Einstein's Diffusion and Probability-Wave Equations of Random Walk and Poisson Processes

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Abstract We derive the probability-wave equations of Random Walk, and of Poisson Processes in Infinitesimal Calculus.

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0.

Einstein's Derivation of the Diffusion Equation

0.1 Einstein's Assumptions for Brownian Motion

In 1905, Einstein analyzed the Brownian Motion.

In [Einstein, p.130], he assumed the following

1) *Each particle moves independently of the other particles*
2) *The motions of a particle over different, not-infinitesimal, time intervals, are mutually independent*
3) τ *is a small but non-infinitesimal time interval so that motions are mutually independent*
4) n *is the number of particles*
5) *Over the time* τ*, a particle moves from* x *to* $x+\Delta$*, where* Δ *depends on the particle, and may be positive or negative*
6) *the number of particles displaced from* Δ*, to* $\Delta + d\Delta$*, over the time* τ *is*

$$dn = n\varphi(\Delta)d\Delta\,,$$

where

$$\int_{\Delta=-\infty}^{\Delta=\infty} \varphi(\Delta)d\Delta = 1,$$

$\varphi(\Delta) \neq 0$, *only for very small* Δ,

$\varphi(\Delta) = \varphi(-\Delta)$.

7) $f(x,t)$ *is the particles' density at* x, *at time* t

We first note that

$\varphi(\Delta)$ ***is the Delta Function*** that was established already in 1882 by Kirchhoff. [Temple, p.158], and was similarly presented without mentioning Kirchhoff by Dirac.

Recently, we established the Delta Function as a hyper-real function in infinitesimal Calculus. [Dan4]. Then,

$$\delta(x) = \begin{cases} \frac{1}{dx}, & x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right] \\ 0, & \text{otherwise} \end{cases}$$

Consequently, according to assumption 6

$$dn = n\delta(\Delta)d\Delta$$

$$= n\begin{cases} \frac{1}{d\Delta}d\Delta, & \Delta \in \left[-\frac{d\Delta}{2}, \frac{d\Delta}{2}\right] \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} n, & \Delta \in \left[-\frac{d\Delta}{2}, \frac{d\Delta}{2}\right] \\ 0, & \text{otherwise} \end{cases}$$

That is, all the particles are in an infinitesimal interval at the origin…

This bizarre conclusion leads to no contradiction since neither n, nor dn, appear in the following Einstein's derivation…

0.2 Einstein's Derivation of the Diffusion Equation

Einstein starts with equation (17), p.131, claiming that

$$f(x,t+\tau) = \int_{\Delta=-\infty}^{\Delta=\infty} f(x+\Delta,t)\delta(\Delta)d\Delta$$

In fact, the sifting by $\delta(\Delta)$ gives $f(x,t)$, and no equality.

Einstein's next claim

$$f(x,t+\tau) = f(x,t) + \tau\frac{\partial f}{\partial t},$$

mandates that τ must be an infinitesimal.

This makes assumptions 2, and 3, meaningless, but leads to no contradiction, since the assumptions are not used in the derivation that follows…

Then, Einstein's expands his integrand

$$f(x+\Delta,t) = f(x,t) + \frac{\partial f(x,t)}{\partial x}\Delta + \frac{1}{2!}\frac{\partial^2 f(x,t)}{\partial x^2}\Delta^2 + ...$$

and writes his integral as

$$f(x,t)\int_{\Delta=-\infty}^{\Delta=\infty}\delta(\Delta)d\Delta+\frac{\partial f}{\partial x}\int_{\Delta=-\infty}^{\Delta=\infty}\Delta\delta(\Delta)d\Delta+\frac{1}{2!}\frac{\partial^2 f}{\partial x^2}\int_{\Delta=-\infty}^{\Delta=\infty}\Delta^2\delta(\Delta)d\Delta+..$$

He observes that the odd integrals vanish

$$\int_{\Delta=-\infty}^{\Delta=\infty}\Delta\delta(\Delta)d\Delta=0,$$

$$\int_{\Delta=-\infty}^{\Delta=\infty}\Delta^3\delta(\Delta)d\Delta=0$$

....

He only misses that the even integrals vanish as well. Indeed, the sifting by $\delta(\Delta)$, gives $\Delta^k\Big|_{\Delta=0}=0$.

In particular, his Drift Coefficient, which is [p.131]

$$D=\frac{1}{2\tau}\int_{\Delta=-\infty}^{\Delta=\infty}\Delta^2\varphi(\Delta)d\Delta,$$

vanishes,

and his Diffusion Equation [equation 18, p. 132]

$$\frac{\partial f}{\partial t}=D\frac{\partial^2 f}{\partial x^2},$$

collapses to

$$\frac{\partial f}{\partial t}=0.\Box$$

0.3 Probabilistic Wave Equations

The diffusion equation is an equation for a probability wave.

