# Infinitesimal Calculus of Random Walk and Poisson Processes

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**Abstract** We set up the Infinitesimal Calculus of Random Processes $X(\zeta, t)$, and apply it to the Random Walk $B(\zeta, t)$, and to the Poisson Process $P(\zeta, t)$.

Both Processes are Continuous, and have Derivative Processes with Delta Function Variance.

The integral $\int_{t=a}^{t=b} f(t)dB(\zeta, t)$, of integrable $f(t)$, with respect to the Random Walk $B(\zeta, t)$, and the integral $\int_{t=a}^{t=b} f(t)dP(\zeta, t)$ of integrable $f(t)$, with respect to the Poisson Process $P(\zeta, t)$ are well-defined Random Variables.

**Keywords:** Infinitesimal, Infinite-Hyper-real, Hyper-real, Calculus, Limit, Continuity, Derivative, Integral, Delta Function, Random Variable, Random Process, Random Signal, Stochastic Process, Stochastic Calculus, Probability Distribution, Bernoulli Random Variables, Binomial Distribution, Gaussian, Normal, Expectation, Variance, Random Walk, Poisson Process

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# Contents

15. $\int_{t=a}^{t=b} f(t)dP(\zeta, t)$

# References

# Introduction

## 0.1 Infinitesimal Calculus

Recently we have shown that when the Real Line is represented as the infinite dimensional space of all the Cauchy sequences of rational numbers, the hyper-reals are spanned by the constant hyper-reals, a family of infinitesimal hyper-reals, and the associated family of infinite hyper-reals.

The infinitesimal hyper-reals are smaller than any real number, yet bigger than zero.

The reciprocals of the infinitesimal hyper-reals are the infinite hyper-reals. They are greater than any real number, yet strictly smaller than infinity.

A neighborhood of infinitesimals separates the zero hyper-real from the reals, and each real number is the center of an interval of hyper-reals, that includes no other real number.

The Hyper-reals are totally ordered, and are lined up on a line, the hyper-real line.

A hyper-real function is a mapping from the hyper-real line into the hyper-real line.

Infinitesimal Calculus is the Calculus of hyper-real functions.

Infinitesimal Calculus is far more effective than the $\varepsilon, \delta$ Calculus, because being based on almost zero numbers, it allows us to deal with their reciprocals, the almost infinite numbers. We have no use for infinity by itself, but to comprehend the effects of singularities, we have use for the almost infinite.

Infinitesimals are a precise tool compared to the vague limit concept, and the awkward $\varepsilon, \delta$ statements.

Random walks are made clearer with infinitesimals.

Poisson Process can be derived only in Infinitesimal Calculus.

## 0.2 Random Processes

Probability Distributions are defined on Random Variables. Random Variables assign numerical vales to outcomes. Thus, maps outcomes into the real line.

Random Variables that evolve in time are called Random Processes, in Mechanics, or Random Signals, in Electricity.

Random Walk is the Random drift of a particle in fluid due to collisions with fluid molecules.

Poisson Process models the Random arrival of radioactive particles at a counter.

# 1.

# Hyper-real Line

The minimal domain and range, needed for the definition and analysis of a hyper-real function, is the hyper-real line.

Each real number $\alpha$ can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of infinitesimal hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the infinite hyper-reals.
4. The infinite hyper-reals are greater than any real number, yet strictly smaller than infinity.

5. The infinite hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a non-constant hyper-real.
7. The Hyper-reals are the totality of constant hyper-reals, a family of infinitesimals, a family of infinitesimals with negative sign, a family of infinite hyper-reals, a family of infinite hyper-reals with negative sign, and non-constant hyper-reals.
8. The hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.
9. That line includes the real numbers separated by the non-constant hyper-reals. Each real number is the center of an interval of hyper-reals, that includes no other real number.
10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.
11. Zero is not an infinitesimal, because zero is not strictly greater than zero.

12. We do not add infinity to the hyper-real line.

13. The infinitesimals, the infinitesimals with negative signs, the infinite hyper-reals, and the infinite hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.

14. The hyper-real line is embedded in $\mathbb{R}^{\infty}$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the hyper-real onto the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal hyper-reals, or to the infinite hyper-reals, or to the non-constant hyper-reals.

16. No neighbourhood of a hyper-real is homeomorphic to an $\mathbb{R}^n$ ball. Therefore, the hyper-real line is not a manifold.

17. The hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

# 2.

# Hyper-real Function

## 2.1 Definition of a hyper-real function

*$f(x)$ is a hyper-real function, iff it is from the hyper-reals into the hyper-reals.*

This means that any number in the domain, or in the range of a hyper-real $f(x)$ is either one of the following

real

real + infinitesimal

real – infinitesimal

infinitesimal

infinitesimal with negative sign

infinite hyper-real

infinite hyper-real with negative sign

Clearly,

**2.2** *Every function from the reals into the reals is a hyper-real function.*

# 3.

# Integral of Hyper-real Function

In [Dan3], we defined the integral of a Hyper-real Function.

Let $f(x)$ be a hyper-real function on the interval $[a,b]$.

The interval may not be bounded.

$f(x)$ may take infinite hyper-real values, and need not be bounded.

At each

$$a \le x \le b,$$

there is a rectangle with base $[x - \frac{dx}{2}, x + \frac{dx}{2}]$, height $f(x)$, and area

$$f(x)dx.$$

We form the **Integration Sum** of all the areas for the $x$'s that start at $x = a$, and end at $x = b$,

$$\sum_{x \in [a,b]} f(x)dx.$$

If for any infinitesimal $dx$, the Integration Sum has the same hyper-real value, then $f(x)$ is integrable over the interval $[a,b]$.

Then, we call the Integration Sum the integral of $f(x)$ from $x = a$, to $x = b$, and denote it by

$$\int_{x=a}^{x=b} f(x)dx\,.$$

If the hyper-real is infinite, then it is the integral over $[a,b]$,

If the hyper-real is finite,

$$\int_{x=a}^{x=b} f(x)dx = \text{ real part of the hyper-real}\,.\Box$$

## 3.1 The countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers, $Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = .... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty\,.$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[a,b]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(x)dx$.

The Lower Integral is the Integration Sum where $f(x)$ is replaced

by its lowest value on each interval $[x-\frac{dx}{2}, x+\frac{dx}{2}]$

**3.2**
$$\sum_{x\in[a,b]}\left(\inf_{x-\frac{dx}{2}\le t\le x+\frac{dx}{2}} f(t)\right)dx$$

The Upper Integral is the Integration Sum where $f(x)$ is replaced by its largest value on each interval $[x-\frac{dx}{2}, x+\frac{dx}{2}]$

**3.3**
$$\sum_{x\in[a,b]}\left(\sup_{x-\frac{dx}{2}\le t\le x+\frac{dx}{2}} f(t)\right)dx$$

If the integral is a finite hyper-real, we have

**3.4** *A hyper-real function has a finite integral if and only if its upper integral and its lower integral are finite, and differ by an infinitesimal.*

# 4.

# Delta Function

In [Dan4], we defined the Delta Function, and established its properties

1. The Delta Function is a hyper-real function defined from the hyper-real line into the set of two hyper-reals $\left\{0, \frac{1}{dx}\right\}$. The hyper-real $0$ is the sequence $\langle 0, 0, 0, ... \rangle$. The infinite hyper-real $\frac{1}{dx}$ depends on our choice of $dx$.
2. We will usually choose the family of infinitesimals that is spanned by the sequences $\left\langle \frac{1}{n} \right\rangle, \left\langle \frac{1}{n^2} \right\rangle, \left\langle \frac{1}{n^3} \right\rangle, \ldots$ It is a semigroup with respect to vector addition, and includes all the scalar multiples of the generating sequences that are non-zero. That is, the family includes infinitesimals with negative sign. Therefore, $\frac{1}{dx}$ will mean the sequence $\langle n \rangle$. Alternatively, we may choose

the family spanned by the sequences $\left\langle \frac{1}{2^n} \right\rangle, \left\langle \frac{1}{3^n} \right\rangle, \left\langle \frac{1}{4^n} \right\rangle$,... Then, $\frac{1}{dx}$ will mean the sequence $\left\langle 2^n \right\rangle$. Once we determined the basic infinitesimal $dx$, we will use it in the Infinite Riemann Sum that defines an Integral in Infinitesimal Calculus.

