Evolution Equations of Random Walk, and Poisson Processes

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March, 2013

Abstract A Random Differential Equation is the evolution equation of a Random Process $X(\zeta, t)$, driven by $\dot{B}(\zeta, t)$, where $B(\zeta, t)$ is Random Walk, or by $\dot{P}(\zeta, t)$, where $P(\zeta, t)$ is a Poisson Process.

For instance, the Langevin equation for the linear oscillator is a first order evolution equation for the linear oscillator process, driven by $\dot{B}(\zeta, t)$. To date, due to the confusion surrounding Random Differential Equations, only that equation was integrated and given a Random Process solution.

Integrating the probability-wave equation associated with the random process $X(\zeta, t)$ is equivalent to integrating the Random Differential Equation for the time-evolution of the

Process $X(\zeta, t)$.

But the evolution equation solution is superior to the setup, and solution of the probability-wave equation:

While the probability-wave equation setup is instructive, [Dan6], it involves Conditional Probabilities that for higher order equations are beyond comprehension, reminding of the Monty Hall Problem [Rosenhouse].

Consequently, it will be difficult to detect an error in the setup of the probability-wave equation.

Furthermore, the probability-wave equation is a partial differential equation, and its solution is more difficult than the solution of the evolution equation which is an ordinary differential equation.

Only the Wiener integral is necessary to integrate the Langevin equation. But instead of applying it, textbooks keep busy with the ill-defined Ito Integral that attempted to generalize the Wiener Integral.

We solve the second order Langevin equation for the harmonic oscillator, driven by $\dot{B}(\zeta, t)$.

To date, all random differential equations were assumed to be driven by $\dot{B}(\zeta, t)$. There is no reason to believe that none are driven by Poisson Processes.

Thus, we develop here the theory of Random differential equations driven by Poisson Processes.

Keywords: Ito Integral, Ito Process, Ito Formula, Stochastic Integration, Infinitesimal, Infinite-Hyper-real, Hyper-real, Calculus, Limit, Continuity, Derivative, Integral, Delta Function, Random Variable, Random Process, Random Signal, Stochastic Process, Stochastic Calculus, Probability Distribution, Bernoulli Random Variables, Binomial Distribution, Gaussian, Normal, Expectation, Variance, Random Walk, Poisson Process, Random Differential Equations, Stochastic Differential Equations

2000 Mathematics Subject Classification 26E35; 26E30; 26E15; 26E20; 26A06; 26A12; 03E10; 03E55; 03E17; 03H15; 46S20; 97I40; 97I30.

Contents

Introduction

0.1 The Evolution of Random Processes

A Random Differential Equation is the evolution equation of a Random Process $X(\zeta,t)$, driven by $\dot{B}(\zeta,t)$, where $B(\zeta,t)$ is Random Walk, or by $\dot{P}(\zeta,t)$, where $P(\zeta,t)$ is a Poisson Process.

For instance, the Langevin equation for the linear oscillator is a first order evolution equation for the linear oscillator process, driven by $\dot{B}(\zeta,t)$. To date, due to the confusion surrounding Random Differential Equations, only that equation was integrated and given a Random Process solution.

We solve here the second order Langevin equation for the harmonic oscillator, driven by Random walk. Our method of Variation of parameters may be applied to Langevin equation of any order.

0.2 Probability-waves associated with $X(\zeta,t)$, and the time-evolution of $X(\zeta,t)$

Random Processes were described by the probability-wave equations associated with them: The Diffusion Equation for Random Walk, and the differential-difference equation for the Poisson Process. [Dan5].

Integrating the probability-wave equation associated with the random process $X(\zeta, t)$ is equivalent to integrating the Random Differential Equation for the time-evolution of the Process $X(\zeta, t)$.

But the evolution equation solution is superior to the setup, and solution of the probability-wave equation:

While the probability-wave equation setup is instructive, [Dan6], it involves Conditional Probabilities that for higher order equations are beyond comprehension, reminding of the Monty Hall Problem [Rosenhouse].

Consequently, it will be difficult to detect an error in the setup of the probability-wave equation.

Furthermore, the probability-wave equation is a partial differential equation, and its solution is more difficult than the solution of the evolution equation which is an ordinary differential equation.

0.3 The Wiener Integral, and the Ito Integral

To integrate the equations of the time-evolution of random walk, Wiener defined the integral of $f(t)$ with respect to the random walk $B(\zeta, t)$.

Wiener's Integral enables the solution of the first order Langevin equation for the linear oscillator, which is equivalent to the Focker-Planck equation for the probability-wave.

Only the Wiener integral is necessary to integrate the Langevin equation. But instead of applying it, authors keep busy with the ill-defined Ito Integral that intended unnecessarily to generalize the Wiener Integral.

In [Dan5], we defined in Infinitesimal Calculus the Integral

$\int_{t=a}^{t=b} f(t)dB(\zeta, t)$, where $f(t)$ is integrable hyper-real function,

and $B(\zeta, t)$ is a Random Walk.

In [Dan7] we showed that $f(t)$ may not be replaced with a Hyper-real Random Process $f(\zeta, t)$, and that the Ito integral that purports to do that is ill-defined, and does not exist.

0.4 Evolution Equations Driven by Poisson Process

To date, all random differential equations are for Random Processes $X_B(\zeta, t)$, driven by Random Walk Process.

But Shot Noise Processes are driven by Poisson Processes.

We develop here the theory of Random differential equations for Processes $X_P(\zeta, t)$ driven by Poisson Processes.

1.

Hyper-real Line

The minimal domain and range, needed for the definition and analysis of a hyper-real function, is the hyper-real line.

Each real number α can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of infinitesimal hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the infinite hyper-reals.
4. The infinite hyper-reals are greater than any real number, yet strictly smaller than infinity.

5. The infinite hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a non-constant hyper-real.
7. The Hyper-reals are the totality of constant hyper-reals, a family of infinitesimals, a family of infinitesimals with negative sign, a family of infinite hyper-reals, a family of infinite hyper-reals with negative sign, and non-constant hyper-reals.
8. The hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.
9. That line includes the real numbers separated by the non-constant hyper-reals. Each real number is the center of an interval of hyper-reals, that includes no other real number.
10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.
11. Zero is not an infinitesimal, because zero is not strictly greater than zero.

12. We do not add infinity to the hyper-real line.

13. The infinitesimals, the infinitesimals with negative signs, the infinite hyper-reals, and the infinite hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.

14. The hyper-real line is embedded in $\mathbb{R}^{\infty}$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the hyper-real onto the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal hyper-reals, or to the infinite hyper-reals, or to the non-constant hyper-reals.

16. No neighbourhood of a hyper-real is homeomorphic to an $\mathbb{R}^{n}$ ball. Therefore, the hyper-real line is not a manifold.

17. The hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

2.

Hyper-real Function

2.1 Definition of a hyper-real function

$f(x)$ is a hyper-real function, iff it is from the hyper-reals into the hyper-reals.

This means that any number in the domain, or in the range of a hyper-real $f(x)$ is either one of the following

real

real + infinitesimal

real – infinitesimal

infinitesimal

infinitesimal with negative sign

infinite hyper-real

infinite hyper-real with negative sign

Clearly,

2.2 *Every function from the reals into the reals is a hyper-real function.*

3.

Integral of Hyper-real Function

In [Dan3], we defined the integral of a Hyper-real Function.

Let $f(x)$ be a hyper-real function on the interval $[a,b]$.

The interval may not be bounded.

$f(x)$ may take infinite hyper-real values, and need not be bounded.

At each

$$a \leq x \leq b,$$

there is a rectangle with base $[x - \frac{dx}{2}, x + \frac{dx}{2}]$, height $f(x)$, and area

$$f(x)dx.$$

We form the **Integration Sum** of all the areas for the x's that start at $x = a$, and end at $x = b$,

$$\sum_{x \in [a,b]} f(x)dx.$$

If for any infinitesimal dx, the Integration Sum has the same hyper-real value, then $f(x)$ is integrable over the interval $[a,b]$.

Then, we call the Integration Sum the integral of $f(x)$ from $x = a$, to $x = b$, and denote it by

$$\int_{x=a}^{x=b} f(x)dx .$$

If the hyper-real is infinite, then it is the integral over $[a,b]$,

If the hyper-real is finite,

$$\int_{x=a}^{x=b} f(x)dx = \text{ real part of the hyper-real}. \square$$

3.1 The countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers, $Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty .$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[a,b]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(x)dx$.

The Lower Integral is the Integration Sum where $f(x)$ is replaced

by its lowest value on each interval $[x-\frac{dx}{2}, x+\frac{dx}{2}]$

3.2 $$\sum_{x\in[a,b]}\left(\inf_{x-\frac{dx}{2}\le t\le x+\frac{dx}{2}} f(t)\right)dx$$

The Upper Integral is the Integration Sum where $f(x)$ is replaced by its largest value on each interval $[x-\frac{dx}{2}, x+\frac{dx}{2}]$

3.3 $$\sum_{x\in[a,b]}\left(\sup_{x-\frac{dx}{2}\le t\le x+\frac{dx}{2}} f(t)\right)dx$$

If the integral is a finite hyper-real, we have

3.4 *A hyper-real function has a finite integral if and only if its upper integral and its lower integral are finite, and differ by an infinitesimal.*

4.

Delta Function

In [Dan4], we defined the Delta Function, and established its properties

1. The Delta Function is a hyper-real function defined from the hyper-real line into the set of two hyper-reals $\left\{0, \frac{1}{dx}\right\}$. The hyper-real 0 is the sequence $\langle 0,0,0,...\rangle$.

   The infinite hyper-real $\frac{1}{dx}$ depends on our choice of dx.

2. We will usually choose the family of infinitesimals that is spanned by the sequences $\left\langle \frac{1}{n}\right\rangle, \left\langle \frac{1}{n^2}\right\rangle, \left\langle \frac{1}{n^3}\right\rangle$,... It is a semigroup with respect to vector addition, and includes all the scalar multiples of the generating sequences that are non-zero. That is, the family includes infinitesimals with negative sign. Therefore, $\frac{1}{dx}$ will mean the sequence $\langle n\rangle$. Alternatively, we may choose

the family spanned by the sequences $\left\langle \frac{1}{2^n} \right\rangle, \left\langle \frac{1}{3^n} \right\rangle, \left\langle \frac{1}{4^n} \right\rangle$,... Then, $\frac{1}{dx}$ will mean the sequence $\left\langle 2^n \right\rangle$. Once we determined the basic infinitesimal dx, we will use it in the Infinite Riemann Sum that defines an Integral in Infinitesimal Calculus.

3. The Delta Function is strictly smaller than ∞

4. We define, $\delta(x) \equiv \frac{1}{dx} \chi_{\left[-\frac{dx}{2}, \frac{dx}{2}\right]}(x)$,

   where $\chi_{\left[-\frac{dx}{2}, \frac{dx}{2}\right]}(x) = \begin{cases} 1, x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right] \\ 0, \text{otherwise} \end{cases}$.

5. Hence,

   - for $x < 0$, $\delta(x) = 0$
   - at $x = -\frac{dx}{2}$, $\delta(x)$ jumps from 0 to $\frac{1}{dx}$,
   - for $x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right]$, $\delta(x) = \frac{1}{dx}$.
   - at $x = 0$, $\delta(0) = \frac{1}{dx}$
   - at $x = \frac{dx}{2}$, $\delta(x)$ drops from $\frac{1}{dx}$ to 0.
   - for $x > 0$, $\delta(x) = 0$.