As such it can be derived by probabilistic considerations.

Here, we use these considerations in infinitesimal calculus to drive the diffusion equation for the Random drift of a particle in fluid due to collisions with fluid molecules.

And the probability-wave equation for the Poisson Process that models the Random arrival of radioactive particles at a counter.

1.

Hyper-real Line

The minimal domain and range, needed for the definition and analysis of a hyper-real function, is the hyper-real line.

Each real number α can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of infinitesimal hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the infinite hyper-reals.
4. The infinite hyper-reals are greater than any real number, yet strictly smaller than infinity.

5. The infinite hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a non-constant hyper-real.
7. The Hyper-reals are the totality of constant hyper-reals, a family of infinitesimals, a family of infinitesimals with negative sign, a family of infinite hyper-reals, a family of infinite hyper-reals with negative sign, and non-constant hyper-reals.
8. The hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.
9. That line includes the real numbers separated by the non-constant hyper-reals. Each real number is the center of an interval of hyper-reals, that includes no other real number.
10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.
11. Zero is not an infinitesimal, because zero is not strictly greater than zero.

12. We do not add infinity to the hyper-real line.

13. The infinitesimals, the infinitesimals with negative signs, the infinite hyper-reals, and the infinite hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.

14. The hyper-real line is embedded in $\mathbb{R}^{\infty}$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the hyper-real onto the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal hyper-reals, or to the infinite hyper-reals, or to the non-constant hyper-reals.

16. No neighbourhood of a hyper-real is homeomorphic to an $\mathbb{R}^n$ ball. Therefore, the hyper-real line is not a manifold.

17. The hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

2.

Hyper-real Function

2.1 Definition of a hyper-real function

$f(x)$ is a hyper-real function, iff it is from the hyper-reals into the hyper-reals.

This means that any number in the domain, or in the range of a hyper-real $f(x)$ is either one of the following

real

real + infinitesimal

real – infinitesimal

infinitesimal

infinitesimal with negative sign

infinite hyper-real

infinite hyper-real with negative sign

Clearly,

2.2 *Every function from the reals into the reals is a hyper-real function.*

3.

Integral of Hyper-real Function

In [Dan3], we defined the integral of a Hyper-real Function.

Let $f(x)$ be a hyper-real function on the interval $[a,b]$.

The interval may not be bounded.

$f(x)$ may take infinite hyper-real values, and need not be bounded.

At each

$$a \le x \le b,$$

there is a rectangle with base $[x - \frac{dx}{2}, x + \frac{dx}{2}]$, height $f(x)$, and area

$$f(x)dx.$$

We form the **Integration Sum** of all the areas for the x's that start at $x = a$, and end at $x = b$,

$$\sum_{x \in [a,b]} f(x)dx.$$

If for any infinitesimal dx, the Integration Sum has the same hyper-real value, then $f(x)$ is integrable over the interval $[a,b]$.

Then, we call the Integration Sum the integral of $f(x)$ from $x = a$, to $x = b$, and denote it by

$$\int_{x=a}^{x=b} f(x)dx .$$

If the hyper-real is infinite, then it is the integral over $[a,b]$,

If the hyper-real is finite,

$$\int_{x=a}^{x=b} f(x)dx = \text{ real part of the hyper-real}. \square$$

3.1 The countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers, $Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty .$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[a,b]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(x)dx$.

The Lower Integral is the Integration Sum where $f(x)$ is replaced

by its lowest value on each interval $[x-\frac{dx}{2}, x+\frac{dx}{2}]$

3.2 $$\sum_{x\in[a,b]}\left(\inf_{x-\frac{dx}{2}\le t\le x+\frac{dx}{2}} f(t)\right)dx$$

The Upper Integral is the Integration Sum where $f(x)$ is replaced by its largest value on each interval $[x-\frac{dx}{2}, x+\frac{dx}{2}]$

3.3 $$\sum_{x\in[a,b]}\left(\sup_{x-\frac{dx}{2}\le t\le x+\frac{dx}{2}} f(t)\right)dx$$

If the integral is a finite hyper-real, we have

3.4 *A hyper-real function has a finite integral if and only if its upper integral and its lower integral are finite, and differ by an infinitesimal.*

4.

