8305,-35.4391,-33.535

The errors of the partial sums are

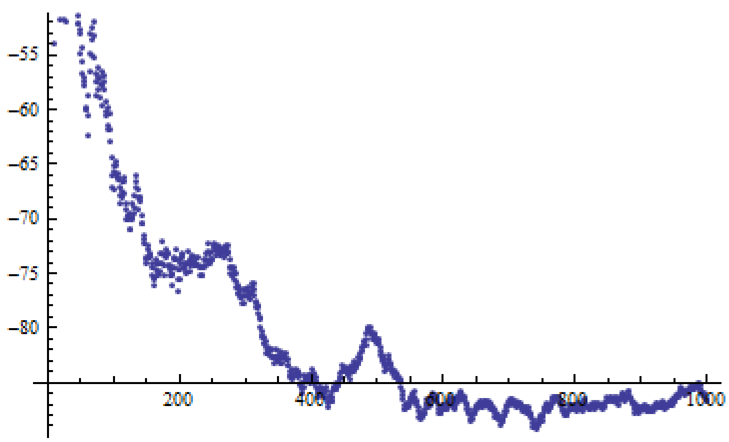


To reduce the error, we need to use more terms of the Hypothesis series.

We next try 1000.

4.3 Approximating the Hypothesis Series for F(10^7) with its first 1000 partial sums

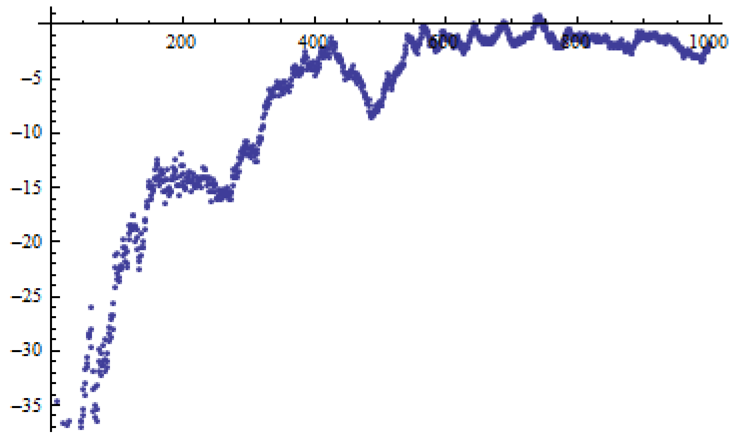
The Plot of the first 1000 partial sums **y** is



The last 10 partial sums y_{991} to y_{1000} are

-85.552,-85.7195,-85.9942,-86.1248,-85.9541,-86.165,-85.9645,-86.2453,
-86.4756,-86.7459

The plot of the errors `yy=Re[-88.4443892473355`-y]` is



We see that the error decreases at first, but starts growing later.

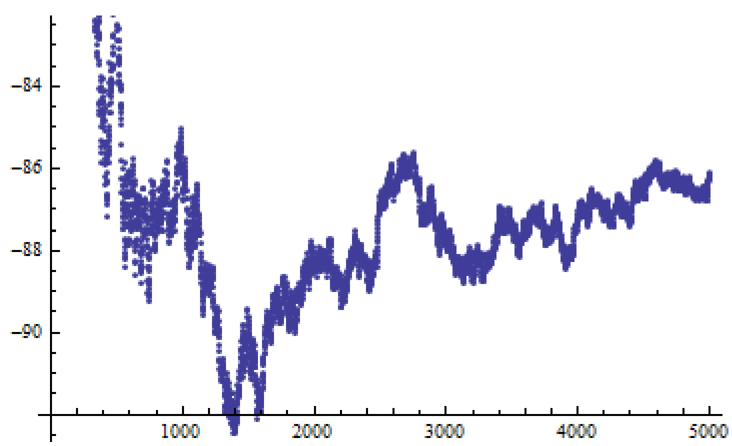
The last 10 errors of the partial sums yy_{991} to yy_{1000} are

-2.8924,-2.72494,-2.45016,-2.31959,-2.49025,-2.27943,-2.47994,-2.19906,
-1.9688,-1.69848

Can we get smaller error with 5000 partial sums?

4.4 Approximating the Hypothesis Series for F(10^7) with its first 5000 partial sums

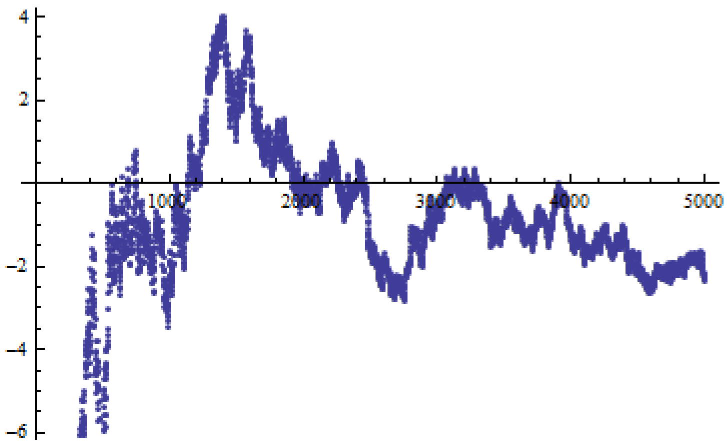
The Plot of the first 5000 partial sums **y** is



The last 10 partial sums y_{4991} to y_{5000} are

-86.3152,-86.2419,-86.2878,-86.2152,-86.209,-86.1451,-86.0839,-86.1528,
-86.1743,-86.247

The Plot of the errors **yy=Re[-88.4443892473355`-y] is**



We see that the fluctuations of the partial sums are not symmetric about the observed Hypothesis series, and do not decay significantly when we increase the number of the partial sums. The partial sum y_{2000} is a better approximation than y_{5000} .

The last 10 errors in the partial sums yy_{4991} to yy_{5000} are

-2.1292,-2.20249,-2.15663,-2.22923,-2.23541,-2.29926,-2.36047,-2.2916,
-2.27005,-2.19742

Can we get better convergence if we average the partial sums?

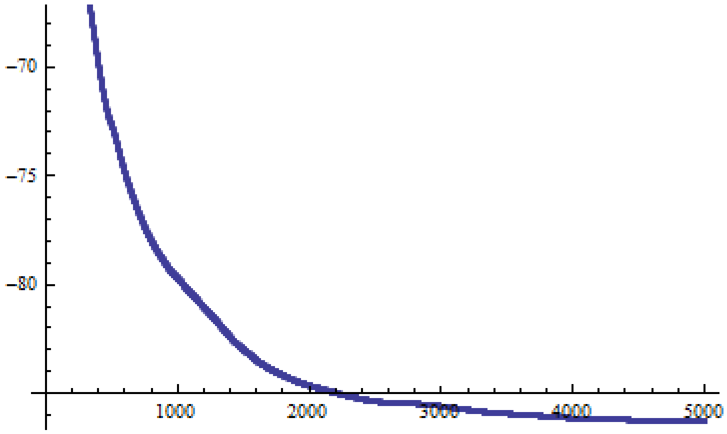
4.5 Approximating the Hypothesis Series for F(10^7) with the first 5000 averaged partial sums

We compute the first 5000 Cesaro-Arithmetic-Means of the partial sums

The following program inputs **y,** and outputs the first 5000 Cesaro Arithmetic Means **u**,

```
z=Range[5000]
u=Range[5000]
z[[1]]=y[[1]]
u[[1]]=z[[1]]
Do [z[[j]] = z[[j- 1]]+y[[j]];
    u[[j]]=z[[j]]/j,{j, 2,5000}]
Print[u]
```

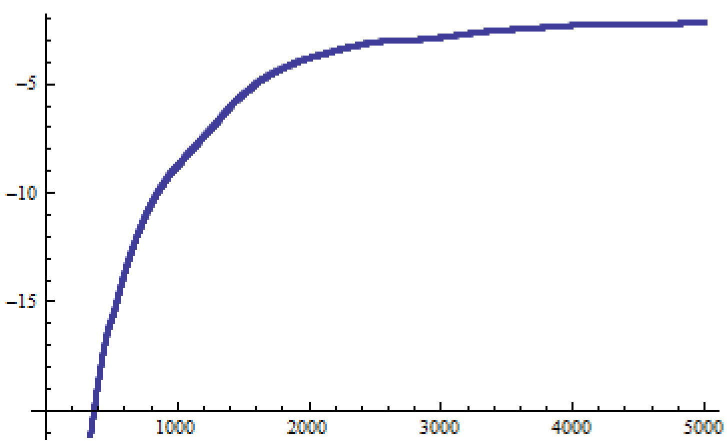
The Arithmetic Means **u** fluctuate less, and have a smooth plot.



The last 10 Arithmetic Means u_{4991} to u_{5000} are

-86.2375,-86.2375,-86.2375,-86.2375,-86.2375,-86.2375,-86.2375,
-86.2374,-86.2374,-86.2374

The error of the Arithmetic Means `uu=Re[-88.4443892473355`-u]` is



The last 10 errors of the Arithmetic Means are

-2.20688,-2.20688,-2.20687,-2.20687,-2.20688,-2.20689,-2.20693,-2.20694,-2.20695,-2.20695

These errors are not significantly smaller than the errors of the partial sums.

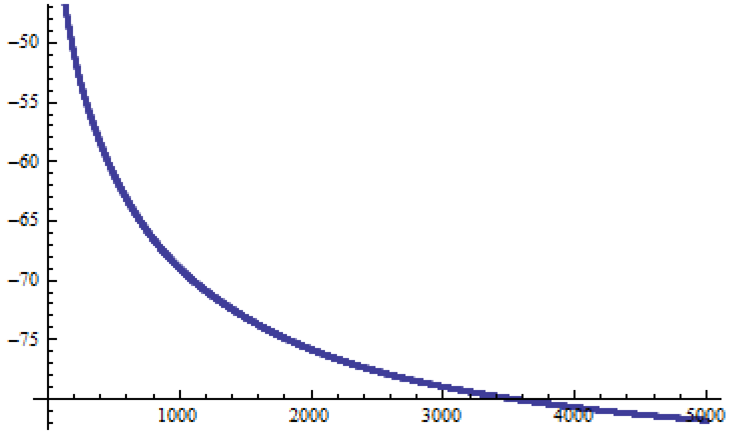
Can we get smaller error by averaging the Cesaro-Arithmetic Means?