3. The Delta Function is strictly smaller than $\infty$

4. We define, $\delta(x) \equiv \frac{1}{dx} \chi_{\left[-\frac{dx}{2}, \frac{dx}{2}\right]}(x)$,

$$\text{where} \quad \chi_{\left[-\frac{dx}{2}, \frac{dx}{2}\right]}(x) = \begin{cases} 1, x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right] \\ 0, \text{otherwise} \end{cases}.$$

5. Hence,

- for $x < 0$, $\delta(x) = 0$
- at $x = -\frac{dx}{2}$, $\delta(x)$ jumps from $0$ to $\frac{1}{dx}$,
- for $x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right]$, $\delta(x) = \frac{1}{dx}$.
- at $x = 0$, $\delta(0) = \frac{1}{dx}$
- at $x = \frac{dx}{2}$, $\delta(x)$ drops from $\frac{1}{dx}$ to $0$.
- for $x > 0$, $\delta(x) = 0$.

❖ $x\delta(x) = 0$

6. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle \chi_{[-\frac{1}{2},\frac{1}{2}]}(x), 2\chi_{[-\frac{1}{4},\frac{1}{4}]}(x), 3\chi_{[-\frac{1}{6},\frac{1}{6}]}(x) ... \right\rangle$

7. If $dx = \left\langle \frac{2}{n} \right\rangle$, $\delta(x) = \left\langle \frac{1}{2\cosh^2 x}, \frac{2}{2\cosh^2 2x}, \frac{3}{2\cosh^2 3x}, ... \right\rangle$

8. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle e^{-x}\chi_{[0,\infty)}, 2e^{-2x}\chi_{[0,\infty)}, 3e^{-3x}\chi_{[0,\infty)}, ... \right\rangle$

9. $$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1.$$

10. $$\delta(\xi - x) = \frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} e^{-ik(\xi - x)} dk$$

# 5.

# Hyper-real Random Variable

A **Random Variable**

$$X(\zeta)$$

is a real-valued function that maps any event (=outcome) $\zeta$, in the Sample space $S$, into a real number $x$, in $\mathbb{R}$.

$S$ includes the non-event $\phi$, and $X(\phi) = 0$.

**Example**

A ball is drawn from a container that has $5$ Red balls, and $4$ Black balls.

The $2$ possible outcomes,

$$\zeta_1 = B, \quad \zeta_2 = R,$$

constitute the sample space,

$$S = \{\zeta_1, \zeta_2\}.$$

The number of Red balls is a Random Variable, $X(\zeta)$ with the values

$$X(\zeta_1) = X(B) = 0,$$

$$X(\zeta_2) = X(R) = 1. \square$$

## 5.1 Hyper-real $X(\zeta)$

$X(\zeta)$ is Hyper-real Random Variable iff its values may include infinitesimals, and infinite hyper-reals.

## 5.2 Hyper-real Probability Distribution of $X(\zeta)$

Let $X(\zeta)$ be Hyper-real, and define,

$$dF(x) = \Pr(x - \tfrac{1}{2}dx \le X(\zeta) \le x + \tfrac{1}{2}dx).$$

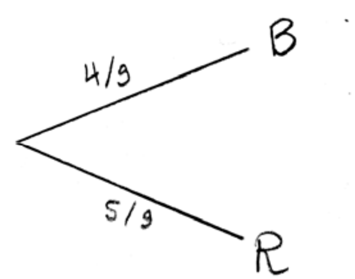
Then,

$$F(x) = \sum_{x = X(\zeta),\ \zeta \in \mathrm{S}} dF(x).$$

is a Hyper-real Probability Distribution of $X(\zeta)$

**Example**

If a ball is drawn from a container that has $5$ Red balls, and $4$ Black balls, and $X(\zeta)$ is the number of Red balls,




$$dF(0) = \Pr(X(\zeta) = 0) = \tfrac{4}{9}$$

$$dF(1) = \Pr(X(\zeta) = 1) = \tfrac{5}{9}.\square$$

## 5.3 Hyper-real Probability Density of $X(\zeta)$

Let $X(\zeta)$ be Hyper-real. If there is Hyper-real $f(x)$ so that

$$dF(x) = f(x)dx,$$

Then

$$f(x) = \frac{dF(x)}{dx}$$

is the Hyper-real Probability Density of $X(\zeta)$.

## 5.4 Expectation of Hyper-real $X(\zeta)$

$$E[X(\zeta)] \equiv \sum_{x=X(\zeta),\ \zeta\in\mathrm{S}} x dF(x),$$

*is a Hyper-real number.*

If $dF(x) = f(x)dx$,

$$E[X(\zeta)] = \sum_{x=X(\zeta),\ \zeta\in\mathrm{S}} x f(x) dx.$$

**Example**

If a ball is drawn from a container that has $5$ Red balls, and $4$ Black balls, and $X(\zeta)$ is the number of Red balls,

$$E[X(\zeta)] = \sum_{x=X(\zeta),\ \zeta\in\mathrm{S}} x dF(x)$$

$$= 0 \cdot \underbrace{dF(0)}_{4/9} + 1 \cdot \underbrace{dF(1)}_{5/9} = \tfrac{5}{9}.\square$$

## 5.5 2nd Moment of Hyper-real $X(\zeta)$

$$E[X^2(\zeta)] \equiv \sum_{x=X(\zeta),\ \zeta\in S} x^2 dF(x)$$

*is a Hyper-real number*.

**Example**

If a ball is drawn from a container that has $5$ Red balls, and $4$ Black balls, and $X(\zeta)$ is the number of Red balls,

$$E[X^2(\zeta)] = \sum_{x=X(\zeta),\ \zeta\in S} x^2 dF(x)$$

$$= 0^2 \cdot \underbrace{dF(0)}_{4/9} + 1^2 \cdot \underbrace{dF(1)}_{5/9} = \tfrac{5}{9}.\square$$

## 5.6 Variance of Hyper-real Random Variable $X(\zeta)$

$$\mathrm{Var}[X(\zeta)] \equiv E[X^2(\zeta)] - (E[X(\zeta)])^2$$

*is a Hyper-real number*.

**Example**

If a ball is drawn from a container that has $5$ Red balls, and $4$ Black balls, and $X(\zeta)$ is the number of Red balls,

$$\mathrm{Var}[X(\zeta)] = E[X^2(\zeta)] - (E[X(\zeta)])^2 = \tfrac{5}{9} - (\tfrac{5}{9})^2 = \tfrac{20}{81}.\square$$

# 6.

# Normal Distribution and Delta Function

A Normal Random Variable $N(\zeta)$, with $E[N(\zeta)] = \mu$, and $\mathrm{Var}[N(\zeta)] = \sigma^2$, has a probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

The Variance of a Hyper-real $N(\zeta)$ may be an infinitesimal, or an infinite hyper real.

