Ito Integral

H. Vic Dannon
vic0@comcast.net
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Abstract We show that Ito's integral $\int_{t=a}^{t=b} X(\zeta,t)dB(\zeta,t)$, where $X(\zeta,t)$ is a Random Process, and $B(\zeta,t)$ is Random Walk is ill defined because it

- violates Riemann's Oscillation Condition for Integrability: The Oscillation Integral is not an infinitesimal.
- violates the Fundamental Theorem of the Integral Calculus: Reversing the integration returns a function different from the original integrand.
- depends on the partitions of $[a,b]$, and on the choice of the intermediate points in them. Thus, the Random Variable has an undetermined Expectation.

Consequently, the Ito Integral does not exist, and formulas that include that integral do not hold.

Keywords: Ito Integral, Ito Process, Ito Formula, Stochastic Integration, Infinitesimal, Infinite-Hyper-real, Hyper-real, Calculus, Limit, Continuity, Derivative, Integral, Delta Function, Random Variable, Random Process, Random Signal, Stochastic Process, Stochastic Calculus, Probability Distribution, Bernoulli Random Variables, Binomial Distribution, Gaussian, Normal, Expectation, Variance, Random Walk, Poisson Process,

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Introduction

In [Dan5], we defined in Infinitesimal Calculus the Wiener Integral $\int_{t=a}^{t=b} f(t)dB(\zeta,t)$, where $f(t)$ is integrable hyper-real function, and $B(\zeta,t)$ is a Random Walk.

In [Ito], Ito argued that $f(t)$ can be replaced with a Hyper-real Random Process $X(\zeta,t)$.

We argue here that his claim fails for $X(\zeta,t) = B(\zeta,t)$.

We show that Ito's approach through Lebesgue Integration, ignores the basics of Integration set by Riemann.

The Ito integration sums do not converge to an Integral.

The Ito integral $\int_{t=a}^{t=b} B(\zeta,t)dB(\zeta,t)$ is ill defined because it

- violates Riemann's Oscillation Condition for Integrability: The Oscillation Integral is not an infinitesimal.
- violates the Fundamental Theorem of the Integral Calculus: Reversing the integration returns a function different from the original integrand.

- depends on the partitions of $[a,b]$, and on the choice of the intermediate points in them. Thus, the Random Variable has an undetermined Expectation.

Any integration is an infinite summation. Then, we cannot tell the difference between

$$\sum_{t=a}^{t=b} X(\zeta,t)[X(\zeta,t+dt)-X(\zeta,t)],$$

$$\sum_{t=a}^{t=b} X(\zeta,t+dt)[X(\zeta,t+dt)-X(\zeta,t)],$$

and any summation in between them

$$\sum_{t=a}^{t=b} \left[\lambda X(\zeta,t)+(1-\lambda)X(\zeta,t+dt)\right][X(\zeta,t+dt)-X(\zeta,t)]$$

It is not up to us to choose the sum with $\lambda = \frac{1}{2}$, and it is meaningless to say that we did that.

The infinite summation yields a definite value, only if all these –impossible to distinguish between- results, are all the same.

Furthermore, when we encounter uncertain outcomes, our first measure of certainty is the average. The average is what we expect, give or take some deviation.

Thus, our first measure is the Expectation of the Random Variable, and our second measure is the Variance, which over time will limit the Expectation within a band.

If we cannot tell the Average of our outcomes, our Random Variable is undefined, and should be discarded.

Thus, a Random Process $X(\zeta, t)$ may not be integrated with respect to a Random Walk $B(\zeta, t)$, and an Integral in which $f(t)$ is replaced by $X(\zeta, t)$ does not exist.

1.

Hyper-real Line

The minimal domain and range, needed for the definition and analysis of a hyper-real function, is the hyper-real line.

Each real number α can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of infinitesimal hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the infinite hyper-reals.
4. The infinite hyper-reals are greater than any real number, yet strictly smaller than infinity.

