Chi-Squared Goodness-of-Fit-Test
of the
Riemann Hypothesis

H. Vic Dannon
vick@adnc.com
February, 2008

Abstract

We propose a high confidence Chi-Squared test for the Riemann formula for the count of the prime numbers, when it includes the term that follows from the Hypothesis. Riemann could not establish the validity of that term, and left it out of his approximation formula.

This statistical testing does not prove the Hypothesis that all Zeta zeros are on $x = \frac{1}{2}$, but it confirms it with high statistical confidence, and allows us to use Riemann's formula for the count of the primes, complete with the Hypothesis Series, with that high statistical confidence.

KeyWords Riemann Hypothesis. Count of Prime Numbers.

2000 Mathematical Subject Classification 11M26

Preface

The computing of the number of the primes up to a number x, started with Gauss who approximated it by the logarithmic integral.

In his 1859 Zeta paper, Riemann obtained an intricate formula that uses all the zeros of the Zeta function on the line $x = \frac{1}{2}$, to solve the problem completely, provided that all the zeros of the Zeta function in $0 < x < 1$, are on the line $x = \frac{1}{2}$.

The Riemann formula has four terms. But only the first and the third of these terms have non-negligible values. The first is a dominant term that can be computed precisely. The third term is smaller and depends on the provision regarding the zeros of the Zeta function.

This provision became known as the Riemann Hypothesis, but it was never hypothesized by Riemann. Not seeing an easy proof for it, Riemann used only the first term of his formula and obtained an approximation far superior to Gauss for the count of the primes. This term is known as the Riemann Approximation term.

We shall refer to the third term that depends on the Hypothesis, and was neglected since Riemann, as the Riemann-Hypothesis-term.

Here, we will confirm that the Hypothesis holds with high statistical certainty, and the Riemann formula can be used with that high certainty.

The uncertainty we found was way under 10^{-16}. Indeed, the software produced certainty of 1.0. Consequently, we did not try to use many more Zeta zeros to obtain a lower uncertainty.

Our computations indicate that if not for the limitations of the software, the Hypothesis can be confirmed to any degree of certainty.

1

Riemann's Formula for the Count of the Primes.

1.1 Riemann's Derivation

Riemann Denoted the count the number of primes < t, by $F(t)$.

By [Ed], [Riem], and [Dan], Riemann used the Euler product formula

$$\zeta(z) \equiv \sum_{n=natural} \frac{1}{n^z} = \prod_{p=prime} \frac{1}{1-1\ /\ p^z}$$

to obtain

$$\frac{\log \zeta(z)}{z} = \int_{t=1}^{t=\infty} \left(F(t) + \frac{1}{2}F(t^{1/2}) + \frac{1}{3}F(t^{1/3}) + \ldots \right) \frac{dt}{t^{z+1}}$$

Riemann denoted

$$f(t) = F(t) + \frac{1}{2}F(t^{1/2}) + \frac{1}{3}F(t^{1/3}) + \frac{1}{4}F(t^{1/4}) + \ldots$$

Inverting the equation,

$$\frac{\log \zeta(z)}{z} = \int_{t=1}^{t=\infty} f(t) \frac{dt}{t^{z+1}}$$

he obtained

$$f(t) = \frac{1}{2\pi} \int_{y=-\infty}^{y=\infty} \frac{\log \zeta(z)}{z} t^z dy.$$

The entire function

$$\xi(z) \equiv (z-1)\pi^{-\frac{z}{2}}\Gamma(\tfrac{z}{2}+1)\zeta(z)$$

has the same zeros as $\zeta(z)$ in $0 < x < 1$, and has the factorization

$$\xi(z) = \xi(0)\prod_{n=1}^{\infty}\left(1 - \frac{z}{\frac{1}{2} - i\alpha_n}\right)\left(1 - \frac{z}{\frac{1}{2} + i\alpha_n}\right) = \xi(0)\prod_{n=1}^{\infty}\left(1 + \frac{z^2 - z}{\alpha_n^2 + \frac{1}{4}}\right)$$

where $\frac{1}{2} - i\alpha_n$, and $\frac{1}{2} + i\alpha_n$ are the zeros of $\xi(z)$, and are sequenced by their size.

Replacing $\zeta(z)$ with $\xi(z)$

$$f(t) = \frac{1}{2\pi i}\int_{y=-\infty}^{y=\infty}\frac{t^z}{\log t}d_z\left[\frac{\log(z-1)}{z} + \frac{\log\Gamma\left(\frac{1}{2}z+1\right)}{z} - \frac{\log\xi(0)}{z}\right.$$

$$\left. -\frac{1}{z}\sum_{n=1}^{\infty}\log\left(1+\frac{z^2 - z}{\alpha_n^2 + \frac{1}{4}}\right)\right].$$

ξ has infinitely many zeros on the line $x = \frac{1}{2}$.

If $\mathrm{Im}(\alpha_n) = 0$ for all n, then all the zeros of ξ are on the line $x = \frac{1}{2}$, and the integration of $-\frac{1}{z}\sum_{n=1}^{\infty}\log\left(1+\frac{z^2 - z}{\alpha_n^2 + \frac{1}{4}}\right)$ yields the sum

$$\sum_{n=1}^{\infty}\left[\mathrm{Li}(t^{\frac{1}{2}+i\alpha_n}) + \mathrm{Li}(t^{\frac{1}{2}-i\alpha_n})\right]$$

where

$$\mathrm{Li}(t) \equiv \int_{u=0}^{u=t}\frac{du}{\log u} \equiv \lim_{\varepsilon\downarrow 0}\left\{\int_{u=0}^{u=1-\varepsilon}\frac{du}{\log u} + \int_{u=1+\varepsilon}^{u=t}\frac{du}{\log u}\right\}$$

is the logarithmic integral.

Then, integrating the rest of the terms for $f(t)$, Riemann derived,

$$\boxed{\begin{aligned} f(t) = &\ \mathrm{LogIntegral}(t) \\ &- \log 2 \\ &- \sum_{n=1}^{\infty}\left[\mathrm{L}\,ogIntegral(t^{\frac{1}{2}+i\alpha_n}) + \mathrm{L}\,ogIntegral(t^{\frac{1}{2}-i\alpha_n})\right] \\ &+ \int_{u=t}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du \end{aligned}}.$$

$F(t)$ is obtained by Mobius Inversion of the definition of $f(t)$

The Mobius Inversion of $f(t) = F(t) + \frac{1}{2}F(t^{1/2}) + \frac{1}{3}F(t^{1/3}) + \ldots$gives

$$\boxed{\begin{aligned} F(t) = &\ f(t) \\ &- \frac{1}{2}f(t^{1/2}) \\ &+ \ldots \\ &+ \frac{(-1)^{\mu(m)}}{m}f(t^{1/m}) \\ &+ \ldots \\ &+ \frac{(-1)^{\mu(m_{j_0})}}{m_{j_0}}f(t^{1/m_{j_0}}) \end{aligned}}$$

1.2 Formula Cut-Off, and the Mobius Inversion Table

While $f(t)$ is an infinite series, the $F(t)$ summation terminates at m_{j_0} so that

$$t^{1/m_{j_0}} > 2,$$

and

$$t^{1/[m_{j_0+1}]} < 2.$$

For instance, if $t = 10$, we find that

$$10^{1/3} = 2.15 > 2,$$

and

$$10^{1/5} = 1.58 < 2$$

Therefore, the formula cut-off is at $m_{j_0} = 3$

m is a natural number that has no prime factors squared.

Any number that has a prime factor squared is skipped.

$\mu(m)$ is the number of prime factors of m.

Mobius Inversion Formula is constructed with the aid of the following table

m	p_1	p_2	p_3	$p_4.....$	$\mu(m)$	$\frac{(-1)^{\mu(m)}}{m}$
1	$-$				0	1
2	2				1	$-1/2$
3	3				1	$-1/3$
$4 = 2^2$						
5	5				1	$-1/5$
6	2	3			2	$1/6$
7	7				1	$-1/7$
$8 = 2^3$						
$9 = 3^2$						
10	2	5			2	$1/10$
11	11				1	$-1/11$
$12 = 2^2 3$						
13	13				1	$-1/13$

Appendix A has the values of m, and μ, up to $m = 113$.

For $t = 10$, we find the admissible $m's$ from the following Mobius Inversion Table

m	p_1	p_2	p_3	$p_4 \ldots\ldots$	$\mu(m)$	$\frac{(-1)^{\mu(m)}}{m}$
1	$-$				0	1
2	2				1	$-1/2$
3	3				1	$-1/3$
$4 = 2^2$						

Therefore, the number of primes up to 10 is

$$F(10) = f(10) - \frac{1}{2} f(10^{1/2}) - \frac{1}{3} f(10^{1/3})$$

Note that $F(10) = 4$.

We proceed to write this formula for $F(10)$ in detail.

1.3 Riemann's Formula for F(10)

If the Riemann Hypothesis holds, then for $t = 10$,

$$f(10) = \text{LogIntegral}(10) - \log 2 - \sum_{\alpha} \left[\text{LogIntegral}(10^{\frac{1}{2}+i\alpha}) + \text{LogIntegral}(10^{\frac{1}{2}-i\alpha}) \right] + \int_{u=10}^{u=\infty} \frac{1}{u \log u} \frac{1}{u^2 - 1} du$$

$$-\tfrac{1}{2}f(\sqrt{10}) = -\tfrac{1}{2}\mathrm{LogIntegral}(\sqrt{10})$$
$$+\tfrac{1}{2}\log 2$$
$$+\tfrac{1}{2}\sum_{\alpha}\left[\mathrm{LogIntegral}(\sqrt{10}^{\frac{1}{2}+i\alpha}) + \mathrm{LogIntegral}(\sqrt{10}^{\frac{1}{2}-i\alpha})\right]$$
$$-\tfrac{1}{2}\int_{u=\sqrt{10}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{3}f(\sqrt[3]{10}) = -\tfrac{1}{3}\mathrm{LogIntegral}(\sqrt[3]{10})$$
$$+\tfrac{1}{3}\log 2$$
$$+\tfrac{1}{3}\sum_{\alpha}\left[\mathrm{LogIntegral}(\sqrt[3]{10}^{\frac{1}{2}+i\alpha}) + \mathrm{LogIntegral}(\sqrt[3]{10}^{\frac{1}{2}-i\alpha})\right]$$
$$-\tfrac{1}{3}\int_{u=\sqrt[3]{10}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

Substituting in the formula for $F(10)$ at the end of section **1.3**,

$$F(10) = \quad \mathrm{LogIntegral}(10) - \tfrac{1}{2}\mathrm{LogIntegral}(\sqrt{10}) - \tfrac{1}{3}\mathrm{LogIntegral}(\sqrt[3]{10})$$

$$+(-1+\tfrac{1}{2}+\tfrac{1}{3})\log 2$$

$$-\sum_{\alpha}\left[\left(\mathrm{LogIntegral}(10^{\frac{1}{2}+i\alpha}) + \mathrm{LogIntegral}(10^{\frac{1}{2}-i\alpha})\right)\right.$$
$$+\tfrac{1}{2}\sum_{\alpha}\left(\mathrm{LogIntegral}(\sqrt{10}^{\frac{1}{2}+i\alpha}) + \mathrm{LogIntegral}(\sqrt{10}^{\frac{1}{2}-i\alpha})\right)$$
$$+\tfrac{1}{3}\sum_{\alpha}\left(\mathrm{LogIntegral}(\sqrt[3]{10}^{\frac{1}{2}+i\alpha}) + \mathrm{LogIntegral}(\sqrt[3]{10}^{\frac{1}{2}-i\alpha})\right)$$

$$+\int_{u=10}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du - \tfrac{1}{2}\int_{u=\sqrt{10}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du - \tfrac{1}{3}\int_{u=\sqrt[3]{10}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

If we aim to compute by hand, this is our final formula.

But in order to ready the sum of the three infinite series for computer run, it is crucial that we write them as one infinite series, each of its terms is the sum of three pairs of logarithmic integrals.

Each term of the series is evaluated at a distinct zero of the Zeta function on the line $x = \frac{1}{2}$, and at its conjugate zero.

Consequently, we have one infinite series that depends on the Hypothesis.

This is how the Hypothesis Term becomes the Hypothesis Series.

$$\boxed{\begin{aligned} F(10) = \quad & \mathrm{LogIntegral}(10) - \tfrac{1}{2}\mathrm{LogIntegral}(\sqrt{10}) - \tfrac{1}{3}\mathrm{LogIntegral}(\sqrt[3]{10}) \\ & +(-1 + \tfrac{1}{2} + \tfrac{1}{3})\log 2 \\ & +\sum_{\alpha}\Big[-\Big(\mathrm{LogIntegral}(10^{\frac{1}{2}+i\alpha}) + \mathrm{LogIntegral}(10^{\frac{1}{2}-i\alpha})\Big) \\ & \quad +\tfrac{1}{2}\Big(\mathrm{LogIntegral}(\sqrt{10}^{\frac{1}{2}+i\alpha}) + \mathrm{LogIntegral}(\sqrt{10}^{\frac{1}{2}-i\alpha})\Big) \\ & \quad +\tfrac{1}{3}\Big(\mathrm{LogIntegral}(\sqrt[3]{10}^{\frac{1}{2}+i\alpha}) + \mathrm{LogIntegral}(\sqrt[3]{10}^{\frac{1}{2}-i\alpha})\Big)\Big] \\ & + \int_{u=10}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du - \tfrac{1}{2}\int_{u=\sqrt{10}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du - \tfrac{1}{3}\int_{u=\sqrt[3]{10}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du \end{aligned}}$$

If the Hypothesis holds, this expression holds true.