❖ $x\delta(x) = 0$

6. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle \chi_{[-\frac{1}{2},\frac{1}{2}]}(x), 2\chi_{[-\frac{1}{4},\frac{1}{4}]}(x), 3\chi_{[-\frac{1}{6},\frac{1}{6}]}(x)... \right\rangle$

7. If $dx = \left\langle \frac{2}{n} \right\rangle$, $\delta(x) = \left\langle \dfrac{1}{2\cosh^2 x}, \dfrac{2}{2\cosh^2 2x}, \dfrac{3}{2\cosh^2 3x}, ... \right\rangle$

8. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle e^{-x}\chi_{[0,\infty)}, 2e^{-2x}\chi_{[0,\infty)}, 3e^{-3x}\chi_{[0,\infty)}, ... \right\rangle$

9. $$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1.$$

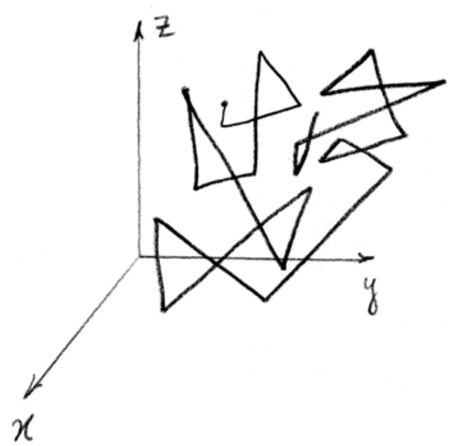
10. $$\delta(\xi - x) = \frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} e^{-ik(\xi - x)} dk$$

5.

Random Walk $B(\zeta, t)$

The Random Walk of small particles in fluid is named after Brown, who first observed it, Brownian Motion. It models other processes, such as the fluctuations of a stock price.

In a volume of fluid, the path of a particle is in any direction in the volume, and of variable size



5.1 Bernoulli Random Variables of the Walk

We restrict the Walk here to the line, in uniform infinitesimal size steps dx:

To the left, with probability

$$p = \tfrac{1}{2},$$

or to the right, with probability

$$q = 1 - p = \tfrac{1}{2}.$$

At time t, after

N infinitesimal time intervals dt,

$N = \frac{t}{dt}$, is an infinite hyper-real,

the particle is at the point

$$x.$$

At the ith step we define the Bernoulli Random Variable,

$$B_i(\text{right step}) = dx, \quad \zeta_1 = \text{right step}.$$

$$B_i(\text{left step}) = -dx, \quad \zeta_2 = \text{left step}.$$

where $i = 1, 2, ..., N$.

$$\Pr(B_i = dx) = \tfrac{1}{2},$$

$$\Pr(B_i = -dx) = \tfrac{1}{2},$$

$$E[B_i] = dx \cdot \tfrac{1}{2} + (-dx) \cdot \tfrac{1}{2} = 0,$$

$$E[B_i^2] = (dx)^2 \cdot \tfrac{1}{2} + (-dx)^2 \cdot \tfrac{1}{2} = (dx)^2$$

$$\text{Var}[B_i] = \underbrace{E[B_i^2]}_{(dx)^2} - (\underbrace{E[B_i]}_{0})^2 = 0$$

5.2 The Random Walk

$$B(\zeta, t) = B_1 + B_2 + ... + B_N$$

is a Random Process with

$$E[B(\zeta,t)] = 0\,,$$

$$\mathrm{Var}[B(\zeta,t)] = N(dx)^2\,.$$

<u>*Proof*</u>: Since the B_i are independent,

$$E[B(\zeta,t)] = \underbrace{E[B_1]}_{0} + ... + \underbrace{E[B_N]}_{0} = 0$$

$$\mathrm{Var}[B(\zeta,t)] = \underbrace{\mathrm{Var}[B_1]}_{(dx)^2} + ... + \underbrace{\mathrm{Var}[B_N]}_{(dx)^2} = N(dx)^2\,.\square$$

5.3 $B(\zeta,t+dt) - B(\zeta,t)$ ***is a Bernoulli Random Variable*** B_i

5.4 $(dx)^2 = (2D)dt \Rightarrow$ **Random Walk is Continuous**

<u>*Proof*</u>:

$$E[\{B(\zeta,t+dt) - B(\zeta,t)\}^2] =$$

$$= \mathrm{Var}[\underbrace{B(\zeta,t+dt) - B(\zeta,t)}_{B_i}] + (E[\underbrace{B(\zeta,t+dt) - B(\zeta,t)}_{B_i}])^2\,,$$

where B_i is a Bernoulli Random Variable,

$$= \underbrace{\mathrm{Var}[B_i]}_{(dx)^2=(2D)dt} + (\underbrace{E[B_i]}_{0})^2 = (2D)dt\,.\square$$

5.5 *<u>If</u>* $(dx)^2 = (2D)dt$

<u>Then</u> The Derivative of Random Walk is

$$\dot{B} = \frac{1}{dt} B_i ,$$

where **(1)** $B_i = B(\zeta, t_0 + dt) - B(\zeta, t_0)$*, is a Bernoulli Random Variable.*

(2) $E[\dot{B}] = 0$,

(3) $\mathrm{Var}[\dot{B}] = 2D\delta(t_0)$,

<u>Proof</u>:

(1) For each $t = t_0$, we need to find a Random Signal $\dot{B}(\zeta, t_0)$, so that for any dt,

$$E\left[\left[\frac{B(\zeta, t_0 + dt) - B(\zeta, t_0)}{dt} - \dot{B}(\zeta, t_0)\right]^2\right] = \text{infinitesimal},$$

Since $B(\zeta, t_0 + dt) - B(\zeta, t_0)$, is a Bernoulli Random Variable B_i,

$$E\left[\left\{\frac{X(B, t + dt) - B(\zeta, t)}{dt} - \dot{B}(\zeta, t)\right\}^2\right] = E\left[\left\{\frac{B_i}{dt} - \dot{B}\right\}^2\right]$$

Therefore, at time $t = t_0$, the Random Variable

$$\frac{1}{dt} B_i ,$$

is the derivative of the Random Walk $B(\zeta, t_0)$.$\square$

(2) $$E[\dot{B}] = \tfrac{1}{dt}\underbrace{E[B_i]}_{0} = 0.\square$$

(3) $$\mathrm{Var}[\dot{B}] = E[\dot{B}^2] - (\underbrace{E[\dot{B}]}_{0})^2$$

$$= \frac{1}{(dt)^2}\underbrace{E[B_i^{\,2}]}_{(dx)^2},$$

$$= \underbrace{\frac{(dx)^2}{dt}}_{2D}\frac{1}{dt}$$

$$= (2D)\delta(t_0).\square$$

6.

Integration Sums of $f(t)$ with respect to $B(\zeta,t)$.

Let $f(t)$ be a hyper-real function on the bounded time interval $[a,b]$. $f(t)$ need not be bounded.

In [Dan5], we defined the Integral $\int\limits_{t=a}^{t=b} f(t)dB(\zeta,t)$:

At each $a \le t \le b$, there is a Bernoulli Random Variable

$$dB(\zeta,t) = B(\zeta,t+dt) - B(\zeta,t) = B_i(\zeta,t) = \dot{B}(\zeta,t)dt.$$

We form the **Integration Sum**

$$\sum_{t=a}^{t=b} f(t)dB(\zeta,t) = \sum_{t=a}^{t=b} f(t)B_i(\zeta,t).$$

For any dt,

(1) the <u>First Moment</u> of the Integration Sum is

$$E\left[\sum_{t=a}^{t=b} f(t)B_i(\zeta,t)\right] = \sum_{t=a}^{t=b} f(t)\underbrace{E[B_i(\zeta,t)]}_{0} = 0.$$

(2) the Second Moment of the Integration sum is

$$E\left[\left(\sum_{t=a}^{t=b} f(t)B_i(\zeta,t)\right)^2\right] = E\left[\left(\sum_{t=a}^{t=b} f(t)B_i(\zeta,t)\right)\left(\sum_{\tau=a}^{\tau=b} f(\tau)B_j(\zeta,\tau)\right)\right]$$

$$= \sum_{t=a}^{t=b}\sum_{\tau=a}^{\tau=b} f(t)f(\tau)E[B_j(\zeta,\tau)B_i(\zeta,t)]$$

Since the Bernoulli Random Variables are independent,

$$E[B_j(\zeta,\tau)B_i(\zeta,t)] = E[B_i^2(\zeta,t)] = (dx)^2$$

only for $t = \tau$. Then,

$$E\left[\left(\sum_{t=a}^{t=b} f(t)B_i(\zeta,t)\right)^2\right] = \sum_{t=a}^{t=b} f^2(t)\underbrace{(dx)^2}_{(2D)dt},$$

$$= 2D\int_{t=a}^{t=b} f^2(t)dt,$$

assuming $(dx)^2 = (2D)dt$, where D is the Drift coefficient of the Random Walk, and assuming that the hyper-real $f(t)$ is integrable.

Thus, for any dt, the Integration Sum is a unique well-defined hyper-real Random Variable

$$I_B(\zeta) = \int_{t=a}^{t=b} f(t)dB(\zeta,t) \quad .$$

7.

Evolution of Linear Oscillator Driven by $\dot{B}(\zeta, t)$

7.1 The Evolution Equation of Linear Oscillator

Process driven by Random Force $\dot{B}(\zeta, t)$

A particle of mass m, affected by a Random Force $\dot{B}(\zeta, t)$, is drifting in a fluid with viscosity γ, with random velocity $v(\zeta, t)$.

The balance of forces on the particle is

$$m\frac{dv(\zeta,t)}{dt} + \gamma v(\zeta, t) = \dot{B}(\zeta, t)$$

$$dv(\zeta, t) = \underbrace{\underbrace{\tfrac{1}{m}}_{\alpha}\underbrace{\dot{B}(\zeta, t)dt}_{dB(\zeta,t)}}_{\text{infinitesimal diffusion}} - \underbrace{\underbrace{\tfrac{\gamma}{m}}_{\beta}v(\zeta, t)dt}_{\text{infinitesimal drift}}$$

Thus, the 1st order Evolution Equation for the Linear Oscillator is

$$dv(\zeta, t) = \alpha dB(\zeta, t) - \beta v(\zeta, t)dt,$$

$$\dot{v}(\zeta, t) = \alpha\dot{B}(\zeta, t) - \beta v(\zeta, t)$$

7.2 The Integrating Factor Solution $X_B(\zeta,t)$

$$X_B(\zeta,t) = e^{-\beta t}X_B(\zeta,0) + \alpha \int_{\tau=0}^{\tau=t} e^{-\beta(t-\tau)}dB(\zeta,\tau)$$

Proof: Multiplying

$$dX_B(\zeta,t) = \alpha dB(\zeta,t) - \beta X_B(\zeta,t)dt$$

by its integrating factor $e^{\beta t}$,

$$e^{\beta t}dX_B(\zeta,t) = \alpha e^{\beta t}dB(\zeta,t) - \beta e^{\beta t}X_B(\zeta,t)dt,$$

$$\underbrace{e^{\beta t}dX_B(\zeta,t) + \beta e^{\beta t}X_B(\zeta,t)dt}_{d\left\{e^{\beta t}X_B(\zeta,t)\right\}} = \alpha e^{\beta t}dB(\zeta,t),$$

$$d\left\{e^{\beta t}X_B(\zeta,t)\right\} = \alpha e^{\beta t}dB(\zeta,t).$$

Summing over time,

$$\underbrace{\sum_{\tau=0}^{\tau=t} d\left\{e^{\beta\tau}X_B(\zeta,\tau)\right\}}_{\left\{e^{\beta\tau}X_B(\zeta,\tau)\right\}_{\tau=0}^{\tau=t}} = \alpha\sum_{\tau=0}^{\tau=t} e^{\beta\tau}dB(\zeta,\tau),$$

$$e^{\beta t}X(\zeta,t) - X(\zeta,0) = \alpha\sum_{\tau=0}^{\tau=t} e^{\beta\tau}dB(\zeta,\tau).$$

Since $e^{\beta\tau}$ is integrable, by section 6, $\sum_{\tau=0}^{\tau=t} e^{\beta\tau}dB(\zeta,\tau)$ sums up

to the Random Variable $I_B(\zeta) = \int_{\tau=0}^{\tau=t} e^{\beta\tau}dB(\zeta,\tau)$, and

$$e^{\beta t} X_B(\zeta,t) - X_B(\zeta,0) = \alpha \int_{\tau=0}^{\tau=t} e^{\beta\tau} dB(\zeta,\tau),$$

$$X_B(\zeta,t) = e^{-\beta t} X_B(\zeta,0) = \alpha \int_{\tau=0}^{\tau=t} e^{-\beta(t-\tau)} dB(\zeta,\tau). \square$$

We obtain the same solution for the 1st order Langevin Equation by the Variation of Parameters Method.