4.6 Approximating the Hypothesis Series for F(10^7) by averaging the first 5000 Arithmetic Means

The following program inputs the Arithmetic Means **u,** and outputs their Arithmetic Means **v**,

```
w=Range[5000]
v=Range[5000]
w[[1]]=u[[1]]
v[[1]]=w[[1]]
Do [w[[j]] = w[[j- 1]]+u[[j]];
    v[[j]]=w[[j]]/j,{j, 2,5000}]
Print[v]
```

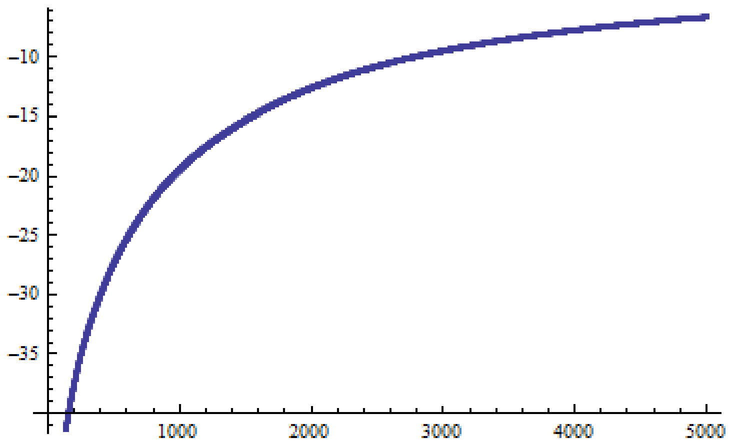
The plot of the first 5000 2^{nd} Arithmetic Means is



The last 10 values of the 2^{nd} Arithmetic Means are

-81.8004,-81.8012,-81.8021,-81.803,-81.8039,-81.8048,-81.8057,-81.8066,-81.8075,-81.8083

The plot of the error of the 2nd Means **vv=Re[-88.4443892473355`-v]** is



The last 10 errors of the 2nd Means are

-6.64403,-6.64314,-6.64226,-6.64137,-6.64048,-6.63959,-6.63871,-6.63782,-6.63693,-6.63605

These errors are far greater than the errors in the partial series sums.

Thus, we renew our attempt to obtain smaller errors by using more partial sums.

4.7 Statistical Analysis of the First 5000 Partial sums of the Hypothesis series for F(10^7)

The Mean of y is

$$\mu = \frac{y_1 + y_2 + \dots y_{5000}}{5000} = \text{-86.23743652782214}$$

The Variance of y is

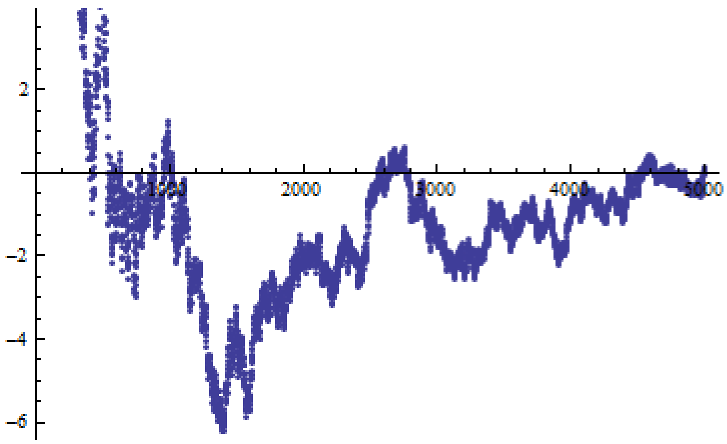
$$\sigma^2 = 37.22306406258638`$$

Thus, the Standard Deviation of y is

$$\sigma = 6.101070730829662`$$

The Plot of the Hypothesis Series values y about their mean here is obtained with

ListPlot[Re[y+86.23743652780221`]]



If we assume that the values of y are normally distributed, we can expect with 95% confidence to find the partial sums of the Hypothesis Series for F(10^7) between

$$\mu - 2\sigma = -86.2374365278 - 2 \times 6.101070731$$

= -98.4

and

$$\mu + 2\sigma = -86.2374365278 + 2 \times 6.101070731$$

= -74

The observed value of the Hypothesis series, -88.4, is in this range

Since the main term in $F(10\,\hat{}\,7)$ is

664,667.45

we can expect with 95% confidence to find the values of $F(10 \hat{} 7)$ between

664,667.45 – 98.4 = 664,569

and

664,667.45 – 74 = $664,593$

This range includes the exact value F(10^7)=664,579.

If we further assume that the Means of y are normally distributed with Standard Deviation

$$\sigma_{5000} = \frac{\sigma}{\sqrt{5000}} = 0.086,$$

we can expect with 95% confidence to find the Means of the partial sums of the Hypothesis Series for $F(10 \hat{} 7)$ between

$$\mu - 2\sigma_{5000} = -86.2374365278 - 2 \times 0.086$$

= -86.4

and

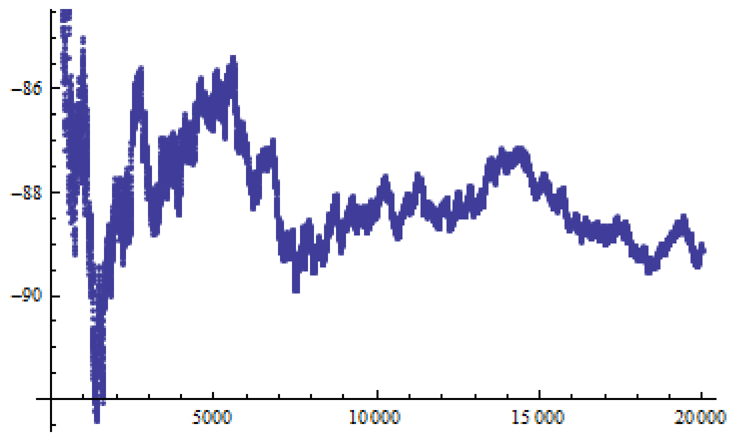
$$\mu + 2\sigma_{5000} = -86.2374365278 + 2 \times 0.086$$

= -86.06.

Since the observed value of the Hypothesis series, -88.4, is out of this range, the assumption must be wrong. The means of the partial sums of the Hypothesis Series are not normally distributed in that small range.

4.8 Approximating the Hypothesis Series for F(10^7) with its first 20,000 partial sums

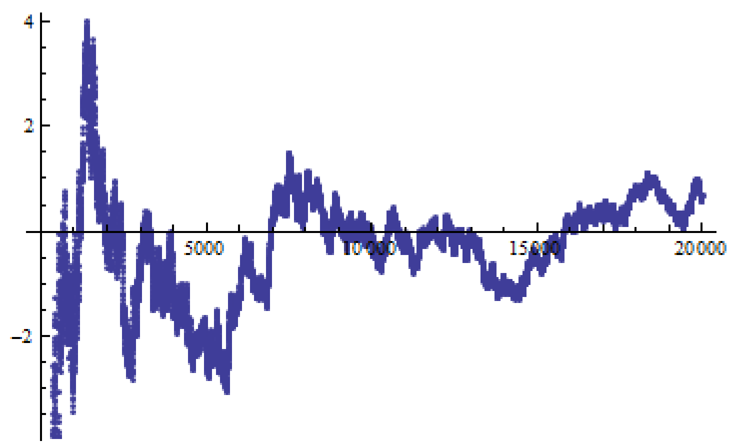
The plot of the first 20,000 partial sums is



The last 10 partial sums $y_{19,991}$ to $y_{20,000}$ are

**-89.0873,-89.0692,-89.0852,-89.0677,-89.0892,-89.0713,-89.0805,
-89.0958,-89.1105,-89.1267**

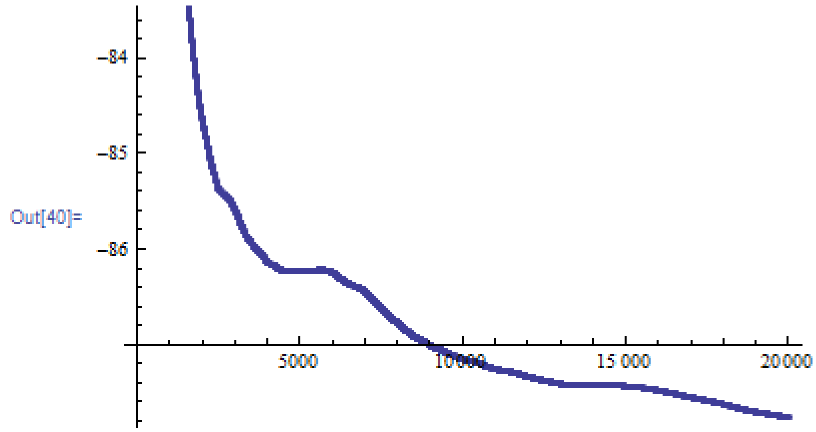
The plot of the error of the partial sums **yy=Re[-88.4443892473355`-y]** is



The errors of the last 10 partial sums are

**0.64294,0.624764,0.640842,0.623359,0.644843,0.626914,0.636071,0.651365,0.666071,
0.682285**

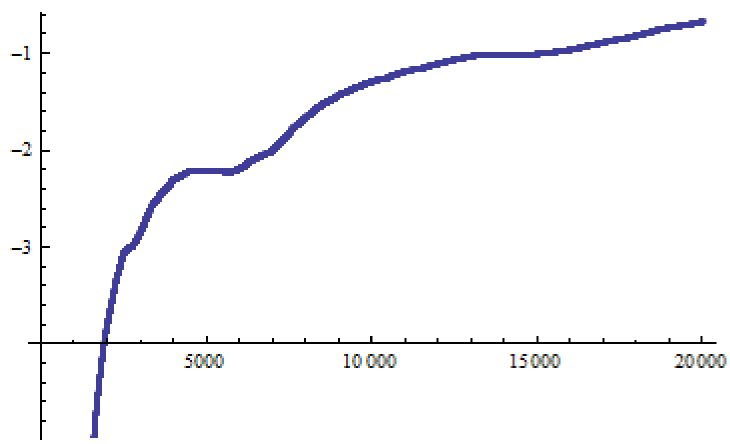
The plot of the first 20,000 Cesaro-Arithmetic Means is



The last 10 Arithmetic Means $u_{19,991}$ to $u_{20,000}$ are

-87.7706,-87.7707,-87.7707,-87.7708,-87.7709,-87.7709,-87.771,-87.7711, -87.7711,-87.7712

The plot of the errors of the first 20,000 Arithmetic Means is



The errors of the last 10 Arithmetic Means u_{19991} till $u_{20,000}$ are

-0.673783,-0.673718,-0.673653,-0.673588,-0.673522,-0.673457,-0.673391, -0.673325,-0.673258,-0.67319

4.9 Statistical Analysis of the First 20,000 Partial sums of the Hypothesis series for F(10^7)

The Mean of y is

$$\mu = \frac{y_1 + y_2 + ...y_{20,000}}{20,000} = \text{-}87.77$$

The Variance of y is

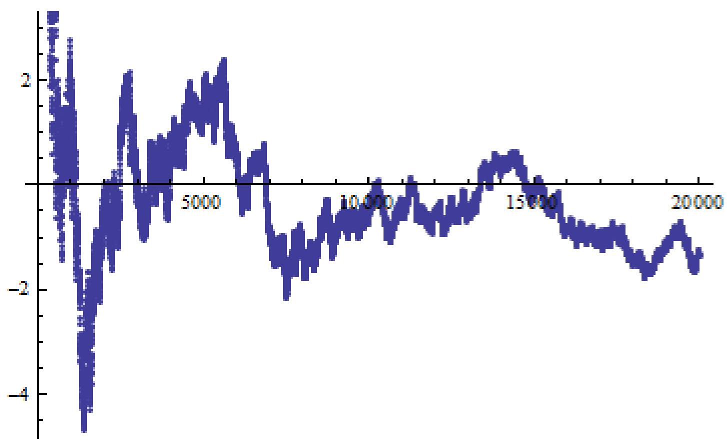
$$\sigma^2 = 10.55$$

Thus, the Standard Deviation of y is

$$\sigma = 3.25$$

The Plot of the Hypothesis Series values y about their mean here is obtained with

```
ListPlot[Re[y+87.77119891580756`]]
```



If we assume that the values of y are normally distributed, we can expect with 95% confidence to find the partial sums of the Hypothesis Series for F(10^7) between

$$\mu - 2\sigma = -87.77 - 2 \times 3.25$$

$$= \text{-}94.2$$

and

$$\mu + 2\sigma = -87.77 + 2 \times 3.25$$

$$= -81.3$$

The observed value of the Hypothesis series, -88.4, is in this range

Since the main term in $F(10 \,\hat{}\, 7)$ is

664,667.45

we can expect with 95% confidence to find the values of $F(10 \,\hat{}\, 7)$ between

664,667.45 – 94.2 = 664,573

and

664,667.45 – 81.3 = $664{,}586$

This range includes the exact value F(10^7)=664,579.