5. The infinite hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a non-constant hyper-real.
7. The Hyper-reals are the totality of constant hyper-reals, a family of infinitesimals, a family of infinitesimals with negative sign, a family of infinite hyper-reals, a family of infinite hyper-reals with negative sign, and non-constant hyper-reals.
8. The hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.
9. That line includes the real numbers separated by the non-constant hyper-reals. Each real number is the center of an interval of hyper-reals, that includes no other real number.
10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.
11. Zero is not an infinitesimal, because zero is not strictly greater than zero.

12. We do not add infinity to the hyper-real line.

13. The infinitesimals, the infinitesimals with negative signs, the infinite hyper-reals, and the infinite hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.

14. The hyper-real line is embedded in $\mathbb{R}^\infty$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the hyper-real onto the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal hyper-reals, or to the infinite hyper-reals, or to the non-constant hyper-reals.

16. No neighbourhood of a hyper-real is homeomorphic to an $\mathbb{R}^n$ ball. Therefore, the hyper-real line is not a manifold.

17. The hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

2.

Hyper-real Function

2.1 Definition of a hyper-real function

$f(x)$ is a hyper-real function, iff it is from the hyper-reals into the hyper-reals.

This means that any number in the domain, or in the range of a hyper-real $f(x)$ is either one of the following

real

real + infinitesimal

real – infinitesimal

infinitesimal

infinitesimal with negative sign

infinite hyper-real

infinite hyper-real with negative sign

Clearly,

2.2 *Every function from the reals into the reals is a hyper-real function.*

3.

Integral of Hyper-real Function

In [Dan3], we defined the integral of a Hyper-real Function.

Let $f(x)$ be a hyper-real function on the interval $[a,b]$.

The interval may not be bounded.

$f(x)$ may take infinite hyper-real values, and need not be bounded.

At each

$$a \leq x \leq b,$$

there is a rectangle with base $[x - \frac{dx}{2}, x + \frac{dx}{2}]$, height $f(x)$, and area

$$f(x)dx.$$

We form the **Integration Sum** of all the areas for the x's that start at $x = a$, and end at $x = b$,

$$\sum_{x \in [a,b]} f(x)dx.$$

If for any infinitesimal dx, the Integration Sum has the same hyper-real value, then $f(x)$ is integrable over the interval $[a,b]$.

Then, we call the Integration Sum the integral of $f(x)$ from $x = a$, to $x = b$, and denote it by

$$\int_{x=a}^{x=b} f(x)dx .$$

If the hyper-real is infinite, then it is the integral over $[a,b]$,

If the hyper-real is finite,

$$\int_{x=a}^{x=b} f(x)dx = \text{ real part of the hyper-real}. \square$$

3.1 The countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers, $Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty .$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[a,b]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(x)dx$.

The Lower Integral is the Integration Sum where $f(x)$ is replaced

by its lowest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

3.2 $$\sum_{x \in [a,b]} \left(\inf_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

The Upper Integral is the Integration Sum where $f(x)$ is replaced by its largest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

3.3 $$\sum_{x \in [a,b]} \left(\sup_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

If the integral is a finite hyper-real, we have

3.4 *A hyper-real function has a finite integral if and only if its upper integral and its lower integral are finite, and differ by an infinitesimal.*

4.

Delta Function

In [Dan4], we defined the Delta Function, and established its properties

1. The Delta Function is a hyper-real function defined from the hyper-real line into the set of two hyper-reals $\left\{0, \frac{1}{dx}\right\}$. The hyper-real 0 is the sequence $\langle 0, 0, 0, ... \rangle$. The infinite hyper-real $\frac{1}{dx}$ depends on our choice of dx.
2. We will usually choose the family of infinitesimals that is spanned by the sequences $\left\langle \frac{1}{n} \right\rangle, \left\langle \frac{1}{n^2} \right\rangle, \left\langle \frac{1}{n^3} \right\rangle, \ldots$ It is a semigroup with respect to vector addition, and includes all the scalar multiples of the generating sequences that are non-zero. That is, the family includes infinitesimals with negative sign. Therefore, $\frac{1}{dx}$ will mean the sequence $\langle n \rangle$. Alternatively, we may choose

the family spanned by the sequences $\left\langle \frac{1}{2^n} \right\rangle, \left\langle \frac{1}{3^n} \right\rangle, \left\langle \frac{1}{4^n} \right\rangle$,... Then, $\frac{1}{dx}$ will mean the sequence $\left\langle 2^n \right\rangle$. Once we determined the basic infinitesimal dx, we will use it in the Infinite Riemann Sum that defines an Integral in Infinitesimal Calculus.