7.3 The Variation of Parameters Solution

$$X_B(\zeta,t) = \underbrace{X_B(\zeta,0)e^{-\beta t}}_{X_{\text{homogeneous}}(\zeta,t)} + \underbrace{\alpha \int_{\tau=0}^{\tau=t} e^{-\beta(t-\tau)} dB(\zeta,\tau)}_{X_{\text{particular}}(\zeta,t)}$$

Proof:

$$dX_B(\zeta,t) = \alpha dB(\zeta,t) - \beta X_B(\zeta,t)dt,$$

$$\dot{X}_B(\zeta,t) + \beta X_B(\zeta,t) = \alpha \dot{B}(\zeta,t).$$

To find the general solution for the homogeneous equation,

$$\dot{X}_B(\zeta,t) + \beta X_B(\zeta,t) = 0,$$

we substitute in it the separation of variables solution

$$X_{\text{hom}}(\zeta,t) = A(\zeta)e^{mt}.$$

$$A(\zeta)me^{mt} + \beta A(\zeta)e^{mt} = 0,$$

$$(m+\beta)A(\zeta)e^{mt} = 0,$$

$$m+\beta = 0,$$

$$m = -\beta,$$

$$X_{\text{hom}}(\zeta,t) = A(\zeta)e^{-\beta t}.$$

To find a particular solution of

$$\dot{X}_B(\zeta,t) + \beta X_B(\zeta,t) = \alpha \dot{B}(\zeta,t),$$

we substitute in it the Variation of Parameter Solution

$$X_B(\zeta,t) = A(\zeta,t)e^{-\beta t}$$

$$\dot{A}e^{-\beta t} \underbrace{-A\beta e^{-\beta t} + \beta A e^{-\beta t}}_{0} = \alpha \dot{B}(\zeta,t),$$

$$\dot{A} = \alpha e^{\beta t}\dot{B}(\zeta,t)$$

$$\underbrace{\dot{A}dt}_{dA} = \alpha e^{\beta t} \underbrace{\dot{B}(\zeta,t)dt}_{dB(\zeta,t)}$$

$$\underbrace{A(\zeta,\tau)\Big|_{\tau=0}^{\tau=t}}_{A(\zeta,t)-A(\zeta,0)} = \alpha \int_{\tau=0}^{\tau=t} e^{\beta\tau} dB(\zeta,\tau),$$

$$A(\zeta,t) = A(\zeta,0) + \alpha \int_{\tau=0}^{\tau=t} e^{\beta\tau} dB(\zeta,\tau),$$

Therefore,

$$X_B(\zeta,t) = A(\zeta,t)e^{-\beta t}$$

$$= A(\zeta, 0)e^{-\beta t} + \alpha e^{-\beta t} \int_{\tau=0}^{\tau=t} e^{\beta\tau} dB(\zeta, \tau)$$

$$= \underbrace{X(\zeta, 0)e^{-\beta t}}_{X_{\text{homogeneous}}(\zeta,t)} + \underbrace{\alpha \int_{\tau=0}^{\tau=t} e^{-\beta(t-\tau)} dB(\zeta, \tau)}_{X_{\text{particular}}(\zeta,t)} . \square$$

8.

Linear Oscillator Process $X_B(\zeta,t)$ Driven by $\dot{B}(\zeta,t)$

8.1 The Normal Random Process

$$X_B(\zeta,t) = e^{-\beta t}X_B(\zeta,0) + \alpha \int_{\tau=0}^{\tau=t} e^{-\beta(t-\tau)} dB(\zeta,\tau)$$

is Normally Distributed with

Mean

$$E[X_B(\zeta,t)] = e^{-\beta t}E[X_B(\zeta,0)],$$

and Variance

$$\mathrm{Var}[X_B(\zeta,t)] = e^{-2\beta t}\,\mathrm{Var}\big[\,X_B(\zeta,0)\big] + D\tfrac{\alpha^2}{\beta}(1-e^{-2\beta t}).$$

<u>*Proof*</u>: Since $dB(\zeta,\tau)$ is Normally Distributed, so is

$I_B(\zeta) = \int_{\tau=0}^{\tau=t} e^{-\beta(t-\tau)} dB(\zeta,\tau)$. Hence, $X_B(\zeta,t)$ has Normal distribution.

$$E[I_B(\zeta)] = E\left[\sum_{\tau=0}^{\tau=t} e^{\beta\tau} dB(\zeta,\tau)\right] = \sum_{\tau=0}^{\tau=t} e^{\beta\tau}\, \underbrace{E[\underbrace{dB(\zeta,\tau)}_{B_i}]}_{0} = 0\,.$$

$$E[X_B(\zeta,t)] = \underbrace{E[e^{-\beta t}X_B(\zeta,0)]}_{e^{-\beta t}E[X_B(\zeta,0)]} + \underbrace{E[\alpha e^{-\beta t}I_B(\zeta)]}_{\alpha e^{-\beta t}\underbrace{E[I_B(\zeta)]}_{0}},$$

$$= e^{-\beta t}E[X_B(\zeta,0)].\square$$

$$\mathrm{Var}[I_B(\zeta)] = E\Big[{I_B}^2(\zeta)\Big] - (\underbrace{E\big[I_B(\zeta)\big]}_{0})^2,$$

$$= \sum_{\tau=0}^{\tau=t} e^{2\beta\tau} \quad \underbrace{E[B_i^2]}_{(dx)^2=(2D)d\tau} \quad ,$$

$$= 2D\sum_{\tau=0}^{\tau=t} e^{2\beta\tau}d\tau = 2D\tfrac{1}{2\beta}(e^{2\beta t}-1).$$

$$\mathrm{Var}[X_B(\zeta,t)] = \underbrace{\mathrm{Var}\Big[e^{-\beta t}X_B(\zeta,0)\Big]}_{e^{-2\beta t}\,\mathrm{Var}\big[X_B(\zeta,0)\big]} + \underbrace{\mathrm{Var}\Big[\alpha e^{-\beta t}I_B(\zeta)\Big]}_{\alpha^2 e^{-2\beta t}\,\mathrm{Var}\big[I_B(\zeta)\big]}$$

$$= e^{-2\beta t}\,\mathrm{Var}\Big[X_B(\zeta,0)\Big] + D\tfrac{\alpha^2}{\beta}(1-e^{-2\beta t}).\square$$

8.2 The Linear Oscillator Equilibrium State

At Equilibrium, t = infinite hyper-real Θ,

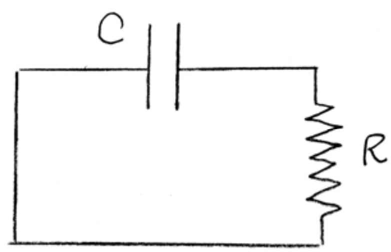
$$E[X_B(\zeta,\Theta)] \approx 0$$

$$\mathrm{Var}[X_B(\zeta,\Theta)] \approx D\tfrac{\alpha^2}{\beta} = \tfrac{D}{m\gamma}.$$

9.

RC Linear Oscillator Process $q_B(\zeta, t)$ Driven by $\dot{B}(\zeta, t)$

9.1 The Evolution Equation of the RC Linear Oscillator Process $q_B(\zeta, t)$ Driven by $\dot{B}(\zeta, t)$



Thermal Noise in the Resistor R, generates Random Voltage $\dot{B}(\zeta, t)$ on the Resistor R, that drives a random current $i_B(\zeta, t) = \dfrac{dq_B(\zeta, t)}{dt}$ through the Circuit.

$q_B(\zeta, t)$ is the random charge on the Capacitor C

The balance of voltages on the circuit components is

$$\frac{dq_B(\zeta, t)}{dt} R + \frac{q_B(\zeta, t)}{C} = \dot{B}(\zeta, t),$$

$$dq_B(\zeta,t) = \underbrace{\tfrac{1}{R}}_{\alpha}\underbrace{\dot{B}(\zeta,t)dt}_{dB(\zeta,t)} - \underbrace{\tfrac{1}{RC}}_{\beta}q_B(\zeta,t)dt\,.$$

9.2 The Random Charge Process $q_B(\zeta,t)$

$$q_B(\zeta,t) = e^{-\frac{1}{RC}t}q_B(\zeta,0) + \tfrac{1}{R}\int_{\tau=0}^{\tau=t} e^{-\frac{1}{RC}(t-\tau)}dB(\zeta,\tau)$$

Proof: By section 8,

$$q_B(\zeta,t) = e^{-\beta t}q_B(\zeta,0) + \alpha\int_{\tau=0}^{\tau=t} e^{-\beta(t-\tau)}dB(\zeta,\tau)\,,$$

$$= e^{-\frac{1}{RC}t}q_B(\zeta,0) + \tfrac{1}{R}\int_{\tau=0}^{\tau=t} e^{-\frac{1}{RC}(t-\tau)}dB(\zeta,\tau).\square$$

9.3 The Normal Distribution of the Random Charge

$$q_B(\zeta,t) = e^{-\frac{1}{RC}t}q_B(\zeta,0) + \tfrac{1}{R}\int_{\tau=0}^{\tau=t} e^{-\frac{1}{RC}(t-\tau)}dB(\zeta,\tau)$$

is Normally Distributed with

Mean

$$E[q_B(\zeta,t)] = e^{-\frac{1}{RC}t}E[q_B(\zeta,0)],$$

and Variance

$$\mathrm{Var}[q_B(\zeta,t)] = e^{-2\frac{1}{RC}t}\,\mathrm{Var}\big[\,q_B(\zeta,0)\big] + D\tfrac{C}{R}(1 - e^{-2\frac{1}{RC}t})$$

Proof: By 8.1,

$$E[q_B(\zeta,t)] = e^{-\beta t}E[q_B(\zeta,0)] = e^{-\frac{1}{RC}t}E[q_B(\zeta,0)].\square$$

$$\mathrm{Var}[q_B(\zeta,t)] = e^{-2\beta t}\,\mathrm{Var}\big[q_B(\zeta,0)\big] + D\tfrac{\alpha^2}{\beta}(1-e^{-2\beta t})$$

$$= e^{-2\frac{1}{RC}t}\,\mathrm{Var}\big[q_B(\zeta,0)\big] + D\tfrac{C}{R}(1-e^{-2\frac{1}{RC}t}).\square$$

9.4 The Random Charge Steady State

At the Steady State, $t =$ infinite hyper-real Θ,

$$E[q_B(\zeta,\Theta)] \approx 0.$$

$$\mathrm{Var}[q_B(\zeta,\Theta)] \approx D\tfrac{\alpha^2}{\beta} = \tfrac{DC}{R}.$$

9.5 The Charge Noise Energy at the Steady State

$$\tfrac{1}{2}kT = \tfrac{1}{2C}E[{q_B}^2(\zeta,\Theta)] = \tfrac{D}{2R},$$

where k is Boltzmann Constant,

and T is the absolute Temperature.

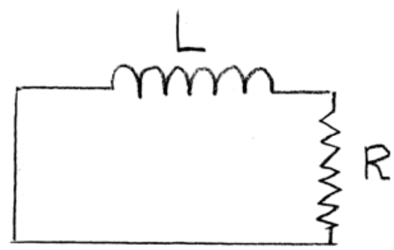
Proof:

$$\frac{E[{q_B}^2(\zeta,\Theta)]}{2C} = \tfrac{1}{2C}\{\underbrace{\mathrm{Var}[q_B(\zeta,\Theta)]}_{\frac{DC}{R}} + (\underbrace{E[q_B(\zeta,\Theta)]}_{0})^2\} = \tfrac{D}{2R}.\square$$

10.