The first term is the Riemann Approximation Term.

In the example above, it is the sum of 3 LogIntegrals.

It is the dominant term, and it is superior to any approximation attempt before Riemann.

The second term is called the Log Term.

In the above example it is made of three summands.

For large numbers, it is negligible.

The third term is the Hypothesis Series.

It is the term that Riemann, not having a proof for the Hypothesis, did not use.

Here, we will establish the validity of this term, and hence, the validity of the Riemann Hypothesis, up to a high statistical certainty.

The fourth term is called the Integral Term.

In the above example, it is the sum of three integrals.

It is negligible for large numbers.

1.4 Riemann's Approximation

Riemann did not use the Hypothesis Series. Riemann wrote

> "…*It is very likely that all of the zeros are real. One would like to have a rigorous proof of this, but after several fleeting attempts to no avail, I have temporarily set aside the search for this proof because it appeared to be unnecessary for the immediate purpose of my investigation*…"

Riemann used

$$f(t) \approx \mathrm{L}\,ogIntegral(t)$$

Then,

$$F(t) \approx \mathrm{Li}(t) - \frac{1}{2}\mathrm{Li}(t^{\frac{1}{2}}) - \frac{1}{3}\mathrm{Li}(t^{\frac{1}{3}}) - \frac{1}{5}\mathrm{Li}(t^{\frac{1}{5}}) + ... + \frac{(-1)^{\mu_{m_{j_0}}}}{m_{j_0}}\mathrm{Li}(t^{m_{j_0}}).$$

That is, only the LogIntegral term, cutoff at m_{j_0}, is used.

Riemann's approximation is better than the Gauss approximation

$$F(t) \approx \mathrm{Li}(t),$$

as the Lehmer table [Ed, p. 35] shows.

t	Riemann's Error	Gauss's Error
$1,000,000$	30	130
$2,000,000$	-9	122
$3,000,000$	0	155
$4,000,000$	33	206
$5,000,000$	-64	125
$6,000,000$	24	228
$7,000,000$	-38	179
$8,000,000$	-6	223
$9,000,000$	-53	187
$10,000,000$	88	339

Riemann was interested in the effect of the Hypothesis Series on the count of the Primes. He wrote

The finite sum of oscillatory terms over zeros of Zeta that are less than t*,*

$$-2\frac{t^{-\frac{1}{2}}}{\log t}\sum_{\alpha=\text{zeros}<t}\cos(\alpha\log t),$$

causes irregular fluctuations in the density of the primes. In a future count, it would be interesting to trace the fluctuations of the density of the primes $F'(t)$ *to the particular oscillatory terms* $f'(t)$.

1.5 Riemann's Formula for F(t)

Assuming that the Riemann Hypothesis holds, we have

$$f(t) = \mathrm{Li}(t) - \log 2 - \sum_{\alpha}\Big[\mathrm{Li}(t^{\frac{1}{2}+i\alpha}) + \mathrm{Li}(t^{\frac{1}{2}-i\alpha})\Big] + \int_{u=t}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}\,du$$

$$-\tfrac{1}{2}f(t^{1/2}) = -\tfrac{1}{2}\mathrm{Li}(t^{1/2}) + \tfrac{1}{2}\log 2 + \tfrac{1}{2}\sum_{\alpha}\Big[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/2}) + \mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/2})\Big]$$

$$-\tfrac{1}{2}\int_{u=t^{1/2}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}\,du$$

$$-\tfrac{1}{3}f(t^{1/3}) = -\tfrac{1}{3}\mathrm{Li}(t^{1/3}) + \tfrac{1}{3}\log 2 + \tfrac{1}{3}\sum_{\alpha}\Big[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/3}) + \mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/3})\Big]$$

$$-\tfrac{1}{3}\int_{u=t^{1/3}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}\,du$$

$$-\tfrac{1}{5}f(t^{1/5}) = -\tfrac{1}{5}\mathrm{Li}(t^{1/5}) + \tfrac{1}{5}\log 2 + \tfrac{1}{5}\sum_{\alpha}\Big[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/5}) + \mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/5})\Big]$$

$$-\tfrac{1}{5}\int_{u=t^{1/5}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}\,du$$

$$\tfrac{1}{6}f(t^{1/6}) = \tfrac{1}{6}\mathrm{Li}(t^{1/6}) - \tfrac{1}{6}\log 2 - \tfrac{1}{6}\sum_{\alpha}\Big[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/6}) + \mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/6})\Big]$$

$$+\tfrac{1}{6}\int_{u=t^{1/6}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}\,du$$

$$-\tfrac{1}{7}f(t^{1/7}) = -\tfrac{1}{7}\mathrm{Li}(t^{1/7}) + \tfrac{1}{7}\log 2 + \tfrac{1}{7}\sum_{\alpha}\Big[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/7}) + \mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/7})\Big]$$

$$-\tfrac{1}{7}\int_{u=t^{1/7}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}\,du$$

$$\tfrac{1}{10}f(t^{1/10}) = \tfrac{1}{10}\mathrm{Li}(t^{1/10}) - \tfrac{1}{10}\log 2 - \tfrac{1}{10}\sum_{\alpha}\Big[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/10}) + \mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/10})\Big]$$

$$+\tfrac{1}{10}\int_{u=t^{1/10}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{11}f(t^{\frac{1}{11}}) = -\tfrac{1}{11}\mathrm{Li}(t^{\frac{1}{11}}) + \tfrac{1}{11}\log 2 + \tfrac{1}{11}\sum_{\alpha}\Big[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/11}) + \mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/11})\Big]$$

$$-\tfrac{1}{11}\int_{u=t^{1/11}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{13}f(t^{\frac{1}{13}}) = -\tfrac{1}{13}\mathrm{Li}(t^{\frac{1}{13}}) + \tfrac{1}{13}\log 2 + \tfrac{1}{13}\sum_{\alpha}\Big[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/13}) + \mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/13})\Big]$$

$$-\tfrac{1}{13}\int_{u=t^{1/13}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

....

Then, the count of the primes that are smaller than t is

$$F(t) = \mathrm{Li}(t) - \log 2 - \sum_{\alpha}\Big[\mathrm{Li}(t^{\frac{1}{2}+i\alpha}) + \mathrm{Li}(t^{\frac{1}{2}-i\alpha})\Big] + \int_{u=t}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{2}\mathrm{Li}(t^{1/2}) + \tfrac{1}{2}\log 2 + \tfrac{1}{2}\sum_{\alpha}\Big[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/2}) + \mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/2})\Big]$$

$$-\tfrac{1}{2}\int_{u=t^{1/2}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{3}\mathrm{Li}(t^{1/3}) + \tfrac{1}{3}\log 2 + \tfrac{1}{3}\sum_{\alpha}\Big[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/3}) + \mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/3})\Big]$$

$$-\tfrac{1}{3}\int_{u=t^{1/3}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

+...

$$+\frac{(-1)^{\mu}}{m_{j_0}}\Bigg(\mathrm{Li}(t^{1/m_{j_0}})-\log 2-\sum_{\alpha}\Big[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/m_{j_0}})+\mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/m_{j_0}})\Big]$$

$$+\int_{u=t^{1/m_{j_0}}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du\Bigg).$$

Grouping like-terms, Riemann's Formula is,

$$\boxed{\begin{aligned}
F(t) &= \left\{\mathrm{Li}(t)-\tfrac{1}{2}\mathrm{Li}(t^{1/2})-\tfrac{1}{3}\mathrm{Li}(t^{1/3})+...+\frac{(-1)^{\mu}}{m_{j_0}}\mathrm{Li}(t^{1/m_{j_0}})\right\}\\
&+\left\{-1+\tfrac{1}{2}+\tfrac{1}{3}+...+\frac{(-1)^{\mu}}{m_{j_0}}\right\}\mathrm{Log}\,2\\
&+\sum_{\alpha}\Bigg(-\Big[\mathrm{Li}(t^{\frac{1}{2}+i\alpha})+\mathrm{Li}(t^{\frac{1}{2}-i\alpha})\Big]\\
&\qquad+\tfrac{1}{2}\Big[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/2})+\mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/2})\Big]\\
&\qquad+\tfrac{1}{3}\Big[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/3})+\mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/3})\Big]+.....\\
&\quad-\frac{(-1)^{\mu}}{m_{j_0}}\Big[\mathrm{Li}(t^{(\frac{1}{2}+i\alpha)/m_{j_0}})+\mathrm{Li}(t^{(\frac{1}{2}-i\alpha)/m_{j_0}})\Big]\\
&+\Bigg\{\int_{u=t}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du-\tfrac{1}{2}\int_{u=t^{1/2}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du\\
&\quad-\tfrac{1}{3}\int_{u=t^{1/3}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du....+\frac{(-1)^{\mu}}{m_{j_0}}\int_{u=t^{1/m_{j_0}}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du\Bigg\}
\end{aligned}}$$

where the summation terminates at m_{j_0} so that

$$t^{1/m_{j_0}}>2,\qquad \text{and } t^{1/[m_{j_0+1}]}<2.$$

The infinite sum over the zeta zeros is the Hypothesis Series.

Appendix B lists the formulas for the $f(t)$ term up to

$$m = 113$$

We proceed to set up a Chi-Squared Goodness-of-fit test for the Riemann-Hypothesis-Series,

and hence, for Riemann's Formula for the Count of the Primes,

and therefore, for the Riemann Hypothesis.

2

Chi-Squared Goodness-of-fit-Test of Riemann Hypothesis Series.

2.1 Random Number Generation.

To set up a Chi-Squared test, with say $\nu = 5$ degrees of freedom, we generate $\nu + 1 = 6$ random numbers $t_1, t_2, ...t_6$ in an interval on the line $x = \frac{1}{2}$.

We rearrange the random numbers by size so that we have $t_1 < t_2 < .... < t_6$.

2.2 Observed values

Each of the t_i's, determines a value for the count of the primes $F_{\text{observed}}(t_i)$.

2.3 Expected values

To compute an expected value $F_{\text{e xpected}}(t_i)$, we need to approximate the logarithmic integral series that sums up over all the zeros of Zeta on $x = 1/2$. We use the tables from [Odlyz].

2.4 The Null-Hypothesis:

Riemann's formula for the count of the primes is valid with Riemann Hypothesis Series.

We aim to show that the distribution expected by the formula that includes the Hypothesis Series, fits well the observed distribution of primes.

2.5 The Chi-Squared Statistic for F(t)

$$\chi^2_{\text{computed}}(\alpha,\nu) = \sum_i \frac{\left[F_{\text{observed}}(t_i) - F_{\text{expected}}(t_i)\right]^2}{F_{\text{expected}}(t_i)}.$$

Here, we will have

$$F_{\text{expected}}(t_i) \approx F_{\text{observed}}(t_i),$$

and approximate the Chi-Squared terms by

$$\frac{\left[F_{\text{observed}}(t_i) - F_{\text{expected}}(t_i)\right]^2}{F_{\text{observed}}(t_i)}.$$

2.6 The Number of Degrees of Freedom

of the Chi-Squared distribution in this example is $\nu = 5$. At least 5 degrees of freedom are recommended for Chi-Squared Goodness-of-Fit testing. Greater confidence of the test, will require greater ν.

As ν increases, the skewed Chi-Squared distribution gets close to the symmetric normal distribution

2.7 Example of 0.99997% Confidence Chi-Squared Test:

We apply a chi-squared test to confirm the Riemann *approximation* formula of 1.4. Suppose that our generated random numbers are

$$10^4 < 10^5 < 10^8 < 10^{11} < 10^{15} < 10^{16}$$

Using the table in [Saut],

t_i	$F_{observed}(t_i)$	$F_{\exp\text{ected}}(t_i)$	$error(t_i)$	$\frac{[error(t_i)]^2}{F_{\exp\text{ected}}}$
10^4	1229	1227	-2	$(3.3)\ /\ 10^3$
10^5	9592	9587	-5	$2.6\ /\ 10^3$
10^8	$5,761,455$	$5,761,552$	97	$1.6\ /\ 10^3$
10^{11}	$4,118,054,813$	$4,118,052,495$	$-2,318$	$1.3\ /\ 10^3$
10^{15}	$29,844,570,422,669$	$29,844,570,495,887$	$73,218$	$0.2\ /\ 10^3$
10^{16}	$279,238,341,033,925$	$279,238,341,360,977$	$327,052$	$0.4\ /\ 10^3$

Therefore,

$$\chi^2{}_{\text{computed}}(5) = \left(3.3 + 2.6 + 1.6 + 1.3 + 0.2 + 0.4\right) / 10^3$$
$$< 0.01 = \chi^2(0.999997, 5)$$

That is, if the t_i were generated at random, we could conclude that the Riemann approximation formula of 1.4, holds with 99.9997% confidence.