3. The Delta Function is strictly smaller than ∞

4. We define, $\delta(x) \equiv \frac{1}{dx} \chi_{\left[-\frac{dx}{2}, \frac{dx}{2}\right]}(x)$,

$$\text{where} \quad \chi_{\left[-\frac{dx}{2}, \frac{dx}{2}\right]}(x) = \begin{cases} 1, x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right] \\ 0, \text{otherwise} \end{cases}.$$

5. Hence,

- for $x < 0$, $\delta(x) = 0$
- at $x = -\frac{dx}{2}$, $\delta(x)$ jumps from 0 to $\frac{1}{dx}$,
- for $x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right]$, $\delta(x) = \frac{1}{dx}$.
- at $x = 0$, $\delta(0) = \frac{1}{dx}$
- at $x = \frac{dx}{2}$, $\delta(x)$ drops from $\frac{1}{dx}$ to 0.
- for $x > 0$, $\delta(x) = 0$.

❖ $x\delta(x) = 0$

6. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle \chi_{[-\frac{1}{2},\frac{1}{2}]}(x), 2\chi_{[-\frac{1}{4},\frac{1}{4}]}(x), 3\chi_{[-\frac{1}{6},\frac{1}{6}]}(x)... \right\rangle$

7. If $dx = \left\langle \frac{2}{n} \right\rangle$, $\delta(x) = \left\langle \frac{1}{2\cosh^2 x}, \frac{2}{2\cosh^2 2x}, \frac{3}{2\cosh^2 3x}, ... \right\rangle$

8. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle e^{-x}\chi_{[0,\infty)}, 2e^{-2x}\chi_{[0,\infty)}, 3e^{-3x}\chi_{[0,\infty)}, ... \right\rangle$

9. $$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1.$$

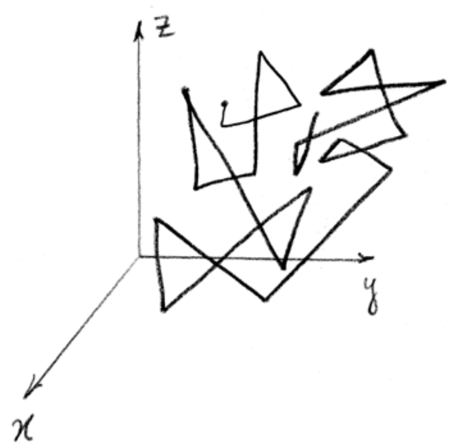
10. $$\delta(\xi - x) = \frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} e^{-ik(\xi - x)}dk$$

5.

Random Walk $B(\zeta, t)$

The Random Walk of small particles in fluid is named after Brown, who first observed it, Brownian Motion. It models other processes, such as the fluctuations of a stock price.

In a volume of fluid, the path of a particle is in any direction in the volume, and of variable size



5.1 Bernoulli Random Variables of the Walk

We restrict the Walk here to the line, in uniform infinitesimal size steps dx:

To the left, with probability

$$p = \tfrac{1}{2},$$

or to the right, with probability

$$q = 1 - p = \tfrac{1}{2}.$$

At time t, after

N infinitesimal time intervals dt,

$N = \frac{t}{dt}$, is an infinite hyper-real,

the particle is at the point

$$x.$$

At the ith step we define the Bernoulli Random Variable,

$$B_i(\text{right step}) = dx, \quad \zeta_1 = \text{right step}.$$

$$B_i(\text{left step}) = -dx, \quad \zeta_2 = \text{left step}.$$

where $i = 1, 2, ..., N$.

$$\Pr(B_i = dx) = \tfrac{1}{2},$$

$$\Pr(B_i = -dx) = \tfrac{1}{2},$$

$$E[B_i] = dx \cdot \tfrac{1}{2} + (-dx) \cdot \tfrac{1}{2} = 0,$$

$$E[B_i^2] = (dx)^2 \cdot \tfrac{1}{2} + (-dx)^2 \cdot \tfrac{1}{2} = (dx)^2$$

$$\mathrm{Var}[B_i] = \underbrace{E[B_i^2]}_{(dx)^2} - (\underbrace{E[B_i]}_{0})^2 = 0$$