If we further assume that the Means of y are normally distributed with Standard Deviation

$$\sigma_{20,000} = \frac{\sigma}{\sqrt{20000}} = 0.02,$$

we can expect with 95% confidence to find the Means of the partial sums of the Hypothesis Series for $F(10 \,\hat{}\, 7)$ between

$$\mu - 2\sigma_{20,000} = -87.77 - 2 \times 0.02$$

$$= -87.81$$

and

$$\mu + 2\sigma_{20,000} = -87.77 + 2 \times 0.02$$

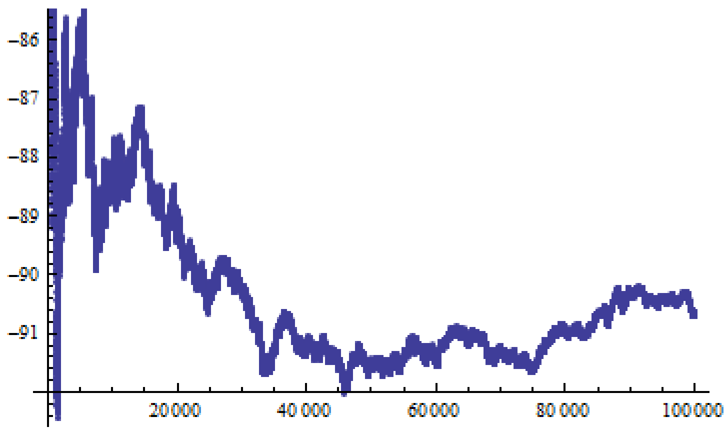
$$= -87.73.$$

Since the observed value of the Hypothesis series, -88.4, is out of this range, the assumption must be wrong. The means of the partial sums of the Hypothesis Series are not normally distributed in that small range.

Finally, we compute with 100,000 partial sums.

4.10 Approximating the Hypothesis Series for F(10^7) with its first 100,000 partial sums

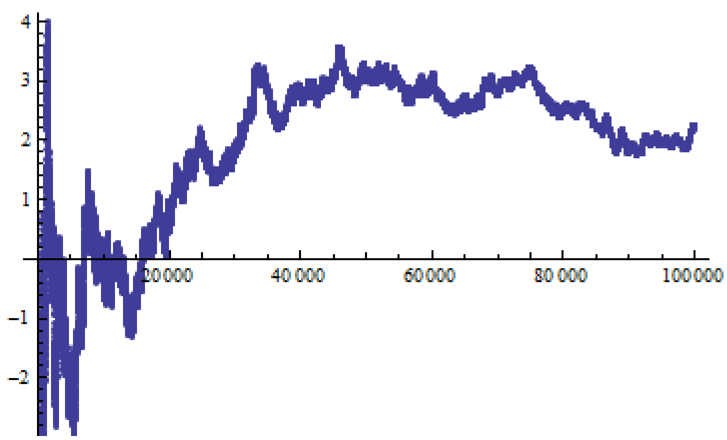
The plot of the first 100,000 partial sums is



The last 10 partial sums $y_{99,991}$ to $y_{100,000}$ are

-90.6765,-90.6802,-90.6755,-90.6713,-90.6705,-90.6753,-90.6758,-90.6758,-90.6771,-90.6771

The error of the partial sums, **yy=Re[-88.4443892473355`-y]** is

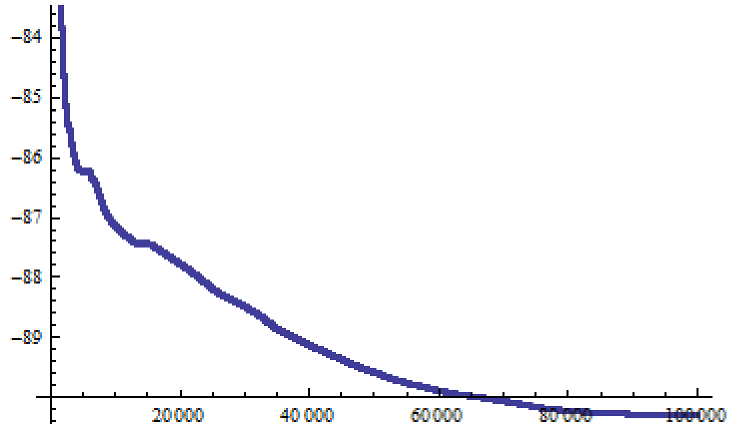


The error grows after 20,000 partial sums and up to 100,000 partial sums.

The errors of the last 10 partial sums are

2.23216,2.23586,2.23106,2.22691,2.22608,2.2309,2.2314,2.23143,2.23272, 2.23275

The plot of the first 100,000 Cesaro-Arithmetic Means is

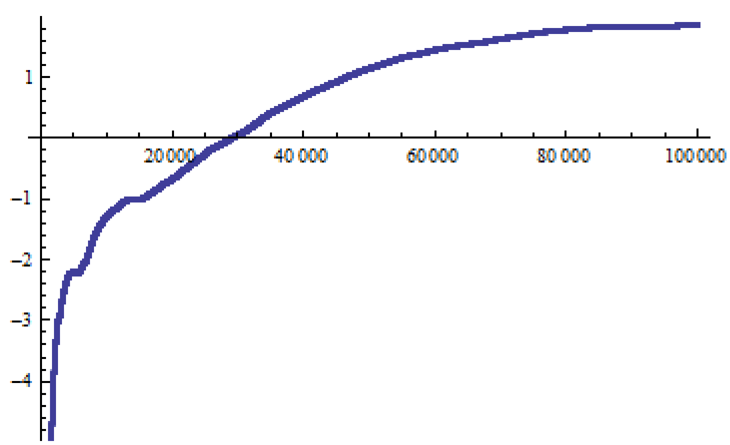


It is smoother than the partial sums plot, but it does not converge faster to $F(10\,\hat{}\,7)$.

The last 10 Arithmetic Means $u_{99,991}$ to $u_{100,000}$ are

-90.2903,-90.2903,-90.2903,-90.2903,-90.2903,-90.2903,-90.2903,-90.2903, -90.2904,-90.2904

The plot of the Arithmetic Means error **uu=Re[88.4443892473355`-u]** is



Beyond 30,000 partial sums of the infinite series, the error grows.

The errors of the last 10 Arithmetic Means are

```
1.84593,1.84593,1.84594,1.84594,1.84595,1.84595,1.84595,1.84596,1.84596,
1.84597
```

4.11 Statistical Analysis of the First 100,000 Partial sums of the Hypothesis series for F(10^7)

The Mean of y is

$$\mu = \frac{y_1 + y_2 + ...y_{100,000}}{100,000} = \text{-90.29}$$

The Variance of y is

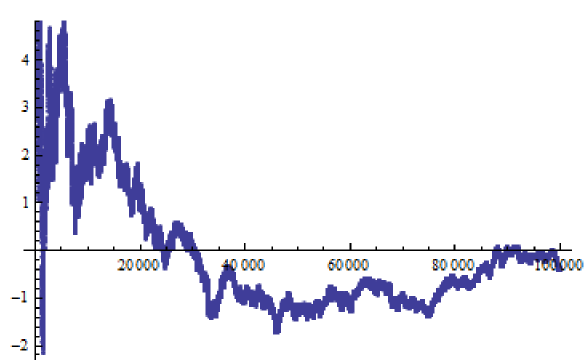
$$\sigma^2 = 3.93655$$

Thus, the Standard Deviation of y is

$$\sigma = 1.98407$$

The Plot of the Hypothesis Series values y about their mean here is obtained with

```
ListPlot[Re[y+90.29035494492261`]]
```



If we assume that the values of y are normally distributed, we can expect with 95% confidence to find the partial sums of the Hypothesis Series for F(10^7) between

$$\mu - 2\sigma = -90.29 - 2 \times 1.984$$

= -94.2

and

$$\mu + 2\sigma = -90.29 + 2 \times 1.984$$

= -86.3

The observed value of the Hypothesis series, -88.4, is in this range

Since the main term in $F(10 \,\hat{}\, 7)$ is

664,667.45

we can expect with 95% confidence to find the values of $F(10 \,\hat{}\, 7)$ between

664,667.45 – 94.2 = 664,573

and

664,667.45 – 86.3 = $664,589$

This range includes the exact value F(10^7)=664,579.

If we further assume that the Means of y are normally distributed with Standard Deviation

$$\sigma_{100,000} = \frac{\sigma}{\sqrt{100000}} = 0.006,$$

we can expect with 95% confidence to find the Means of the partial sums of the Hypothesis Series for $F(10 \,\hat{}\, 7)$ between

$$\mu - 2\sigma_{100,000} = -90.29 - 2 \times 0.006$$

$$= -90.3$$

and

$$\mu + 2\sigma_{100,000} = -90.29 + 2 \times 0.006$$

$$= -90.2.$$

Since the observed value of the Hypothesis series, -88.4, is out of this range, the assumption must be wrong. The means of the partial sums of the Hypothesis Series are not normally distributed in that small range.

5

F(10^21), and its exact terms

5.1 F(10^21)

We know not the value of $F(10 \text{ ^ } 21)$.

According to [Cran p.152] , it was approximated in [Lag] by

21,127,269,486,018,731,928.

But [Lag] do not present their computation, and their claim can not be reproduced independently. We discuss this further in 6.8.

5.2 m(10^21)

We sum up to $m(10^{21}) = 69$, because

$$\sqrt[69]{10^{21}} = 2.015 > 2$$

and

$$\sqrt[70]{10^{21}} = 1.995 < 2 .$$

By Appendix A, we use the 43 numbers

1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, 33, 34, 35, 37, 38, 39, 41, 42, 43, 46, 47, 51, 53, 55, 57, 58, 59, 61, 62, 65, 66, 67, 69.

That is, each of the four terms in Riemann's Formula is the sum of 43 terms.