## 6.1 Infinite Hyper-real Variance

$$\sigma = \frac{1}{dx} \quad \Rightarrow \quad f(x) = \text{infinitesimal}$$

*Proof*:

$$f(x) = \frac{1}{\sqrt{2\pi}}(dx)e^{-\frac{1}{2}(x-\mu)^2(dx)^2}$$

$\underline{x = \text{finite hyper-real}}$ Then,

$(x - \mu)$ is finite hyper-real,

$(x-\mu)dx$ is at most infinitesimal (it vanishes at $x=\mu$),

$\frac{1}{2}(x-\mu)^2(dx)^2$ is at most infinitesimal,

$$e^{-\frac{1}{2}(x-\mu)^2(dx)^2} \approx 1 - \underbrace{\tfrac{1}{2}(x-\mu)^2(dx)^2}_{\text{infinitesimal}} \approx 1,$$

$$f(x) \approx \tfrac{1}{\sqrt{2\pi}}(dx) = \text{infinitesimal}.$$

$\underline{x = \text{infinite hyper-real}}$ Then,

$x = \alpha\dfrac{1}{dx}$, where $\alpha$ is finite hyper-real,

$$\tfrac{1}{2}(x-\mu)^2(dx)^2 = \tfrac{1}{2}\left(\alpha\tfrac{1}{dx} - \mu\right)^2(dx)^2,$$

$$= \tfrac{1}{2}\alpha^2 - \alpha\mu(dx) + \tfrac{1}{2}\mu^2(dx)^2,$$

$$\approx \tfrac{1}{2}\alpha^2,$$

$$e^{-\frac{1}{2}(x-\mu)^2(dx)^2} \approx e^{-\frac{1}{2}\alpha^2},$$

$$f(x) \approx \tfrac{1}{\sqrt{2\pi}}(dx)e^{-\frac{1}{2}\alpha^2} = \text{infinitesimal}.\square$$

## 6.2 Infinitesimal Variance

$\sigma = dx \quad \Rightarrow \quad f(x) = \text{Delta Function}$

*Proof*: We'll show that

$$f(x) = \frac{1}{\sqrt{2\pi}dx}e^{-\frac{1}{2}\left(\frac{x-\mu}{dx}\right)^2}$$

is a Delta Function.

$\underline{x = \mu}$ Then, $e^{-\frac{1}{2}(\frac{x-\mu}{dx})^2} = e^0 = 1$,

$$f(\mu) = \frac{1}{\sqrt{2\pi}dx}.$$

That is, at $x = \mu$, the density function peaks to $\dfrac{1}{\sqrt{2\pi}dx}$.

$\underline{x \neq \mu}$ Substituting $e^{-\frac{1}{2}(\frac{x-\mu}{dx})^2} = 1 + \frac{1}{2}(\frac{x-\mu}{dx})^2 + \frac{1}{2!}\frac{1}{2^2}(\frac{x-\mu}{dx})^4 + \ldots$,

$$f(x) = \frac{1}{\sqrt{2\pi}}\frac{1}{dx}\left\{\frac{1}{1 + \frac{1}{2}(\frac{x-\mu}{dx})^2 + \frac{1}{2!}\frac{1}{2^2}(\frac{x-\mu}{dx})^4 + \ldots}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{1}{dx + \frac{1}{2}(x-\mu)^2\frac{1}{dx} + \frac{1}{2!}\frac{1}{2^2}(x-\mu)^4\frac{1}{(dx)^3} + \ldots}\right\}$$

$$\approx \frac{1}{\sqrt{2\pi}}\left\{\frac{1}{\frac{1}{2}(x-\mu)^2\frac{1}{dx} + \frac{1}{2!}\frac{1}{2^2}(x-\mu)^4\frac{1}{(dx)^3} + \ldots}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{\text{infinite hyper-real}} = \text{infinitesimal}$$

Finally, for a normal density function,

$$\int_{x=-\infty}^{x=\infty} f(x)dx = \int_{x=-\infty}^{x=\infty} \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}dx = 1. \square$$

# 7.

# Hyper-real Random Signal

A **Random Signal** (=Random Process) is a Random Variable that depends also on the time $t$:

$$X(\zeta, t).$$

Then, the outcome of a Black ball,

$$\zeta = B$$

is identified with the outcome of drawing one Black ball, and one Red ball successively,

$$BR, \text{ and } RB,$$

and with the drawing of one Black ball, and two Red balls successively,

$$BRR,\ RBR,\ RRB,$$

etc.

For a given outcome $\zeta_0$,

$$X(\zeta_0, t) = x_{\zeta_0}(t),$$

is a function of $t$, a <u>Sample Function</u>, or <u>Process Realization</u>.

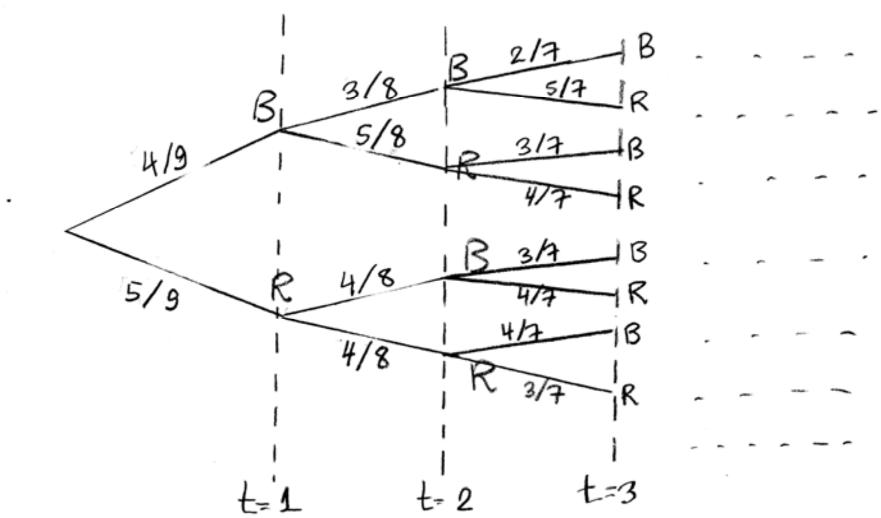
<u>**Example**</u>

At time $t = 1$, a ball is drawn from a container that has $5$ Red balls, and $4$ Black balls, and $X(\zeta, 1)$ is the number of

Red balls at $t = 1$.

At time $t = 2$, another ball is drawn from the container that now has $8$ Red, and Black balls, and $X(\zeta, 2)$ is the number of Red balls at $t = 2$.

At time $t = 3$, another ball is drawn from the container that now has $7$ Red, and Black balls, and $X(\zeta(3))$ is the number of Red balls at $t = 3$.




The outcome of no Red balls, appears once at $t = 1$, once at $t = 2$, and once at $t = 3$:

$$X(\text{no}R, 1) = X(\text{no}R, 2) = X(\text{no}R, 3) = 1.$$

The outcome of one Red ball, appears once at $t = 1$, twice at $t = 2$, and 3 times at $t = 3$,

$$X(1R, 1) = 1,$$

$$X(1R,2) = 2,$$

$$X(1R,3) = 3.$$

The outcome of two Red balls, appears once at $t = 2$, and four times at $t = 3$,

$$X(2R,1) = 0,$$

$$X(2R,2) = 1,$$

$$X(2R,3) = 4.$$

The outcome of three Red balls, appears once at $t = 3$,

$$X(3R,1) = 0,$$

$$X(3R,2) = 0,$$

$$X(3R,3) = 1.$$

The sample space of the process is

$$\{0R, 1R, 2R, 3R\}.\square$$

## 7.1 Hyper-real $X(\zeta, t)$

A Random Signal is Hyper-real <u>iff</u> the time variable $t$, and the values of $X(\zeta, t)$ may include infinitesimals, and infinite hyper-reals.

## 7.2 Hyper-real Probability Distribution of $X(\zeta, t)$

Let $X(\zeta, t)$ be Hyper-real, fix $t = t_0$, and define,

$$dF(x, t_0) = \Pr(x - \tfrac{1}{2}dx \leq X(\zeta, t_0) < x + \tfrac{1}{2}dx).$$

Then,

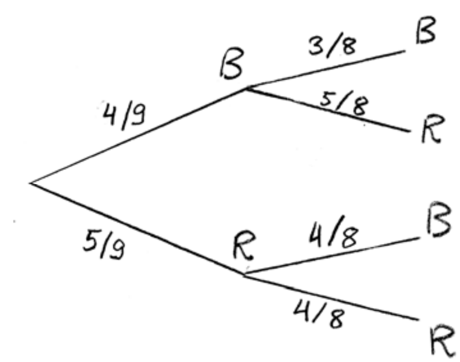
$$F(x, t_0) = \sum_{x = X(\zeta, t_0),\ \zeta \in \mathrm{S}} dF(x, t_0).$$

is a Hyper-real Probability Distribution of $X(\zeta, t_0)$.