RL Linear Oscillator Process $i_B(\zeta, t)$ Driven by $\dot{B}(\zeta, t)$

10.1 The Evolution Equation of the RL Linear Oscillator Process $i_B(\zeta, t)$ Driven by $\dot{B}(\zeta, t)$



Thermal Noise in the Resistor R, generates Random Voltage $\dot{B}(\zeta, t)$ on the Resistor R, that drives a random current $i_B(\zeta, t)$ through the Circuit.

$L\dfrac{di_B(\zeta, t)}{dt}$ is the Random Voltage on the Solenoid L.

The balance of voltages on the circuit components is

$$L\frac{di_B(\zeta, t)}{dt} + i_B(\zeta, t)R = \dot{B}(\zeta, t),$$

$$di_B(\zeta,t) = \underbrace{\tfrac{1}{L}}_{\alpha}\underbrace{\dot{B}(\zeta,t)dt}_{dB(\zeta,t)} - \underbrace{\tfrac{R}{L}}_{\beta} i_B(\zeta,t)dt\,.$$

10.2 The Random Current Process $i_B(\zeta,t)$

$$i_B(\zeta,t) = e^{-\frac{R}{L}t} i_B(\zeta,0) + \frac{1}{L}\int_{\tau=0}^{\tau=t} e^{-\frac{R}{L}(t-\tau)} dB(\zeta,\tau)$$

Proof: By 7.2,

$$i_B(\zeta,t) = e^{-\beta t} i_B(\zeta,0) + \alpha\int_{\tau=0}^{\tau=t} e^{-\beta(t-\tau)} dB(\zeta,\tau),$$

$$= e^{-\frac{R}{L}t} i_B(\zeta,0) + \frac{1}{L}\int_{\tau=0}^{\tau=t} e^{-\frac{R}{L}(t-\tau)} dB(\zeta,\tau).\square$$

10.3 The Normal Distribution of $i_B(\zeta,t)$

$$i_B(\zeta,t) = e^{-\frac{R}{L}t} i_B(\zeta,0) + \frac{1}{L}\int_{\tau=0}^{\tau=t} e^{-\frac{R}{L}(t-\tau)} dB(\zeta,\tau)$$

is Normally Distributed with

Mean

$$E[i_B(\zeta,t)] = e^{-\frac{R}{L}t} E[i_B(\zeta,0)],$$

and Variance

$$\mathrm{Var}[i_B(\zeta,t)] = e^{-2\frac{R}{L}t}\,\mathrm{Var}\Big[\,i_B(\zeta,0)\Big] + D\,\tfrac{1}{RL}(1-e^{-2\frac{R}{L}t})$$

<u>*Proof*</u>: By 8.1,

$$E[i_B(\zeta,t)] = e^{-\beta t}E[i_B(\zeta,0)] = e^{-\frac{R}{L}t}E[i_B(\zeta,0)].\square$$

$$\mathrm{Var}[i_B(\zeta,t)] = e^{-2\beta t}\,\mathrm{Var}\Big[\,i_B(\zeta,0)\Big] + D\,\tfrac{\alpha^2}{\beta}(1-e^{-2\beta t})$$

$$= e^{-2\frac{R}{L}t}\,\mathrm{Var}\Big[\,i_B(\zeta,0)\Big] + D\,\tfrac{1}{RL}(1-e^{-2\frac{R}{L}t}).\square$$

10.4 The Random Current Steady State

At the Steady State, t = infinite hyper-real Θ,

$$E[i_B(\zeta,\Theta)] \approx 0\,.$$

$$\mathrm{Var}[i_B(\zeta,\Theta)] \approx D\,\tfrac{\alpha^2}{\beta} = \tfrac{D}{RL}.$$

10.5 The Current Noise Energy at the Steady State

$$\tfrac{1}{2}kT = \tfrac{1}{2}E[{i_B}^2(\zeta,\Theta)]L = \tfrac{D}{2R},$$

where k *is Boltzmann Constant,*

and T *is the absolute Temperature.*

<u>*Proof*</u>:

$$\tfrac{1}{2}E[{i_B}^2(\zeta,\Theta)]L = \tfrac{1}{2}\{\underbrace{\mathrm{Var}[i_B(\zeta,\Theta)]}_{\frac{D}{RL}} + (\underbrace{E[i_B(\zeta,\Theta)]}_{0})^2\}L = \tfrac{D}{2R}.\square$$

11. Evolution of Harmonic Oscillator Driven by $\dot{B}(\zeta, t)$

11.1 Evolution equation of Harmonic Oscillator driven by Random Force $\dot{B}(\zeta, t)$

A particle of mass m, affected by a Random Force $\dot{B}(\zeta, t)$, is drifting in a fluid with viscosity γ, and Hooke's Law Constant k, with random velocity $\dot{x}(\zeta, t)$.

The balance of forces on the particle is

$$m\frac{d\dot{x}(\zeta,t)}{dt} + \gamma\dot{x}(\zeta,t) + kx(\zeta,t) = \dot{B}(\zeta,t),$$

$$d\dot{x}(\zeta,t) = \underbrace{\underbrace{\tfrac{1}{m}}_{\alpha}\underbrace{\dot{B}(\zeta,t)dt}_{dB(\zeta,t)}}_{\text{infinitesimal diffusion}} - \underbrace{\underbrace{\tfrac{\gamma}{m}}_{\beta}\dot{x}(\zeta,t)dt}_{\text{infinitesimal drift}} - \underbrace{\underbrace{\tfrac{k}{m}}_{\omega^2}x(\zeta,t)dt}_{\text{infinitesimal Hooke's}}$$

Thus, the evolution Equation for the Harmonic Oscillator driven by a Random Walk $B(\zeta, t)$ is

$$d\dot{x}_B(\zeta,t) = \alpha dB(\zeta,t) - \beta\dot{x}_B(\zeta,t)dt - \omega^2 x_B(\zeta,t)dt.$$

or

$$\ddot{x}_B(\zeta,t) = \alpha\dot{B}(\zeta,t) - \beta\dot{x}_B(\zeta,t) - \omega^2 x_B(\zeta,t).$$

11.2 Variation of Parameters Solution

for the Harmonic Oscillator Process

$$d\dot{X}_B(\zeta,t) = \alpha dB(\zeta,t) - \beta\dot{X}_B(\zeta,t)dt - \omega^2 X_B(\zeta,t)dt\,,$$

is $$X_B(\zeta,t) = \underbrace{e^{-\frac{1}{2}\beta t}\{A_1(\zeta,0)e^{\frac{1}{2}\kappa t} + A_2(\zeta,0)e^{-\frac{1}{2}\kappa t}\}}_{X_{\text{homogeneous}}(\zeta,t)}$$

$$+\underbrace{\frac{\alpha}{\kappa}\int_{\tau=0}^{\tau=t}\{e^{m_1(t-\tau)} - e^{m_2(t-\tau)}\}dB(\zeta,\tau)}_{X_{\text{particular}}(\zeta,t)},$$

provided $$\kappa = \sqrt{\beta^2 - 4\omega^2} > 0\,,$$

$$m_1 = -\tfrac{1}{2}\beta + \tfrac{1}{2}\kappa\,,$$

$$m_2 = -\tfrac{1}{2}\beta - \tfrac{1}{2}\kappa$$

Proof:

$$d\dot{X}_B(\zeta,t) = \alpha dB(\zeta,t) - \beta\dot{X}_B(\zeta,t)dt - \omega^2 X_B(\zeta,t)dt\,,$$

$$\ddot{X}_B(\zeta,t) + \beta\dot{X}_B(\zeta,t) + \omega^2 X_B(\zeta,t) = \alpha\dot{B}(\zeta,t).$$

To find the general solution for the homogeneous equation,

$$\ddot{X}_B(\zeta,t) + \beta\dot{X}_B(\zeta,t) + \omega^2 X_B(\zeta,t) = 0\,,$$

we substitute in it the separation of variables solution

$$X_{\text{hom}}(\zeta,t) = A(\zeta)e^{mt}.$$

$$A(\zeta)m^2e^{mt} + \beta A(\zeta)me^{mt} + \omega^2 A(\zeta)e^{mt} = 0\,,$$

$$(m^2 + \beta m + \omega^2)A(\zeta)e^{mt} = 0,$$

$$m^2 + \beta m + \omega^2 = 0,$$

$$m_1 = -\tfrac{1}{2}\beta + \tfrac{1}{2}\underbrace{\sqrt{\beta^2 - 4\omega^2}}_{\kappa>0},$$

$$m_2 = -\tfrac{1}{2}\beta - \tfrac{1}{2}\sqrt{\beta^2 - 4\omega^2}$$

$$X_{\text{hom}}(\zeta,t) = A_1(\zeta)e^{m_1 t} + A_2(\zeta)e^{m_2 t}.$$

To find a particular solution of

$$\ddot{X}_B(\zeta,t) + \beta\dot{X}_B(\zeta,t) + \omega^2 X_B(\zeta,t) = \alpha\dot{B}(\zeta,t),$$

we substitute in it the Variation of Parameter Solution

$$X_B(\zeta,t) = A_1(\zeta,t)e^{m_1 t} + A_2(\zeta,t)e^{m_2 t},$$

so that

$$\dot{A}_1 e^{m_1 t} + \dot{A}_2 e^{m_2 t} = 0.$$

Then

$$\dot{X}_B(\zeta,t) = A_1 m_1 e^{m_1 t} + A_2 m_2 e^{m_2 t} + \underbrace{\left[\dot{A}_1 e^{m_1 t} + \dot{A}_2 e^{m_2 t}\right]}_{0},$$

$$\ddot{X}_B(\zeta,t) = \dot{A}_1 m_1 e^{m_1 t} + \dot{A}_2 m_2 e^{m_2 t} + A_1 {m_1}^2 e^{m_1 t} + A_2 {m_2}^2 e^{m_2 t},$$

$$\alpha\dot{B} = \ddot{X} + \beta\dot{X} + \omega^2 X$$

$$= \dot{A}_1 m_1 e^{m_1 t} + \dot{A}_2 m_2 e^{m_2 t} + {m_1}^2 A_1 e^{m_1 t} + {m_2}^2 A_2 e^{m_2 t} +$$

$$+\beta A_1 m_1 e^{m_1 t} + \beta A_2 m_2 e^{m_2 t} +$$

$$+\omega^2 A_1 e^{m_1 t} + \omega^2 A_2 e^{m_2 t}$$

$$= \dot{A}_1(\zeta,t)m_1e^{m_1t} + \dot{A}_2(\zeta,t)m_2e^{m_2t} +$$

$$+\underbrace{({m_1}^2 + \beta m_1 + \omega^2)A_1e^{m_1t}}_{0} + \underbrace{({m_2}^2 + \beta m_2 + \omega^2)A_2e^{m_2t}}_{0}.$$