5.2 The Random Walk

$$B(\zeta, t) = B_1 + B_2 + ... + B_N$$

is a Random Process with

$$E[B(\zeta,t)] = 0,$$

$$\mathrm{Var}[B(\zeta,t)] = N(dx)^2.$$

Proof: Since the B_i are independent,

$$E[B(\zeta,t)] = \underbrace{E[B_1]}_{0} + ... + \underbrace{E[B_N]}_{0} = 0$$

$$\mathrm{Var}[B(\zeta,t)] = \underbrace{\mathrm{Var}[B_1]}_{(dx)^2} + ... + \underbrace{\mathrm{Var}[B_N]}_{(dx)^2} = N(dx)^2. \square$$

5.3 $B(\zeta,t+dt) - B(\zeta,t)$ *is a Bernoulli Random Variable* B_i

6.

Integration Sums of $B(\zeta,t)$ with respect to $B(\zeta,t)$.

In [Dan5], we defined the Integral $\int\limits_{t=a}^{t=b} f(t)dB(\zeta,t)$:

6.1 Integration Sums of $f(t)$ with respect to $B(\zeta,t)$.

Let $f(t)$ be a hyper-real function on the bounded time interval $[a,b]$. $f(t)$ need not be bounded.

At each $a \le t \le b$, there is a Bernoulli Random Variable

$$dB(\zeta,t) = B(\zeta,t+dt) - B(\zeta,t) = B_i(\zeta,t) = \dot{B}(\zeta,t)dt.$$

We form the **Integration Sum**

$$\sum_{t=a}^{t=b} f(t)dB(\zeta,t) = \sum_{t=a}^{t=b} f(t)B_i(\zeta,t).$$

For any dt,

(1) the First Moment of the Integration Sum is

$$E\left[\sum_{t=a}^{t=b} f(t)B_i(\zeta,t)\right] = \sum_{t=a}^{t=b} f(t)\underbrace{E[B_i(\zeta,t)]}_{0} = 0.$$

(2) the Second Moment of the Integration sum is

$$E\left[\left(\sum_{t=a}^{t=b} f(t)B_i(\zeta,t)\right)^2\right] = E\left[\left(\sum_{t=a}^{t=b} f(t)B_i(\zeta,t)\right)\left(\sum_{\tau=a}^{\tau=b} f(\tau)B_j(\zeta,\tau)\right)\right]$$

$$= \sum_{t=a}^{t=b}\sum_{\tau=a}^{\tau=b} f(t)f(\tau)E[B_j(\zeta,\tau)B_i(\zeta,t)]$$

Since the Bernoulli Random Variables are independent,

$$E[B_j(\zeta,\tau)B_i(\zeta,t)] = E[B_i^{\,2}(\zeta,t)] = (dx)^2$$

only for $t = \tau$. Then,

$$E\left[\left(\sum_{t=a}^{t=b} f(t)B_i(\zeta,t)\right)^2\right] = \sum_{t=a}^{t=b} f^2(t)\underbrace{(dx)^2}_{(2D)dt},$$

$$= 2D\sum_{t=a}^{t=b} f^2(t)dt,$$

$$= 2D\int_{t=a}^{t=b} f^2(t)dt,$$

assuming $(dx)^2 = (2D)dt$, and $f(t)$ integrable.

Thus, for any dt, the Integration Sum is a unique well-defined hyper-real Random Variable $I(\zeta)$.

We call $I(\zeta)$ the integral of $f(t)$, with respect to $B(\zeta,t)$ from $x = a$, to $x = b$, and denote it by $\int\limits_{t=a}^{t=b} f(t)dB(\zeta,t)$.