**Example**

At time $t = 1$, a ball is drawn from a container that has $5$ Red balls, and $4$ Black balls, and $X(\zeta, 1)$ is the number of Red balls at $t = 1$.

At time $t = 2$, another ball is drawn from the container that now has $8$ Red, and Black balls, and $X(\zeta, 2)$ is the number of Red balls at $t = 2$.




$$dF(0,2) = \Pr(X(\zeta,2) = 0) = \tfrac{4}{9} \cdot \tfrac{3}{8} = \tfrac{1}{6}$$

$$dF(1,2) = \Pr(X(\zeta,2) = 1) = \tfrac{4}{9} \cdot \tfrac{5}{8} + \tfrac{5}{9} \cdot \tfrac{4}{8} = \tfrac{5}{9}$$

$$dF(2,2) = \Pr(X(\zeta,2) = 2) = \tfrac{5}{9} \cdot \tfrac{4}{8} = \tfrac{5}{18}.\square$$

### 7.3 Hyper-real Probability Density of $X(\zeta, t)$

Let $X(\zeta, t)$ be Hyper-real, and fix $t = t_0$. If there is Hyper-real $f(x, t_0)$ so that

$$dF(x, t_0) = f(x, t_0)dx,$$

Then

$$f(x, t_0) = \frac{dF(x, t_0)}{dx}$$

is the Hyper-real Probability Density of $X(\zeta, t_0)$.

### 7.4 Expectation of Hyper-real $X(\zeta, t)$

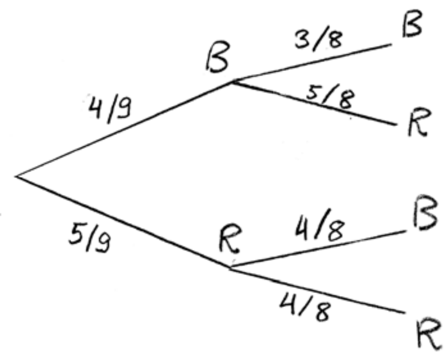
Let $X(\zeta, t)$ be Hyper-real, fix $t = t_0$, and define

$$E[X(\zeta, t_0)] \equiv \sum_{x = X(\zeta, t_0),\ \zeta \in \mathrm{S}} x dF(x, t_0),$$

If $dF(x, t_0) = f(x, t_0)dx$,

$$E[X(\zeta, t_0)] = \sum_{x = X(\zeta, t_0),\ \zeta \in \mathrm{S}} x f(x, t_0)dx.$$

**Example**




$$E[X(\zeta,2)] = \sum_{x=X(\zeta,2),\ \zeta\in \mathrm{S}} x dF(x,2)$$

$$= 0\cdot \underbrace{dF(0,2)}_{1/6} + 1\cdot \underbrace{dF(1,2)}_{5/9} + 2\cdot \underbrace{dF(2,2)}_{5/18} = \tfrac{10}{9}.\square$$

## 7.5 2nd Moment of Hyper-real $X(\zeta,t)$

$$E[X^2(\zeta,t)] \equiv \sum_{x=X(\zeta,t),\ \zeta\in \mathrm{S}} x^2 dF(x,t).$$

**Example**

$$E[X^2(\zeta,t)] = \sum_{x=X(\zeta,t),\ \zeta\in \mathrm{S}} x^2 dF(x,t)$$

$$= 0^2\cdot \underbrace{dF(0)}_{1/6} + 1^2\cdot \underbrace{dF(1)}_{5/9} + 2^2\cdot \underbrace{dF(2)}_{5/18} = \tfrac{5}{3}.\square$$

## 7.6 Variance of Hyper-real Random Variable

$$\mathrm{Var}[X(\zeta,t)] \equiv E[X^2(\zeta,t)] - (E[X(\zeta,t)])^2.$$

## Example

$$\mathrm{Var}[X(\zeta,t)] = E[X^2(\zeta,t)] - (E[X(\zeta,t)])^2$$

$$= \tfrac{9}{5} - (\tfrac{6}{5})^2 = \tfrac{9}{25}\,.\square$$

# 8.

# Continuity of $X(\zeta, t)$

**8.1** *Hyper-real* $X(\zeta, t)$ *is continuous at* $t = t_0$ *iff for any* $dt$,

$$E\{[X(\zeta, t_0 + dt) - X(\zeta, t_0)]^2\} = \text{infinitesimal},$$

$$\Leftrightarrow \quad \sum_{X(\zeta, t_0),\ \zeta \in S} [X(\zeta, t_0 + dt) - X(\zeta, t_0)]^2 dF(x, t_0) = \text{infinitesimal}$$

If $dF(x, t_0) = f(x, t_0)dx$,

$$\Leftrightarrow \quad \sum_{X(\zeta, t_0),\ \zeta \in S} [X(\zeta, t_0 + dt) - X(\zeta, t_0)]^2 f(x, t_0)dx = \text{infinitesimal}$$

**8.2** $X(\zeta, t)$ *is continuous at* $t = t_0 \Rightarrow E[X(\zeta, t_0)]$ *is continuous*

*Proof*:

$$0 \le E[\{[X(\zeta, t_0 + dt) - X(\zeta, t_0)] - E[X(\zeta, t_0 + dt) - X(\zeta, t_0)]\}^2]$$

$$= E\{[X(\zeta, t_0 + dt) - X(\zeta, t_0)]^2\}$$

$$-2E\{[X(\zeta, t_0 + dt) - X(\zeta, t_0)]E[X(\zeta, t_0 + dt) - X(\zeta, t_0)]\}$$

$$+\{E[X(\zeta, t_0 + dt) - X(\zeta, t_0)]\}^2$$

$$= E\{[X(\zeta, t_0 + dt) - X(\zeta, t_0)]^2\} - \{E[X(\zeta, t_0 + dt) - X(\zeta, t_0)]\}^2$$

Therefore,

$$\underbrace{\{E[X(\zeta,t_0+dt)-X(\zeta,t_0)]\}^2}_{\geq 0} \leq \underbrace{E\{[X(\zeta,t_0+dt)-X(\zeta,t_0)]^2\}}_{\text{ifinitesimal}}$$

Hence,

$$\{E[X(\zeta,t_0+dt)-X(\zeta,t_0)]\}^2 = \text{infinitesimal},$$

$$E[X(\zeta,t_0+dt)-X(\zeta,t_0)] = \text{infinitesimal}.$$

# 9.

# Derivative of $X(\zeta, t)$

**9.1** ***Hyper-real*** $X(\zeta, t)$ ***has derivative with respect to*** $t$ ***at*** $t = t_0$ ***<u>iff</u> there is a Random Signal*** $X'(\zeta, t) = \partial_t X(\zeta, t)$, ***so that for any*** $dt$,

$$E\left[\left[\frac{X(\zeta, t_0 + dt) - X(\zeta, t_0)}{dt} - X'(\zeta, t_0)\right]^2\right] = \text{infinitesimal},$$

$$\Leftrightarrow \quad \sum_{x = X(\zeta, t_0),\ \zeta \in S}\left[\frac{x(\zeta, t_0 + dt) - x(\zeta, t_0)}{dt} - x'(\zeta, t_0)\right]^2 dF(x, t_0) = \text{infinitesimal}$$

If $dF(x, t_0) = f(x, t_0)dx$,

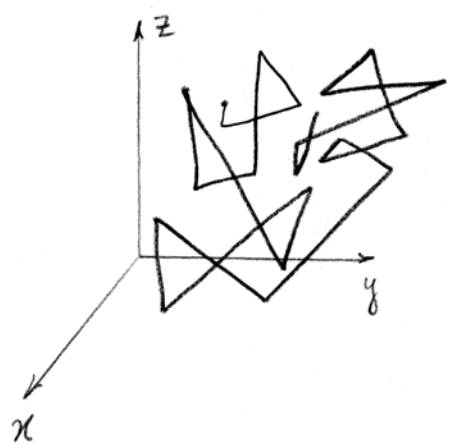
$$\Leftrightarrow \quad \sum_{x = X(\zeta, t_0),\ \zeta \in S}\left[\frac{x(\zeta, t_0 + dt) - x(\zeta, t_0)}{dt} - x'(\zeta, t_0)\right]^2 f(x, t_0)dx = \text{infinitesimal}$$

# 10.

# Random Walk

The Random Walk of small particles in fluid is named after Brown, who first observed it, Brownian Motion. It models other processes, such as the fluctuations of a stock price.