We seek higher confidence for the test of the Riemann-Hypothesis-term

3

The Chi-Squared test value

3.1 Chi-Squared, and Gamma

By [Abram], the Chi-Squared distribution is the Cumulative Probability Function

$$Q(\chi^2;\nu) = \frac{1}{2^{\nu/2}\Gamma(\nu\ /\ 2)} \int_{t=\chi^2}^{\infty} e^{-t/2} t^{\nu/2-1} dt .$$

To use the MATHEMATICA Gamma function

$$\mathrm{Gamma}[a,z] = \int_{u=z}^{u=\infty} e^{-u} u^{a-1} du ,$$

we identify $a = \frac{\nu}{2}$, $u = \frac{t}{2}$, $z = \frac{\chi^2}{2}$. Then,

$$Q(\chi^2;\nu) = \frac{1}{\Gamma(\nu\ /\ 2)} \int_{u=\chi^2/2}^{\infty} e^{-u} u^{\nu/2-1} du = \frac{1}{\Gamma(\nu\ /\ 2)} \mathrm{Gamma}[\nu/2,\chi^2\ /\ 2]$$

For $\nu = 8$, and $\chi^2 = 0.00044$,

$$Q(0.00044;8) = N[\frac{1}{3!}\mathrm{Gamma}[4{,}0.00022]] = 0.9999999999999998`$$

This means an uncertainty

$$P(\chi^2,8) = P(0.00044,8) = 2\ /\ 10^{16}.$$

3.2 Uncertainty Limitation

We use here the MATHEMATICA software because the MINITAB software does not return a χ^2 value already for $P(\chi^2) = 10^{-13}$.

However, for $\chi^2 = 0.0004$,

$$Q(0.0004;8) = N[\frac{1}{3!}\mathrm{Gamma}[4,0.0002]] = 1.0$$

This means an uncertainty

$$P(\chi^2, 8) = P(0.0004, 8) = 0.$$

Or, that we have reached the limit on the accuracy of the MATHEMATICA software. The smallest uncertainty that the software will detect is

$$P(\chi^2, 8) = 2 / 10^{16}.$$

3.3 Uncertainty Used

Consequently, to obtain 99.99999999999998% confidence in a Chi-Squared test for $F(t)$, we choose $\nu = 8$, generate at random the 9 numbers

$$t_1 < t_2 < t_3 < t_4 < t_5 < t_6 < t_7 < t_8 < t_9,$$

compute at each one the term $\frac{\left[F_{\text{observed}}(t_i) - F_{\text{expected}}(t_i)\right]^2}{F_{\text{expected}}(t_i)}$, and sum those terms up to obtain $\chi^2_{\text{computed}}(8)$.

If $\chi^2_{\text{computed}}(8) < 0.00044$, we accept the null Hypothesis with at least 99.99999999999998% confidence.

4

The random numbers

4.1 The choice of the Random numbers

Computing with the table in [Saut], we see that while $\left|error(t_i)\right|$ of the Riemann approximation for the count of the primes, may be larger for larger t_i, the Chi-Squared term $\dfrac{[error(t_i)]^2}{F_{\text{expected}}(t_i)}$ is likely to be smaller for larger t_i.

N	$\pi(n) = F_{observed}(N)$	$F_{\text{expected}}(N)$	$error(N)$	$\dfrac{[error(N)]^2}{F_{\text{expected}}(N)}$
10^2	25	26	1	$385 / 10^4$
10^3	168	168	0	0
10^4	1229	1227	-2	$33 / 10^4$
10^5	9592	9587	-5	$26 / 10^4$
10^6	$78,498$	$78,527$	29	$107 / 10^4$
10^7	$664,579$	$664,667$	88	$116 / 10^4$
10^8	$5,761,455$	$5,761,552$	97	$16 / 10^4$
10^9	$50,847,534$	$50,847,455$	-79	$1.2 / 10^4$
10^{10}	$455,052,511$	$455,050,683$	-1828	$73 / 10^4$
10^{11}	$4,118,054,813$	$4,118,052,495$	-2318	$13 / 10^4$
10^{12}	$37,607,912,018$	$37,607,910,542$	-1476	$0.6 / 10^4$
10^{13}	$346,065,536,839$	$346,065,531,066$	-5773	$1 / 10^4$
10^{14}	$3,204,941,750,802$	$3,204,941,731,602$	$-19,200$	$1.1 / 10^4$
10^{15}	$29,844,570,422,669$	$29,844,570,349,451$	$73,218$	$1.8 / 10^4$
10^{16}	$279,238,341,033,925$	$279,238,340,706,873$	$327,052$	$3.8 / 10^4$

The observed decrease of $\frac{[error(t_i)]^2}{F_{\text{expected}}(t_i)}$ is not monotonic, and we do not know of a proof for it. The statement that

$$\frac{\left[\pi(t) - \text{Li}(t)\right]^2}{\text{Li}(t)} \to 0 \text{, as } t \to \infty,$$

implies the Prime Number Theorem

$$\frac{\pi(t) - \text{Li}(t)}{\text{Li}(t)} \to 0 \text{, as } t \to \infty,$$

but is not implied by the prime Number Theorem.

Nevertheless, the observed decrease suggests that at a smaller t_i, the Chi-Squared term is larger, and smaller t_i will be effective for our test. Consequently, we shall choose for our test, random numbers

$$100 < t_i < 10,000$$

4.2 The Random Numbers

The MINITAB software generated from a uniform distribution nine random numbers that we rounded, and arranged by size,

$$\begin{aligned} t_1 &= 381 \\ t_2 &= 923 \\ t_3 &= 1674 \\ t_4 &= 2024 \\ t_5 &= 2247 \\ t_6 &= 5458 \\ t_7 &= 6349 \\ t_8 &= 8802 \\ t_9 &= 9016 \end{aligned}$$

From the table of primes up to 10,000 in [Rib], we find $F_{observed}(t_i)$, for the nine random numbers,

t_i	381	923	1674	2024	2247	5458	6349	8802	9016
$F_{observed}(t_i)$	75	157	263	306	334	721	826	1095	1121

The tables in [Lehm] list the prime numbers up to $10,000,000$.

5

The Chi-squared term for F(381)

5.1 m(381)

We sum up to $m(381) = 7$, because

$$\sqrt[7]{381} = 2.33 > 2 \text{ and } \sqrt[10]{381} = 1.811 < 2.$$

5.2 The integral term

s4 =

$$\mathrm{N}[\int_{381}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx - \frac{1}{2}\int_{\sqrt{381}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx - \frac{1}{3}\int_{\sqrt[3]{381}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx - \frac{1}{5}\int_{\sqrt[5]{381}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx + \frac{1}{6}\int_{\sqrt[6]{381}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx - \frac{1}{7}\int_{\sqrt[7]{381}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx]$$

= -0.010618753735023375`

5.3 The Log2 term

s2 = N[(-1+1/2+1/3+1/5-1/6+1/7)*Log[2]]

= 0.006601401719618527`

5.4 The Riemann approximation term

```
s1 =
  N[LogIntegral[381] -
    1/2 LogIntegral[√381] -
    1/3 LogIntegral[∛381] -
    1/5 LogIntegral[381^(1/5)] +
    1/6 LogIntegral[381^(1/6)] -
    1/7 LogIntegral[381^(1/7)]]
    =75.35134682594668`
```

5.5 The Observed Count of the Primes for 381

$$F_{0bserved}(381) = 75$$

5.6 The observed Hypothesis Series for 381

is `v =Total[{75,-s1,-s2,-s4}]`

`=-0.3473294739312728``

5.7 The expected Hypothesis Series for 381

We aim to approximate the expected Hypothesis Series with partial sums of its infinite series.

By [Derb, p. 390, endpoint 128], for complex numbers we have to use in MATHEMATICA

$$\mathrm{Li}(t^{1/2+ir}) = \mathrm{ExpIntegralEi}[(1\ /\ 2 + ir)\mathrm{Log}[t]$$

Therefore, the term of the Hypothesis Series for $F(381)$ at the zero α is

t =

$$
\begin{aligned}
&-\Big(\mathrm{Li}(381^{1/2+i\alpha})+\mathrm{Li}(381^{1/2-i\alpha})\Big)\\
&+\tfrac{1}{2}\Big(\mathrm{Li}(381^{(1/2+i\alpha)/2})+\mathrm{Li}(381^{(1/2-i\alpha)/2})\Big)\\
&+\tfrac{1}{3}\Big(\mathrm{Li}(381^{(1/2+i\alpha)/3})+\mathrm{Li}(381^{(1/2-i\alpha)/3})\Big)\\
&+\tfrac{1}{5}\Big(\mathrm{Li}(381^{(1/2+i\alpha)/5})+\mathrm{Li}(381^{(1/2-i\alpha)/5})\Big)\\
&-\tfrac{1}{6}\Big(\mathrm{Li}(381^{(1/2+i\alpha)/6})+\mathrm{Li}(381^{(1/2-i\alpha)/6})\Big)\\
&+\tfrac{1}{7}\Big(\mathrm{Li}(381^{(1/2+i\alpha)/7})+\mathrm{Li}(381^{(1/2-i\alpha)/7})\Big)
\end{aligned}
$$

= -(ExpIntegralEi[(1/2+ δ)Log[381]]+

ExpIntegralEi[(1/2- δ)Log[381]])

+(1/2)(ExpIntegralEi[(1/2+ δ)Log[381]/2]+

ExpIntegralEi[(1/2- δ)Log[381]/2])

+(1/3)(ExpIntegralEi[(1/2+ δ)Log[381]/3]+

ExpIntegralEi[(1/2- δ)Log[381]/3])

+(1/5)(ExpIntegralEi[(1/2+ δ)Log[381]/5]+

ExpIntegralEi[(1/2- δ)Log[381]/5])

-(1/6)(ExpIntegralEi[(1/2+ δ)Log[381]/6]+

ExpIntegralEi[(1/2- δ)Log[381]/6])

+(1/7)(ExpIntegralEi[(1/2+ δ)Log[381]/7]+

ExpIntegralEi[(1/2- δ)Log[381]/7])

The first 50 zeros of the zeta function multiplied by i are

δ ={14.1347 ⅈ,21.022 ⅈ,25.0109 ⅈ,30.4249 ⅈ,32.9351 ⅈ,37.5862 ⅈ,40.9187 ⅈ,43.3271 ⅈ,48.0052 ⅈ,49.7738 ⅈ,52.9703 ⅈ,56.4462 ⅈ,59.347 ⅈ,60.8318 ⅈ,65.1125 ⅈ,67.0798 ⅈ,69.5464 ⅈ,72.0672 ⅈ,75.7047 ⅈ,77.1448 ⅈ,79.3374 ⅈ,82.9104 ⅈ,84.7355 ⅈ,87.4253 ⅈ,88.8091 ⅈ,92.4919 ⅈ,94.6513 ⅈ,95.8706 ⅈ,98.8312 ⅈ,101.318 ⅈ,103.726 ⅈ,105.447 ⅈ,107.169 ⅈ,111.03 ⅈ,111.875 ⅈ,114.32 ⅈ,116.227 ⅈ,118.791 ⅈ,121.37 ⅈ,122.947 ⅈ,124.257 ⅈ,127.517 ⅈ,129.579 ⅈ,131.088 ⅈ,133.498 ⅈ,134.757 ⅈ,138.116 ⅈ,139.736 ⅈ,141.124 ⅈ,143.112 ⅈ}

At each of these zeros, we compute **t**. We obtain

```
{-0.509338+0. i,0.0997343 +0. i,0.142347 +0. i,0.2355 +0. i,-0.157656+0. i,0.020229 +0. i,0.148675 +0. i,0.0173037 +0. i,-0.080168+0. i,-0.0814755+0. i,-0.0804875+0. i,-0.124942+0. i,-0.0829943+0. i,0.0267704 +0. i,0.0199384 +0. i,-0.0450854+0. i,0.0870017 +0. i,-0.0995605+0. i,0.0423838 +0. i,0.0232718 +0. i,-0.0277735+0. i,-0.016244+0. i,-0.0660189+0. i,0.0790202 +0. i,-0.00599591+0. i,-0.0126983+0. i,-0.0192838+0. i,0.083498 +0. i,-0.0155991+0. i,0.0477536 +0. i,-0.0509095+0. i,0.0667422 +0. i,-0.0577721+0. i,-0.00581459+0. i,0.0660742 +0. i,-0.0369921+0. i,0.00597762 +0. i,-0.0231158+0. i,0.0645874 +0. i,-0.050692+0. i,0.00123119 +0. i,0.0551029 +0. i,0.0217123 +0. i,0.00518654 +0. i,-0.0327757+0. i,-0.0235986+0. i,0.0474598 +0. i,-0.0345629+0. i,-0.0228729+0. i,-0.0342391+0. i}
```