6.2 Integration Suns of $B(\zeta,t)$, with respect to $B(\zeta,t)$

To define $\int_{t=a}^{t=b} B(\zeta,t)dB(\zeta,t)$, we form the **Integration Sum**

$$\sum_{t=a}^{t=b} B(\zeta,t)dB(\zeta,t) = \sum_{t=a}^{t=b} B(\zeta,t)B_i(\zeta,t).$$

(1) the First Moment of the Integration Sum is

$$E\left[\sum_{t=a}^{t=b} B(\zeta,t)B_i(\zeta,t)\right] = \sum_{t=a}^{t=b} E[B(\zeta,t)B_i(\zeta,t)]$$

$$= \sum_{t=a}^{t=b} \underbrace{E[B_i^{\,2}(\zeta,t)]}_{(dx)^2=(2D)dt} = 2D(b-a).$$

(2) the Second Moment of the Integration sum is

$$E\left[\left(\sum_{t=a}^{t=b} B(\zeta,t)B_i(\zeta,t)\right)^2\right] = E\left[\left(\sum_{t=a}^{t=b} B(\zeta,t)B_i(\zeta,t)\right)\left(\sum_{\tau=a}^{\tau=b} B(\zeta,\tau)B_j(\zeta,\tau)\right)\right]$$

This expression cannot be easily resolved, and we are not able to conclude that for any dt, the Integration Sum is a unique well-defined hyper-real Random Variable $I(\zeta)$.

We shall see that for a variety of reasons the integral of $B(\zeta,t)$, with respect to $B(\zeta,t)$ from $x=a$, to $x=b$, cannot be defined.

7.

$\sum_{t=a}^{t=b} B(\zeta,t)dB(\zeta,t)$ **is ill-defined**

Proof: The Expectation of the Oscillation Integral [Dan3, p.46], of $B(\zeta,t)$ with respect to $B(\zeta,t)$ is

$$E\left[\sum_{t=a}^{t=b}\big[B(\zeta,t+dt)-B(\zeta,t)\big]dB(\zeta,t)\right].$$

Since $B(\zeta,t+dt)-B(\zeta,t)$ is a Bernoulli Random Variable B_i, the Oscillation Integral of $B(\zeta,t)$ with respect to $B(\zeta,t)$ equals

$$\sum_{t=a}^{t=b}\underbrace{E[B_i^2]}_{(dx)^2=(2D)dt} = (2D)(b-a)\neq \text{infinitesimal},$$

violating Riemann's Oscillation Condition for Integrability of

$$E\left[\sum_{t=a}^{t-b} B(\zeta,t)dB(\zeta,t)\right].$$

Thus, $E\left[\sum_{t=a}^{t-b} B(\zeta,t)dB(\zeta,t)\right]$ diverges, and is not an Integral.

That is, the first Moment of the Random Variable

$$\sum_{t=a}^{t-b} B(\zeta,t)dB(\zeta,t)$$

does not exist, and the Random Variable is ill-defined.

Consequently, the Random Variable diverges, and is not an integral.

8. $\sum_{t=a}^{t=b} B(\zeta,t)dB(\zeta,t)$ depends on the choice of the intermediate points in the partition

Proof:

To converge to an Integral, the Integration sum

$$E\left[\sum_{t=a}^{t=b} B(\zeta,t)dB(\zeta,t)\right] = E\left[\sum_{t=a}^{t=b} B(\zeta,t)\Big(B(\zeta,t+dt) - B(\zeta,t)\Big)\right]$$

should remain unchanged for any choice of the intermediate points between $B(\zeta,t)$, and $B(\zeta,t+dt)$.

We check three different sets of intermediate points

(1) Intermediate points at $\underline{\tfrac{1}{2}\Big(B(\zeta,t+dt) + B(\zeta,t)\Big)|}$.