In a volume of fluid, the path of a particle is in any direction in the volume, and of variable size




## 10.1 The Bernoulli Random Variables of the Walk

We restrict the Walk here to the line, in uniform infinitesimal size steps $dx$:

To the left, with probability

$$p = \tfrac{1}{2},$$

or to the right, with probability

$$q = \tfrac{1}{2}.$$

At fixed time $t$, after

$N$ infinitesimal time intervals $dt$,

$N = \frac{t}{dt}$, is a fixed infinite hyper-real,

the particle have made

$K$ infinitesimal steps of size $dx$ to the right,

and

$L$ infinitesimal steps of size $dx$ to the left,

and is at the point

$$x = (\underbrace{K - L}_{M})dx = Mdx.$$

$K, L, M$, are infinite hyper-reals.

At the $i$th step we define the Bernoulli Random Variable,

$$B_i(\text{right step}) = dx, \quad \zeta_1 = \text{right step}.$$

$$B_i(\text{left step}) = -dx, \quad \zeta_2 = \text{left step}.$$

where $i = 1, 2, ..., N$.

$$\Pr(B_i = dx) = p = \tfrac{1}{2},$$

$$\Pr(B_i = -dx) = q = \tfrac{1}{2},$$

$$E[B_i] = dx \cdot \tfrac{1}{2} + (-dx) \cdot \tfrac{1}{2} = 0,$$

$$E[B_i^{\,2}] = (dx)^2 \cdot \tfrac{1}{2} + (-dx)^2 \cdot \tfrac{1}{2} = (dx)^2$$

$$\mathrm{Var}[B_i] = \underbrace{E[B_i^{\,2}]}_{(dx)^2} - (\underbrace{E[B_i]}_{0})^2 = (dx)^2.$$

## 10.2 The Binomial Distribution of the Walk

$$B(\zeta, t) = B_1 + B_2 + ... + B_N$$

*is a Random Process with*

$$E[B(\zeta, t)] = 0,$$

$$\mathrm{Var}[B(\zeta, t)] = N(dx)^2,$$

*distributed Binomially*

$$\Pr\left(x - \tfrac{1}{2}dx \le B(\zeta, t) \le x + \tfrac{1}{2}dx\right) = \begin{pmatrix} N \\ \frac{M+N}{2} \end{pmatrix} \frac{1}{2^N}$$

<u>*Proof*</u>: Since the $B_i$ are independent,

$$E[B(\zeta, t)] = \underbrace{E[B_1]}_{0} + ... + \underbrace{E[B_N]}_{0} = 0$$

$$\mathrm{Var}[B(\zeta, t)] = \underbrace{\mathrm{Var}[B_1]}_{(dx)^2} + ... + \underbrace{\mathrm{Var}[B_N]}_{(dx)^2} = N(dx)^2$$

$B(\zeta, t)$ has a Binomial distribution,

$$\Pr\left(x - \tfrac{1}{2}dx \le X(\zeta, t) \le x + \tfrac{1}{2}dx\right) = \begin{pmatrix} N \\ K \end{pmatrix} p^K q^{N-K},$$

$$= \begin{pmatrix} N \\ K \end{pmatrix} \left(\tfrac{1}{2}\right)^{K} \left(\tfrac{1}{2}\right)^{N-K},$$

$$= \begin{pmatrix} N \\ K \end{pmatrix} \tfrac{1}{2^N}.$$

From

$$N = K + L,$$

$$M = K - L,$$

we have

$$K = \tfrac{N+M}{2},$$

$$L = \tfrac{N-M}{2}.$$

Thus,

$$\Pr\left(x - \tfrac{1}{2}dx \le B(\zeta,t) \le x + \tfrac{1}{2}dx\right) = \begin{pmatrix} N \\ \frac{M+N}{2} \end{pmatrix} \tfrac{1}{2^N}. \square$$

## 10.3 The Gaussian Distribution of the Walk

*__If__* $(dx)^2 = 2D(dt)$, *where the __Drift Coefficient__* $D$ *is a constant*

*__Then__, the Binomial distribution of* $B(\zeta,t)$ *is infinitesimally close to a Gaussian distribution of a Random Signal with*

$$\mu = 0,$$

$$\sigma = \sqrt{t2D} = \sqrt{N}dx.$$

$$f(x,t) \approx \frac{1}{\sqrt{2\pi}\sqrt{t2D}} e^{-\frac{1}{2}\frac{x^2}{t2D}}$$

*Proof*:

$$\underbrace{\Pr\Big(x - \tfrac{1}{2}dx \le X(\zeta,t) \le x + \tfrac{1}{2}dx\Big)}_{dF(x,t)} = \frac{N!}{\Big(\frac{N+M}{2}\Big)!\Big(\frac{N-M}{2}\Big)!}\frac{1}{2^N}.$$

Substituting $N! \approx \sqrt{2\pi N}N^N e^{-N}$ from Sterling's Formula for infinite hyper-real $N$,

$$\approx \frac{\sqrt{2\pi N}N^N e^{-N}}{\sqrt{2\pi\frac{N+M}{2}}(\frac{N+M}{2})^{\frac{N+M}{2}}e^{-\frac{N+M}{2}}\sqrt{2\pi\frac{N-M}{2}}(\frac{N-M}{2})^{\frac{N-M}{2}}e^{-\frac{N-M}{2}}}\frac{1}{2^N},$$

$$= \sqrt{\frac{2}{\pi}}\frac{N^{N+\frac{1}{2}}}{N^{\frac{N+M+1}{2}}(1+\frac{M}{N})^{\frac{N+M+1}{2}}N^{\frac{N-M+1}{2}}(1-\frac{M}{N})^{\frac{N-M+1}{2}}},$$

$$= \sqrt{\frac{2}{\pi N}}\frac{1}{(1+\frac{M}{N})^{\frac{N+M+1}{2}}(1-\frac{M}{N})^{\frac{N-M+1}{2}}}.$$

Then, up to an infinitesimal,

$$\log\Big[dF(x,t)\Big] \approx \log\sqrt{\tfrac{2}{\pi N}} - \tfrac{N+M+1}{2}\log(1+\tfrac{M}{N}) - \tfrac{N-M+1}{2}\log(1-\tfrac{M}{N})$$

Since $0 < \frac{M}{N} < 1$,

$$\log(1+\tfrac{M}{N}) \approx \tfrac{M}{N} - \tfrac{1}{2}\tfrac{M^2}{N^2},$$

$$\log(1-\tfrac{M}{N}) \approx -\tfrac{M}{N} - \tfrac{1}{2}\tfrac{M^2}{N^2},$$

$$\log\Big[dF(x,t)\Big] \approx \log\sqrt{\tfrac{2}{\pi N}}$$

$$-\tfrac{N+M+1}{2}\tfrac{M}{N} + \tfrac{N+M+1}{4}\tfrac{M^2}{N^2}$$

$$+\tfrac{N-M+1}{2}\tfrac{M}{N}+\tfrac{N-M+1}{4}\tfrac{M^2}{N^2}$$

$$=\log\sqrt{\tfrac{2}{\pi N}}$$

$$-\tfrac{M}{2}-\tfrac{M^2}{2N}-\tfrac{M}{2N}+\tfrac{M^2}{4N}+\tfrac{M^3}{4N^2}+\tfrac{M^2}{4N^2}$$

$$+\tfrac{M}{2}-\tfrac{M^2}{2N}+\tfrac{M}{2N}+\tfrac{M^2}{4N}-\tfrac{M^3}{4N^2}+\tfrac{M^2}{4N^2}$$

$$=\log\sqrt{\tfrac{2}{\pi N}}-\tfrac{M^2}{2N}+\tfrac{M^2}{2N^2}$$

$$=\log\sqrt{\tfrac{2}{\pi N}}-\tfrac{M^2}{2N}\underbrace{(1-\tfrac{1}{N})}_{\approx 1}$$

$$=\log\tfrac{1}{\sqrt{N}}2\tfrac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\frac{M^2}{N}}.$$