5.3 The Riemann approximation term for F(10^21)

This is the main term.

s1= $N[LogIntegral[10 \text{ ^ } 21]$

$$-\tfrac{1}{2}\,LogIntegral[\sqrt{10\;\hat{}\;21}]$$
$$-\tfrac{1}{3}\,LogIntegral[\sqrt[3]{10\;\hat{}\;21}]$$
$$-\tfrac{1}{5}\,LogIntegral[\sqrt[5]{10\;\hat{}\;21}]$$
$$+\tfrac{1}{6}\,LogIntegral[\sqrt[6]{10\;\hat{}\;21}]$$
$$-\tfrac{1}{7}\,LogIntegral[\sqrt[7]{10\;\hat{}\;21}]$$
$$+\tfrac{1}{10}\,LogIntegral[\sqrt[10]{10\;\hat{}\;21}]$$
$$-\tfrac{1}{11}\,LogIntegral[\sqrt[11]{10\;\hat{}\;21}]$$
$$-\tfrac{1}{13}\,LogIntegral[\sqrt[13]{10\;\hat{}\;21}]$$
$$+\tfrac{1}{14}\,LogIntegral[\sqrt[14]{10\;\hat{}\;21}]$$
$$+\tfrac{1}{15}\,LogIntegral[\sqrt[15]{10\;\hat{}\;21}]$$
$$-\tfrac{1}{17}\,LogIntegral[\sqrt[17]{10\;\hat{}\;21}]$$
$$-\tfrac{1}{19}\,LogIntegral[\sqrt[19]{10\;\hat{}\;21}]$$
$$+\tfrac{1}{21}\,LogIntegral[\sqrt[21]{10\;\hat{}\;21}]$$
$$+\tfrac{1}{22}\,LogIntegral[\sqrt[22]{10\;\hat{}\;21}]$$
$$-\tfrac{1}{23}\,LogIntegral[\sqrt[23]{10\;\hat{}\;21}]$$
$$+\tfrac{1}{26}\,LogIntegral[\sqrt[26]{10\;\hat{}\;21}]$$
$$-\tfrac{1}{29}\,LogIntegral[\sqrt[29]{10\;\hat{}\;21}]$$
$$-\tfrac{1}{30}\,LogIntegral[\sqrt[30]{10\;\hat{}\;21}]$$
$$-\tfrac{1}{31}\,LogIntegral[\sqrt[31]{10\;\hat{}\;21}]$$
$$+\tfrac{1}{33}\,LogIntegral[\sqrt[33]{10\;\hat{}\;21}]$$
$$+\tfrac{1}{34}\,LogIntegral[\sqrt[34]{10\;\hat{}\;21}]$$
$$+\tfrac{1}{35}\,LogIntegral[\sqrt[35]{10\;\hat{}\;21}]$$
$$-\tfrac{1}{37}\,LogIntegral[\sqrt[37]{10\;\hat{}\;21}]$$

$$+\frac{1}{38}LogIntegral[\sqrt[38]{10\ \hat{}\ 21}]$$
$$+\frac{1}{39}LogIntegral[\sqrt[39]{10\ \hat{}\ 21}]$$
$$-\frac{1}{41}LogIntegral[\sqrt[41]{10\ \hat{}\ 21}]$$
$$-\frac{1}{42}LogIntegral[\sqrt[42]{10\ \hat{}\ 21}]$$
$$-\frac{1}{43}LogIntegral[\sqrt[43]{10\ \hat{}\ 21}]$$
$$+\frac{1}{46}LogIntegral[\sqrt[46]{10\ \hat{}\ 21}]$$
$$-\frac{1}{47}LogIntegral[\sqrt[47]{10\ \hat{}\ 21}]$$
$$+\frac{1}{51}LogIntegral[\sqrt[51]{10\ \hat{}\ 21}]$$
$$-\frac{1}{53}LogIntegral[\sqrt[53]{10\ \hat{}\ 21}]$$
$$+\frac{1}{55}LogIntegral[\sqrt[55]{10\ \hat{}\ 21}]$$
$$+\frac{1}{57}LogIntegral[\sqrt[57]{10\ \hat{}\ 21}]$$
$$+\frac{1}{58}LogIntegral[\sqrt[58]{10\ \hat{}\ 58}]$$
$$-\frac{1}{59}LogIntegral[\sqrt[59]{10\ \hat{}\ 21}]$$
$$-\frac{1}{61}LogIntegral[\sqrt[61]{10\ \hat{}\ 21}]$$
$$+\frac{1}{62}LogIntegral[\sqrt[62]{10\ \hat{}\ 21}]$$
$$+\frac{1}{65}LogIntegral[\sqrt[65]{10\ \hat{}\ 21}]$$
$$-\frac{1}{66}LogIntegral[\sqrt[66]{10\ \hat{}\ 21}]$$
$$-\frac{1}{67}LogIntegral[\sqrt[67]{10\ \hat{}\ 21}]$$
$$+\frac{1}{69}LogIntegral[\sqrt[69]{10\ \hat{}\ 21}]$$

= 21,127,269,485,932,299,723.753244052

5.4 The Log2 term for F(10^21)

The log2 term is

s2=

N[(-1+1/2+1/3+1/5-1/6+1/7-1/10+1/11+1/13-1/14-1/15+1/17+1/19-1/21-1/22+1/23-1/26+1/29+1/30+1/31-1/33-1/34-1/35+1/37-1/38-1/39+1/41+1/42+1/43-1/46+1/47-1/51+1/53-1/55-1/57-1/58+1/59+1/61-1/62-1/65+1/66+1/67-1/69)*Log[2]]

= -0.010940444366836378`

This turns out to be negligible compared with the first term.

5.5 The Integral term for F(10^21)

$$\textbf{s4=}\,\mathrm{N}[\int_{10\hat{}21}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx$$

$$-\frac{1}{2}\int_{\sqrt{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx$$

$$-\frac{1}{3}\int_{\sqrt[3]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx$$

$$-\frac{1}{5}\int_{\sqrt[5]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx$$

$$+\frac{1}{6}\int_{\sqrt[6]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx$$

$$-\frac{1}{7}\int_{\sqrt[7]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx$$

$$+\frac{1}{10}\int_{\sqrt[10]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx$$

$$-\frac{1}{11}\int_{\sqrt[11]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx$$

$$-\frac{1}{13}\int_{\sqrt[13]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx$$

$$+\frac{1}{14}\int_{\sqrt[14]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx$$

$$+\frac{1}{15}\int_{\sqrt[15]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx$$

$$-\frac{1}{17}\int_{\sqrt[17]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx$$

$$-\frac{1}{19}\int_{\sqrt[19]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx$$

$$+\frac{1}{21}\int_{\sqrt[21]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx$$

$$+\frac{1}{22}\int_{\sqrt[22]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx$$

$$-\frac{1}{23}\int_{\sqrt[23]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx$$

$$+\frac{1}{26}\int_{\sqrt[26]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx$$

$$-\frac{1}{29}\int_{\sqrt[29]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx$$

$$-\frac{1}{30}\int_{\sqrt[30]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx$$

$$-\frac{1}{31}\int_{\sqrt[31]{10\hat{}7}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx$$

$$+\frac{1}{33}\int_{\sqrt[33]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx$$

$$+\frac{1}{34}\int_{\sqrt[34]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx$$

$$+\frac{1}{35}\int_{\sqrt[35]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx$$

$$-\frac{1}{37}\int_{\sqrt[37]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx$$

$$+\frac{1}{38}\int_{\sqrt[38]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx$$

$$-\frac{1}{41}\int_{\sqrt[41]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx$$

$$-\frac{1}{42}\int_{\sqrt[42]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx$$

$$-\frac{1}{43}\int_{\sqrt[43]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx$$

$$+\frac{1}{46}\int_{\sqrt[46]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx$$

$$-\frac{1}{47}\int_{\sqrt[47]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx$$

$$+\frac{1}{51}\int_{\sqrt[51]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx$$

$$-\frac{1}{53}\int_{\sqrt[53]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx$$

$$+\frac{1}{55}\int_{\sqrt[55]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx$$

$$+\frac{1}{57}\int_{\sqrt[57]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx$$

$$+\frac{1}{58}\int_{\sqrt[58]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx$$

$$-\frac{1}{59}\int_{\sqrt[59]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx$$

$$-\frac{1}{61}\int_{\sqrt[61]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx$$

$$+\frac{1}{62}\int_{\sqrt[62]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx$$

$$+\frac{1}{65}\int_{\sqrt[65]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx$$

$$-\frac{1}{66}\int_{\sqrt[66]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx$$

$$-\frac{1}{67}\int_{\sqrt[67]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx$$

$$+\frac{1}{69}\int_{\sqrt[69]{10\hat{}21}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx$$

= 0.0017008980754812085`

This too, turns out to be negligible compared with the first term.

6

The Hypothesis Series for F(10^21)

6.1 The Hypothesis Series for F(10^21)

Each term of the Hypothesis series is the sum of 43 pairs of logarithmic integrals computed at the Zeta-zeros $i\alpha$, and $-i\alpha$.

$$\tau(\alpha) = \mathrm{Li}(10^{21(\frac{1}{2}-\alpha)}) + \mathrm{Li}(10^{21(\frac{1}{2}+\alpha)})$$

$$+\tfrac{1}{2}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/2}) + \mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/2})]$$

$$+\tfrac{1}{3}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/3}) + \mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/3})]$$

$$+\tfrac{1}{5}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/5}) + \mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/5})]$$

$$-\tfrac{1}{6}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/6}) + \mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/6})]$$

$$+\tfrac{1}{7}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/7}) + \mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/7})]$$

$$-\tfrac{1}{10}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/10}) + \mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/10})]$$

$$+\tfrac{1}{11}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/11}) + \mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/11})]$$

$$+\tfrac{1}{13}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/13}) + \mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/13})]$$

$$-\tfrac{1}{14}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/14}) + \mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/14})]$$

$$-\tfrac{1}{15}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/15}) + \mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/15})]$$

$$+\tfrac{1}{17}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/17}) + \mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/17})]$$

$$+\tfrac{1}{19}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/19}) + \mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/19})]$$

$$-\tfrac{1}{21}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/21})+\mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/21})]$$

$$-\tfrac{1}{22}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/22})+\mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/22})]$$

$$+\tfrac{1}{23}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/23})+\mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/23})]$$

$$-\tfrac{1}{26}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/26})+\mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/26})]$$

$$+\tfrac{1}{29}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/29})+\mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/29})]$$

$$+\tfrac{1}{30}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/30})+\mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/30})]$$

$$+\tfrac{1}{31}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/31})+\mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/31})]$$

$$-\tfrac{1}{33}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/33})+\mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/33})]$$

$$-\tfrac{1}{34}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/34})+\mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/34})]$$

$$+\tfrac{1}{35}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/35})+\mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/35})]$$

$$+\tfrac{1}{37}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/37})+\mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/37})]$$

$$-\tfrac{1}{38}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/38})+\mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/38})]$$

$$-\tfrac{1}{39}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/39})+\mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/39})]$$

$$+\tfrac{1}{41}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/41})+\mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/41})]$$

$$+\tfrac{1}{42}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/42})+\mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/42})]$$

$$+\tfrac{1}{43}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/43})+\mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/43})]$$

$$-\tfrac{1}{46}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/46})+\mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/46})]$$

$$+\tfrac{1}{47}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/47}) + \mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/47})]$$

$$-\tfrac{1}{51}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/51}) + \mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/51})]$$

$$+\tfrac{1}{53}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/53}) + \mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/53})]$$

$$-\tfrac{1}{55}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/55}) + \mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/55})]$$

$$-\tfrac{1}{57}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/57}) + \mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/57})]$$

$$-\tfrac{1}{58}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/58}) + \mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/58})]$$

$$+\tfrac{1}{59}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/59}) + \mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/59})]$$

$$+\tfrac{1}{61}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/61}) + \mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/61})]$$

$$-\tfrac{1}{62}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/62}) + \mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/62})]$$

$$-\tfrac{1}{65}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/65}) + \mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/65})]$$

$$+\tfrac{1}{66}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/66}) + \mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/66})]$$

$$+\tfrac{1}{67}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/67}) + \mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/67})]$$