The Integration Sum becomes

$$\sum_{t=a}^{t=b-dt} \tfrac{1}{2}\Big(B(\zeta,t+dt) + B(\zeta,t)\Big)\Big(B(\zeta,t+dt) - B(\zeta,t)\Big) =$$

$$= \sum_{t=a}^{t=b-dt} \tfrac{1}{2}\Big(B^2(\zeta, t+dt) - B^2(\zeta, t)\Big),$$

$$= \tfrac{1}{2}\Big(B^2(\zeta, b) - B^2(\zeta, a)\Big).$$

(2) Intermediate points at $\underline{B(\zeta, t)}|$

The Integration Sum is,

$$\sum_{t=a}^{t=b} B(\zeta, t)\Big(B(\zeta, t+dt) - B(\zeta, t)\Big) =$$

$$\underbrace{\sum_{t=a}^{t=b-dt} \tfrac{1}{2}\Big(B(\zeta, t+dt) + B(\zeta, t)\Big)\Big(B(\zeta, t+dt) - B(\zeta, t)\Big)}_{\frac{1}{2}\left(B^2(\zeta,b) - B^2(\zeta,a)\right)}$$

$$-\tfrac{1}{2}\sum_{t=a}^{t=b-dt} \underbrace{\Big(B(\zeta, t+dt) - B(\zeta, t)\Big)\Big(B(\zeta, t+dt) - B(\zeta, t)\Big)}_{B_i^{\,2}}$$

Then,

$$E\left[\sum_{t=a}^{t=b-dt} B(\zeta, t)\Big(B(\zeta, t+dt) - B(\zeta, t)\Big)\right] =$$

$$= \tfrac{1}{2} E\Big[B^2(\zeta, b) - B^2(\zeta, a)\Big] - \tfrac{1}{2}\sum_{t=a}^{t=b} \underbrace{E[B_i^{\,2}]}_{(dx)^2 = (2D)dt}$$

$$= \tfrac{1}{2} E\Big[B^2(\zeta, b) - B^2(\zeta, a)\Big] - D(b-a).$$

(3) Intermediate points at $\underline{B(\zeta, t+dt)}|$,

the Integration Sum is,

$$\sum_{t=a}^{t=b-dt} B(\zeta,t+dt)\Big(B(\zeta,t+dt)-B(\zeta,t)\Big) =$$

$$\underbrace{\sum_{t=a}^{t=b-dt} \tfrac{1}{2}\Big(B(\zeta,t+dt)+B(\zeta,t)\Big)\Big(B(\zeta,t+dt)-B(\zeta,t)\Big)}_{\frac{1}{2}\left(B^2(\zeta,b)-B^2(\zeta,a)\right)}$$

$$+\tfrac{1}{2}\sum_{t=a}^{t=b-dt} \underbrace{\Big(B(\zeta,t+dt)-B(\zeta,t)\Big)\Big(B(\zeta,t+dt)-B(\zeta,t)\Big)}_{B_i^{\ 2}}$$

Then,

$$E\left[\sum_{t=a}^{t=b-dt} B(\zeta,t+dt)\Big(B(\zeta,t+dt)-B(\zeta,t)\Big)\right] =$$

$$= \tfrac{1}{2}E\Big[B^2(\zeta,b)-B^2(\zeta,a)\Big] + \tfrac{1}{2}\sum_{t=a}^{t=b} \underbrace{E[B_i^{\ 2}]}_{(dx)^2=(2D)dt}$$

$$= \tfrac{1}{2}E\Big[B^2(\zeta,b)-B^2(\zeta,a)\Big] + D(b-a)\,.$$

Each of the three cases, has $E\left[\sum_{t=a}^{t=b} B(\zeta,t)dB(\zeta,t)\right]$ that differs from the others by non-infinitesimal number.

Thus, the First Moment of the Random Variable $\sum_{t=a}^{t=b} B(\zeta,t)dB(\zeta,t)$ is undefined.

That is, the Integration Sum is ill-defined, and does not exist.

Indeed, for any intermediate point

$$\lambda B(\zeta, t+dt) + (1-\lambda)B(\zeta, t), \quad 0 \le \lambda \le 1,$$

the integration sum is different

$$E\left[\sum_{t=a}^{t=b}\left[\lambda B(\zeta, t+dt) + (1-\lambda)B(\zeta, t)\right]\left(B(\zeta, t+dt) - B(\zeta, t)\right)\right]$$

$$= \lambda \underbrace{E\left[\sum_{t=a}^{t=b} B(\zeta, t+dt)\left(B(\zeta, t+dt) - B(\zeta, t)\right)\right]}_{\frac{1}{2}E\left[B^2(\zeta,b)-B^2(\zeta,a)\right]+D(b-a)} +$$

$$+(1-\lambda)\underbrace{E\left[\sum_{t=a}^{t=b} B(\zeta, t)\left(B(\zeta, t+dt) - B(\zeta, t)\right)\right]}_{\frac{1}{2}E\left[B^2(\zeta,b)-B^2(\zeta,a)\right]-D(b-a)}.$$

$$= \tfrac{1}{2}E\left[B^2(\zeta, b) - B^2(\zeta, a)\right] + (\lambda - \tfrac{1}{2})D(b-a). \square$$

9.