This yields two equations for $\dot{A}_1$, and $\dot{A}_2$,

$$\dot{A}_1e^{m_1t} + \dot{A}_2e^{m_2t} = 0,$$

$$\dot{A}_1m_1e^{m_1t} + \dot{A}_2m_2e^{m_2t} = \alpha\dot{B}.$$

In matrix form,

$$\begin{bmatrix} e^{m_1t} & e^{m_2t} \\ m_1e^{m_1t} & m_2e^{m_2t} \end{bmatrix}\begin{bmatrix} \dot{A}_1 \\ \dot{A}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha\dot{B} \end{bmatrix}.$$

$$\dot{A}_1 = \frac{\begin{vmatrix} 0 & e^{m_2t} \\ \alpha\dot{B} & m_2e^{m_2t} \end{vmatrix}}{\begin{vmatrix} e^{m_1t} & e^{m_2t} \\ m_1e^{m_1t} & m_2e^{m_2t} \end{vmatrix}} = \frac{-\alpha\dot{B}e^{m_2t}}{(\underbrace{m_2 - m_1}_{-\kappa})e^{(m_1+m_2)t}} = \frac{\alpha\dot{B}}{\kappa}e^{-m_1t},$$

$$\underbrace{\dot{A}_1dt}_{dA_1} = \frac{\alpha}{\kappa}e^{-m_1t}\underbrace{\dot{B}dt}_{dB(\zeta,t)}$$

$$\underbrace{A_1(\zeta,\tau)\Big|_{\tau=0}^{\tau=t}}_{A_1(\zeta,t)-A_1(\zeta,0)} = \frac{\alpha}{\kappa}\int_{\tau=0}^{\tau=t} e^{-m_1\tau}dB(\zeta,\tau)$$

$$A_1(\zeta,t) = A_1(\zeta,0) + \frac{\alpha}{\kappa}\int_{\tau=0}^{\tau=t} e^{-m_1\tau} dB(\zeta,\tau)$$

$$\dot{A}_2 = \frac{\begin{vmatrix} e^{m_1 t} & 0 \\ m_1 e^{m_1 t} & \alpha\dot{B} \end{vmatrix}}{\begin{vmatrix} e^{m_1 t} & e^{m_2 t} \\ m_1 e^{m_1 t} & m_2 e^{m_2 t} \end{vmatrix}} = \frac{\alpha\dot{B}e^{m_1 t}}{(\underbrace{m_2 - m_1}_{-\kappa})e^{(m_1+m_2)t}} = -\frac{\alpha\dot{B}}{\kappa}e^{-m_2 t}$$

$$\underbrace{\dot{A}_2 dt}_{dA_2} = -\frac{\alpha}{\kappa}e^{-m_2 t}\underbrace{\dot{B}dt}_{dB(\zeta,t)}$$

$$\underbrace{A_2(\zeta,\tau)\Big|_{\tau=0}^{\tau=t}}_{A_2(\zeta,t)-A_2(\zeta,0)} = -\frac{\alpha}{\kappa}\int_{\tau=0}^{\tau=t} e^{-m_2\tau} dB(\zeta,\tau)$$

$$A_2(\zeta,t) = A_2(\zeta,0) - \frac{\alpha}{\kappa}\int_{\tau=0}^{\tau=t} e^{-m_2\tau} dB(\zeta,\tau),$$

Therefore,

$$X_B(\zeta,t) = A_1(\zeta,t)e^{m_1 t} + A_2(\zeta,t)e^{m_2 t}$$

$$= A_1(\zeta,0)e^{m_1 t} + \frac{\alpha}{\kappa}\int_{\tau=0}^{\tau=t} e^{m_1(t-\tau)} dB(\zeta,\tau) +$$

$$+A_2(\zeta,0)e^{m_2 t} - \frac{\alpha}{\kappa}\int_{\tau=0}^{\tau=t} e^{m_2(t-\tau)} dB(\zeta,\tau)$$

$$= A_1(\zeta,0)e^{m_1 t} + A_2(\zeta,0)e^{m_2 t} + \frac{\alpha}{\kappa}\int_{\tau=0}^{\tau=t}\{e^{m_1(t-\tau)} - e^{m_2(t-\tau)}\}dB(\zeta,\tau),$$

$$= e^{-\frac{1}{2}\beta t}\{A_1(\zeta,0)e^{\frac{1}{2}\kappa t} + A_2(\zeta,0)e^{-\frac{1}{2}\kappa t}\}$$

$$+\frac{\alpha}{\kappa}\int_{\tau=0}^{\tau=t}\{e^{m_1(t-\tau)} - e^{m_2(t-\tau)}\}dB(\zeta,\tau).\square$$

11.3 The Time-Rate Random Process $\dot{X}(\zeta,t)$

$$\dot{X}_B(\zeta,t) = e^{-\frac{1}{2}\beta t}\{m_1 A_1(\zeta,0)e^{\frac{1}{2}\kappa t} + m_2 A_2(\zeta,0)e^{-\frac{1}{2}\kappa t}\}$$

$$+\frac{\alpha}{\kappa}\int_{\tau=0}^{\tau=t}\{m_1 e^{m_1(t-\tau)} - m_2 e^{m_2(t-\tau)}\}dB(\zeta,\tau),$$

provided $\kappa = \sqrt{\beta^2 - 4\omega^2} > 0$,

$$m_1 = -\tfrac{1}{2}\beta + \tfrac{1}{2}\kappa,$$

$$m_2 = -\tfrac{1}{2}\beta - \tfrac{1}{2}\kappa$$

Proof: From the proof of 11.2,

$$\dot{X}_B(\zeta,t) = A_1(\zeta,t)m_1 e^{m_1 t} + A_2(\zeta,t)m_2 e^{m_2 t}$$

$$= e^{-\frac{1}{2}\beta t}\{m_1 A_1(\zeta,0)e^{\frac{1}{2}\kappa t} + m_2 A_2(\zeta,0)e^{-\frac{1}{2}\kappa t}\}$$

$$+\frac{\alpha}{\kappa}\int_{\tau=0}^{\tau=t}\{m_1 e^{m_1(t-\tau)} - m_2 e^{m_2(t-\tau)}\}dB(\zeta,\tau).\square$$

12.

Harmonic Oscillator Process Driven by $\dot{B}(\zeta, t)$

12.1 The Normal Distribution of the Harmonic Oscillator Process

$$X_B(\zeta,t) = e^{-\frac{1}{2}\beta t}\{A_1(\zeta,0)e^{\frac{1}{2}\kappa t} + A_2(\zeta,0)e^{-\frac{1}{2}\kappa t}\}$$

$$+\frac{\alpha}{\kappa}\int_{\tau=0}^{\tau=t}\{e^{m_1(t-\tau)} - e^{m_2(t-\tau)}\}dB(\zeta,\tau)$$

is Normally Distributed with

Mean

$$E[X_B(\zeta,t)] = E[A_1(\zeta,0)]e^{m_1 t} + E[A_2(\zeta,0)]e^{m_2 t},$$

and Variance

$$\mathrm{Var}[X_B(\zeta,t)] = e^{-(\beta-\kappa)t}\,\mathrm{Var}\left[A_1(\zeta,0)\right] + e^{-(\beta+\kappa)t}\,\mathrm{Var}\left[A_2(\zeta,0)\right] +$$

$$+\frac{\alpha^2}{\kappa^2}2D\left\{-\frac{e^{-(\beta-\kappa)t}}{\beta-\kappa} + \frac{e^{-\beta t}}{\beta} - \frac{e^{-(\beta+\kappa)t}}{\beta+\kappa}\right\} +$$

$$+D\frac{\alpha^2}{\beta\omega^2}\frac{\beta^2-2\omega^2}{\beta^2-4\omega^2},$$

provided $\kappa = \sqrt{\beta^2 - 4\omega^2} > 0$,

$$m_1 = -\tfrac{1}{2}\beta + \tfrac{1}{2}\kappa,$$

$$m_2 = -\tfrac{1}{2}\beta - \tfrac{1}{2}\kappa.$$

Proof: Since $dB(\zeta,\tau)$ is Normally Distributed, so is

$I_B(\zeta) = \int\limits_{\tau=0}^{\tau=t} \{e^{m_1(t-\tau)} - e^{m_2(t-\tau)}\} dB(\zeta,\tau)$. Hence, $X_B(\zeta,t)$ has Normal Distribution.

$$E[I_B(\zeta)] = E\left[\sum_{\tau=0}^{\tau=t}\{e^{m_1(t-\tau)} - e^{m_2(t-\tau)}\}dB(\zeta,\tau)\right],$$

$$= \sum_{\tau=0}^{\tau=t}\{e^{m_1(t-\tau)} - e^{m_2(t-\tau)}\}\underbrace{E[\underbrace{dB(\zeta,\tau)}_{B_i}]}_{0} = 0.$$

$$E[X_B(\zeta,t)] = \underbrace{E[A_1(\zeta,0)e^{m_1 t} + A_2(\zeta,0)e^{m_2 t}]}_{\left\{E[A_1(\zeta,0)]e^{m_1 t} + E[A_2(\zeta,0)]e^{m_2 t}\right\}} + \underbrace{E[\tfrac{\alpha}{\kappa} I_B(\zeta)]}_{\frac{\alpha}{\kappa}\underbrace{E[I_B(\zeta)]}_{0}}$$

$$= E[A_1(\zeta,0)]e^{m_1 t} + E[A_2(\zeta,0)]e^{m_2 t}.\square$$

$$\mathrm{Var}[I_B(\zeta)] = E\left[I_B{}^2(\zeta)\right] - (\underbrace{E\left[I_B(\zeta)\right]}_{0})^2,$$

$$= \sum_{\tau=0}^{\tau=t}\{e^{m_1(t-\tau)} - e^{m_2(t-\tau)}\}^2 \quad \underbrace{E[B_i^2]}_{(dx)^2=(2D)d\tau} \quad ,$$

$$= 2D\int_{\tau=0}^{\tau=t}\{e^{2m_1(t-\tau)} - e^{-\beta(t-\tau)} + e^{2m_2(t-\tau)}\}d\tau\,.$$

$$= 2D\left\{\frac{e^{2m_1 t}-1}{2m_1} + \frac{e^{-\beta t}-1}{\beta} + \frac{e^{2m_2 t}-1}{2m_2}\right\}$$

$$\operatorname{Var}[X_B(\zeta,t)] = \underbrace{\operatorname{Var}\Big[\{A_1(\zeta,0)e^{m_1 t} + A_2(\zeta,0)e^{m_2 t}\}\Big]}_{e^{2m_1 t}\operatorname{Var}[A_1(\zeta,0)]+e^{2m_2 t}\operatorname{Var}[A_2(\zeta,0)]} + \underbrace{\operatorname{Var}\Big[\tfrac{\alpha}{\kappa}I_B(\zeta)\Big]}_{\frac{\alpha^2}{\kappa^2}\operatorname{Var}[I_B(\zeta)]}$$

$$= e^{2m_1 t}\operatorname{Var}\big[A_1(\zeta,0)\big] + e^{2m_2 t}\operatorname{Var}\big[A_2(\zeta,0)\big] +$$

$$+\frac{\alpha^2}{\kappa^2}2D\left\{\frac{e^{2m_1 t}-1}{2m_1} + \frac{e^{-\beta t}-1}{\beta} + \frac{e^{2m_2 t}-1}{2m_2}\right\}$$

$$= e^{-(\beta-\kappa)t}\operatorname{Var}\big[A_1(\zeta,0)\big] + e^{-(\beta+\kappa)t}\operatorname{Var}\big[A_2(\zeta,0)\big] +$$

$$+\frac{\alpha^2}{\kappa^2}2D\left\{-\frac{e^{-(\beta-\kappa)t}}{\beta-\kappa} + \frac{e^{-\beta t}}{\beta} - \frac{e^{-(\beta+\kappa)t}}{\beta+\kappa}\right\}$$

$$\underbrace{+\frac{\alpha^2}{\kappa^2}2D\underbrace{\left\{\frac{1}{\beta-\kappa} - \frac{1}{\beta} + \frac{1}{\beta+\kappa}\right\}}_{\frac{2\beta}{\beta^2-\kappa^2}-\frac{1}{\beta}=\frac{\beta}{2\omega^2}-\frac{1}{\beta}}}_{D\frac{\alpha^2}{\beta\omega^2}\frac{\beta^2-2\omega^2}{\beta^2-4\omega^2}}.\square$$

12.2 The Harmonic Process Equilibrium State

At Equilibrium, $t =$ infinite hyper-real Θ,

$$E[X_B(\zeta,\Theta)] \approx 0\,,$$

$$\mathrm{Var}[X_B(\zeta,\Theta)] \approx D\frac{\alpha^2}{\beta\omega^2}\frac{\beta^2-2\omega^2}{\beta^2-4\omega^2} > 0\,,$$

provided $\beta^2 - 4\omega^2 > 0$.