Delta Function

In [Dan4], we defined the Delta Function, and established its properties

1. The Delta Function is a hyper-real function defined from the hyper-real line into the set of two hyper-reals $\left\{0, \frac{1}{dx}\right\}$. The hyper-real 0 is the sequence $\langle 0, 0, 0, ... \rangle$. The infinite hyper-real $\frac{1}{dx}$ depends on our choice of dx.
2. We will usually choose the family of infinitesimals that is spanned by the sequences $\left\langle \frac{1}{n} \right\rangle, \left\langle \frac{1}{n^2} \right\rangle, \left\langle \frac{1}{n^3} \right\rangle, \ldots$ It is a semigroup with respect to vector addition, and includes all the scalar multiples of the generating sequences that are non-zero. That is, the family includes infinitesimals with negative sign. Therefore, $\frac{1}{dx}$ will mean the sequence $\langle n \rangle$. Alternatively, we may choose

the family spanned by the sequences $\left\langle \frac{1}{2^n} \right\rangle, \left\langle \frac{1}{3^n} \right\rangle, \left\langle \frac{1}{4^n} \right\rangle, \ldots$ Then, $\frac{1}{dx}$ will mean the sequence $\left\langle 2^n \right\rangle$. Once we determined the basic infinitesimal dx, we will use it in the Infinite Riemann Sum that defines an Integral in Infinitesimal Calculus.

3. The Delta Function is strictly smaller than ∞
4. We define, $\delta(x) \equiv \frac{1}{dx} \chi_{\left[-\frac{dx}{2}, \frac{dx}{2}\right]}(x)$,

   where $\chi_{\left[-\frac{dx}{2}, \frac{dx}{2}\right]}(x) = \begin{cases} 1, x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right] \\ 0, \text{otherwise} \end{cases}$.
5. Hence,
   - for $x < 0$, $\delta(x) = 0$
   - at $x = -\frac{dx}{2}$, $\delta(x)$ jumps from 0 to $\frac{1}{dx}$,
   - for $x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right]$, $\delta(x) = \frac{1}{dx}$.
   - at $x = 0$, $\delta(0) = \frac{1}{dx}$
   - at $x = \frac{dx}{2}$, $\delta(x)$ drops from $\frac{1}{dx}$ to 0.
   - for $x > 0$, $\delta(x) = 0$.

❖ $x\delta(x) = 0$

6. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle \chi_{[-\frac{1}{2},\frac{1}{2}]}(x), 2\chi_{[-\frac{1}{4},\frac{1}{4}]}(x), 3\chi_{[-\frac{1}{6},\frac{1}{6}]}(x) ... \right\rangle$

7. If $dx = \left\langle \frac{2}{n} \right\rangle$, $\delta(x) = \left\langle \frac{1}{2\cosh^2 x}, \frac{2}{2\cosh^2 2x}, \frac{3}{2\cosh^2 3x}, ... \right\rangle$

8. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle e^{-x}\chi_{[0,\infty)}, 2e^{-2x}\chi_{[0,\infty)}, 3e^{-3x}\chi_{[0,\infty)}, ... \right\rangle$

9. $$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1.$$

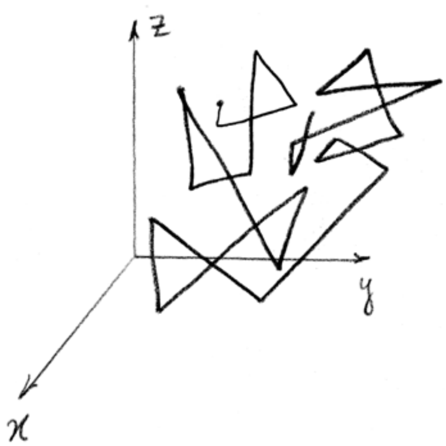
10. $$\delta(\xi - x) = \frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} e^{-ik(\xi - x)} dk$$

5.

Probability-Wave Equation for Random Walk

The Random Walk of small particles in fluid is named after Brown, who first observed it, Brownian Motion. It models other processes, such as the fluctuations of a stock price.

In a volume of fluid, the path of a particle is in any direction in the volume, and of variable size



5.1 Bernoulli Random Variables of the Walk

We restrict the Walk here to the line, in uniform infinitesimal size steps dx:

To the left, with probability

$$p\,,$$

or to the right, with probability

$$q = 1 - p\,.$$

At time t, after

$$N \text{ infinitesimal time intervals } dt\,,$$

$$N = \tfrac{t}{dt}, \text{ is an infinite hyper-real,}$$

the particle is at the point

$$x\,.$$

At the ith step we define the Bernoulli Random Variable,

$$B_i(\text{right step}) = dx\,, \qquad \zeta_1 = \text{right step}\,.$$

$$B_i(\text{left step}) = -dx\,, \qquad \zeta_2 = \text{left step}\,.$$

where $i = 1, 2, ..., N$.