The following program computes the first 50 partial sums of the infinite series

```
y=Range[50]
y[[1]]=t[[1]]
Do [y[[j]] = y[[j-1]]+t[[j]],{j, 2,50}]
```

As the graph in [Derb] shows, the Hypothesis series has oscillatory convergence. The partial sum **y** is likely to deviate the least from **v** after, or before a change of sign of

$$p = v - y.$$

The values of **p** are

```
{0.162008 +0. i,0.062274 +0. i,-0.0800731+0. i,-0.315573+0. i,-0.157918+0. i,-0.178147+0. i,-0.326822+0. i,-0.344126+0. i,-0.263958+0. i,-0.182482+0. i,-0.101995+0. i,0.0229478 +0. i,0.105942 +0. i,0.0791717 +0. i,0.0592333 +0. i,0.104319 +0. i,0.0173171 +0. i,0.116878 +0. i,0.0744938 +0. i,0.051222 +0. i,0.0789955 +0. i,0.0952395 +0. i,0.161258 +0. i,0.0822382 +0. i,0.0882341 +0. i,0.100932 +0. i,0.120216 +0. i,0.0367182 +0. i,0.0523173 +0. i,0.00456369 +0. i,0.0554732 +0. i,-0.011269+0. i,0.0465031 +0. i,0.0523177 +0. i,-0.0137565+0. i,0.0232355 +0. i,0.0172579 +0.
```

```
i,0.0403737 +0. i,-0.0242137+0. i,0.0264783 +0. i,0.0252471 +0. i,-
0.0298558+0. i,-0.0515681+0. i,-0.0567547+0. i,-0.023979+0. i,-
0.000380325+0. i,-0.0478401+0. i,-0.0132773+0. i,0.00959561 +0.
i,0.0438347 +0. i}
```

In this list,

Min[Abs[p]]= 0.000380325

Thus, The minimal error in this list is obtained with the 46th partial sum.

5.8 The Chi-Squared term for 381

Running the program with the first 2800 zeta zeros, the minimal error is

Min[Abs[p]]= 7.29391×10^{-6}

We take

$$error(381) = (7.29391)10^{-6}$$

and the Chi-Squared error term is

$$\left|error(381)\right|^2 \;/\; F_{\text{expected}}(381) = (7.09348)10^{-13}$$

6

The Chi-squared term for F(923)

6.1 We sum up to $m(923) = 7$, because

$$\sqrt[7]{923} = 2.65 > 2 \text{ and } \sqrt[10]{923} = 1.97 < 2.$$

6.2 The integral term is

s4=

$$\begin{aligned}&\mathrm{N}[\int_{923}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx\\&-\tfrac{1}{2}\int_{\sqrt{923}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx\\&-\tfrac{1}{3}\int_{\sqrt[3]{923}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx\\&-\tfrac{1}{5}\int_{\sqrt[5]{923}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx\\&+\tfrac{1}{6}\int_{\sqrt[6]{923}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx\\&-\tfrac{1}{7}\int_{\sqrt[7]{923}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx]\end{aligned}$$

=-0.00676033302597897 4

6.3 The $\log 2$ term is

s2=N[(-1+1/2+1/3+1/5-1/6+1/7)*Log[2]]

=0.00660140171961852 7

6.4 The Riemann approximation term is

s1=

$$\begin{aligned}&\mathrm{N}[\mathrm{LogIntegral}[923]\text{-}\\&\tfrac{1}{2}\mathrm{LogIntegral}[\sqrt{923}]\text{-}\\&\tfrac{1}{3}\mathrm{LogIntegral}[\sqrt[3]{923}]\text{-}\\&\tfrac{1}{5}\mathrm{LogIntegral}[\sqrt[5]{923}]+\\&\tfrac{1}{6}\mathrm{LogIntegral}[\sqrt[6]{923}]\text{-}\\&\tfrac{1}{7}\mathrm{LogIntegral}[\sqrt[7]{923}]]\end{aligned}$$

= 157.3502496845243

6.5

$$F_{0bserved}(923) = 157$$

6.6 **v=Total[{157,-s1,-s2,-s4}]**

=-0.3500907532179397

6.7 **t=**

$$\begin{aligned}&-\Big(\mathrm{Li}(923^{1/2+i\alpha})+\mathrm{Li}(923^{1/2-i\alpha})\Big)\\&+\tfrac{1}{2}\Big(\mathrm{Li}(923^{(1/2+i\alpha)/2})+\mathrm{Li}(923^{(1/2-i\alpha)/2})\Big)\\&+\tfrac{1}{3}\Big(\mathrm{Li}(923^{(1/2+i\alpha)/3})+\mathrm{Li}(923^{(1/2-i\alpha)/3})\Big)\\&+\tfrac{1}{5}\Big(\mathrm{Li}(923^{(1/2+i\alpha)/5})+\mathrm{Li}(923^{(1/2-i\alpha)/5})\Big)\\&-\tfrac{1}{6}\Big(\mathrm{Li}(923^{(1/2+i\alpha)/6})+\mathrm{Li}(923^{(1/2-i\alpha)/6})\Big)\\&+\tfrac{1}{7}\Big(\mathrm{Li}(923^{(1/2+i\alpha)/7})+\mathrm{Li}(923^{(1/2-i\alpha)/7})\Big)\end{aligned}$$

6.8 The minimal error obtained with the first 2800 zeta zeros is

Min[Abs[p]]=(8.822) 10^{-6}

We take

$$error(923)=(8.8220)10^{-6}$$

Then, the Chi-Squared term for 923 is

$$\left|error(923)\right|^2 \ / \ F_{\text{expected}}(923) = (4.94617)10^{-13}.$$

7

The Chi-squared term for F(1674)

7.1 We sum up to $m(1674) = 10$, because

$$\sqrt[10]{1674} = 2.1 > 2 \text{ and } \sqrt[11]{1674} = 1.96 < 2.$$

7.2 The integral term is

s4 =

$$\mathrm{N}[\int_{1674}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx$$
$$-\tfrac{1}{2}\int_{\sqrt{1674}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx$$
$$-\tfrac{1}{3}\int_{\sqrt[3]{1674}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx$$
$$-\tfrac{1}{5}\int_{\sqrt[5]{1674}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx$$
$$+\tfrac{1}{6}\int_{\sqrt[6]{1674}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx$$
$$-\tfrac{1}{7}\int_{\sqrt[7]{1674}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx$$
$$+\tfrac{1}{10}\int_{\sqrt[10]{1674}}^{\infty}\frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}dx]$$

= 0.006773572185215815`

7.3 The $\log 2$ term is

s2 = N[(-1+1/2+1/3+1/5-1/6+1/7-1/10)*Log[2]]

=-0.062713316336376`

7.4 The Riemann approximation term is **`s1=`**

$$\begin{aligned}&\text{N[LogIntegral[1674]-}\\&\tfrac{1}{2}\text{LogIntegral}[\sqrt{1674}]\text{-}\\&\tfrac{1}{3}\text{LogIntegral}[\sqrt[3]{1674}]\text{-}\\&\tfrac{1}{5}\text{LogIntegral}[\sqrt[5]{1674}]+\\&\tfrac{1}{6}\text{LogIntegral}[\sqrt[6]{1674}]\text{-}\\&\tfrac{1}{7}\text{LogIntegral}[\sqrt[7]{1674}]+\\&\tfrac{1}{10}\text{LogIntegral}[\sqrt[10]{1674}]]\end{aligned}$$

`= 260.6624479608718``

7.5

$$F_{0observed}(1674) = 263$$

7.6

`v=Total[{263,-s1,-s2,-s4}]`

`= 2.393491783279344`

7.7 **`t=`**

$$\begin{aligned}&-\left(\mathrm{Li}(1674^{1/2+i\alpha})+\mathrm{Li}(1674^{1/2-i\alpha})\right)\\&+\tfrac{1}{2}\left(\mathrm{Li}(1674^{(1/2+i\alpha)/2})+\mathrm{Li}(1674^{(1/2-i\alpha)/2})\right)\\&+\tfrac{1}{3}\left(\mathrm{Li}(1674^{(1/2+i\alpha)/3})+\mathrm{Li}(1674^{(1/2-i\alpha)/3})\right)\\&+\tfrac{1}{5}\left(\mathrm{Li}(1674^{(1/2+i\alpha)/5})+\mathrm{Li}(1674^{(1/2-i\alpha)/5})\right)\\&-\tfrac{1}{6}\left(\mathrm{Li}(1674^{(1/2+i\alpha)/6})+\mathrm{Li}(1674^{(1/2-i\alpha)/6})\right)\\&+\tfrac{1}{7}\left(\mathrm{Li}(1674^{(1/2+i\alpha)/7})+\mathrm{Li}(1674^{(1/2-i\alpha)/7})\right)\\&-\tfrac{1}{10}\left(\mathrm{Li}(1674^{(1/2+i\alpha)/10})+\mathrm{Li}(1674^{(1/2-i\alpha)/10})\right)\end{aligned}$$

7.8 The minimal error obtained with the first 2800 zeta zeros is

`Min[Abs[p]]= 0.0000533725`

We take

$$error(1674) = 0.0000533725$$

and the Chi-Squared error term is

$$\left|error(1674)\right|^2 \;/\; F_{\text{expected}}(1674) = (1.08313)10^{-11}$$

8

The Chi-squared term for F(2024)

8.1 We sum up to $m(2024) = 10$, because

$$\sqrt[10]{2024} = 2.14 > 2 \text{ and } \sqrt[11]{2024} = 1.997 < 2.$$

8.2 The integral term is

s4=

$$\begin{aligned}
&\mathrm{N}[\int_{2024}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\tfrac{1}{2}\int_{\sqrt{2024}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\tfrac{1}{3}\int_{\sqrt[3]{2024}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\tfrac{1}{5}\int_{\sqrt[5]{2024}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&+\tfrac{1}{6}\int_{\sqrt[6]{2024}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\tfrac{1}{7}\int_{\sqrt[7]{2024}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&+\tfrac{1}{10}\int_{\sqrt[10]{2024}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx]
\end{aligned}$$

= 0.006480300815245885`

8.3 The $\log 2$ term is

s2 = N[(-1+1/2+1/3+1/5-1/6+1/7-1/10)*Log[2]]

=-0.062713316336376`

8.4 The Riemann approximation term is

s1=

$$\begin{aligned}&\text{N[LogIntegral[2024]-}\\&\tfrac{1}{2}\,\text{LogIntegral}[\sqrt{2024}]\text{-}\\&\tfrac{1}{3}\,\text{LogIntegral}[\sqrt[3]{2024}]\text{-}\\&\tfrac{1}{5}\,\text{LogIntegral}[\sqrt[5]{2024}]+\\&\tfrac{1}{6}\,\text{LogIntegral}[\sqrt[6]{2024}]\text{-}\\&\tfrac{1}{7}\,\text{LogIntegral}[\sqrt[7]{2024}]+\\&\tfrac{1}{10}\,\text{LogIntegral}[\sqrt[10]{2024}]]\end{aligned}$$

=306.54364361954345`

8.5 $$F_{0bserved}(2024) = 306$$

8.6 **v =Total[{306,-s1,-s2,-s4}]**

= -0.487410604022315 2`

8.7 **t=**

$$\begin{aligned}&-\Big(\mathrm{Li}(2024^{1/2+i\alpha})+\mathrm{Li}(2024^{1/2-i\alpha})\Big)\\&+\tfrac{1}{2}\Big(\mathrm{Li}(2024^{(1/2+i\alpha)/2})+\mathrm{Li}(2024^{(1/2-i\alpha)/2})\Big)\\&+\tfrac{1}{3}\Big(\mathrm{Li}(2024^{(1/2+i\alpha)/3})+\mathrm{Li}(2024^{(1/2-i\alpha)/3})\Big)\\&+\tfrac{1}{5}\Big(\mathrm{Li}(2024^{(1/2+i\alpha)/5})+\mathrm{Li}(2024^{(1/2-i\alpha)/5})\Big)\\&-\tfrac{1}{6}\Big(\mathrm{Li}(2024^{(1/2+i\alpha)/6})+\mathrm{Li}(2024^{(1/2-i\alpha)/6})\Big)\\&+\tfrac{1}{7}\Big(\mathrm{Li}(2024^{(1/2+i\alpha)/7})+\mathrm{Li}(2024^{(1/2-i\alpha)/7})\Big)\\&-\tfrac{1}{10}\Big(\mathrm{Li}(2024^{(1/2+i\alpha)/10})+\mathrm{Li}(2024^{(1/2-i\alpha)/10})\Big)\end{aligned}$$