<u>*Proof*</u>:

$$E[X_B(\zeta,\Theta)] = E[A_1(\zeta,0)]e^{m_1\Theta} + E[A_2(\zeta,0)]e^{m_2\Theta}$$

$$= E[A_1(\zeta,0)]e^{-\frac{1}{2}(\beta-\sqrt{\beta^2-4\omega^2})\Theta} + E[A_2(\zeta,0)]e^{-\frac{1}{2}(\beta+\sqrt{\beta^2-4\omega^2})\Theta}.$$

Since we assume $\beta^2 > 4\omega^2$, then,

$\beta - \sqrt{\beta^2-4\omega^2} > 0$, and $e^{-\frac{1}{2}(\beta-\sqrt{\beta^2-4\omega^2})\Theta} \approx 0$

$\beta + \sqrt{\beta^2-4\omega^2} > 0$, and $e^{-\frac{1}{2}(\beta+\sqrt{\beta^2-4\omega^2})\Theta} \approx 0$

Hence, $E[X_B(\zeta,\Theta)] \approx 0.\square$

$$\mathrm{Var}[X_B(\zeta,\Theta)] = \underbrace{e^{-(\beta-\kappa)\Theta}\,\mathrm{Var}\big[A_1(\zeta,0)\big]}_{\approx 0,\ \text{since}\ \beta-\kappa>0} + \underbrace{e^{-(\beta+\kappa)\Theta}\,\mathrm{Var}\big[A_2(\zeta,0)\big]}_{\approx 0,\ \text{since}\ \beta+\kappa>0} +$$

$$+\frac{\alpha^2}{\kappa^2}2D\underbrace{\left\{-\frac{e^{-(\beta-\kappa)\Theta}}{\beta-\kappa}+\frac{e^{-\beta\Theta}}{\beta}-\frac{e^{-(\beta+\kappa)\Theta}}{\beta+\kappa}\right\}}_{\approx 0}+$$

$$+D\frac{\alpha^2}{\beta\omega^2}\frac{\beta^2-2\omega^2}{\beta^2-4\omega^2}$$

$\approx D\dfrac{\alpha^2}{\beta\omega^2}\dfrac{\beta^2-2\omega^2}{\beta^2-4\omega^2} > 0$, since $\beta > 0$, and $\beta^2 - 4\omega^2 > 0.\square$

12.3 Normal Distribution of the Process $\dot{X}_B(\zeta, t)$

$$\dot{X}_B(\zeta,t) = A_1(\zeta,t)m_1e^{m_1t} + A_2(\zeta,t)m_2e^{m_2t} +$$

$$+\frac{\alpha}{\kappa}\int_{\tau=0}^{\tau=t}\{m_1e^{m_1(t-\tau)} - m_2e^{m_2(t-\tau)}\}dB(\zeta,\tau)$$

is Normally Distributed with

Mean

$$E[\dot{X}_B(\zeta,t)] = m_1E[A_1(\zeta,0)]e^{m_1t} + m_2E[A_2(\zeta,0)]e^{m_2t},$$

and Variance

$$\mathrm{Var}[\dot{X}_B(\zeta,t)] = {m_1}^2e^{-(\beta-\kappa)t}\,\mathrm{Var}\big[A_1(\zeta,0)\big] +$$

$$+{m_2}^2e^{-(\beta+\kappa)t}\,\mathrm{Var}\big[A_2(\zeta,0)\big] +$$

$$+\tfrac{\alpha^2}{\kappa^2}D\Big\{-\tfrac{\beta-\kappa}{2}e^{-(\beta-\kappa)t} + 4\tfrac{\omega^2}{\beta}e^{-\beta t} - \tfrac{\beta+\kappa}{2}e^{-(\beta+\kappa)t}\Big\} + D\tfrac{\alpha^2}{2\beta},$$

provided $\kappa = \sqrt{\beta^2 - 4\omega^2} > 0$,

$$m_1 = -\tfrac{1}{2}\beta + \tfrac{1}{2}\kappa,$$

$$m_2 = -\tfrac{1}{2}\beta - \tfrac{1}{2}\kappa.$$

Proof: Since $dB(\zeta,\tau)$ is Normally Distributed, so is

$$J_B(\zeta) = \int_{\tau=0}^{\tau=t}\{m_1e^{m_1(t-\tau)} - m_2e^{m_2(t-\tau)}\}dB(\zeta,\tau).$$ Hence, $X_B(\zeta,t)$

has Normal Distribution.

$$E[J_B(\zeta)] = E\left[\sum_{\tau=0}^{\tau=t}\{m_1 e^{m_1(t-\tau)} - m_2 e^{m_2(t-\tau)}\}dB(\zeta,\tau)\right],$$

$$= \sum_{\tau=0}^{\tau=t}\{m_1 e^{m_1(t-\tau)} - m_2 e^{m_2(t-\tau)}\}\underbrace{E[\underbrace{dB(\zeta,\tau)}_{B_i}]}_{0} = 0.$$

$$E[\dot{X}_B(\zeta,t)] = \underbrace{E[m_1 A_1(\zeta,0)e^{m_1 t} + m_2 A_2(\zeta,0)e^{m_2 t}]}_{\left\{m_1 E[A_1(\zeta,0)]e^{m_1 t} + m_2 E[A_2(\zeta,0)]e^{m_2 t}\right\}} + \underbrace{E[2\tfrac{\alpha}{\kappa}J_B(\zeta)]}_{2\frac{\alpha}{\kappa}\underbrace{E[J_B(\zeta)]}_{0}}$$

$$= m_1 E[A_1(\zeta,0)]e^{m_1 t} + m_2 E[A_2(\zeta,0)]e^{m_2 t}.\square$$

$$\mathrm{Var}[J_B(\zeta)] = E\left[J_B{}^2(\zeta)\right] - (\underbrace{E\left[J_B(\zeta)\right]}_{0})^2,$$

$$= \sum_{\tau=0}^{\tau=t}\{m_1 e^{m_1(t-\tau)} - m_2 e^{m_2(t-\tau)}\}^2 \underbrace{E[B_i^2]}_{(dx)^2=(2D)d\tau},$$

$$= 2D\int_{\tau=0}^{\tau=t}\{m_1{}^2 e^{2m_1(t-\tau)} - 2m_1 m_2 e^{-\beta(t-\tau)} + m_2{}^2 e^{2m_2(t-\tau)}\}d\tau.$$

$$= 2D\left\{m_1\frac{e^{2m_1 t}-1}{2} + 2\underbrace{m_1 m_2}_{\omega^2}\frac{e^{-\beta t}-1}{\beta} + m_2\frac{e^{2m_2 t}-1}{2}\right\}$$

$$\mathrm{Var}[\dot{X}_B(\zeta,t)] = \underbrace{\mathrm{Var}\left[\{m_1 A_1(\zeta,0)e^{m_1 t} + m_2 A_2(\zeta,0)e^{m_2 t}\}\right]}_{m_1{}^2 e^{2m_1 t}\mathrm{Var}\left[A_1(\zeta,0)\right] + m_2{}^2 e^{2m_2 t}\mathrm{Var}\left[A_2(\zeta,0)\right]} + \underbrace{\mathrm{Var}\left[\tfrac{\alpha}{\kappa}J_B(\zeta)\right]}_{\frac{\alpha^2}{\kappa^2}\mathrm{Var}\left[J_B(\zeta)\right]}$$

$$= {m_1}^2 e^{2m_1 t}\,\mathrm{Var}\big[A_1(\zeta,0)\big] + {m_2}^2 e^{2m_2 t}\,\mathrm{Var}\big[A_2(\zeta,0)\big] +$$

$$+\frac{\alpha^2}{\kappa^2} 2D\left\{m_1 \frac{e^{2m_1 t}-1}{2} + 2\omega^2 \frac{e^{-\beta t}-1}{\beta} + m_2 \frac{e^{2m_2 t}-1}{2}\right\}$$

$$= {m_1}^2 e^{-(\beta-\kappa)t}\,\mathrm{Var}\big[A_1(\zeta,0)\big] + {m_2}^2 e^{-(\beta+\kappa)t}\,\mathrm{Var}\big[A_2(\zeta,0)\big] +$$

$$+D\tfrac{\alpha^2}{\kappa^2} e^{-\beta t}\left\{-\tfrac{1}{2}(\beta-\kappa)e^{\kappa t} + 4\tfrac{\omega^2}{\beta} - \tfrac{1}{2}(\beta+\kappa)e^{-\kappa t}\right\}$$

$$\underbrace{+D\tfrac{\alpha^2}{\kappa^2}\underbrace{\left\{\tfrac{1}{2}(\beta-\kappa) - 4\tfrac{\omega^2}{\beta} + \tfrac{1}{2}(\beta+\kappa)\right\}}_{\frac{\beta^2-4\omega^2}{\beta}}}_{D\frac{\alpha^2}{\beta}}.\square$$

12.4 The Time-Rate Process Equilibrium State

At Equilibrium, $t =$ infinite hyper-real Θ,

$$E[\dot{X}_B(\zeta,\Theta)] \approx 0,$$

$$\mathrm{Var}[\dot{X}_B(\zeta,\Theta)] \approx D\frac{\alpha^2}{\beta} > 0,$$

provided $\beta^2 - 4\omega^2 > 0$.

<u>*Proof*</u>:

$$E[\dot{X}_B(\zeta,\Theta)] = m_1 E[A_1(\zeta,0)]e^{m_1\Theta} + m_2 E[A_2(\zeta,0)]e^{m_2\Theta}$$

$$= m_1 E[A_1(\zeta,0)]e^{-\frac{1}{2}(\beta-\sqrt{\beta^2-4\omega^2})\Theta} + m_2 E[A_2(\zeta,0)]e^{-\frac{1}{2}(\beta+\sqrt{\beta^2-4\omega^2})\Theta}.$$

Since we assume $\beta^2 > 4\omega^2$, then,

$$\beta - \sqrt{\beta^2 - 4\omega^2} > 0, \text{ and } e^{-\frac{1}{2}(\beta - \sqrt{\beta^2 - 4\omega^2})\Theta} \approx 0$$

$$\beta + \sqrt{\beta^2 - 4\omega^2} > 0, \text{ and } e^{-\frac{1}{2}(\beta + \sqrt{\beta^2 - 4\omega^2})\Theta} \approx 0$$

Hence, $E[\dot{X}_B(\zeta,\Theta)] \approx 0.\square$

$$\mathrm{Var}[\dot{X}_B(\zeta,\Theta)] = \underbrace{m_1{}^2 e^{-(\beta-\kappa)\Theta}\,\mathrm{Var}\big[A_1(\zeta,0)\big]}_{\approx 0,\text{ since } \beta-\kappa>0} +$$

$$+\underbrace{m_2{}^2 e^{-(\beta+\kappa)\Theta}\,\mathrm{Var}\big[A_2(\zeta,0)\big]}_{\approx 0,\text{ since } \beta+\kappa>0} +$$

$$+\frac{\alpha^2}{\kappa^2} 2D \underbrace{\left\{-\frac{\beta-\kappa}{4} e^{-(\beta-\kappa)\Theta} + 2\frac{\omega^2}{\beta} e^{-\beta\Theta} - \frac{\beta+\kappa}{4} e^{-(\beta+\kappa)\Theta}\right\}}_{\approx 0} +$$

$$+D\frac{\alpha^2}{\beta}$$

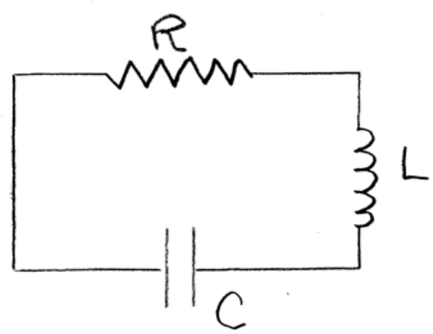
$$\approx D\frac{\alpha^2}{\beta} > 0,$$

since $\beta > 0.\square$

13.