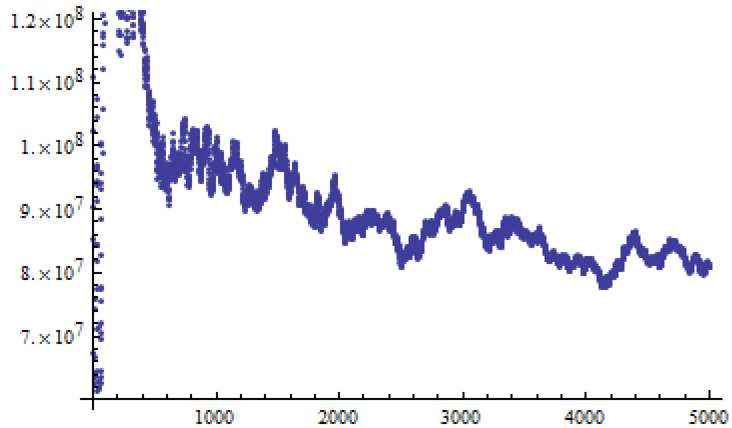
$$-\tfrac{1}{69}[\mathrm{Li}(10^{21(\frac{1}{2}-\alpha)/69}) + \mathrm{Li}(10^{21(\frac{1}{2}+\alpha)/69})]$$

By [Derb, p. 390, endpoint 128], for complex numbers we have to use in MATHEMATICA

$$\mathrm{Li}\Big((10\ \hat{}\ 21)^{1/2+i\alpha}\Big) = \mathrm{ExpIntegralEi}[(1\ /\ 2 + i\alpha)\mathrm{Log}[10\ \hat{}\ 21]].$$

6.2 Approximating the Hypothesis Series for F(10^21) with its first 5000 partial sums

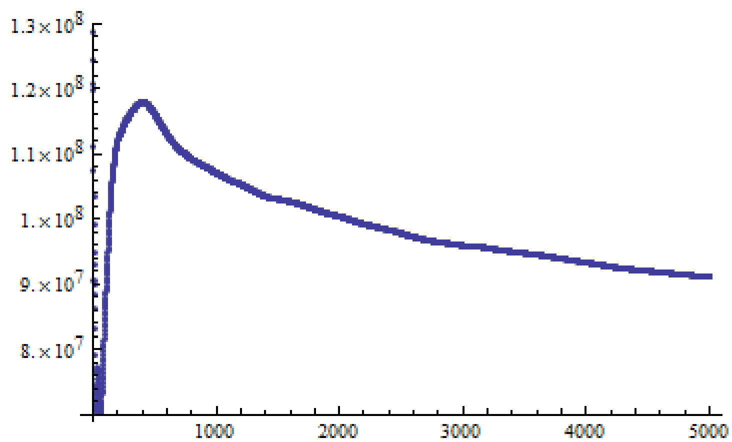
The plot of the first 5000 partial sums is



The last 10 partial sums y_{4991} to y_{5000} are

8.15401×10^7,8.12998×10^7,8.10862×10^7,8.08623×10^7,8.0938×10^7, 8.09201×10^7,8.09295×10^7,8.10946×10^7,8.08965×10^7,8.11305×10^7

The plot of the first 5000 Cesaro-Arithmetic Means is



The last 10 Arithmetic Means u_{4991} to u_{5000} are

9.10462×10^7,9.10442×10^7,9.10422×10^7,9.10402×10^7,9.10382×10^7,

9.10361×10^7,9.10341×10^7,9.10321×10^7,9.10301×10^7,9.10281×10^7

6.3 Statistical Analysis of the first 5000 Partial Sums of the Hypothesis Series for F(10^21)

The Mean of y is

$$\mu = \frac{y_1 + y_2 + ...y_{5000}}{5000} = 91{,}028{,}129$$

The Variance of y is

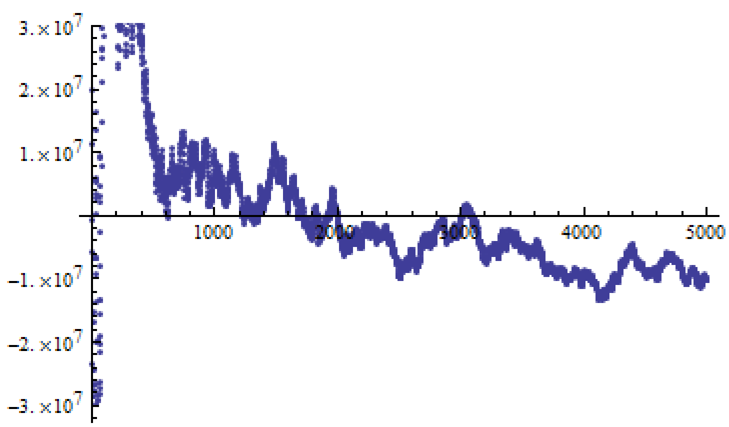
$$\sigma^2 = 1.5215573326889834` * 10^{14}$$

Thus, the Standard Deviation of y is

$$\sigma = 12{,}335{,}142$$

The Plot of the Hypothesis Series values y about their mean here is obtained with

```
ListPlot[Re[y+91028129]]
```



If we assume that the values of y are normally distributed, we can expect with 95% confidence to find the partial sums of the Hypothesis Series for F(10^21) between

$$\mu - 2\sigma = 91,028,129 - 2 \times 12,335,142$$

= 66,357,845

and

$$\mu + 2\sigma = 91,028,129 + 2 \times 12,335,142$$

=115,698,413

Then, since the main term in $F(10 \,\hat{}\, 21)$ is

21,127,269,485,932,299,723

we can expect with 95% confidence to find the values of $F(10 \,\hat{}\, 21)$ between

21,127,269,485,932,299,723+66,357,845

=21,127,269,485,998,657,568

and

21,127,269,485,932,299,723+115,698,413

=21,127,269,486,047,998,136

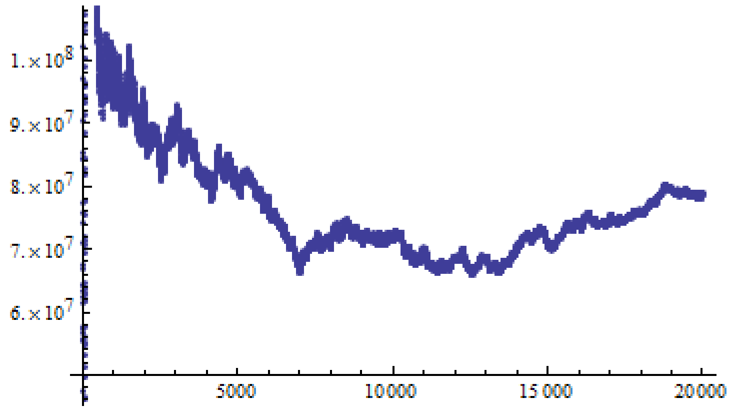
Due to the unpredictable convergence of the Hypothesis Series, we cannot claim that its values are distributed according to the normal distribution, or according to any other parametric distribution.

To obtain smaller error bounds, under the assumption of normal distribution, we need more partial sums.

Next, we compute with 20,000 partial sums of the Hypothesis Series.

6.4 Approximating the Hypothesis Series for F(10^21) with its first 20,000 partial sums

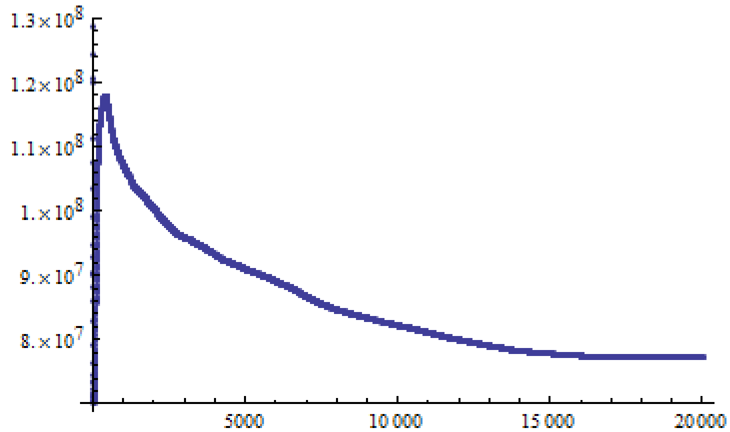
The plot of the first 20000 partial sums is



The last 10 partial sums $y_{19,991}$ to $y_{20,000}$ are

**7.89212×10^7,7.89386×10^7,7.88941×10^7,7.89233×10^7,7.8976×10^7,
7.89865×10^7,7.89157×10^7,7.88628×10^7,7.88027×10^7,7.87636×10^7**

The plot of the first 20,000 Cesaro-Arithmetic Means is



The last 10 Arithmetic Means $u_{19,991}$ to $u_{20,000}$ are

**7.72821×10^7,7.72822×10^7,7.72823×10^7,7.72824×10^7,7.72824×10^7,
7.72825×10^7,7.72826×10^7,7.72827×10^7,7.72828×10^7,7.72828×10^7**

6.5 Statistical Analysis of the first 20,000 Partial Sums of the Hypothesis Series for F(10^21)

The Mean of y is

$$\mu = \frac{y_1 + y_2 + ...y_{20,000}}{20,000} = 77{,}282{,}843$$

The Variance of y is

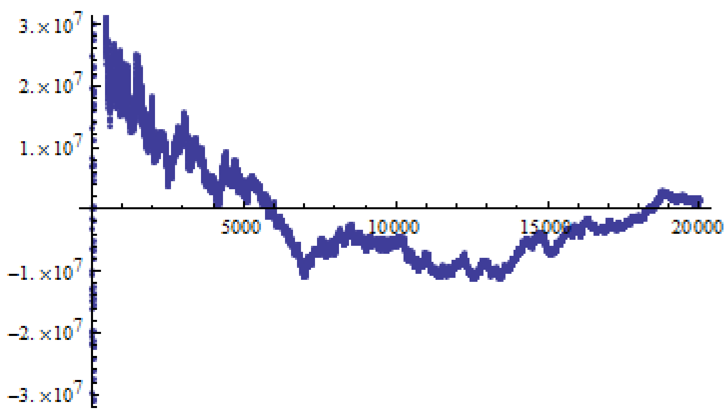
$$\sigma^2 = 1.1267934615584983` * 10^{14}$$