$$\Pr(B_i = dx) = p\,,$$

$$\Pr(B_i = -dx) = q\,,$$

$$E[B_i] = dx \cdot p + (-dx)\cdot q = (p-q)dx\,,$$

$$E[B_i^{\,2}] = (dx)^2 \cdot p + (-dx)^2 \cdot q = (dx)^2$$

$$\mathrm{Var}[B_i] = \underbrace{E[B_i^{\,2}]}_{(dx)^2} - (\underbrace{E[B_i]}_{(p-q)dx})^2$$

$$= (\underbrace{1+p-q}_{2p})(\underbrace{1-p}_{q} + q)(dx)^2 = 4pq(dx)^2$$

5.2 The Random Walk

$$B(\zeta,t) = B_1 + B_2 + ... + B_N$$

is a Random Process with

$$E[B(\zeta,t)] = (p-q)Ndx\,,$$

$$\mathrm{Var}[B(\zeta,t)] = 4pqN(dx)^2.$$

<u>*Proof*</u>: Since the B_i are independent,

$$E[B(\zeta,t)] = \underbrace{E[B_1]}_{(p-q)dx} + ... + \underbrace{E[B_N]}_{(p-q)dx} = (p-q)Ndx$$

$$\mathrm{Var}[B(\zeta,t)] = \underbrace{\mathrm{Var}[B_1]}_{4pq(dx)^2} + ... + \underbrace{\mathrm{Var}[B_N]}_{4pq(dx)^2} = 4pqN(dx)^2\,.\square$$

5.3 The Focker-Planck Probability-Wave Equation of the Walk

<u>*Let*</u> (1) $$(dx)^2 = 2D(dt)\,,$$

where the <u>Drift Coefficient</u> D is a constant,

(2) $$(p-q)dx = 2Cdt\,,$$

where the Speed C is a constant

(3) $$\Pr(x - \tfrac{1}{2}dx \le B(\zeta,t) \le x + \tfrac{1}{2}dx) = f(x,t)dx$$

<u>*Then,*</u> *the Probability-Wave Equation of $B(\zeta,t)$ for $f(x,t)$ is infinitesimally close to the Diffusion Equation*

$$\partial_t f = -2C\partial_x f + (D)\partial_x^2 f$$

Proof: We'll denote

$$\Pr\Big(x - \tfrac{1}{2}dx \le B(\zeta, t) \le x + \tfrac{1}{2}dx\Big) = \Pr\Big(B(\zeta, t) \triangleq x\Big)$$

Then, by Bayes' Theorem,

$$\underbrace{\Pr\Big(B(\zeta, t+dt) \triangleq x\Big)}_{f(x,t+dt)dx} =$$

$$= \underbrace{\Pr\Big(B(\zeta, t+dt) \triangleq x \,/\, B(\zeta, t) \triangleq x - dx\Big)}_{p}\underbrace{\Pr\Big(B(\zeta, t) \triangleq x - dx\Big)}_{f(x-dx,t)dx} +$$

$$+ \underbrace{\Pr\Big(B(\zeta, t+dt) \triangleq x \,/\, B(\zeta, t) \triangleq x + dx\Big)}_{q}\underbrace{\Pr\Big(B(\zeta, t) \triangleq x + dx\Big)}_{f(x+dx,t)dx}$$

That is,

$$f(x, t+dt) = pf(x - dx, t) + qf(x + dx, t).$$

Substituting

$$f(x, t+dt) \approx f(x, t) + (\partial_t f(x, t))dt,$$

$$f(x - dx, t) \approx f(x, t) - (\partial_x f(x, t))dx + \tfrac{1}{2}(\partial_x^2 f(x, t))(dx)^2,$$

$$f(x + dx, t) \approx f(x, t) + (\partial_x f(x, t))dx + \tfrac{1}{2}(\partial_x^2 f(x, t))(dx)^2,$$

we obtain

$$(\partial_t f(x, t))dt \approx \underbrace{(q - p)dx}_{-2Cdt}(\partial_x f(x, t)) + \underbrace{\tfrac{1}{2}(dx)^2}_{(D)dt}(\partial_x^2 f(x, t)),$$

$$\partial_t f(x,t) \approx -2C\partial_x f(x,t) + (D)\partial_x^2 f(x,t),$$

which is the Diffusion Equation.□

5.4 $$f(x.t) = \frac{1}{\sqrt{2\pi}\sqrt{(4pq)2Dt}} e^{-\frac{1}{2}\frac{(x-2Ct)^2}{(4pq)2Dt}}$$

solves the Diffusion Equation,

$$\partial_t f(x,t) = -2C\partial_x f(x,t) + (D)\partial_x^2 f(x,t).$$

Proof: By substitution.□

6.