The Myth that Lebesgue's Integral generalizes Riemann's

Ito's use of Lebesgue's Integration, is consistent with his ignorance of the basics of Integration, and suggests a belief in the Myth that Lebesgue's Integration generalizes Riemann's . But in [Dan6] we observed that

- Riemann's Function [Dan6] is Riemann-Integrable over a Non-Measurable set of Discontinuities,
- the Lebesgue integral cannot be defined over the non-measurable rationals, while the Riemann Integral may be defined over the rationals,

Thus, we concluded that

Riemann Integral generalizes Lebesgue's, and the Lebesgue-integrable functions are a subset of the Riemann-integrable functions.

Riemann's requirements that

- the oscillation integral must be infinitesimal, and

- the integral may not depend on the choice of the intermediate points,

are the foundations of any treatment of integration.

The misinterpretation of the ill-defined Random Variable

$$\sum_{t=a}^{t=b} B(\zeta,t)dB(\zeta,t)$$

as different Integrals, named after Ito, for $\lambda = 1$, and after Stratonovich for $\lambda = \frac{1}{2}$,.., misses the point that these different values prove that there is no integral.

Ito never comprehended why $f(t)$ of the Wiener Integral cannot be replaced by a Random Process $f(\zeta,t)$, and believed that this can be resolved by applying the "more general" Lebesgue Integral.

As demonstrated in [Dan7], Lebesgue's theories of measure and integration are conducive to delusionary claims.

10. $\sum_{t=a}^{t=b} B(\zeta,t)dB(\zeta,t)$ violates the Fundamental Theorem of the Integral Calculus.

The Fundamental Theorem of Calculus guarantees that Integration and Differentiation are well defined inverse operations, that when applied consecutively yield the original function.

It is well known to hold in the Calculus of Limits under given conditions. In the Infinitesimal Calculus, the Fundamental Theorem holds with almost no conditions.

In [Dan5], we established

10.1 *Let* $f(x)$ *be Hyper-real Integrable on* $[a,b]$

$$\textit{Then, for any } x \in [a,b], \qquad \text{p.v.D} \int_{u=a}^{u=x} f(u)du = f(x)$$

Consequently, if $F(x) = \int_{t=a}^{t=x} f(t)dt$,

$$\int_{t=a}^{t=b} f(t)dt = \int_{t=a}^{t=b} dF(t) = F(b) - F(a)$$

Thus, the Fundamental Theorem of Calculus requires that we have

$$\int_{t=a}^{t=b} B(\zeta,t)dB(\zeta,t) = \int_{t=a}^{t=b} d\left(\tfrac{1}{2}B^2(\zeta,t)\right) = \tfrac{1}{2}B^2(\zeta,b) - \tfrac{1}{2}B^2(\zeta,a),$$

and

$$E\left[\int_{t=a}^{t=b} B(\zeta,t)dB(\zeta,t)\right] = \tfrac{1}{2}E\left[B^2(\zeta,b) - B^2(\zeta,a)\right].$$

Instead, Ito has an additional λ-dependent term

$$E\left[\int_{t=a}^{t=b} B(\zeta,t)dB(\zeta,t)\right] = \tfrac{1}{2}E\left[B^2(\zeta,b) - B^2(\zeta,a)\right] + (\lambda - \tfrac{1}{2})D(b-a)$$

Thus, Ito's ill-defined Integral violates the Fundamental Theorem of the integral Calculus.

11.