Thus, the Standard Deviation of y is

$$\sigma = 10,615,054$$

The Plot of the Hypothesis Series values y about their mean here is obtained with

ListPlot[Re[y+77282843]]



If we assume that the values of y are normally distributed, we can expect with 95% confidence to find the partial sums of the Hypothesis Series for F(10^21) between

$$\mu - 2\sigma = 77,282,843 - 2 \times 10,615,054$$

$$= \underline{56{,}052{,}735}$$

and

$$\mu + 2\sigma = 77,282,843 + 2 \times 10,615,054$$

=98,512,951

Since the main term in $F(10\,\hat{}\,21)$ is

21,127,269,485,932,299,723

we can expect with 95% confidence to find the values of $F(10\,\hat{}\,21)$ between

21,127,269,485,932,299,723+56,052,735

= 21,127,269,485,988,352,458

and

21,127,269,485,932,299,723+98,512,951

= 21,127,269,486,030,812,674

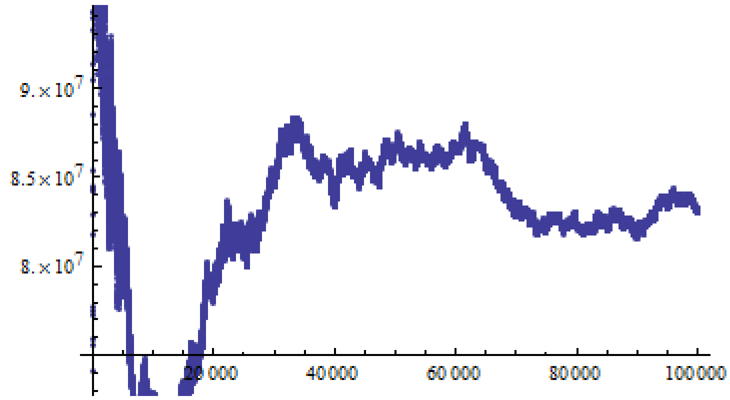
Since these bounds are within the bounds obtained with 5000 partial sums of the Hypothesis Series, we will keep assuming a normal distribution for the partial sums of the Hypothesis Series.

Then, to obtain smaller error bounds, we need more partial sums.

Next, we compute with 100,000 partial sums of the Hypothesis Series.

6.6 Approximating the Hypothesis Series for F(10^21) with its first 100,000 partial sums

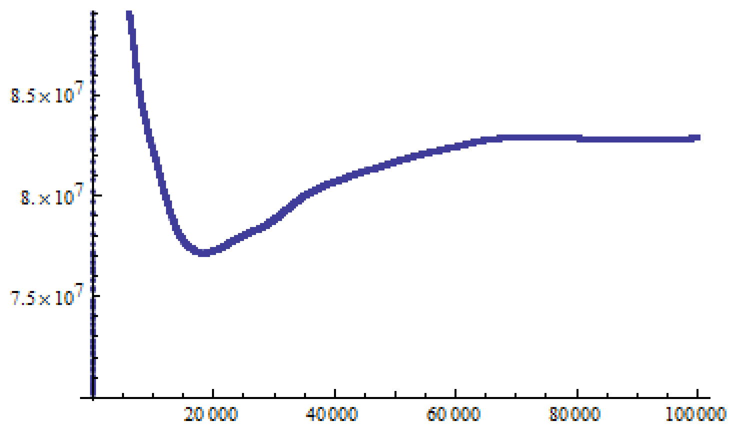
The plot of the first 100,000 partial sums is



The last 10 partial sums $y_{99,991}$ to $y_{100,000}$ are

**8.30262×10^7,8.30143×10^7,8.30103×10^7,8.30153×10^7,8.30225×10^7,
8.30307×10^7,8.30256×10^7,8.30255×10^7,8.30137×10^7,8.30122×10^7**

The plot of the first 100,000 Cesaro-Arithmetic Means is



The last 10 Arithmetic Means $u_{99,991}$ to $u_{100,000}$ are

**8.28479×10^7,8.28479×10^7,8.28479×10^7,8.28479×10^7,8.28479×10^7,
8.28479×10^7,8.28479×10^7,8.28479×10^7,8.28479×10^7,8.28479×10^7**

6.7 Statistical Analysis of the first 100,000 Partial Sums of the Hypothesis Series for F(10^21)

The Mean of y is

$$\mu = \frac{y_1 + y_2 + \ldots y_{100,000}}{100,000} = 82,847,940$$

The Variance of y is

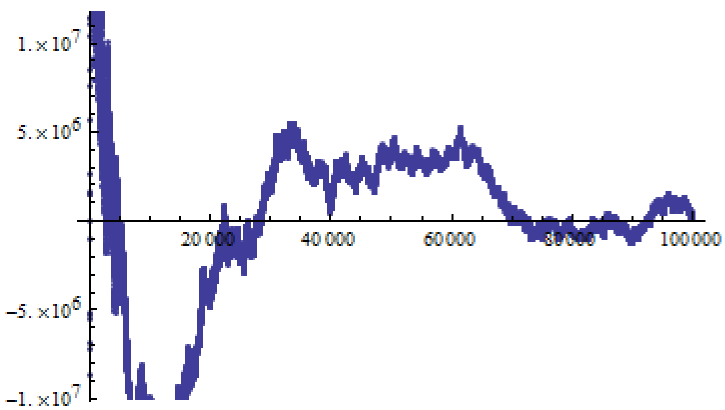
$$\sigma^2 = 3.330365217435684` * 10^{13}$$

Thus,

$$\sigma = 5,770,932$$

The Plot of the Hypothesis Series values y about their mean here is obtained with

```
ListPlot[Re[y+82847940]]
```



If we assume that the values of y are normally distributed, we can expect with 95% confidence to find the partial sums of the Hypothesis Series for F(10^21) between

$$\mu - 2\sigma = 82,847,940 - 2 \times 5,770,932$$

= <u>71,306,076</u>

and

$$\mu + 2\sigma = 82,847,940 + 2 \times 5,770,932$$

= <u>94,389,804</u>

Since the main term in $F(10 \hat{} 21)$ is

21,127,269,485,932,299,723

we can expect with 95% confidence to find the values of $F(10 \hat{} 21)$ between

21,127,269,485,932,299,723+71,306,076

= 21,127,269,486,003,605,799

and

21,127,269,485,932,299,723+94,389,804

= 21,127,269,486,026,689,527

6.8 The Value of the Hypothesis Series for F(10^21)

The statistical treatment of the Hypothesis Series value y, supplies us with a mean value, and with error bounds.

Both depend on the number of the partial series that we use, and change as we increase that number.

The error bounds seem to get smaller as we increase the number of partial sums.

It seems that on a larger computer, using more Zeta function zeros, the error bounds will be smaller

[Lag] claims to have obtained by a combinatorial method

$$\mathrm{F}(10\hat{}21) = 21,127,269,486,018,731,928$$

exactly, with no error bounds

This indicates Hypothesis Series precise value of

86,432,205

Such exact results, without error bounds, may be obtained on a super-computer, with either many more Zeta function zeros -by 2002, 50 Billion zeros were known- or by super-computer counting.

[Cran], and [Ed] believe that [Lag] have discovered new methods for the count of the primes that bypass Riemann's Formula 1.5.

This means, amongst other things, that

1) Riemann's Formula of 1.5 can be replaced with some other methods of equal ingenuity.
2) The Riemann Hypothesis, that is necessary for the Riemann Formula 1.5, is obsolete for the count of the primes.

If so, [Lag] contribution have been unfairly unrecognized for over 20 years.

But [Lag] does not seem to be aware of the fact that Riemann's Formula holds only under the Hypothesis, and they seem to be using the formula for Riemann's auxiliary function $f(t)$, rather than the formula for Riemann's count of the primes $F(t)$, of 1.5.

It is more likely that [Lag] "combinatorial" method that produced their 86,432,205 result, was obtained by super-computer counting.

Clearly, 86,432,205 is within the range of our estimation.

It is close to our last partial sum

$$y_{100,000} = 83,012,200,$$

and to our last arithmetic mean

$$u_{100,000} = 82,847,900$$

We plan not to confirm it by counting.

Appendix A:

Mobius m, and μ for expanding $F(t)$ in $\frac{(-1)^{\mu}}{m}f(t^{1/m})$

m	p_1	p_2	p_3	p_4.....	μ
1	–				0
2	2				1
3	3				1
5	5				1
6	2	3			2
7	7				1
10	2	5			2
11	11				1
13	13				1
14	2	7			2
15	3	5			2
17	17				1
19	19				1
21	3	7			2
22	2	11			2
23	23				1
26	2	13			2
29	29				1
30	2	3	5		3
31	31				1
33	3	11			2
34	2	17			2
35	5	7			2
37	37				1
38	2	19			2
39	3	13			2
41	41				1
42	2	3	7		3
43	43				1
46	2	23			2

m	p_1	p_2	p_3	p_4.....	μ
47	47				1
51	3	17			2
53	53				1
55	5	11			2
57	3	19			2
58	2	29			2
59	59				1
61	61				1
62	2	31			2
65	5	13			2
66	2	3	11		3
67	67				1
69	3	23			2
70	2	5	7		3
71	71				1
73	73				1
74	2	37			2
77	7	11			2
78	2	3	13		3
79	79				1
82	2	41			2
83	83				1
85	5	17			2
86	2	43			2
87	3	29			2
89	89				1
91	7	13			2
93	3	31			2
94	2	47			2
95	5	19			2

m	p_1	p_2	p_3	p_4.....	μ
97	97				1
101	101				1
102	2	3	17		3
103	103				1
105	3	5	7		3
106	2	53			2
107	107				1
109	109				1
110	2	5	11		3
111	3	37			2
113	113				1