RLC Harmonic Oscillator Driven by Thermal Noise Voltage $\dot{B}(\zeta, t)$

13.1 The Evolution Equation of RLC Harmonic Process driven by Thermal Noise Voltage $\dot{B}(\zeta, t)$



Thermal Noise in the Resistor R, generates Random Voltage $\dot{B}(\zeta, t)$ on the Resistor R, that drives a Random Current $i_B(\zeta, t) = \dfrac{dq_B(\zeta, t)}{dt}$ through the Circuit.

$i_B(\zeta, t)R = \dot{q}_B(\zeta, t)R$ is the Random Voltage on R.

$L\dfrac{di_B(\zeta, t)}{dt} = L\ddot{q}_B(\zeta, t)$ is the Random Voltage on L.

$\dfrac{q_B(\zeta,t)}{C}$ is the Random Voltage on C.

The balance of voltages on the circuit components is

$$L\ddot{q}_B(\zeta,t) + \dot{q}_B(\zeta,t)R + \tfrac{1}{C}q_B(\zeta,t) = \dot{B}(\zeta,t),$$

$$\ddot{q}_B(\zeta,t) = \underbrace{\tfrac{1}{L}}_{\alpha}\dot{B}(\zeta,t) - \underbrace{\tfrac{R}{L}}_{\beta}\dot{q}_B(\zeta,t) - \underbrace{\tfrac{1}{LC}}_{\omega^2}q_B(\zeta,t).$$

13.2 The Thermal Noise Charge $q_B(\zeta,t)$

$$q_B(\zeta,t) = e^{-\frac{R}{2L}t}\{A_1(\zeta,0)e^{\frac{\sqrt{R^2C^2-4LC}}{2LC}t} + A_2(\zeta,0)e^{-\frac{\sqrt{R^2C^2-4LC}}{2LC}t}\} +$$

$$+\frac{C}{\sqrt{R^2C^2-4LC}}\int_{\tau=0}^{\tau=t}\{e^{-\left\{\frac{R}{2L}-\frac{\sqrt{R^2C^2-4LC}}{2LC}\right\}(t-\tau)} - e^{-\left\{\frac{R}{2L}+\frac{\sqrt{R^2C^2-4LC}}{2LC}\right\}(t-\tau)}\}dB(\zeta,\tau)$$

<u>*Proof*</u>: By 11.2, the Harmonic Oscillator evolution equation

$$d\dot{q}_B(\zeta,t) = \alpha dB(\zeta,t) - \beta\dot{q}_B(\zeta,t)dt - \omega^2 q_B(\zeta,t)dt.$$

is solved by the Process

$$q_B(\zeta,t) = e^{-\frac{1}{2}\beta t}\{A_1(\zeta,0)e^{\frac{1}{2}\kappa t} + A_2(\zeta,0)e^{-\frac{1}{2}\kappa t}\} +$$

$$+\frac{\alpha}{\kappa}\int_{\tau=0}^{\tau=t}\{e^{m_1(t-\tau)} - e^{m_2(t-\tau)}\}dB(\zeta,\tau),$$

provided $\kappa = \sqrt{\beta^2 - 4\omega^2} > 0$,

$$m_1 = -\tfrac{1}{2}\beta + \tfrac{1}{2}\kappa,$$

$$m_2 = -\tfrac{1}{2}\beta - \tfrac{1}{2}\kappa.$$

Substituting

$$\alpha = \frac{1}{L}, \qquad \beta = \frac{R}{L}, \qquad \omega^2 = \frac{1}{LC},$$

then,

$$\kappa = \sqrt{\tfrac{R^2}{L^2} - 4\tfrac{1}{LC}} = \tfrac{1}{LC}\sqrt{R^2C^2 - 4LC},$$

$$m_1 = -\tfrac{R}{2L} + \tfrac{\sqrt{R^2C^2-4LC}}{2LC},$$

$$m_1 = -\tfrac{R}{2L} - \tfrac{\sqrt{R^2C^2-4LC}}{2LC},$$

$$q_B(\zeta,t) = e^{-\frac{R}{2L}t}\left\{A_1(\zeta,0)e^{\frac{\sqrt{R^2C^2-4LC}}{2LC}t} + A_2(\zeta,0)e^{-\frac{\sqrt{R^2C^2-4LC}}{2LC}t}\right\} +$$

$$+\frac{C}{\sqrt{R^2C^2-4LC}}\int_{\tau=0}^{\tau=t}\{e^{-\left\{\frac{R}{2L}-\frac{\sqrt{R^2C^2-4LC}}{2LC}\right\}(t-\tau)} - e^{-\left\{\frac{R}{2L}+\frac{\sqrt{R^2C^2-4LC}}{2LC}\right\}(t-\tau)}\}dB(\zeta,\tau).$$

13.3 Normal Distribution of the Thermal Noise Charge

$$q_B(\zeta,t) = e^{-\frac{R}{2L}t}\left\{A_1(\zeta,0)e^{\frac{\sqrt{R^2C^2-4LC}}{2LC}t} + A_2(\zeta,0)e^{-\frac{\sqrt{R^2C^2-4LC}}{2LC}t}\right\} +$$

$$+\frac{C}{\sqrt{R^2C^2-4LC}}\int_{\tau=0}^{\tau=t}\{e^{-\left\{\frac{R}{2L}-\frac{\sqrt{R^2C^2-4LC}}{2LC}\right\}(t-\tau)} - e^{-\left\{\frac{R}{2L}+\frac{\sqrt{R^2C^2-4LC}}{2LC}\right\}(t-\tau)}\}dB(\zeta,\tau)$$

is Normally Distributed with

Mean

$$E[q_B(\zeta,t)] = e^{-\frac{R}{2L}t}\{E[A_1(\zeta,0)]e^{\frac{\sqrt{R^2C^2-4LC}}{2LC}t} + E[A_2(\zeta,0)]e^{-\frac{\sqrt{R^2C^2-4LC}}{2LC}t}\},$$

and Variance

$$\mathrm{Var}[q_B(\zeta,t)] = e^{-\frac{R}{L}t}\{\mathrm{Var}[A_1(\zeta,0)]e^{\frac{\sqrt{R^2C^2-4LC}}{LC}t} + \mathrm{Var}[A_2(\zeta,0)]e^{-\frac{\sqrt{R^2C^2-4LC}}{LC}t}\}$$

$$+\frac{2DC^2LCe^{-\frac{R}{L}t}}{R^2C^2-4LC}\left\{\frac{e^{\frac{\sqrt{R^2C^2-4LC}}{LC}t}}{-RC+\sqrt{R^2C^2-4LC}} + \frac{1}{RC} - \frac{e^{-\frac{\sqrt{R^2C^2-4LC}}{LC}t}}{RC+\sqrt{R^2C^2-4LC}}\right\}$$

$$+D\frac{C}{R}\frac{R^2C-2L}{R^2C-4L}.$$

<u>Proof</u>: By 12.1,

$$E[q_B(\zeta,t)] = E[A_1(\zeta,0)]e^{m_1t} + E[A_2(\zeta,0)]e^{m_2t}$$

$$= e^{-\frac{R}{2L}t}\{E[A_1(\zeta,0)]e^{\frac{\sqrt{R^2C^2-4LC}}{2LC}t} + E[A_2(\zeta,0)]e^{-\frac{\sqrt{R^2C^2-4LC}}{2LC}t}\}.\square$$

$$\mathrm{Var}[q(\zeta,t)] = e^{-(\beta-\kappa)t}\,\mathrm{Var}\left[A_1(\zeta,0)\right] + e^{-(\beta+\kappa)t}\,\mathrm{Var}\left[A_2(\zeta,0)\right] +$$

$$+2D\frac{\alpha^2}{\kappa^2}\left\{-\frac{e^{-(\beta-\kappa)t}}{\beta-\kappa} + \frac{e^{-\beta t}}{\beta} - \frac{e^{-(\beta+\kappa)t}}{\beta+\kappa}\right\} +$$

$$+D\frac{\alpha^2}{\beta\omega^2}\frac{\beta^2-2\omega^2}{\beta^2-4\omega^2},$$

$$= e^{-\frac{R}{L}t}\{\mathrm{Var}[A_1(\zeta,0)]e^{\frac{\sqrt{R^2C^2-4LC}}{LC}t} + \mathrm{Var}[A_2(\zeta,0)]e^{-\frac{\sqrt{R^2C^2-4LC}}{LC}t}\} +$$

$$+\frac{2DC^2LCe^{-\frac{R}{L}t}}{R^2C^2-4LC}\left\{\frac{e^{\frac{\sqrt{R^2C^2-4LC}}{LC}t}}{-RC+\sqrt{R^2C^2-4LC}}+\frac{1}{RC}-\frac{e^{-\frac{\sqrt{R^2C^2-4LC}}{LC}t}}{RC+\sqrt{R^2C^2-4LC}}\right\}$$

$$+D\frac{C}{R}\frac{R^2C-2L}{R^2C-4L}.\square$$

13.4 The Thermal Noise Charge Steady State

At the Steady State, $t =$ infinite hyper-real Θ,

$$E[q_B(\zeta,\Theta)] \approx 0.$$

$$\mathrm{Var}[q_B(\zeta,\Theta)] \approx D\frac{C}{R}\frac{R^2C-2L}{R^2C-4L}.$$

13.5 The Noise Energy of $q_B(\zeta,t)$ at the Steady State

$$\tfrac{1}{2}kT = \tfrac{1}{2C}E[{q_B}^2(\zeta,\Theta)] = \frac{D}{2R}\frac{R^2C-2L}{R^2C-4L},$$

where k is Boltzmann Constant,

and T is the absolute Temperature.

Proof:

$$\tfrac{1}{2C}E[{q_B}^2(\zeta,\Theta)] = \tfrac{1}{2C}\{\underbrace{\mathrm{Var}[q_B(\zeta,\Theta)]}_{D\frac{C}{R}\frac{R^2C-2L}{R^2C-4L}} + (\underbrace{E[q_B(\zeta,\Theta)]}_{0})^2\}$$

$$= \tfrac{D}{2R}\tfrac{R^2C-2L}{R^2C-4L}.\square$$

13.6 The Thermal Noise Current $i_B(\zeta, t)$

$$\dot{q}_B(\zeta,t) = e^{-\frac{R}{2L}t}\left(-\frac{R}{2L} + \frac{\sqrt{R^2C^2-4LC}}{2LC}\right)A_1(\zeta,0)e^{\frac{\sqrt{R^2C^2-4LC}}{2LC}t} +$$

$$-e^{-\frac{R}{2L}t}\left(\frac{R}{2L} + \frac{\sqrt{R^2C^2-4LC}}{2LC}\right)A_2(\zeta,0)e^{-\frac{\sqrt{R^2C^2-4LC}}{2LC}t} +$$

$$-\frac{C}{\sqrt{R^2C^2-4LC}}\left(\frac{R}{2L} - \frac{\sqrt{R^2C^2-4LC}}{2LC}\right)\int_{\tau=0}^{\tau=t} e^{-[\frac{R}{2L}-\frac{\sqrt{R^2C^2-4LC}}{2LC}](t-\tau)}dB(\zeta,\tau) +$$

$$+\frac{C}{\sqrt{R^2C^2-4LC}}\left(\frac{R}{2L} + \frac{\sqrt{R^2C^2-4LC}}{2LC}\right)\int_{\tau=0}^{\tau=t} e^{-[\frac{R}{2L}+\frac{\sqrt{R^2C^2-4LC}}{2LC}](t-\tau)}dB(\zeta,\tau).$$