This would give $dF(x,t)\approx 2\dfrac{1}{\sqrt{2\pi}\sqrt{N}}e^{-\frac{1}{2}\frac{M^2}{N}}$, but accounting for negative $M$, and $x$, we have

$$dF(x,t)\approx\frac{1}{\sqrt{2\pi}\sqrt{N}}e^{-\frac{1}{2}\frac{M^2}{N}}$$

$$=\frac{1}{\sqrt{2\pi}\sqrt{\frac{t}{dt}}}e^{-\frac{1}{2}\frac{\frac{x^2}{(dx)^2}}{\frac{t}{dt}}}$$

$$=\frac{\sqrt{dt}}{\sqrt{2\pi}\sqrt{t}}e^{-\frac{1}{2}\frac{dt}{(dx)^2}\frac{x^2}{t}}$$

Thus, we <u>need to assume</u> that $(dx)^2$, and $dt$ are proportional,

$$(dx)^2 = 2D(dt),$$

where the Drift Coefficient $D$ is a constant of the Walk.

Then,

$$dF(x,t) \approx \frac{1}{\sqrt{2\pi}\sqrt{t2D}} e^{-\frac{1}{2}\frac{x^2}{t2D}} dx.$$

Hence, the probability density of the Walk is

$$f(x,t) = \frac{dF(x,t)}{dx} \approx \frac{1}{\sqrt{2\pi}\sqrt{t2D}} e^{-\frac{1}{2}\frac{x^2}{t2D}},$$

with

$$\mu = 0,$$

$$\sigma = \sqrt{2tD} = \sqrt{N}dx.\square$$

**10.4** $f(x,t)$ *solves the parabolic wave equation* $\partial_t f = D\partial_x^2 f$.

<u>*Proof*</u>: By substitution.□

## 10.5 Increments of Random Walk

<u>*If*</u> $(dx)^2 = 2D(dt)$,

<u>*Then*</u>

**1)** *For any* $\tau > 0$, *the distribution of* $B(\zeta, t+\tau) - B(\zeta, t)$ *is infinitesimally close to a Gaussian distribution that has*

$$\mu = 0,$$

$$\sigma^2 = \tau 2D,$$

*and depends only on* $\tau$ *(Stationary Process).*

**2)** *For fixed* $t$*, and any* $dt$*, the Random Variables*

$$B(\zeta,t) - B(\zeta,t-dt),$$

$$B(\zeta,t-dt) - B(\zeta,t-2dt),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$B(\zeta,dt) - B(\zeta,0),$$

*are independent, random variables.*

<u>*Proof*</u>:

1) Let $T = \frac{\tau}{dt}$. Then, as in 10.2, the Binomial distribution of

$$B(\zeta,t+\tau) - B(\zeta,t) = B_{N+1} + B_{N+2} + \ldots + B_{N+T},$$

is infinitesimally close to a Gaussian distribution with $\mu = 0$, and $\sigma^2 = \tau 2D$, that depends only on $\tau$.

2) $$B(\zeta,t) - B(\zeta,t-dt)$$

is precisely one Bernoulli Random Variable that is statistically independent of the precisely one Bernoulli Random Variable that equals

$$B(\zeta,t-dt) - B(\zeta,t-2dt)$$

# 11.

# Random Walk is Continuous, has a Derivative with Delta Function Variance, and $E[B(\zeta,t)]$ has unbounded Variation

**11.1** $(dx)^2 = (2D)dt \Rightarrow$ **Random Walk is Continuous**

*Proof*:

$E[\{B(\zeta,t+dt)-B(\zeta,t)\}^2] =$

$$= \mathrm{Var}[\underbrace{B(\zeta,t+dt)-B(\zeta,t)}_{B_i}] + (E[\underbrace{B(\zeta,t+dt)-B(\zeta,t)}_{B_i}])^2,$$

where $B_i$ is a Bernoulli Random Variable,

$$= \underbrace{\mathrm{Var}[B_i]}_{(dx)^2=(2D)dt} + (\underbrace{E[B_i]}_{0})^2 = (2D)dt\,.\square$$

**11.2** *If* $(dx)^2 = (2D)dt$

*<u>Then</u> The Derivative of Random Walk is*

$$\dot{B} = \frac{1}{dt} B_i ,$$

*where* (1) $B_i = B(\zeta, t_0 + dt) - B(\zeta, t_0)$*, is a Bernoulli Random Variable.*

(2) $E[\dot{B}] = 0$,

(3) $\mathrm{Var}[\dot{B}] = 2D\delta(t_0)$,

*<u>Proof</u>*:

(1) For each $t = t_0$, we need to find a Random Signal $\dot{B}(\zeta, t_0)$, so that for any $dt$,

$$E\left[\left[\frac{B(\zeta, t_0 + dt) - B(\zeta, t_0)}{dt} - \dot{B}(\zeta, t_0)\right]^2\right] = \text{infinitesimal},$$

Since $B(\zeta, t_0 + dt) - B(\zeta, t_0)$, is a Bernoulli Random Variable $B_i$,

$$E\left[\left\{\frac{X(B, t + dt) - B(\zeta, t)}{dt} - \dot{B}(\zeta, t)\right\}^2\right] = E\left[\left\{\frac{B_i}{dt} - \dot{B}\right\}^2\right]$$

Therefore, at time $t = t_0$, the Random Variable

$$\frac{1}{dt} B_i ,$$

is the derivative of the Random Walk $B(\zeta, t_0)$.□

**(2)** $$E[\dot{B}] = \frac{1}{dt}\underbrace{E[B_i]}_{0} = 0.\square$$

**(3)** $$\mathrm{Var}[\dot{B}] = E[\dot{B}^2] - (\underbrace{E[\dot{B}]}_{0})^2$$

$$= \frac{1}{(dt)^2}\underbrace{E[B_i^{\,2}]}_{(dx)^2},$$

$$= \underbrace{\frac{(dx)^2}{dt}}_{2D}\frac{1}{dt}$$

$$= (2D)\delta(t_0).\square$$

## **11.3** $E[B(\zeta,t)]$ *has unbounded Variation in* $[a,b]$

*Proof*:

$$2D(b-a) = \underbrace{(2D)dt}_{(dx)^2} + \underbrace{(2D)dt}_{(dx)^2} + ... + \underbrace{(2D)dt}_{(dx)^2}$$

$$= E\left[\left\{B(\zeta,b) - B(\zeta,b-dt)\right\}^2\right] + .. + E\left[\left\{B(\zeta,a+dt) - B(\zeta,a)\right\}^2\right]$$

$$\leq \underbrace{\max_{a\leq t\leq b}\left|B(\zeta,t+dt) - B(\zeta,t)\right|}_{\text{infinitesimal}} E\left[\left|B(\zeta,b) - B(\zeta,b-dt)\right|\right] + ..$$

$$... + \max_{a\leq t\leq b}\left|B(\zeta,t+dt) - B(\zeta,t)\right| E\left[\left|B(\zeta,a+dt) - B(\zeta,a)\right|\right] =$$

$$= \text{infinitesimal}\{E\left|B(\zeta,b) - B(\zeta,b-dt)\right| + ... + E\left|B(\zeta,a+dt) - B(\zeta,a)\right|\},$$

$$=\text{infinitesimal}\left\{E\left[\left|B(\zeta,b)-B(\zeta,b-dt)\right|+...+\left|B(\zeta,a+dt)-B(\zeta,a)\right|\right]\right\},$$

since the Bernoulli Random Variables are independent.