This Table allows computing the Hypothesis term for $F(N)$, with

$$N < 2^{113} = (1.038459372)\bullet 10^{34}$$

Appendix B: Expanding $F(t)$ in $\frac{(-1)^{\mu}}{m} f(t^{1/m})$

$$f(t) = \text{Li}(t) - \log 2 - \sum_{\alpha}\left[\text{Li}(t^{1/2+i\alpha}) + \text{Li}(t^{1/2-i\alpha})\right] + \int_{u=t}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{2} f(t^{1/2}) = -\tfrac{1}{2}\text{Li}(t^{1/2}) + \tfrac{1}{2}\log 2 + \tfrac{1}{2}\sum_{\alpha}\left[\text{Li}(t^{(1/2+i\alpha)/2}) + \text{Li}(t^{(1/2-i\alpha)/2})\right] - \tfrac{1}{2}\int_{u=t^{1/2}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{3} f(t^{1/3}) = -\tfrac{1}{3}\text{Li}(t^{1/3}) + \tfrac{1}{3}\log 2 + \tfrac{1}{3}\sum_{\alpha}\left[\text{Li}(t^{(1/2+i\alpha)/3}) + \text{Li}(t^{(1/2-i\alpha)/3})\right] - \tfrac{1}{3}\int_{u=t^{1/3}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{5} f(t^{1/5}) = -\tfrac{1}{5}\text{Li}(t^{1/5}) + \tfrac{1}{5}\log 2 + \tfrac{1}{5}\sum_{\alpha}\left[\text{Li}(t^{(1/2+i\alpha)/5}) + \text{Li}(t^{(1/2-i\alpha)/5})\right] - \tfrac{1}{5}\int_{u=t^{1/5}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{6} f(t^{1/6}) = \tfrac{1}{6}\text{Li}(t^{1/6}) - \tfrac{1}{6}\log 2 - \tfrac{1}{6}\sum_{\alpha}\left[\text{Li}(t^{(1/2+i\alpha)/6}) + \text{Li}(t^{(1/2-i\alpha)/6})\right] + \tfrac{1}{6}\int_{u=t^{1/6}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{7} f(t^{1/7}) = -\tfrac{1}{7}\text{Li}(t^{1/7}) + \tfrac{1}{7}\log 2 + \tfrac{1}{7}\sum_{\alpha}\left[\text{Li}(t^{(1/2+i\alpha)/7}) + \text{Li}(t^{(1/2-i\alpha)/7})\right] - \tfrac{1}{7}\int_{u=t^{1/7}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{10} f(t^{1/10}) = \tfrac{1}{10}\text{Li}(t^{1/10}) - \tfrac{1}{10}\log 2 - \tfrac{1}{10}\sum_{\alpha}\left[\text{Li}(t^{(1/2+i\alpha)/10}) + \text{Li}(t^{(1/2-i\alpha)/10})\right] + \tfrac{1}{10}\int_{u=t^{1/10}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{11} f(t^{1/11}) = -\tfrac{1}{11}\text{Li}(t^{1/11}) + \tfrac{1}{11}\log 2 + \tfrac{1}{11}\sum_{\alpha}\left[\text{Li}(t^{(1/2+i\alpha)/11}) + \text{Li}(t^{(1/2-i\alpha)/11})\right] - \tfrac{1}{11}\int_{u=t^{1/11}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{13}f(t^{1/13})=-\tfrac{1}{13}\mathrm{Li}(t^{1/13})+\tfrac{1}{13}\log 2+\tfrac{1}{13}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/13})+\mathrm{Li}(t^{(1/2-i\alpha)/13})\right]-\tfrac{1}{13}\int_{u=t^{1/13}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{14}f(t^{1/14})=\tfrac{1}{14}\mathrm{Li}(t^{1/14})-\tfrac{1}{14}\log 2-\tfrac{1}{14}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/14})+\mathrm{Li}(t^{(1/2-i\alpha)/14})\right]+\tfrac{1}{14}\int_{u=t^{1/14}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{15}f(t^{1/15})=\tfrac{1}{15}\mathrm{Li}(t^{1/15})-\tfrac{1}{15}\log 2-\tfrac{1}{15}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/15})+\mathrm{Li}(t^{(1/2-i\alpha)/15})\right]+\tfrac{1}{15}\int_{u=t^{1/15}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{17}f(t^{1/17})=-\tfrac{1}{17}\mathrm{Li}(t^{1/17})+\tfrac{1}{17}\log 2+\tfrac{1}{17}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/17})+\mathrm{Li}(t^{(1/2-i\alpha)/17})\right]-\tfrac{1}{17}\int_{u=t^{1/17}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{19}f(t^{1/19})=-\tfrac{1}{19}\mathrm{Li}(t^{1/19})+\tfrac{1}{19}\log 2+\tfrac{1}{19}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/19})+\mathrm{Li}(t^{(1/2-i\alpha)/19})\right]-\tfrac{1}{19}\int_{u=t^{1/19}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{21}f(t^{1/21})=\tfrac{1}{21}\mathrm{Li}(t^{1/21})-\tfrac{1}{21}\log 2-\tfrac{1}{21}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/21})+\mathrm{Li}(t^{(1/2-i\alpha)/21})\right]+\tfrac{1}{21}\int_{u=t^{1/21}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{23}f(t^{1/23})=-\tfrac{1}{23}\mathrm{Li}(t^{1/23})+\tfrac{1}{23}\log 2+\tfrac{1}{23}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/23})+\mathrm{Li}(t^{(1/2-i\alpha)/23})\right]-\tfrac{1}{23}\int_{u=t^{1/23}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{26}f(t^{1/26})=\tfrac{1}{26}\mathrm{Li}(t^{1/26})-\tfrac{1}{26}\log 2-\tfrac{1}{26}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/26})+\mathrm{Li}(t^{(1/2-i\alpha)/26})\right]+\tfrac{1}{26}\int_{u=t^{1/26}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{29}f(t^{1/29})=-\tfrac{1}{29}\mathrm{Li}(t^{1/29})+\tfrac{1}{29}\log 2+\tfrac{1}{29}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/29})+\mathrm{Li}(t^{(1/2-i\alpha)/29})\right]-\tfrac{1}{29}\int_{u=t^{1/29}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{30}f(t^{1/30})=-\tfrac{1}{30}\mathrm{Li}(t^{1/30})+\tfrac{1}{30}\log 2+\tfrac{1}{30}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/30})+\mathrm{Li}(t^{(1/2-i\alpha)/30})\right]-\tfrac{1}{30}\int_{u=t^{1/30}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{31} f(t^{1/31}) = -\tfrac{1}{31}\mathrm{Li}(t^{1/31}) + \tfrac{1}{31}\log 2 + \tfrac{1}{31}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/31}) + \mathrm{Li}(t^{(1/2-i\alpha)/31})\Big] - \tfrac{1}{31}\int_{u=t^{1/31}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{33} f(t^{1/33}) = \tfrac{1}{33}\mathrm{Li}(t^{1/33}) - \tfrac{1}{33}\log 2 - \tfrac{1}{33}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/33}) + \mathrm{Li}(t^{(1/2-i\alpha)/33})\Big] + \tfrac{1}{33}\int_{u=t^{1/33}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{34} f(t^{1/34}) = \tfrac{1}{34}\mathrm{Li}(t^{1/34}) - \tfrac{1}{34}\log 2 - \tfrac{1}{34}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/34}) + \mathrm{Li}(t^{(1/2-i\alpha)/34})\Big] + \tfrac{1}{34}\int_{u=t^{1/34}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{35} f(t^{1/35}) = \tfrac{1}{35}\mathrm{Li}(t^{1/35}) - \tfrac{1}{35}\log 2 - \tfrac{1}{35}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/35}) + \mathrm{Li}(t^{(1/2-i\alpha)/35})\Big] + \tfrac{1}{35}\int_{u=t^{1/35}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{37} f(t^{1/37}) = -\tfrac{1}{37}\mathrm{Li}(t^{1/37}) + \tfrac{1}{37}\log 2 + \tfrac{1}{37}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/37}) + \mathrm{Li}(t^{(1/2-i\alpha)/37})\Big] - \tfrac{1}{37}\int_{u=t^{1/37}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{38} f(t^{1/38}) = \tfrac{1}{38}\mathrm{Li}(t^{1/38}) - \tfrac{1}{38}\log 2 - \tfrac{1}{38}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/38}) + \mathrm{Li}(t^{(1/2-i\alpha)/38})\Big] + \tfrac{1}{38}\int_{u=t^{1/38}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{39} f(t^{1/39}) = \tfrac{1}{39}\mathrm{Li}(t^{1/39}) - \tfrac{1}{39}\log 2 - \tfrac{1}{39}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/39}) + \mathrm{Li}(t^{(1/2-i\alpha)/39})\Big] + \tfrac{1}{39}\int_{u=t^{1/39}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{41} f(t^{1/41}) = -\tfrac{1}{41}\mathrm{Li}(t^{1/41}) + \tfrac{1}{41}\log 2 + \tfrac{1}{41}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/41}) + \mathrm{Li}(t^{(1/2-i\alpha)/41})\Big] - \tfrac{1}{41}\int_{u=t^{1/41}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{42} f(t^{1/42}) = -\tfrac{1}{42}\mathrm{Li}(t^{1/42}) + \tfrac{1}{42}\log 2 + \tfrac{1}{42}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/42}) + \mathrm{Li}(t^{(1/2-i\alpha)/42})\Big] - \tfrac{1}{42}\int_{u=t^{1/42}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{43} f(t^{1/43}) = -\tfrac{1}{43}\mathrm{Li}(t^{1/43}) + \tfrac{1}{43}\log 2 + \tfrac{1}{43}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/43}) + \mathrm{Li}(t^{(1/2-i\alpha)/43})\Big] - \tfrac{1}{43}\int_{u=t^{1/43}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{46} f(t^{1/46}) = \tfrac{1}{46}\mathrm{Li}(t^{1/46}) - \tfrac{1}{46}\log 2 - \tfrac{1}{46}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/46}) + \mathrm{Li}(t^{(1/2-i\alpha)/46})\Big] + \tfrac{1}{46}\int_{u=t^{1/46}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{47} f(t^{1/47}) = -\tfrac{1}{47}\mathrm{Li}(t^{1/47}) + \tfrac{1}{47}\log 2 + \tfrac{1}{47}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/47}) + \mathrm{Li}(t^{(1/2-i\alpha)/47})\Big] - \tfrac{1}{47}\int_{u=t^{1/47}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{51} f(t^{1/51}) = \tfrac{1}{51}\mathrm{Li}(t^{1/51}) - \tfrac{1}{51}\log 2 - \tfrac{1}{51}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/51}) + \mathrm{Li}(t^{(1/2-i\alpha)/51})\Big] + \tfrac{1}{51}\int_{u=t^{1/51}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{53} f(t^{1/53}) = -\tfrac{1}{53}\mathrm{Li}(t^{1/53}) + \tfrac{1}{53}\log 2 + \tfrac{1}{53}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/53}) + \mathrm{Li}(t^{(1/2-i\alpha)/53})\Big] - \tfrac{1}{53}\int_{u=t^{1/53}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{55} f(t^{1/55}) = \tfrac{1}{55}\mathrm{Li}(t^{1/55}) - \tfrac{1}{55}\log 2 - \tfrac{1}{55}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/55}) + \mathrm{Li}(t^{(1/2-i\alpha)/55})\Big] + \tfrac{1}{55}\int_{u=t^{1/51}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{57} f(t^{1/57}) = \tfrac{1}{57}\mathrm{Li}(t^{1/57}) - \tfrac{1}{57}\log 2 - \tfrac{1}{57}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/57}) + \mathrm{Li}(t^{(1/2-i\alpha)/57})\Big] + \tfrac{1}{51}\int_{u=t^{1/57}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{58} f(t^{1/58}) = \tfrac{1}{58}\mathrm{Li}(t^{1/58}) - \tfrac{1}{58}\log 2 - \tfrac{1}{58}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/58}) + \mathrm{Li}(t^{(1/2-i\alpha)/58})\Big] + \tfrac{1}{58}\int_{u=t^{1/58}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{59} f(t^{1/59}) = -\tfrac{1}{59}\mathrm{Li}(t^{1/59}) + \tfrac{1}{59}\log 