8.8 The minimal error obtained with the first 2800 zeta zeros is

```
Min[Abs[p]]= 0.0000145234
```

We take

$$error(2024) = 0.0000145234$$

and the Chi-Squared error term is

$$\left|error(2024)\right|^2 \;/\; F_{\text{expected}}(2024) = (6.89314)10^{-13}$$

9

The Chi-squared term for F(2247)

9.1 We sum up to $m(2247) = 11$, because

$$\sqrt[11]{2247} = 2.01 > 2 \text{ and } \sqrt[13]{2247} = 1.81 < 2\,.$$

9.2 The integral term is

s4=

$$\begin{aligned}
&\mathrm{N}[\int_{2247}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\tfrac{1}{2}\int_{\sqrt{2247}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\tfrac{1}{3}\int_{\sqrt[3]{2247}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\tfrac{1}{5}\int_{\sqrt[5]{2247}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&+\tfrac{1}{6}\int_{\sqrt[6]{2247}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\tfrac{1}{7}\int_{\sqrt[7]{2247}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&+\tfrac{1}{10}\int_{\sqrt[10]{2247}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\tfrac{1}{11}\int_{\sqrt[11]{2247}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx]
\end{aligned}$$

=-0.006042898086798278`

9.3 The $\log 2$ term is

s2 = N[(-1+1/2+1/3+1/5-1/6+1/7-1/10+1/11)*Log[2]]

=0.00030006371452811483`

9.4 The Riemann approximation term is

s1=

$$\begin{aligned}&\mathrm{N[LogIntegral[2247]}\text{-}\\&\tfrac{1}{2}\mathrm{LogIntegral}[\sqrt{2247}]\text{-}\\&\tfrac{1}{3}\mathrm{LogIntegral}[\sqrt[3]{2247}]\text{-}\\&\tfrac{1}{5}\mathrm{LogIntegral}[\sqrt[5]{2247}]+\\&\tfrac{1}{6}\mathrm{LogIntegral}[\sqrt[6]{2247}]\text{-}\\&\tfrac{1}{7}\mathrm{LogIntegral}[\sqrt[7]{2247}]+\\&\tfrac{1}{10}\mathrm{LogIntegral}[\sqrt[10]{2247}]\text{-}\\&\tfrac{1}{11}\mathrm{LogIntegral}[\sqrt[11]{2247}]]\end{aligned}$$

=335.15560434467403

9.5

$$F_{0bserved}(2247) = 334$$

9.6 **v =Total[{334,-s1,-s2,-s4}]**

= -1.1498615103017618

9.7 **t=**

$$
\begin{aligned}
&-\Big(\mathrm{Li}(2247^{1/2+i\alpha})+\mathrm{Li}(2247^{1/2-i\alpha})\Big)\\
&+\tfrac{1}{2}\Big(\mathrm{Li}(2247^{(1/2+i\alpha)/2})+\mathrm{Li}(2247^{(1/2-i\alpha)/2})\Big)\\
&+\tfrac{1}{3}\Big(\mathrm{Li}(2247^{(1/2+i\alpha)/3})+\mathrm{Li}(2247^{(1/2-i\alpha)/3})\Big)\\
&+\tfrac{1}{5}\Big(\mathrm{Li}(2247^{(1/2+i\alpha)/5})+\mathrm{Li}(2247^{(1/2-i\alpha)/5})\Big)\\
&-\tfrac{1}{6}\Big(\mathrm{Li}(2247^{(1/2+i\alpha)/6})+\mathrm{Li}(2247^{(1/2-i\alpha)/6})\Big)\\
&+\tfrac{1}{7}\Big(\mathrm{Li}(2247^{(1/2+i\alpha)/7})+\mathrm{Li}(2247^{(1/2-i\alpha)/7})\Big)\\
&-\tfrac{1}{10}\Big(\mathrm{Li}(2247^{(1/2+i\alpha)/10})+\mathrm{Li}(2247^{(1/2-i\alpha)/10})\Big)\\
&+\tfrac{1}{11}\Big(\mathrm{Li}(2247^{(1/2+i\alpha)/11})+\mathrm{Li}(2247^{(1/2-i\alpha)/11})\Big)
\end{aligned}
$$

9.8 The minimal error obtained with the first 2800 zeta zeros is

`Min[Abs[p]]= 0.0000104842`

We take

$$error(2247) = 0.0000104842$$

and the Chi-Squared error term is

$$\left|error(2247)\right|^2 \,/\, F_{\text{expected}}(2247) = (3.29096)10^{-13}$$

10

The Chi-squared term for F(5458)

10.1 We sum up to $m(5458) = 11$, because

$$\sqrt[11]{5458} = 2.18 > 2 \text{ and } \sqrt[13]{5458} = 1.93 < 2.$$

10.2 The integral term is

s4=

$$\begin{aligned}
&\mathrm{N}[\int_{5458}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\tfrac{1}{2}\int_{\sqrt{5458}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\tfrac{1}{3}\int_{\sqrt[3]{5458}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\tfrac{1}{5}\int_{\sqrt[5]{5458}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&+\tfrac{1}{6}\int_{\sqrt[6]{5458}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\tfrac{1}{7}\int_{\sqrt[7]{5458}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&+\tfrac{1}{10}\int_{\sqrt[10]{5458}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\tfrac{1}{11}\int_{\sqrt[11]{5458}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx]
\end{aligned}$$

= -0.004340260436657939

10.3 The $\log 2$ term is

s2 = N[(-1+1/2+1/3+1/5-1/6+1/7-1/10+1/11)*Log[2]]

= 0.0003000637145281148 3

10.4 The Riemann approximation term is

s1=

$$\begin{aligned}&\text{N[LogIntegral[5458]-}\\&\tfrac{1}{2}\,\text{LogIntegral}[\sqrt{5458}]\text{-}\\&\tfrac{1}{3}\,\text{LogIntegral}[\sqrt[3]{5458}]\text{-}\\&\tfrac{1}{5}\,\text{LogIntegral}[\sqrt[5]{5458}]+\\&\tfrac{1}{6}\,\text{LogIntegral}[\sqrt[6]{5458}]\text{-}\\&\tfrac{1}{7}\,\text{LogIntegral}[\sqrt[7]{5458}]+\\&\tfrac{1}{10}\,\text{LogIntegral}[\sqrt[10]{5458}]\text{-}\\&\tfrac{1}{11}\,\text{LogIntegral}[\sqrt[11]{5458}]]\end{aligned}$$

= 722.0044424749316

10.5 $$F_{0bserved}(5458) = 721$$

10.6 **v =Total[{721,-s1,-s2,-s4}]**

= -1.0004022782094497

10.7 **t=**

$$
\begin{aligned}
&-\Big(\mathrm{Li}(5458^{1/2+i\alpha}) + \mathrm{Li}(5458^{1/2-i\alpha})\Big)\\
&+\tfrac{1}{2}\Big(\mathrm{Li}(5458^{(1/2+i\alpha)/2}) + \mathrm{Li}(5458^{(1/2-i\alpha)/2})\Big)\\
&+\tfrac{1}{3}\Big(\mathrm{Li}(5458^{(1/2+i\alpha)/3}) + \mathrm{Li}(5458^{(1/2-i\alpha)/3})\Big)\\
&+\tfrac{1}{5}\Big(\mathrm{Li}(5458^{(1/2+i\alpha)/5}) + \mathrm{Li}(5458^{(1/2-i\alpha)/5})\Big)\\
&-\tfrac{1}{6}\Big(\mathrm{Li}(5458^{(1/2+i\alpha)/6}) + \mathrm{Li}(5458^{(1/2-i\alpha)/6})\Big)\\
&+\tfrac{1}{7}\Big(\mathrm{Li}(5458^{(1/2+i\alpha)/7}) + \mathrm{Li}(5458^{(1/2-i\alpha)/7})\Big)\\
&-\tfrac{1}{10}\Big(\mathrm{Li}(5458^{(1/2+i\alpha)/10}) + \mathrm{Li}(5458^{(1/2-i\alpha)/10})\Big)\\
&+\tfrac{1}{11}\Big(\mathrm{Li}(5458^{(1/2+i\alpha)/11}) + \mathrm{Li}(5458^{(1/2-i\alpha)/11})\Big)
\end{aligned}
$$

10.8 The minimal error obtained with the first 2800 zeta zeros is

`Min[Abs[p]]= 0.0000539328`

We take

$$error(5458) = 0.0000539328$$

and the Chi-Squared error term is

$$\Big|error(5458)\Big|^2 \;/\; F_{\text{expected}}(5458) = (4.03433)10^{-12}$$

11

The Chi-squared term for F(6349)

11.1 We sum up to $m(6349) = 11$, because

$$\sqrt[11]{6349} = 2.21 > 2 \text{ and } \sqrt[13]{6349} = 1.96 < 2.$$

11.2 The integral term is

s4=

$$\begin{aligned}
&\mathrm{N}[\int_{6349}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\tfrac{1}{2}\int_{\sqrt{6349}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\tfrac{1}{3}\int_{\sqrt[3]{6349}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\tfrac{1}{5}\int_{\sqrt[5]{6349}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&+\tfrac{1}{6}\int_{\sqrt[6]{6349}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\tfrac{1}{7}\int_{\sqrt[7]{6349}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&+\tfrac{1}{10}\int_{\sqrt[10]{6349}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\tfrac{1}{11}\int_{\sqrt[11]{6349}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx]
\end{aligned}$$

= -0.004112932363175427

11.3 The $\log 2$ term is

s2 = N[(-1+1/2+1/3+1/5-1/6+1/7-1/10+1/11)*Log[2]]

= 0.0003000637145281148 3

11.4 The Riemann approximation term is

s1=

$$
\begin{aligned}
&\mathrm{N}[\mathrm{LogIntegral}[6349]- \\
&\tfrac{1}{2}\mathrm{LogIntegral}[\sqrt{6349}]- \\
&\tfrac{1}{3}\mathrm{LogIntegral}[\sqrt[3]{6349}]- \\
&\tfrac{1}{5}\mathrm{LogIntegral}[\sqrt[5]{6349}]+ \\
&\tfrac{1}{6}\mathrm{LogIntegral}[\sqrt[6]{6349}]- \\
&\tfrac{1}{7}\mathrm{LogIntegral}[\sqrt[7]{6349}]+ \\
&\tfrac{1}{10}\mathrm{LogIntegral}[\sqrt[10]{6349}]- \\
&\tfrac{1}{11}\mathrm{LogIntegral}[\sqrt[11]{6349}]]
\end{aligned}
$$

= 823.8404154923783

11.5 $$F_{0bserved}(6349) = 826$$

11.6 **v =Total[{826,-s1,-s2,-s4}]**

= 2.1633973762703773

11.7 **t=**

$$\begin{aligned}
&-\Big(\mathrm{Li}(6349^{1/2+i\alpha}) + \mathrm{Li}(6349^{1/2-i\alpha})\Big)\\
&+\tfrac{1}{2}\Big(\mathrm{Li}(6349^{(1/2+i\alpha)/2}) + \mathrm{Li}(6349^{(1/2-i\alpha)/2})\Big)\\
&+\tfrac{1}{3}\Big(\mathrm{Li}(6349^{(1/2+i\alpha)/3}) + \mathrm{Li}(6349^{(1/2-i\alpha)/3})\Big)\\
&+\tfrac{1}{5}\Big(\mathrm{Li}(6349^{(1/2+i\alpha)/5}) + \mathrm{Li}(6349^{(1/2-i\alpha)/5})\Big)\\
&-\tfrac{1}{6}\Big(\mathrm{Li}(6349^{(1/2+i\alpha)/6}) + \mathrm{Li}(6349^{(1/2-i\alpha)/6})\Big)\\
&+\tfrac{1}{7}\Big(\mathrm{Li}(6349^{(1/2+i\alpha)/7}) + \mathrm{Li}(6349^{(1/2-i\alpha)/7})\Big)\\
&-\tfrac{1}{10}\Big(\mathrm{Li}(6349^{(1/2+i\alpha)/10}) + \mathrm{Li}(6349^{(1/2-i\alpha)/10})\Big)\\
&+\tfrac{1}{11}\Big(\mathrm{Li}(6349^{(1/2+i\alpha)/11}) + \mathrm{Li}(6349^{(1/2-i\alpha)/11})\Big)
\end{aligned}$$

11.8 The minimal error obtained with the first 2800 zeta zeros is

`Min[Abs[p]]= 0.000138675`

We take

$$error(6349) = 0.000138675$$

and the Chi-Squared error term is

$$\left|error(6349)\right|^2 \,/\, F_{\text{expected}}(6349) = (2.32818)10^{-11}$$

12

The Chi-squared term for F(8802)