Therefore,

$$E\left[\left|B(\zeta,b)-B(\zeta,b-dt)\right|+...+\left|B(\zeta,a+dt)-B(\zeta,a)\right|\right]=\frac{(2D)(b-a)}{\text{infinitesimal}},$$

is infinite hyper-real, and $E[B(\zeta,t)]$ has unbounded variation in $[a,b]$.$\square$

# 12.

$$\int_{t=a}^{t=b} f(t)dB(\zeta,t)$$

While $E[B(\zeta,t)]$ has unbounded Variation in $[a,b]$, integration with respect to $B(\zeta,t)$ is possible.

Let $f(t)$ be a hyper-real function on the bounded time interval $[a,b]$. $f(t)$ need not be bounded.

At each $a \le t \le b$, there is a Bernoulli Random Variable

$$dB(\zeta,t) = B(\zeta,t+dt) - B(\zeta,t) = B_i(\zeta,t) = \dot{B}(\zeta,t)dt.$$

We form the **Integration Sum**

$$\sum_{t=a}^{t=b} f(t)dB(\zeta,t) = \sum_{t=a}^{t=b} f(t)B_i(\zeta,t) = \sum_{t=a}^{t=b} f(t)\dot{B}(\zeta,t)dt$$

For any $dt$,

(1) the <u>First Moment</u> of the Integration Sum is

$$E\left[\sum_{t=a}^{t=b} f(t)\dot{B}(\zeta,t)dt\right] = \sum_{t=a}^{t=b} f(t)\underbrace{E[\dot{B}(\zeta,t)]}_{0}dt = 0.$$

(2) the Second Moment of the Integration sum is

$$E\left[\left(\sum_{t=a}^{t=b} f(t)B_i(\zeta,t)\right)^2\right] = E\left[\left(\sum_{t=a}^{t=b} f(t)B_i(\zeta,t)\right)\left(\sum_{\tau=a}^{\tau=b} f(\tau)B_j(\zeta,\tau)\right)\right]$$

$$= \sum_{t=a}^{t=b}\sum_{\tau=a}^{\tau=b} f(t)f(\tau)E[B_j(\zeta,\tau)B_i(\zeta,t)]$$

Since the Bernoulli Random Variables are independent,

$$E[B_j(\zeta,\tau)B_i(\zeta,t)] = E[B_i^2(\zeta,t)] = (dx)^2$$

only for $t = \tau$. Then,

$$E\left[\left(\sum_{t=a}^{t=b} f(t)B_i(\zeta,t)\right)^2\right] = \sum_{t=a}^{t=b} f^2(t)\underbrace{(dx)^2}_{(2D)dt},$$

$$= 2D\sum_{t=a}^{t=b} f^2(t)dt,$$

$$= 2D\int_{t=a}^{t=b} f^2(t)dt.$$

assuming $(dx)^2 = (2D)dt$, and $f(t)$ integrable

Thus, for any $dt$, the Integration Sum is a unique well-defined hyper-real Random Variable $I(\zeta)$.

We call $I(\zeta)$ the integral of $f(t)$, with respect to $B(\zeta,t)$ from $x = a$, to $x = b$, and denote it by $\int_{t=a}^{t=b} f(t)dB(\zeta,t)$.

# 13.

# Poisson Process

The arrival at rate $\lambda$, of radioactive particles at a counter is modeled by the Poisson Process. It models other processes, such as the arrival of phone calls at rate $\lambda$, to an operator.

## 13.1 The Bernoulli Random Variables of the Process

We assume that

*an arrival probability in time* $dt$ *is*

$$p = \lambda dt,$$

*and no arrival probability in time* $dt$ *is*

$$q = 1 - \lambda dt.$$

At fixed time $t$, after

$N$ infinitesimal time intervals $dt$,

$N = \frac{t}{dt}$, is an infinite hyper-real,

there are

$k$ arrivals,

$k$ is a finite hyper-real

and

$N - k$ no arrivals,

$$N - k \text{ is an infinite Hyper-real}$$

At the $i$th step we define the Bernoulli Random Variable,

$$P_i(\text{arrival}) = 1, \qquad \zeta_1 = \text{arrival}$$

$$P_i(\text{no-arrival}) = 0, \qquad \zeta_2 = \text{no-arrival}$$

where $i = 1, 2, ..., N$.

$$\Pr(P_i = 1) = p = \lambda dt,$$

$$\Pr(P_i = 0) = q = 1 - \lambda dt,$$

$$E[P_i] = 1 \cdot \lambda dt + 0 \cdot (1 - \lambda dt) = \lambda dt,$$

$$E[P_i^2] = 1^2 \cdot \lambda dt + 0^2 \cdot (1 - \lambda dt) = \lambda dt,$$

$$\mathrm{Var}[P_i] = \underbrace{E[P_i^2]}_{\lambda dt} - (\underbrace{E[P_i]}_{\lambda dt})^2,$$

$$= \lambda dt \underbrace{(1 - \lambda dt)}_{\approx 1} \approx \lambda dt.$$

## 13.2 The Binomial Distribution of the Process

$$P(\zeta, t) = P_1 + P_2 + ... + P_N$$

*is a Random Process with*

$$E[P(\zeta, t)] = \lambda t,$$

$$\mathrm{Var}[P(\zeta, t)] = \lambda t,$$

*distributed Binomially*

$$\Pr\left(P(\zeta,t)=k\right)=\begin{pmatrix}N\\k\end{pmatrix}\left(\lambda dt\right)^{k}\left(1-\lambda dt\right)^{N-k}$$

***Proof***: Since the $P_i$ are independent,

$$E[P(\zeta,t)]=\underbrace{E[P_1]}_{\lambda dt}+...+\underbrace{E[P_N]}_{\lambda dt}=\lambda\underbrace{Ndt}_{t}$$

$$\mathrm{Var}[P(\zeta,t)]=\underbrace{\mathrm{Var}[P_1]}_{\approx\lambda dt}+...+\underbrace{\mathrm{Var}[P_N]}_{\approx\lambda dt}\approx\lambda\underbrace{Ndt}_{t}$$

$P(\zeta,t)$ has a Binomial distribution,

$$\begin{aligned}\Pr\left(P(\zeta,t)=k\right)&=\begin{pmatrix}N\\k\end{pmatrix}p^k q^{N-k},\\&=\begin{pmatrix}N\\k\end{pmatrix}\left(\lambda dt\right)^{k}\left(1-\lambda dt\right)^{N-k}.\end{aligned}$$

## 13.3 The Poisson Distribution of the Process

*The Binomial distribution of $P(\zeta,t)$ is infinitesimally close to a Poisson distribution of a Random Signal with*

$$\mu=\lambda t,$$

$$\sigma^2=\lambda t.$$

$$\Pr[P(\zeta,t)=k]\approx\frac{1}{k!}(\lambda t)^k e^{-\lambda t}$$

***Proof***:

$$\Pr\Big(P(\zeta,t)=k\Big)=\frac{N!}{k!(N-k)!}\Big(\lambda dt\Big)^{k}\Big(1-\lambda dt\Big)^{N-k}.$$