Probability-Wave Equation for Poisson Process

The arrival at rate λ, of radioactive particles at a counter is modeled by the Poisson Process. It models other processes, such as the arrival of phone calls at rate λ, to an operator.

6.1 Bernoulli Random Variables of the Process

We assume that

an arrival probability in time dt is

$$p = \lambda dt,$$

and no arrival probability in time dt is

$$q = 1 - \lambda dt.$$

At time t, after

N infinitesimal time intervals dt,

$N = \frac{t}{dt}$, is an infinite hyper-real,

there are

k arrivals,

k is a finite hyper-real

and

$$N-k \text{ no arrivals,}$$

$$N-k \text{ is an infinite Hyper-real}$$

At the ith step we define the Bernoulli Random Variable,

$$P_i(\text{arrival}) = 1, \qquad \zeta_1 = \text{arrival}$$

$$P_i(\text{no-arrival}) = 0, \qquad \zeta_2 = \text{no-arrival}$$

where $i = 1,2,...,N$.

$$\Pr(P_i = 1) = p = \lambda dt,$$

$$\Pr(P_i = 0) = q = 1 - \lambda dt,$$

$$E[P_i] = 1 \cdot \lambda dt + 0 \cdot (1-\lambda dt) = \lambda dt,$$

$$E[P_i^2] = 1^2 \cdot \lambda dt + 0^2 \cdot (1-\lambda dt) = \lambda dt$$

$$\mathrm{Var}[P_i] = \underbrace{E[P_i^2]}_{\lambda dt} - (\underbrace{E[P_i]}_{\lambda dt})^2,$$

$$= \lambda dt \underbrace{(1-\lambda dt)}_{\approx 1} \approx \lambda dt.$$

6.2 The Poisson Process

$$P(\zeta, t) = P_1 + P_2 + ... + P_N$$

is a Random Process with

$$E[P(\zeta,t)] = \lambda t,$$

$$\mathrm{Var}[P(\zeta,t)] \approx \lambda t$$

Proof: Since the P_i are independent,

$$E[P(\zeta,t)] = \underbrace{E[P_1]}_{\lambda dt} + ... + \underbrace{E[P_N]}_{\lambda dt} = \lambda \underbrace{Ndt}_{t}$$

$$\mathrm{Var}[P(\zeta,t)] = \underbrace{\mathrm{Var}[P_1]}_{\approx\lambda dt} + ... + \underbrace{\mathrm{Var}[P_N]}_{\approx\lambda dt} \approx \lambda \underbrace{Ndt}_{t}$$

6.3 The Probability-Wave Equation of the Process

Let $\Pr\Big(P(\zeta,t) = k\Big) = p(k,t)$

Then *The Probability-Wave Equation of* $X(\zeta,t)$ *for* $p(k,t)$ *is*

the first order differential-difference wave equation

$$\partial_t p(k,t) = -\lambda\Delta_{k-1}p(k,t)$$

Proof: By Bayes' Theorem,

$$\underbrace{\Pr\Big(P(\zeta,t+dt) = k\Big)}_{p(k,t+dt)} =$$

$$= \underbrace{\Pr\Big(P(\zeta,t+dt) = k \;/\; P(\zeta,t) = k-1\Big)}_{p=\lambda dt}\underbrace{\Pr\Big(P(\zeta,t) = k-1\Big)}_{p(k-1,t)} +$$

$$+\underbrace{\Pr\Big(P(\zeta,t+dt) = k \;/\; P(\zeta,t) = k\Big)}_{q=1-\lambda dt}\underbrace{\Pr\Big(P(\zeta,t) = k\Big)}_{p(k,t)}$$

That is,

$$p(k,t+dt) = p(k-1,t)\lambda dt + p(k,t)(1-\lambda dt),$$

$$\underbrace{\frac{p(k,t+dt)-p(k,t)}{dt}}_{\partial_t p(k,t)} = -\lambda \underbrace{\left[p(k,t)-p(k-1,t)\right]}_{\Delta_{k-1}p(k,t)}$$

$$\partial_t p(k,t) = -\lambda \Delta_{k-1} p(k,t),$$

which is the Poisson Probability-Wave Equation.□

6.4 $$p(k,t) = \frac{1}{k!}(\lambda t)^k e^{-\lambda t}$$

solves the Poisson Probability-Wave Equation

$$\partial_t p(k,t) = -\lambda \Delta_{k-1} p(k,t).$$

Proof: By substitution.□

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