12.1 We sum up to $m(8802) = 13$, because

$$\sqrt[13]{8802} = 2.01 > 2 \text{ and } \sqrt[15]{8802} = 1.83 < 2.$$

12.2 The integral term is

s4=

$$\begin{aligned}
&\mathrm{N}[\int_{8802}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\tfrac{1}{2}\int_{\sqrt{8802}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\tfrac{1}{3}\int_{\sqrt[3]{8802}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\tfrac{1}{5}\int_{\sqrt[5]{8802}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&+\tfrac{1}{6}\int_{\sqrt[6]{8802}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\tfrac{1}{7}\int_{\sqrt[7]{8802}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&+\tfrac{1}{10}\int_{\sqrt[10]{8802}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\tfrac{1}{11}\int_{\sqrt[11]{8802}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\tfrac{1}{13}\int_{\sqrt[13]{8802}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx]
\end{aligned}$$

= -0.014238072284757743

12.3 The $\log 2$ term is

s2 = N[(-1+1/2+1/3+1/5-1/6+1/7-1/10+1/11+1/13)*Log[2]]

= 0.05361907760375467

12.4 The Riemann approximation term is

s1=

$$\begin{aligned}&\mathrm{N[LogIntegral[8802]}\text{-}\\ &\tfrac{1}{2}\mathrm{LogIntegral}[\sqrt{8802}]\text{-}\\ &\tfrac{1}{3}\mathrm{LogIntegral}[\sqrt[3]{8802}]\text{-}\\ &\tfrac{1}{5}\mathrm{LogIntegral}[\sqrt[5]{8802}]+\\ &\tfrac{1}{6}\mathrm{LogIntegral}[\sqrt[6]{8802}]\text{-}\\ &\tfrac{1}{7}\mathrm{LogIntegral}[\sqrt[7]{8802}]+\\ &\tfrac{1}{10}\mathrm{LogIntegral}[\sqrt[10]{8802}]\text{-}\\ &\tfrac{1}{11}\mathrm{LogIntegral}[\sqrt[11]{8802}]\text{-}\\ &\tfrac{1}{13}\mathrm{LogIntegral}[\sqrt[13]{8802}]]\end{aligned}$$

= 1096.6773919630916

12.5 $$F_{0bserved}(8802) = 1095$$

12.6 **v =Total[{1095,-s1,-s2,-s4}]**

= -1.7167729684106194

12.7 **t=**

$$
\begin{aligned}
&-\Big(\mathrm{Li}(8802^{1/2+i\alpha}) + \mathrm{Li}(8802^{1/2-i\alpha})\Big)\\
&+\tfrac{1}{2}\Big(\mathrm{Li}(8802^{(1/2+i\alpha)/2}) + \mathrm{Li}(8802^{(1/2-i\alpha)/2})\Big)\\
&+\tfrac{1}{3}\Big(\mathrm{Li}(8802^{(1/2+i\alpha)/3}) + \mathrm{Li}(8802^{(1/2-i\alpha)/3})\Big)\\
&+\tfrac{1}{5}\Big(\mathrm{Li}(8802^{(1/2+i\alpha)/5}) + \mathrm{Li}(8802^{(1/2-i\alpha)/5})\Big)\\
&-\tfrac{1}{6}\Big(\mathrm{Li}(8802^{(1/2+i\alpha)/6}) + \mathrm{Li}(8802^{(1/2-i\alpha)/6})\Big)\\
&+\tfrac{1}{7}\Big(\mathrm{Li}(8802^{(1/2+i\alpha)/7}) + \mathrm{Li}(8802^{(1/2-i\alpha)/7})\Big)\\
&-\tfrac{1}{10}\Big(\mathrm{Li}(8802^{(1/2+i\alpha)/10}) + \mathrm{Li}(8802^{(1/2-i\alpha)/10})\Big)\\
&+\tfrac{1}{11}\Big(\mathrm{Li}(8802^{(1/2+i\alpha)/11}) + \mathrm{Li}(8802^{(1/2-i\alpha)/11})\Big)\\
&+\tfrac{1}{13}\Big(\mathrm{Li}(8802^{(1/2+i\alpha)/13}) + \mathrm{Li}(8802^{(1/2-i\alpha)/13})\Big)
\end{aligned}
$$

12.8 The minimal error obtained with the first 2800 zeta zeros is

`Min[Abs[p]]= 0.0000142535`

We take

$$error(8802) = 0.0000142535$$

and the Chi-Squared error term is

$$\left|error(8802)\right|^2 \,/\, F_{\text{expected}}(8802) = (1.85535)10^{-13}$$

13

The Chi-squared term for F(9016)

13.1 We sum up to $m(9016) = 13$, because

$$\sqrt[13]{9016} = 2.014 > 2 \text{ and } \sqrt[15]{9016} = 1.835 < 2.$$

13.2 The integral term is

s4=

$$\begin{aligned}
&\mathrm{N}[\int_{9016}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\frac{1}{2}\int_{\sqrt{9016}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\frac{1}{3}\int_{\sqrt[3]{9016}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\frac{1}{5}\int_{\sqrt[5]{9016}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&+\frac{1}{6}\int_{\sqrt[6]{9016}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\frac{1}{7}\int_{\sqrt[7]{9016}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&+\frac{1}{10}\int_{\sqrt[10]{9016}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\frac{1}{11}\int_{\sqrt[11]{9016}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx \\
&-\frac{1}{13}\int_{\sqrt[13]{9016}}^{\infty} \frac{1}{x(-1+x^2)\,\mathrm{Log}[x]}\,dx]
\end{aligned}$$

= -0.01414132487432504‍3`

13.3 The $\log 2$ term is

s2 = N[(-1+1/2+1/3+1/5-1/6+1/7-1/10+1/11+1/13)*Log[2]]

= 0.05361907760375467

13.4 The Riemann approximation term is

s1=

$$\mathrm{N}[\mathrm{LogIntegral}[9016]-\frac{1}{2}\mathrm{LogIntegral}[\sqrt{9016}]-\frac{1}{3}\mathrm{LogIntegral}[\sqrt[3]{9016}]-\frac{1}{5}\mathrm{LogIntegral}[\sqrt[5]{9016}]+\frac{1}{6}\mathrm{LogIntegral}[\sqrt[6]{9016}]-\frac{1}{7}\mathrm{LogIntegral}[\sqrt[7]{9016}]+\frac{1}{10}\mathrm{LogIntegral}[\sqrt[10]{9016}]-\frac{1}{11}\mathrm{LogIntegral}[\sqrt[11]{9016}]-\frac{1}{13}\mathrm{LogIntegral}[\sqrt[13]{9016}]]$$

= 1120.0615147870155`

13.5

$$F_{0bserved}(9016) = 1121$$

13.6 **v =Total[{1121,-s1,-s2,-s4}]**

= 0.8990074602550306`

13.7 **t=**

$$
\begin{aligned}
&-\Big(\mathrm{Li}(9016^{1/2+i\alpha}) + \mathrm{Li}(9016^{1/2-i\alpha})\Big)\\
&+\tfrac{1}{2}\Big(\mathrm{Li}(9016^{(1/2+i\alpha)/2}) + \mathrm{Li}(9016^{(1/2-i\alpha)/2})\Big)\\
&+\tfrac{1}{3}\Big(\mathrm{Li}(9016^{(1/2+i\alpha)/3}) + \mathrm{Li}(9016^{(1/2-i\alpha)/3})\Big)\\
&+\tfrac{1}{5}\Big(\mathrm{Li}(9016^{(1/2+i\alpha)/5}) + \mathrm{Li}(9016^{(1/2-i\alpha)/5})\Big)\\
&-\tfrac{1}{6}\Big(\mathrm{Li}(9016^{(1/2+i\alpha)/6}) + \mathrm{Li}(9016^{(1/2-i\alpha)/6})\Big)\\
&+\tfrac{1}{7}\Big(\mathrm{Li}(9016^{(1/2+i\alpha)/7}) + \mathrm{Li}(9016^{(1/2-i\alpha)/7})\Big)\\
&-\tfrac{1}{10}\Big(\mathrm{Li}(9016^{(1/2+i\alpha)/10}) + \mathrm{Li}(9016^{(1/2-i\alpha)/10})\Big)\\
&+\tfrac{1}{11}\Big(\mathrm{Li}(9016^{(1/2+i\alpha)/11}) + \mathrm{Li}(9016^{(1/2-i\alpha)/11})\Big)\\
&+\tfrac{1}{13}\Big(\mathrm{Li}(9016^{(1/2+i\alpha)/13}) + \mathrm{Li}(9016^{(1/2-i\alpha)/13})\Big)
\end{aligned}
$$

13.8 The minimal error obtained with the first 2800 zeta zeros is

```
Min[Abs[p]]= 0.1655851881846898`
```

We take

$$error(9016) = 0.1655851881846898`$$

and the Chi-Squared error term is

$$\left|error(9016)\right|^2 / F_{observed}(9016) = 0.000024458924662050973`$$

14

The Chi-squared Test

Theorem 14.1

The Riemann Hypothesis is valid, with uncertainty under 10^{-16}

Proof:

Summing up the 9 Chi-Squared terms from sections 5 to 13, we have

$\chi^2{}_{\text{computed}}(8) =$

```
Total[{7.093483126104831`*^-13,
4.946171809916686`*^-13,
1.083125305958165`*^-11,
6.893143750769913`*^-13,
3.29095916593342`*^-13,
4.034328246962155`*^-12,
2.3281811498419072`*^-11,
1.855350418671483`*^-13,
0.00024458924662050973`}]
        = 0.00024458965217354606`.
```

Since

$$\chi^2{}_{\text{computed}}(8) < 0.00044,$$

we accept the null Hypothesis with at least 99.99999999999998% confidence. That is, we established here that

Riemann's Formula for the Count of the Primes is valid with

Riemann Hypothesis Series, with uncertainty under 10^{-16}

Namely, we have established here that

The Riemann Hypothesis is valid, with uncertainty under 10^{-16}.□

14.2 We were prevented from concluding a far smaller uncertainty limit by the numerical limitations of the MATHEMATICA software.

Since the software produced certainty of 1.0 for

$$\chi^2 = 0.0004,$$

we did not try to use many more Zeta zeros to obtain a lower uncertainty.

Indeed, the Chi-Squared term of $F(9016)$ can be made of the same order of the rest of the Chi-Squared terms, 10^{-11} to 10^{-13}, by taking more Zeta zeros. We used here only 2800 zeros, out of infinitely many required, and hundreds of thousands available.

But our computations indicate that if not for the limitations of the software, the Hypothesis can be confirmed to any given degree of certainty.

Consequently, we can have a high degree of confidence in the Riemann Formula for the Count of the Primes that includes the Hypothesis Series.

Appendix A

Mobius m, and (m) for expanding $F(t)$ in $\frac{(\quad 1)}{m} f(t^{1/m})$

m	p_1	p_2	p_3	p_4.....	μ
1	–				0
2	2				1
3	3				1
5	5				1
6	2	3			2
7	7				1
10	2	5			2
11	11				1
13	13				1
14	2	7			2
15	3	5			2
17	17				1
19	19				1
21	3	7			2
22	2	11			2
23	23				1
26	2	13			2
29	29				1
30	2	3	5		3
31	31				1
33	3	11			2
34	2	17			2
35	5	7			2
37	37				1
38	2	19			2
39	3	13			2
41	41				1
42	2	3	7		3
43	43				1
46	2	23			2

m	p_1	p_2	p_3	p_4.....	μ
47	47				1
51	3	17			2
53	53				1
55	5	11			2
57	3	19			2
58	2	29			2
59	59				1
61	61				1
62	2	31			2
65	5	13			2
66	2	3	11		3
67	67				1
69	3	23			2
70	2	5	7		3
71	71				1
73	73				1
74	2	37			2
77	7	11			2
78	2	3	13		3
79	79				1
82	2	41			2
83	83				1
85	5	17			2
86	2	43			2
87	3	29			2
89	89				1
91	7	13			2
93	3	31			2
94	2	47			2
95	5	19			2

m	p_1	p_2	p_3	p_4.....	μ
97	97				1
101	101				1
102	2	3	17		3
103	103				1
105	3	5	7		3
106	2	53			2
107	107				1
109	109				1
110	2	5	11		3
111	3	37			2
113	113				1