Substituting $N!\approx\sqrt{2\pi}N^{N+\frac{1}{2}}e^{-N}$ from Sterling's Formula for infinite hyper-real $N$,

$$\approx\frac{\sqrt{2\pi}N^{N+\frac{1}{2}}e^{-N}}{k!\sqrt{2\pi}(N-k)^{N-k+\frac{1}{2}}e^{-N+k}}\Big(\lambda dt\Big)^{k}\Big(1-\lambda dt\Big)^{N-k},$$

$$=\frac{1}{k!}\frac{N^{N+\frac{1}{2}}}{N^{N+\frac{1}{2}-k}(1-\frac{k}{N})^{N-k+\frac{1}{2}}e^{k}}\Big(\lambda\tfrac{t}{N}\Big)^{k}\Big(1-\lambda\tfrac{t}{N}\Big)^{N-k},$$

$$=\frac{1}{k!}\Big(\lambda t\Big)^{k}\underbrace{\frac{1}{(1-\frac{k}{N})^{N}}}_{\approx e^{-k}}e^{-k}\underbrace{\underbrace{(1-\tfrac{k}{N})}_{\approx 1}{}^{k-\frac{1}{2}}}_{\approx 1}\underbrace{\Big(1-\tfrac{\lambda t}{N}\Big)^{N}}_{\approx e^{-\lambda t}}\underbrace{\frac{1}{\underbrace{\Big(1-\frac{\lambda t}{N}\Big)^{k}}_{\approx 1}}}_{\approx 1},$$

since $N$ is an infinite Hyper-real.□

## 13.4 Increment of Poisson Process

**1)** *For any* $\tau>0$, *the distribution of* $P(\zeta,t+\tau)-P(\zeta,t)$ *is infinitesimally close to a Poisson distribution of a Random Signal with*

$$\mu=\lambda\tau,$$

$$\sigma^{2}=\lambda\tau.$$

$$\Pr[P(\zeta,t+\tau)-P(\zeta,t)=k]\approx\frac{1}{k!}(\lambda\tau)^{k}e^{-\lambda\tau}$$

*that depends only on* $\tau$ *(Stationary Process).*

**2)** *For fixed* $t$*, and any* $dt$*, the Random Variables*

$$P(\zeta,t)-P(\zeta,t-dt),$$

$$P(\zeta,t-dt)-P(\zeta,t-2dt),$$

$$\dots\dots\dots\dots\dots\dots,$$

$$P(\zeta,dt)-P(\zeta,0),$$

*are independent, random variables.*

Proof:

1) Let $T=\frac{\tau}{dt}$. Then, as in 12.2, the Binomial distribution of

$$P(\zeta,t+\tau)-P(\zeta,t)=P_{N+1}+P_{N+2}+...+P_{N+T},$$

is infinitesimally close to a Poisson distribution with $\mu=\lambda\tau$, and $\sigma^2=\lambda\tau$, that depends only on $\tau$.□

2) $$P(\zeta,t)-P(\zeta,t-dt)$$

is precisely one Bernoulli Random Variable that is statistically independent of the precisely one Bernoulli Random Variable that equals $P(\zeta,t-dt)-P(\zeta,t-2dt)$.□

# 14.

# Poisson Process is Continuous and has a Derivative with Delta Function Variance

## 14.1 Poisson Process is Continuous

*Proof*:

$$E[\{P(\zeta,t+dt)-P(\zeta,t)\}^2]=$$

$$=\mathrm{Var}[\underbrace{P(\zeta,t+dt)-P(\zeta,t)}_{P_i}]+(E[\underbrace{P(\zeta,t+dt)-P(\zeta,t)}_{P_i}])^2,$$

where $X_i$ is a Bernoulli Random Variable,

$$=\underbrace{\mathrm{Var}[P_i]}_{\approx\lambda dt}+(\underbrace{E[P_i]}_{\lambda dt})^2=\text{infinitesimal}.\square$$

## 14.2 The Derivative of the Poisson process is

$$\dot{P}=\frac{1}{dt}P_i,$$

*where* (1) $P_i=P(\zeta,t_0+dt)-P(\zeta,t_0)$, *is a Bernoulli*

*Random Variable.*

(2) $E[\dot{P}] = \lambda$,

(3) $\mathrm{Var}[\dot{P}] = \lambda\delta(t_0)$

<u>*Proof*</u>:

(1) For each $t = t_0$, we need to find a Random Signal $\dot{P}(\zeta, t_0)$, so that for any $dt$,

$$E\left[\left[\frac{P(\zeta, t_0 + dt) - P(\zeta, t_0)}{dt} - \dot{P}(\zeta, t_0)\right]^2\right] = \text{infinitesimal},$$

Since $P(\zeta, t_0 + dt) - P(\zeta, t_0)$, is a Bernoulli Random Variable $P_i$,

$$E\left[\left\{\frac{P(\zeta, t + dt) - P(\zeta, t)}{dt} - \dot{P}(\zeta, t)\right\}^2\right] = E\left[\left\{\frac{P_i}{dt} - \dot{P}\right\}^2\right]$$

Therefore, at time $t = t_0$, the Random Variable

$$\frac{1}{dt}P_i,$$

is the derivative of the Random Walk $P(\zeta, t_0)$.□

(2) $E[\dot{P}] = \frac{1}{dt}\underbrace{E[P_i]}_{\lambda dt} = \lambda$.□

(3) $\mathrm{Var}[\dot{P}] = E[\dot{P}^2] - (\underbrace{E[\dot{P}]}_{\lambda})^2$

$$= \frac{1}{(dt)^2} \underbrace{E[P_i^2]}_{\lambda dt + \lambda^2 (dt)^2} - \lambda^2$$

$$= \lambda \frac{1}{dt}$$

$$= \lambda \delta(t_0),$$

By [Dan4].□

# 15.

$$\int_{t=a}^{t=b} f(t)dP(\zeta,t)$$

Let $f(t)$ be a hyper-real function on the bounded time interval $[a,b]$. $f(t)$ need not be bounded.

At each $a \le t \le b$, there is a Bernoulli Random Variable

$$dP(\zeta,t) = P(\zeta,t+dt) - P(\zeta,t) = P_i(\zeta,t) = \dot{P}(\zeta,t)dt\,.$$

We form the **Integration Sum**

$$\sum_{t=a}^{t=b} f(t)dP(\zeta,t) = \sum_{t=a}^{t=b} f(t)P_i(\zeta,t) = \sum_{t=a}^{t=b} f(t)\dot{P}(\zeta,t)dt$$

For any $dt$,

(1) the First Moment of the Integration Sum is

$$E\left[\sum_{t=a}^{t=b} f(t)\dot{P}(\zeta,t)dt\right] = \sum_{t=a}^{t=b} f(t)\underbrace{E[\dot{P}(\zeta,t)]}_{\lambda}dt = \lambda\int_{t=a}^{t=b} f(t)dt\,,$$

assuming $f(t)$ integrable.

(2) the Second Moment of the Integration sum is

$$E\left[\left(\sum_{t=a}^{t=b} f(t)P_i(\zeta,t)\right)^2\right] = E\left[\left(\sum_{t=a}^{t=b} f(t)P_i(\zeta,t)\right)\left(\sum_{\tau=a}^{\tau=b} f(\tau)P_j(\zeta,\tau)\right)\right]$$

$$= \sum_{t=a}^{t=b}\sum_{\tau=a}^{\tau=b} f(t)f(\tau)E[P_j(\zeta,\tau)P_i(\zeta,t)]$$

Since the Bernoulli Random Variables are independent,

$$E[P_j(\zeta,\tau)P_i(\zeta,t)] = E[P_i^2(\zeta,t)] = \lambda dt(\underbrace{1+\lambda dt}_{\approx 1})$$

only for $t = \tau$. Then,

$$E\left[\left(\sum_{t=a}^{t=b} f(t)P_i(\zeta,t)\right)^2\right] = \lambda\sum_{t=a}^{t=b} f^2(t)dt\,,$$

$$= \lambda\int_{t=a}^{t=b} f^2(t)dt\,,$$

assuming $f(t)$ integrable.

Thus, assuming $f(t)$ integrable, for any $dt$, the Integration Sum is a unique well-defined hyper-real Random Variable $I(\zeta)$. We call $I(\zeta)$ the integral of $f(t)$, with respect to $P(\zeta,t)$ from $x = a$, to $x = b$, and denote it by

$$\int_{t=a}^{t=b} f(t)dP(\zeta,t)\,.$$

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