Ito's Process and Ito's Formula

11.1 Ito's Process

is defined as the solution $X(\zeta, t)$ of the differential Equations

$$dX(\zeta, t) = f(\zeta, t)dB(\zeta, t) + g(\zeta, t)dt \,.$$

Then,

$$E\Big[\, dX(\zeta, t) - f(\zeta, t)dB(\zeta, t) - g(\zeta, t)dt \,\Big] = 0 \,,$$

and

$$E\Big[\Big|\, dX(\zeta, t) - f(\zeta, t)dB(\zeta, t) - g(\zeta, t)dt \Big|^2\,\Big] = 0$$

Summation over time of the first Moment equation yields the meaningless,

$$E\left[X(\zeta, t) - X(\zeta, 0) - \underbrace{\sum_{\tau=0}^{\tau=t} f(\zeta, \tau)dB(\zeta, \tau)}_{\text{ill-defined Ito Integral}} - \int_{\tau=0}^{\tau=t} g(\zeta, \tau)d\tau \right] \approx 0 \,.$$

11.2 Ito's Formula

Let $B = B(\zeta, t)$ be a Random Walk

$X = X(\zeta, t)$ be an Ito Process,

and $F = F(X,t)$.

Then,

$$dX = f(\zeta,t)dB(\zeta,t) + g(\zeta,t)dt\,,$$

$$(dX)^2 = f^2(dB)^2 + 2fg(dB)dt + g^2(dt)^2,$$

$$E[(dX)^2] = E\Big[\,f^2(dB)^2 + 2fg(dB)dt + g^2(dt)^2\,\Big]$$

$$= E[f^2]\,\underbrace{E[(dB)^2]}_{(dx)^2=(2D)dt} + 2E[fg]\underbrace{E[dB]}_{0}dt + E[g^2](dt)^2$$

$$\approx E[f^2](2D)dt$$

Substituting into

$$E\left[dF - \frac{\partial F}{\partial X}dX - \frac{1}{2}\frac{\partial^2 F}{\partial X^2}(dX)^2 - \frac{\partial F}{\partial t}dt\right] \approx 0\,,$$

we have, Ito's Formula

$$E\left[dF - \frac{\partial F}{\partial X}fdB - \left\{\frac{\partial F}{\partial X}g + \frac{\partial^2 F}{\partial X^2}f^2D + \frac{\partial F}{\partial t}\right\}dt\right] \approx 0\,.$$

The Formula is stated loosely, ([Oksendal], [Kuo]), dropping the Expectation operator.

In particular, the Variance of the Random Process has to be infinitesimal. That is,

$$E\left[\left|dF - \frac{\partial F}{\partial X}fdB - \left\{\frac{\partial F}{\partial X}g + \frac{\partial^2 F}{\partial X^2}f^2D + \frac{\partial F}{\partial t}\right\}dt\right|^2\right] \approx 0\,.$$

So far, the Formula was applied after a summation over time that generated an Integral Formula with the meaningless Ito Integral, $\sum_t \frac{\partial F}{\partial X} f(\zeta,t) dB(\zeta,t)$.

That Integral Formula is referred to as "The Fundamental Theorem of the Stochastic Integral Calculus"...

11.2 Example of the Ito Formula

Taking
$$F(X(\zeta,t),t) = B^2(\zeta,t),$$
$$X(\zeta,t) = B(\zeta,t),$$
$$f(\zeta,t) = 1; \quad g(\zeta,t) = 0$$

$$E\left[d(B^2) - \underbrace{\frac{\partial F}{\partial X}}_{2B} \underbrace{f}_{1} dB - \left\{ \frac{\partial F}{\partial X} \underbrace{g}_{0} + \underbrace{\frac{\partial^2 F}{\partial X^2}}_{2} \underbrace{f^2}_{1} D + \underbrace{\frac{\partial F}{\partial t}}_{0} \right\} dt \right] \approx 0$$

$$E\left[d(B^2) - 2BdB - 2Ddt \right] \approx 0 .$$

This leads the books to $d(B^2) = 2BdB + 2Ddt$

which is summed up over time to yield the "Fundamental etc.", $B^2(\zeta,t) = \underbrace{\sum_{\tau=0}^{\tau=t} B(\zeta,\tau) dB(\zeta,\tau)}_{\text{ill-defined Ito Integral}} + 2(D)t$.

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