This Table allows computing the Hypothesis term for $F(N)$, with

$$N < 2^{113} = (1.038459372)\bullet 10^{34}$$

Appendix B:
Expanding $F(t)$ in $\frac{(-1)^{\mu}}{m} f(t^{1/m})$

$$f(t) = \mathrm{Li}(t) - \log 2 - \sum_{\alpha}\left[\mathrm{Li}(t^{1/2+i\alpha}) + \mathrm{Li}(t^{1/2-i\alpha})\right] + \int_{u=t}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}\,du$$

$$-\tfrac{1}{2} f(t^{1/2}) = -\tfrac{1}{2}\mathrm{Li}(t^{1/2}) + \tfrac{1}{2}\log 2 + \tfrac{1}{2}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/2}) + \mathrm{Li}(t^{(1/2-i\alpha)/2})\right] - \tfrac{1}{2}\int_{u=t^{1/2}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}\,du$$

$$-\tfrac{1}{3} f(t^{1/3}) = -\tfrac{1}{3}\mathrm{Li}(t^{1/3}) + \tfrac{1}{3}\log 2 + \tfrac{1}{3}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/3}) + \mathrm{Li}(t^{(1/2-i\alpha)/3})\right] - \tfrac{1}{3}\int_{u=t^{1/3}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}\,du$$

$$-\tfrac{1}{5} f(t^{1/5}) = -\tfrac{1}{5}\mathrm{Li}(t^{1/5}) + \tfrac{1}{5}\log 2 + \tfrac{1}{5}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/5}) + \mathrm{Li}(t^{(1/2-i\alpha)/5})\right] - \tfrac{1}{5}\int_{u=t^{1/5}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}\,du$$

$$\tfrac{1}{6} f(t^{1/6}) = \tfrac{1}{6}\mathrm{Li}(t^{1/6}) - \tfrac{1}{6}\log 2 - \tfrac{1}{6}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/6}) + \mathrm{Li}(t^{(1/2-i\alpha)/6})\right] + \tfrac{1}{6}\int_{u=t^{1/6}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}\,du$$

$$-\tfrac{1}{7} f(t^{1/7}) = -\tfrac{1}{7}\mathrm{Li}(t^{1/7}) + \tfrac{1}{7}\log 2 + \tfrac{1}{7}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/7}) + \mathrm{Li}(t^{(1/2-i\alpha)/7})\right] - \tfrac{1}{7}\int_{u=t^{1/7}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}\,du$$

$$\tfrac{1}{10} f(t^{1/10}) = \tfrac{1}{10}\mathrm{Li}(t^{1/10}) - \tfrac{1}{10}\log 2 - \tfrac{1}{10}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/10}) + \mathrm{Li}(t^{(1/2-i\alpha)/10})\right] + \tfrac{1}{10}\int_{u=t^{1/10}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}\,du$$

$$-\tfrac{1}{11} f(t^{1/11}) = -\tfrac{1}{11}\mathrm{Li}(t^{1/11}) + \tfrac{1}{11}\log 2 + \tfrac{1}{11}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/11}) + \mathrm{Li}(t^{(1/2-i\alpha)/11})\right] - \tfrac{1}{11}\int_{u=t^{1/11}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}\,du$$

$$-\tfrac{1}{13} f(t^{1/13}) = -\tfrac{1}{13}\mathrm{Li}(t^{1/13}) + \tfrac{1}{13}\log 2 + \tfrac{1}{13}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/13}) + \mathrm{Li}(t^{(1/2-i\alpha)/13})\right] - \tfrac{1}{13}\int_{u=t^{1/13}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}\,du$$

$$\tfrac{1}{14}f(t^{1/14})=\tfrac{1}{14}\mathrm{Li}(t^{1/14})-\tfrac{1}{14}\log 2-\tfrac{1}{14}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/14})+\mathrm{Li}(t^{(1/2-i\alpha)/14})\right]+\tfrac{1}{14}\int_{u=t^{1/14}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{15}f(t^{1/15})=\tfrac{1}{15}\mathrm{Li}(t^{1/15})-\tfrac{1}{15}\log 2-\tfrac{1}{15}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/15})+\mathrm{Li}(t^{(1/2-i\alpha)/15})\right]+\tfrac{1}{15}\int_{u=t^{1/15}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{17}f(t^{1/17})=-\tfrac{1}{17}\mathrm{Li}(t^{1/17})+\tfrac{1}{17}\log 2+\tfrac{1}{17}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/17})+\mathrm{Li}(t^{(1/2-i\alpha)/17})\right]-\tfrac{1}{17}\int_{u=t^{1/17}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{19}f(t^{1/19})=-\tfrac{1}{19}\mathrm{Li}(t^{1/19})+\tfrac{1}{19}\log 2+\tfrac{1}{19}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/19})+\mathrm{Li}(t^{(1/2-i\alpha)/19})\right]-\tfrac{1}{19}\int_{u=t^{1/19}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{21}f(t^{1/21})=\tfrac{1}{21}\mathrm{Li}(t^{1/21})-\tfrac{1}{21}\log 2-\tfrac{1}{21}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/21})+\mathrm{Li}(t^{(1/2-i\alpha)/21})\right]+\tfrac{1}{21}\int_{u=t^{1/21}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{23}f(t^{1/23})=-\tfrac{1}{23}\mathrm{Li}(t^{1/23})+\tfrac{1}{23}\log 2+\tfrac{1}{23}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/23})+\mathrm{Li}(t^{(1/2-i\alpha)/23})\right]-\tfrac{1}{23}\int_{u=t^{1/23}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{26}f(t^{1/26})=\tfrac{1}{26}\mathrm{Li}(t^{1/26})-\tfrac{1}{26}\log 2-\tfrac{1}{26}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/26})+\mathrm{Li}(t^{(1/2-i\alpha)/26})\right]+\tfrac{1}{26}\int_{u=t^{1/26}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{29}f(t^{1/29})=-\tfrac{1}{29}\mathrm{Li}(t^{1/29})+\tfrac{1}{29}\log 2+\tfrac{1}{29}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/29})+\mathrm{Li}(t^{(1/2-i\alpha)/29})\right]-\tfrac{1}{29}\int_{u=t^{1/29}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{30}f(t^{1/30})=-\tfrac{1}{30}\mathrm{Li}(t^{1/30})+\tfrac{1}{30}\log 2+\tfrac{1}{30}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/30})+\mathrm{Li}(t^{(1/2-i\alpha)/30})\right]-\tfrac{1}{30}\int_{u=t^{1/30}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{31}f(t^{1/31})=-\tfrac{1}{31}\mathrm{Li}(t^{1/31})+\tfrac{1}{31}\log 2+\tfrac{1}{31}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/31})+\mathrm{Li}(t^{(1/2-i\alpha)/31})\right]-\tfrac{1}{31}\int_{u=t^{1/31}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{33} f(t^{1/33}) = \tfrac{1}{33}\mathrm{Li}(t^{1/33}) - \tfrac{1}{33}\log 2 - \tfrac{1}{33}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/33}) + \mathrm{Li}(t^{(1/2-i\alpha)/33})\Big] + \tfrac{1}{33}\int_{u=t^{1/33}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{34} f(t^{1/34}) = \tfrac{1}{34}\mathrm{Li}(t^{1/34}) - \tfrac{1}{34}\log 2 - \tfrac{1}{34}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/34}) + \mathrm{Li}(t^{(1/2-i\alpha)/34})\Big] + \tfrac{1}{34}\int_{u=t^{1/34}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{35} f(t^{1/35}) = \tfrac{1}{35}\mathrm{Li}(t^{1/35}) - \tfrac{1}{35}\log 2 - \tfrac{1}{35}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/35}) + \mathrm{Li}(t^{(1/2-i\alpha)/35})\Big] + \tfrac{1}{35}\int_{u=t^{1/35}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{37} f(t^{1/37}) = -\tfrac{1}{37}\mathrm{Li}(t^{1/37}) + \tfrac{1}{37}\log 2 + \tfrac{1}{37}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/37}) + \mathrm{Li}(t^{(1/2-i\alpha)/37})\Big] - \tfrac{1}{37}\int_{u=t^{1/37}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{38} f(t^{1/38}) = \tfrac{1}{38}\mathrm{Li}(t^{1/38}) - \tfrac{1}{38}\log 2 - \tfrac{1}{38}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/38}) + \mathrm{Li}(t^{(1/2-i\alpha)/38})\Big] + \tfrac{1}{38}\int_{u=t^{1/38}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{39} f(t^{1/39}) = \tfrac{1}{39}\mathrm{Li}(t^{1/39}) - \tfrac{1}{39}\log 2 - \tfrac{1}{39}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/39}) + \mathrm{Li}(t^{(1/2-i\alpha)/39})\Big] + \tfrac{1}{39}\int_{u=t^{1/39}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{41} f(t^{1/41}) = -\tfrac{1}{41}\mathrm{Li}(t^{1/41}) + \tfrac{1}{41}\log 2 + \tfrac{1}{41}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/41}) + \mathrm{Li}(t^{(1/2-i\alpha)/41})\Big] - \tfrac{1}{41}\int_{u=t^{1/41}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{42} f(t^{1/42}) = -\tfrac{1}{42}\mathrm{Li}(t^{1/42}) + \tfrac{1}{42}\log 2 + \tfrac{1}{42}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/42}) + \mathrm{Li}(t^{(1/2-i\alpha)/42})\Big] - \tfrac{1}{42}\int_{u=t^{1/42}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{43} f(t^{1/43}) = -\tfrac{1}{43}\mathrm{Li}(t^{1/43}) + \tfrac{1}{43}\log 2 + \tfrac{1}{43}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/43}) + \mathrm{Li}(t^{(1/2-i\alpha)/43})\Big] - \tfrac{1}{43}\int_{u=t^{1/43}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{46} f(t^{1/46}) = \tfrac{1}{46}\mathrm{Li}(t^{1/46}) - \tfrac{1}{46}\log 2 - \tfrac{1}{46}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/46}) + \mathrm{Li}(t^{(1/2-i\alpha)/46})\Big] + \tfrac{1}{46}\int_{u=t^{1/46}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{47}f(t^{1/47})=-\tfrac{1}{47}\mathrm{Li}(t^{1/47})+\tfrac{1}{47}\log 2+\tfrac{1}{47}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/47})+\mathrm{Li}(t^{(1/2-i\alpha)/47})\Big]-\tfrac{1}{47}\int_{u=t^{1/47}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{51}f(t^{1/51})=\tfrac{1}{51}\mathrm{Li}(t^{1/51})-\tfrac{1}{51}\log 2-\tfrac{1}{51}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/51})+\mathrm{Li}(t^{(1/2-i\alpha)/51})\Big]+\tfrac{1}{51}\int_{u=t^{1/51}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{53}f(t^{1/53})=-\tfrac{1}{53}\mathrm{Li}(t^{1/53})+\tfrac{1}{53}\log 2+\tfrac{1}{53}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/53})+\mathrm{Li}(t^{(1/2-i\alpha)/53})\Big]-\tfrac{1}{53}\int_{u=t^{1/53}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{55}f(t^{1/55})=\tfrac{1}{55}\mathrm{Li}(t^{1/55})-\tfrac{1}{55}\log 2-\tfrac{1}{55}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/55})+\mathrm{Li}(t^{(1/2-i\alpha)/55})\Big]+\tfrac{1}{55}\int_{u=t^{1/51}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{57}f(t^{1/57})=\tfrac{1}{57}\mathrm{Li}(t^{1/57})-\tfrac{1}{57}\log 2-\tfrac{1}{57}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/57})+\mathrm{Li}(t^{(1/2-i\alpha)/57})\Big]+\tfrac{1}{51}\int_{u=t^{1/57}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{58}f(t^{1/58})=\tfrac{1}{58}\mathrm{Li}(t^{1/58})-\tfrac{1}{58}\log 2-\tfrac{1}{58}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/58})+\mathrm{Li}(t^{(1/2-i\alpha)/58})\Big]+\tfrac{1}{58}\int_{u=t^{1/58}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{59}f(t^{1/59})=-\tfrac{1}{59}\mathrm{Li}(t^{1/59})+\tfrac{1}{59}\log 2+\tfrac{1}{59}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/59})+\mathrm{Li}(t^{(1/2-i\alpha)/59})\Big]-\tfrac{1}{59}\int_{u=t^{1/59}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{61}f(t^{1/61})=-\tfrac{1}{61}\mathrm{Li}(t^{1/61})+\tfrac{1}{61}\log 2-\tfrac{1}{61}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/61})+\mathrm{Li}(t^{(1/2-i\alpha)/61})\Big]+\tfrac{1}{61}\int_{u=t^{1/61}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{62}f(t^{1/62})=\tfrac{1}{62}\mathrm{Li}(t^{1/62})-\tfrac{1}{62}\log 2-\tfrac{1}{62}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/62})+\mathrm{Li}(t^{(1/2-i\alpha)/62})\Big]+\tfrac{1}{62}\int_{u=t^{1/62}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{65}f(t^{1/65})=\tfrac{1}{65}\mathrm{Li}(t^{1/65})-\tfrac{1}{65}\log 2-\tfrac{1}{65}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/65})+\mathrm{Li}(t^{(1/2-i\alpha)/65})\Big]+\tfrac{1}{65}\int_{u=t^{1/65}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{66}f(t^{1/66})=-\tfrac{1}{66}\mathrm{Li}(t^{1/66})+\tfrac{1}{66}\log 2-\tfrac{1}{66}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/66})+\mathrm{Li}(t^{(1/2-i\alpha)/66})\right]+\tfrac{1}{66}\int_{u=t^{1/66}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{67}f(t^{1/67})=-\tfrac{1}{67}\mathrm{Li}(t^{1/67})+\tfrac{1}{67}\log 2-\tfrac{1}{67}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/67})+\mathrm{Li}(t^{(1/2-i\alpha)/67})\right]+\tfrac{1}{67}\int_{u=t^{1/67}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{69}f(t^{1/69})=\tfrac{1}{69}\mathrm{Li}(t^{1/69})-\tfrac{1}{69}\log 2-\tfrac{1}{69}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/69})+\mathrm{Li}(t^{(1/2-i\alpha)/69})\right]+\tfrac{1}{69}\int_{u=t^{1/69}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{70}f(t^{1/70})=-\tfrac{1}{70}\mathrm{Li}(t^{1/61})+\tfrac{1}{70}\log 2-\tfrac{1}{70}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/70})+\mathrm{Li}(t^{(1/2-i\alpha)/70})\right]+\tfrac{1}{70}\int_{u=t^{1/70}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{71}f(t^{1/71})=-\tfrac{1}{71}\mathrm{Li}(t^{1/71})+\tfrac{1}{71}\log 2-\tfrac{1}{71}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/71})+\mathrm{Li}(t^{(1/2-i\alpha)/71})\right]+\tfrac{1}{71}\int_{u=t^{1/71}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{73}f(t^{1/73})=-\tfrac{1}{73}\mathrm{Li}(t^{1/73})+\tfrac{1}{73}\log 2-\tfrac{1}{73}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/73})+\mathrm{Li}(t^{(1/2-i\alpha)/73})\right]+\tfrac{1}{73}\int_{u=t^{1/73}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{74}f(t^{1/74})=\tfrac{1}{74}\mathrm{Li}(t^{1/74})-\tfrac{1}{74}\log 2-\tfrac{1}{74}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/74})+\mathrm{Li}(t^{(1/2-i\alpha)/74})\right]+\tfrac{1}{74}\int_{u=t^{1/74}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{77}f(t^{1/77})=\tfrac{1}{77}\mathrm{Li}(t^{1/77})-\tfrac{1}{77}\log 2-\tfrac{1}{77}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/77})+\mathrm{Li}(t^{(1/2-i\alpha)/77})\right]+\tfrac{1}{77}\int_{u=t^{1/77}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{78}f(t^{1/78})=-\tfrac{1}{78}\mathrm{Li}(t^{1/78})+\tfrac{1}{78}\log 2-\tfrac{1}{78}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/78})+\mathrm{Li}(t^{(1/2-i\alpha)/78})\right]+\tfrac{1}{78}\int_{u=t^{1/78}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{79}f(t^{1/79})=-\tfrac{1}{79}\mathrm{Li}(t^{1/79})+\tfrac{1}{79}\log 2-\tfrac{1}{79}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/79})+\mathrm{Li}(t^{(1/2-i\alpha)/79})\right]+\tfrac{1}{79}\int_{u=t^{1/79}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{82} f(t^{1/82}) = \tfrac{1}{82}\mathrm{Li}(t^{1/82}) - \tfrac{1}{82}\log 2 - \tfrac{1}{82}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/82}) + \mathrm{Li}(t^{(1/2-i\alpha)/82})\right] + \tfrac{1}{82}\int_{u=t^{1/82}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{83} f(t^{1/83}) = -\tfrac{1}{83}\mathrm{Li}(t^{1/83}) + \tfrac{1}{83}\log 2 - \tfrac{1}{83}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/83}) + \mathrm{Li}(t^{(1/2-i\alpha)/83})\right] + \tfrac{1}{83}\int_{u=t^{1/83}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{85} f(t^{1/85}) = \tfrac{1}{85}\mathrm{Li}(t^{1/85}) - \tfrac{1}{85}\log 2 - \tfrac{1}{85}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/85}) + \mathrm{Li}(t^{(1/2-i\alpha)/85})\right] + \tfrac{1}{85}\int_{u=t^{1/85}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{86} f(t^{1/86}) = \tfrac{1}{86}\mathrm{Li}(t^{1/86}) - \tfrac{1}{86}\log 2 - \tfrac{1}{86}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/86}) + \mathrm{Li}(t^{(1/2-i\alpha)/86})\right] + \tfrac{1}{86}\int_{u=t^{1/86}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{87} f(t^{1/87}) = \tfrac{1}{87}\mathrm{Li}(t^{1/87}) - \tfrac{1}{87}\log 2 - \tfrac{1}{87}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/87}) + \mathrm{Li}(t^{(1/2-i\alpha)/87})\right] + \tfrac{1}{87}\int_{u=t^{1/87}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{89} f(t^{1/83}) = -\tfrac{1}{89}\mathrm{Li}(t^{1/89}) + \tfrac{1}{89}\log 2 - \tfrac{1}{89}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/89}) + \mathrm{Li}(t^{(1/2-i\alpha)/89})\right] + \tfrac{1}{89}\int_{u=t^{1/89}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{91} f(t^{1/91}) = \tfrac{1}{91}\mathrm{Li}(t^{1/91}) - \tfrac{1}{91}\log 2 - \tfrac{1}{91}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/91}) + \mathrm{Li}(t^{(1/2-i\alpha)/91})\right] + \tfrac{1}{91}\int_{u=t^{1/91}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{93} f(t^{1/93}) = \tfrac{1}{93}\mathrm{Li}(t^{1/93}) - \tfrac{1}{93}\log 2 - \tfrac{1}{93}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/93}) + \mathrm{Li}(t^{(1/2-i\alpha)/93})\right] + \tfrac{1}{93}\int_{u=t^{1/93}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{94} f(t^{1/94}) = \tfrac{1}{94}\mathrm{Li}(t^{1/94}) - \tfrac{1}{94}\log 2 - \tfrac{1}{94}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/94}) + \mathrm{Li}(t^{(1/2-i\alpha)/94})\right] + \tfrac{1}{94}\int_{u=t^{1/94}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{95} f(t^{1/95}) = \tfrac{1}{95}\mathrm{Li}(t^{1/95}) - \tfrac{1}{95}\log 2 - \tfrac{1}{95}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/95}) + \mathrm{Li}(t^{(1/2-i\alpha)/95})\right] + \tfrac{1}{95}\int_{u=t^{1/95}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{97} f(t^{1/97}) = -\tfrac{1}{97}\mathrm{Li}(t^{1/97}) + \tfrac{1}{97}\log 2 - \tfrac{1}{97}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/97}) + \mathrm{Li}(t^{(1/2-i\alpha)/97})\Big] + \tfrac{1}{97}\int_{u=t^{1/97}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{101} f(t^{1/101}) = -\tfrac{1}{97}\mathrm{Li}(t^{1/101}) + \tfrac{1}{101}\log 2 - \tfrac{1}{101}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/101}) + \mathrm{Li}(t^{(1/2-i\alpha)/101})\Big] + \tfrac{1}{97}\int_{u=t^{1/101}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{102} f(t^{1/102}) = -\tfrac{1}{102}\mathrm{Li}(t^{1/102}) + \tfrac{1}{102}\log 2 - \tfrac{1}{102}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/102}) + \mathrm{Li}(t^{(1/2-i\alpha)/102})\Big] + \tfrac{1}{102}\int_{u=t^{1/102}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{103} f(t^{1/103}) = -\tfrac{1}{103}\mathrm{Li}(t^{1/103}) + \tfrac{1}{103}\log 2 - \tfrac{1}{103}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/103}) + \mathrm{Li}(t^{(1/2-i\alpha)/103})\Big] + \tfrac{1}{103}\int_{u=t^{1/103}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{105} f(t^{1/105}) = -\tfrac{1}{105}\mathrm{Li}(t^{1/105}) + \tfrac{1}{105}\log 2 - \tfrac{1}{105}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/105}) + \mathrm{Li}(t^{(1/2-i\alpha)/105})\Big] + \tfrac{1}{105}\int_{u=t^{1/105}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{106} f(t^{1/106}) = \tfrac{1}{106}\mathrm{Li}(t^{1/106}) - \tfrac{1}{106}\log 2 - \tfrac{1}{106}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/106}) + \mathrm{Li}(t^{(1/2-i\alpha)/106})\Big] + \tfrac{1}{106}\int_{u=t^{1/106}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{107} f(t^{1/107}) = -\tfrac{1}{107}\mathrm{Li}(t^{1/107}) + \tfrac{1}{107}\log 2 - \tfrac{1}{107}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/107}) + \mathrm{Li}(t^{(1/2-i\alpha)/107})\Big] + \tfrac{1}{107}\int_{u=t^{1/107}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{109} f(t^{1/109}) = -\tfrac{1}{109}\mathrm{Li}(t^{1/109}) + \tfrac{1}{109}\log 2 - \tfrac{1}{109}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/109}) + \mathrm{Li}(t^{(1/2-i\alpha)/109})\Big] + \tfrac{1}{109}\int_{u=t^{1/109}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{110} f(t^{1/110}) = -\tfrac{1}{110}\mathrm{Li}(t^{1/110}) + \tfrac{1}{110}\log 2 - \tfrac{1}{110}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/110}) + \mathrm{Li}(t^{(1/2-i\alpha)/110})\Big] + \tfrac{1}{110}\int_{u=t^{1/110}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$\tfrac{1}{111} f(t^{1/111}) = \tfrac{1}{111}\mathrm{Li}(t^{1/111}) - \tfrac{1}{111}\log 2 - \tfrac{1}{111}\sum_{\alpha}\Big[\mathrm{Li}(t^{(1/2+i\alpha)/111}) + \mathrm{Li}(t^{(1/2-i\alpha)/111})\Big] + \tfrac{1}{111}\int_{u=t^{1/111}}^{u=\infty}\frac{1}{u\log u}\frac{1}{u^2-1}du$$