2 + \tfrac{1}{59}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/59}) + \mathrm{Li}(t^{(1/2-i\alpha)/59})\Big] - \tfrac{1}{59}\int_{u=t^{1/59}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{61} f(t^{1/61}) = -\tfrac{1}{61}\mathrm{Li}(t^{1/61}) + \tfrac{1}{61}\log 2 - \tfrac{1}{61}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/61}) + \mathrm{Li}(t^{(1/2-i\alpha)/61})\Big] + \tfrac{1}{61}\int_{u=t^{1/61}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{62} f(t^{1/62}) = \tfrac{1}{62}\mathrm{Li}(t^{1/62}) - \tfrac{1}{62}\log 2 - \tfrac{1}{62}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/62}) + \mathrm{Li}(t^{(1/2-i\alpha)/62})\Big] + \tfrac{1}{62}\int_{u=t^{1/62}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{65}f(t^{1/65})=\tfrac{1}{65}\mathrm{Li}(t^{1/65})-\tfrac{1}{65}\log 2-\tfrac{1}{65}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/65})+\mathrm{Li}(t^{(1/2-i\alpha)/65})\Big]+\tfrac{1}{65}\int_{u=t^{1/65}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{66}f(t^{1/66})=-\tfrac{1}{66}\mathrm{Li}(t^{1/66})+\tfrac{1}{66}\log 2-\tfrac{1}{66}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/66})+\mathrm{Li}(t^{(1/2-i\alpha)/66})\Big]+\tfrac{1}{66}\int_{u=t^{1/66}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{67}f(t^{1/67})=-\tfrac{1}{67}\mathrm{Li}(t^{1/67})+\tfrac{1}{67}\log 2-\tfrac{1}{67}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/67})+\mathrm{Li}(t^{(1/2-i\alpha)/67})\Big]+\tfrac{1}{67}\int_{u=t^{1/67}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{69}f(t^{1/69})=\tfrac{1}{69}\mathrm{Li}(t^{1/69})-\tfrac{1}{69}\log 2-\tfrac{1}{69}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/69})+\mathrm{Li}(t^{(1/2-i\alpha)/69})\Big]+\tfrac{1}{69}\int_{u=t^{1/69}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{70}f(t^{1/70})=-\tfrac{1}{70}\mathrm{Li}(t^{1/61})+\tfrac{1}{70}\log 2-\tfrac{1}{70}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/70})+\mathrm{Li}(t^{(1/2-i\alpha)/70})\Big]+\tfrac{1}{70}\int_{u=t^{1/70}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{71}f(t^{1/71})=-\tfrac{1}{71}\mathrm{Li}(t^{1/71})+\tfrac{1}{71}\log 2-\tfrac{1}{71}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/71})+\mathrm{Li}(t^{(1/2-i\alpha)/71})\Big]+\tfrac{1}{71}\int_{u=t^{1/71}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{73}f(t^{1/73})=-\tfrac{1}{73}\mathrm{Li}(t^{1/73})+\tfrac{1}{73}\log 2-\tfrac{1}{73}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/73})+\mathrm{Li}(t^{(1/2-i\alpha)/73})\Big]+\tfrac{1}{73}\int_{u=t^{1/73}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{74}f(t^{1/74})=\tfrac{1}{74}\mathrm{Li}(t^{1/74})-\tfrac{1}{74}\log 2-\tfrac{1}{74}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/74})+\mathrm{Li}(t^{(1/2-i\alpha)/74})\Big]+\tfrac{1}{74}\int_{u=t^{1/74}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{77}f(t^{1/77})=\tfrac{1}{77}\mathrm{Li}(t^{1/77})-\tfrac{1}{77}\log 2-\tfrac{1}{77}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/77})+\mathrm{Li}(t^{(1/2-i\alpha)/77})\Big]+\tfrac{1}{77}\int_{u=t^{1/77}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{78}f(t^{1/78})=-\tfrac{1}{78}\mathrm{Li}(t^{1/78})+\tfrac{1}{78}\log 2-\tfrac{1}{78}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/78})+\mathrm{Li}(t^{(1/2-i\alpha)/78})\Big]+\tfrac{1}{78}\int_{u=t^{1/78}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{79}f(t^{1/79})=-\tfrac{1}{79}\mathrm{Li}(t^{1/79})+\tfrac{1}{79}\log 2-\tfrac{1}{79}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/79})+\mathrm{Li}(t^{(1/2-i\alpha)/79})\Big]+\tfrac{1}{79}\int_{u=t^{1/79}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{82}f(t^{1/82})=\tfrac{1}{82}\mathrm{Li}(t^{1/82})-\tfrac{1}{82}\log 2-\tfrac{1}{82}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/82})+\mathrm{Li}(t^{(1/2-i\alpha)/82})\Big]+\tfrac{1}{82}\int_{u=t^{1/82}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{83}f(t^{1/83})=-\tfrac{1}{83}\mathrm{Li}(t^{1/83})+\tfrac{1}{83}\log 2-\tfrac{1}{83}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/83})+\mathrm{Li}(t^{(1/2-i\alpha)/83})\Big]+\tfrac{1}{83}\int_{u=t^{1/83}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{85}f(t^{1/85})=\tfrac{1}{85}\mathrm{Li}(t^{1/85})-\tfrac{1}{85}\log 2-\tfrac{1}{85}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/85})+\mathrm{Li}(t^{(1/2-i\alpha)/85})\Big]+\tfrac{1}{85}\int_{u=t^{1/85}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{86}f(t^{1/86})=\tfrac{1}{86}\mathrm{Li}(t^{1/86})-\tfrac{1}{86}\log 2-\tfrac{1}{86}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/86})+\mathrm{Li}(t^{(1/2-i\alpha)/86})\Big]+\tfrac{1}{86}\int_{u=t^{1/86}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{87}f(t^{1/87})=\tfrac{1}{87}\mathrm{Li}(t^{1/87})-\tfrac{1}{87}\log 2-\tfrac{1}{87}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/87})+\mathrm{Li}(t^{(1/2-i\alpha)/87})\Big]+\tfrac{1}{87}\int_{u=t^{1/87}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{89}f(t^{1/83})=-\tfrac{1}{89}\mathrm{Li}(t^{1/89})+\tfrac{1}{89}\log 2-\tfrac{1}{89}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/89})+\mathrm{Li}(t^{(1/2-i\alpha)/89})\Big]+\tfrac{1}{89}\int_{u=t^{1/89}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{91}f(t^{1/91})=\tfrac{1}{91}\mathrm{Li}(t^{1/91})-\tfrac{1}{91}\log 2-\tfrac{1}{91}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/91})+\mathrm{Li}(t^{(1/2-i\alpha)/91})\Big]+\tfrac{1}{91}\int_{u=t^{1/91}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{93}f(t^{1/93})=\tfrac{1}{93}\mathrm{Li}(t^{1/93})-\tfrac{1}{93}\log 2-\tfrac{1}{93}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/93})+\mathrm{Li}(t^{(1/2-i\alpha)/93})\Big]+\tfrac{1}{93}\int_{u=t^{1/93}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{94}f(t^{1/94})=\tfrac{1}{94}\mathrm{Li}(t^{1/94})-\tfrac{1}{94}\log 2-\tfrac{1}{94}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/94})+\mathrm{Li}(t^{(1/2-i\alpha)/94})\Big]+\tfrac{1}{94}\int_{u=t^{1/94}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{95}f(t^{1/95})=\tfrac{1}{95}\mathrm{Li}(t^{1/95})-\tfrac{1}{95}\log 2-\tfrac{1}{95}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/95})+\mathrm{Li}(t^{(1/2-i\alpha)/95})\Big]+\tfrac{1}{95}\int_{u=t^{1/95}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{97}f(t^{1/97})=-\tfrac{1}{97}\mathrm{Li}(t^{1/97})+\tfrac{1}{97}\log 2-\tfrac{1}{97}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/97})+\mathrm{Li}(t^{(1/2-i\alpha)/97})\Big]+\tfrac{1}{97}\int_{u=t^{1/97}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{101}f(t^{1/101})=-\tfrac{1}{97}\mathrm{Li}(t^{1/101})+\tfrac{1}{101}\log 2-\tfrac{1}{101}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/101})+\mathrm{Li}(t^{(1/2-i\alpha)/101})\Big]+\tfrac{1}{97}\int_{u=t^{1/101}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{102}f(t^{1/102})=-\tfrac{1}{102}\mathrm{Li}(t^{1/102})+\tfrac{1}{102}\log 2-\tfrac{1}{102}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/102})+\mathrm{Li}(t^{(1/2-i\alpha)/102})\Big]+\tfrac{1}{102}\int_{u=t^{1/102}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{103}f(t^{1/103})=-\tfrac{1}{103}\mathrm{Li}(t^{1/103})+\tfrac{1}{103}\log 2-\tfrac{1}{103}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/103})+\mathrm{Li}(t^{(1/2-i\alpha)/103})\Big]+\tfrac{1}{103}\int_{u=t^{1/103}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{105}f(t^{1/105})=-\tfrac{1}{105}\mathrm{Li}(t^{1/105})+\tfrac{1}{105}\log 2-\tfrac{1}{105}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/105})+\mathrm{Li}(t^{(1/2-i\alpha)/105})\Big]+\tfrac{1}{105}\int_{u=t^{1/105}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{106}f(t^{1/106})=\tfrac{1}{106}\mathrm{Li}(t^{1/106})-\tfrac{1}{106}\log 2-\tfrac{1}{106}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/106})+\mathrm{Li}(t^{(1/2-i\alpha)/106})\Big]+\tfrac{1}{106}\int_{u=t^{1/106}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{107}f(t^{1/107})=-\tfrac{1}{107}\mathrm{Li}(t^{1/107})+\tfrac{1}{107}\log 2-\tfrac{1}{107}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/107})+\mathrm{Li}(t^{(1/2-i\alpha)/107})\Big]+\tfrac{1}{107}\int_{u=t^{1/107}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{109}f(t^{1/109})=-\tfrac{1}{109}\mathrm{Li}(t^{1/109})+\tfrac{1}{109}\log 2-\tfrac{1}{109}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/109})+\mathrm{Li}(t^{(1/2-i\alpha)/109})\Big]+\tfrac{1}{109}\int_{u=t^{1/109}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{110}f(t^{1/110})=-\tfrac{1}{110}\mathrm{Li}(t^{1/110})+\tfrac{1}{110}\log 2-\tfrac{1}{110}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/110})+\mathrm{Li}(t^{(1/2-i\alpha)/110})\Big]+\tfrac{1}{110}\int_{u=t^{1/110}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{111} f(t^{1/111}) = \tfrac{1}{111}\mathrm{Li}(t^{1/111}) - \tfrac{1}{111}\log 2 - \tfrac{1}{111}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/111}) + \mathrm{Li}(t^{(1/2-i\alpha)/111})\right] + \tfrac{1}{111}\int_{u=t^{1/111}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{113} f(t^{1/113}) = -\tfrac{1}{113}\mathrm{Li}(t^{1/113}) + \tfrac{1}{113}\log 2 - \tfrac{1}{113}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/113}) + \mathrm{Li}(t^{(1/2-i\alpha)/113})\right] + \tfrac{1}{113}\int_{u=t^{1/113}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

This table gives the Hypothesis term for $F(N)$, with

$$N < 2^{113} = (1.038459372)\bullet 10^{34}$$

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