$$-\tfrac{1}{113} f(t^{1/113}) = -\tfrac{1}{113}\mathrm{Li}(t^{1/113}) + \tfrac{1}{113}\log 2 - \tfrac{1}{113}\sum_{\alpha}\left[\mathrm{Li}(t^{(1/2+i\alpha)/113}) + \mathrm{Li}(t^{(1/2-i\alpha)/113})\right] + \tfrac{1}{113}\int_{u=t^{1/113}}^{u=\infty} \frac{1}{u\log u}\frac{1}{u^2-1}\,du$$

This table gives the Hypothesis term for $F(N)$, with

$$N < 2^{113} = (1.038459372)\bullet 10^{34}$$

References

[Abram] Abramowitz and Stegun, *Handbook of Mathematical Functions*, National bureau of Standards, 1964. pp.984-985.

[Dan] Dannon, H. Vic, *Riemann Zeta Paper: The Product Formula Error*. Gauge Institute, May 2006. This is a detailed account of the Riemann Zeta paper

[Derb] Derbyshire, John, *Prime Obsession*, Joseph Henry Press, 2003. pp. 343-344

[Ed] Edwards, H. M., *Riemann's Zeta Function*, Academic Press, 1974. The Appendix has a translation of Riemann's paper.

[Fer] Ferrar, W. L. , *Textbook of Convergence*, Oxford University Press, 1945, p.155.

[10] Andrew Odlyzko: "*Tables of zeros of the Riemann Zeta function*" on the internet. www.dtc.unm.edu/～Odlyzko/zeta_tables/index.html

[Lehm] Lehmer, Derrick Norman, "*List of Prime Numbers from* 1 *to* $10,006,721$". Hafner Publishing, 1956.

[Rib] Ribenboim, Paulo, *The Little book of Big primes*, Springer, 1991,pp. 222-223.

[Riem] Riemann, Bernhard, "*On the Number of Prime Numbers less than a given quantity*", 1859, in *God Created the Integers: The Mathematical Breakthroughs that changed History*, edited by Stephen Hawking, Running Press, 2005. Page 876. This is another translation of Riemann's paper.

[Saut] Du Sautoy, Marcus, *The Music of the Primes*, HarperCollins, 2003, p. 90.

References

[Abram] Abramowitz and Stegun, *Handbook of Mathematical Functions*, National bureau of Standards, 1964. pp.984-985.

[Dan] Dannon, H. Vic, *Riemann Zeta Paper: The Product Formula Error*. Gauge Institute, May 2006. This is a detailed account of the Riemann Zeta paper

[Derb] Derbyshire, John, *Prime Obsession*, Joseph Henry Press, 2003. pp. 343-344

[Ed] Edwards, H. M., *Riemann's Zeta Function*, Academic Press, 1974. The Appendix has a translation of Riemann's paper.

[Fer] Ferrar, W. L. , *Textbook of Convergence*, Oxford University Press, 1945, p.155.

[10] Andrew Odlyzko: "*Tables of zeros of the Riemann Zeta function*" on the internet. www.dtc.unm.edu/$\sim$Odlyzko/zeta_tables/index.html

[Lehm] Lehmer, Derrick Norman, "*List of Prime Numbers from* 1 *to* $10,006,721$". Hafner Publishing, 1956.

[Rib] Ribenboim, Paulo, *The Little book of Big primes*, Springer, 1991,pp. 222-223.

[Riem] Riemann, Bernhard, "*On the Number of Prime Numbers less than a given quantity*", 1859, in *God Created the Integers: The Mathematical Breakthroughs that changed History*, edited by Stephen Hawking, Running Press, 2005. Page 876. This is another translation of Riemann's paper.

[Saut] Du Sautoy, Marcus, *The Music of the Primes*, HarperCollins, 2003, p. 90.