Proof: By 11.3,

$$\dot{q}_{_B}(\zeta,t) = e^{-\frac{1}{2}\beta t}\{m_1A_1(\zeta,0)e^{\frac{1}{2}\kappa t} + m_2A_2(\zeta,0)e^{-\frac{1}{2}\kappa t}\}$$

$$+\frac{\alpha}{\kappa}\int_{\tau=0}^{\tau=t}\{m_1e^{m_1(t-\tau)} - m_2e^{m_2(t-\tau)}\}dB(\zeta,\tau),$$

$$= e^{-\frac{R}{2L}t}\left(-\frac{R}{2L} + \frac{\sqrt{R^2C^2-4LC}}{2LC}\right)A_1(\zeta,0)e^{\frac{\sqrt{R^2C^2-4LC}}{2LC}t} +$$

$$-e^{-\frac{R}{2L}t}\left(\frac{R}{2L} + \frac{\sqrt{R^2C^2-4LC}}{2LC}\right)A_2(\zeta,0)e^{-\frac{\sqrt{R^2C^2-4LC}}{2LC}t} +$$

$$-\frac{C}{\sqrt{R^2C^2-4LC}}\left(\frac{R}{2L} - \frac{\sqrt{R^2C^2-4LC}}{2LC}\right)\int_{\tau=0}^{\tau=t} e^{-[\frac{R}{2L}-\frac{\sqrt{R^2C^2-4LC}}{2LC}](t-\tau)}dB(\zeta,\tau) +$$

$$+\frac{C}{\sqrt{R^2C^2-4LC}}\left(\frac{R}{2L} + \frac{\sqrt{R^2C^2-4LC}}{2LC}\right)\int_{\tau=0}^{\tau=t} e^{-[\frac{R}{2L}+\frac{\sqrt{R^2C^2-4LC}}{2LC}](t-\tau)}dB(\zeta,\tau).\square$$

13.7 Normal Distribution of Thermal Noise Current

$i_B(\zeta,t)=\dot{q}_B(\zeta,t)$ is Normally Distributed with

Mean

$$E[i_B(\zeta,t)]=e^{-\frac{R}{2L}t}\left(-\tfrac{R}{2L}+\tfrac{\sqrt{R^2C^2-4LC}}{2LC}\right)E[A_1(\zeta,0)]e^{\frac{\sqrt{R^2C^2-4LC}}{2LC}t}+$$

$$-e^{-\frac{R}{2L}t}\left(\tfrac{R}{2L}+\tfrac{\sqrt{R^2C^2-4LC}}{2LC}\right)E[A_2(\zeta,0)]e^{-\frac{\sqrt{R^2C^2-4LC}}{2LC}t},$$

and Variance

$$\mathrm{Var}[i_B(\zeta,t)]=e^{-\frac{R}{L}t}\left(-\tfrac{R}{2L}+\tfrac{\sqrt{R^2C^2-4LC}}{2LC}\right)^2e^{\frac{\sqrt{R^2C^2-4LC}}{LC}t}\,\mathrm{Var}\left[A_1(\zeta,0)\right]+$$

$$+e^{-\frac{R}{L}t}\left(\tfrac{R}{2L}+\tfrac{\sqrt{R^2C^2-4LC}}{2LC}\right)^2e^{-\frac{\sqrt{R^2C^2-4LC}}{LC}t}\,\mathrm{Var}\left[A_2(\zeta,0)\right]+$$

$$-e^{-\frac{R}{L}t}D\tfrac{C^2}{R^2C^2-4LC}\tfrac{RC-\sqrt{R^2C^2-4LC}}{2LC}e^{\frac{\sqrt{R^2C^2-4LC}}{LC}t}+$$

$$+e^{-\frac{R}{L}t}D\tfrac{C^2}{R^2C^2-4LC}\tfrac{4}{RC}+$$

$$-e^{-\frac{R}{L}t}D\tfrac{C^2}{R^2C^2-4LC}\tfrac{RC+\sqrt{R^2C^2-4LC}}{2LC}e^{-\frac{\sqrt{R^2C^2-4LC}}{LC}t}+\tfrac{D}{2RL}$$

Proof: By 12.3,

$$i_B(\zeta,t)=A_1(\zeta,t)m_1e^{m_1t}+A_2(\zeta,t)m_2e^{m_2t}+$$

$$+\frac{\alpha}{\kappa}\int_{\tau=0}^{\tau=t}\{m_1e^{m_1(t-\tau)}-m_2e^{m_2(t-\tau)}\}dB(\zeta,\tau)$$

is Normally Distributed with

Mean

$$E[i_B(\zeta,t)] = m_1 E[A_1(\zeta,0)]e^{m_1 t} + m_2 E[A_2(\zeta,0)]e^{m_2 t},$$

$$= e^{-\frac{R}{2L}t}\left(-\frac{R}{2L} + \frac{\sqrt{R^2C^2-4LC}}{2LC}\right)E[A_1(\zeta,0)]e^{\frac{\sqrt{R^2C^2-4LC}}{2LC}t} +$$

$$-e^{-\frac{R}{2L}t}\left(\frac{R}{2L} + \frac{\sqrt{R^2C^2-4LC}}{2LC}\right)E[A_2(\zeta,0)]e^{-\frac{\sqrt{R^2C^2-4LC}}{2LC}t}.\square$$

and Variance

$$\operatorname{Var}[i_B(\zeta,t)] = m_1^{\,2} e^{-(\beta-\kappa)t}\operatorname{Var}\left[A_1(\zeta,0)\right] +$$

$$+m_2^{\,2} e^{-(\beta+\kappa)t}\operatorname{Var}\left[A_2(\zeta,0)\right] +$$

$$+\frac{\alpha^2}{\kappa^2}2D\left\{-\frac{\beta-\kappa}{4}e^{-(\beta-\kappa)t} + 2\frac{\omega^2}{\beta}e^{-\beta t} - \frac{\beta+\kappa}{4}e^{-(\beta+\kappa)t}\right\} + D\frac{\alpha^2}{2\beta},$$

$$= e^{-\frac{R}{L}t}\left(-\frac{R}{2L} + \frac{\sqrt{R^2C^2-4LC}}{2LC}\right)^2 e^{\frac{\sqrt{R^2C^2-4LC}}{LC}t}\operatorname{Var}\left[A_1(\zeta,0)\right] +$$

$$+e^{-\frac{R}{L}t}\left(\frac{R}{2L} + \frac{\sqrt{R^2C^2-4LC}}{2LC}\right)^2 e^{-\frac{\sqrt{R^2C^2-4LC}}{LC}t}\operatorname{Var}\left[A_2(\zeta,0)\right] +$$

$$-e^{-\frac{R}{L}t}D\frac{C^2}{R^2C^2-4LC}\frac{RC-\sqrt{R^2C^2-4LC}}{2LC}e^{\frac{\sqrt{R^2C^2-4LC}}{LC}t} +$$

$$+e^{-\frac{R}{L}t}D\frac{C^2}{R^2C^2-4LC}\frac{4}{RC} +$$

$$-e^{-\frac{R}{L}t}D\frac{C^2}{R^2C^2-4LC}\frac{RC+\sqrt{R^2C^2-4LC}}{2LC}e^{-\frac{\sqrt{R^2C^2-4LC}}{LC}t} + \frac{D}{2RL}.\square$$

13.8 The Thermal Noise Current Steady State

At the Steady State, $t =$ infinite hyper-real Θ,

$$E[i_B(\zeta,\Theta)] \approx 0.$$

$$\mathrm{Var}[i_B(\zeta,\Theta)] \approx \frac{D}{2RL}.$$

13.9 Thermal Noise Energy of $i_B(\zeta,t)$ at Steady State

$$\tfrac{1}{2}kT = \tfrac{1}{2}E[{i_B}^2(\zeta,\Theta)]L = \frac{D}{4R},$$

where k is Boltzmann Constant,

and T is the absolute Temperature.

Proof:

$$\tfrac{1}{2}E[{i_B}^2(\zeta,\Theta)]L = \tfrac{1}{2}\{\underbrace{\mathrm{Var}[i_B(\zeta,\Theta)]}_{\frac{D}{2RL}} + (\underbrace{E[i_B(\zeta,\Theta)]}_{0})^2\}L$$

$$= \frac{D}{4R}.\square$$

13.10 RLC Harmonic Oscillator Thermal Noise Energy

$$\frac{D}{2R}\left\{\frac{R^2C-2L}{R^2C-4L}+\frac{1}{2}\right\}.$$

14.

Poisson Process $P(\zeta, t)$

The arrival at rate λ, of radioactive particles at a counter is modeled by the Poisson Process. It models other processes, such as the arrival of phone calls at rate λ, to an operator.

14.1 The Bernoulli Random Variables of the Process

We assume that

an arrival probability in time dt *is*

$$p = \lambda dt,$$

and no arrival probability in time dt *is*

$$q = 1 - \lambda dt.$$

At fixed time t, after

N infinitesimal time intervals dt,

$N = \frac{t}{dt}$, is an infinite hyper-real,

there are

k arrivals,

k is a finite hyper-real

and

$N - k$ no arrivals,

$$N - k \text{ is an infinite Hyper-real}$$

At the ith step we define the Bernoulli Random Variable,

$$P_i(\text{arrival}) = 1, \qquad \zeta_1 = \text{arrival}$$

$$P_i(\text{no-arrival}) = 0, \qquad \zeta_2 = \text{no-arrival}$$

where $i = 1, 2, ..., N$.

$$\Pr(P_i = 1) = p = \lambda dt,$$

$$\Pr(P_i = 0) = q = 1 - \lambda dt,$$

$$E[P_i] = 1 \cdot \lambda dt + 0 \cdot (1 - \lambda dt) = \lambda dt,$$

$$E[P_i^2] = 1^2 \cdot \lambda dt + 0^2 \cdot (1 - \lambda dt) = \lambda dt,$$

$$\mathrm{Var}[P_i] = \underbrace{E[P_i^2]}_{\lambda dt} - (\underbrace{E[P_i]}_{\lambda dt})^2,$$

$$= \lambda dt \underbrace{(1 - \lambda dt)}_{\approx 1} \approx \lambda dt.$$

14.2 The Binomial Distribution of the Process

$$P(\zeta, t) = P_1 + P_2 + ... + P_N$$

is a Random Process with

$$E[P(\zeta, t)] = \lambda t,$$

$$\mathrm{Var}[P(\zeta, t)] = \lambda t,$$

Proof: Since the P_i are independent,

$$E[P(\zeta,t)] = \underbrace{E[P_1]}_{\lambda dt} + ... + \underbrace{E[P_N]}_{\lambda dt} = \lambda \underbrace{Ndt}_{t}$$

$$\mathrm{Var}[P(\zeta,t)] = \underbrace{\mathrm{Var}[P_1]}_{\approx\lambda dt} + ... + \underbrace{\mathrm{Var}[P_N]}_{\approx\lambda dt} \approx \lambda \underbrace{Ndt}_{t}$$

14.3 $P(\zeta,t+dt) - P(\zeta,t)$ ***is a Bernoulli Random Variable***

14.4 Poisson Process is Continuous

Proof:

$$E[\{P(\zeta,t+dt) - P(\zeta,t)\}^2] =$$

$$= \mathrm{Var}[\underbrace{P(\zeta,t+dt) - P(\zeta,t)}_{P_i}] + (E[\underbrace{P(\zeta,t+dt) - P(\zeta,t)}_{P_i}])^2,$$

where X_i is a Bernoulli Random Variable,

$$= \underbrace{\mathrm{Var}[P_i]}_{\approx\lambda dt} + (\underbrace{E[P_i]}_{\lambda dt})^2 = \text{infinitesimal}.\square$$

14.5 The Derivative of the Poisson process

$$\dot{P} = \frac{1}{dt}P_i,$$

where (1) $P_i = P(\zeta,t_0+dt) - P(\zeta,t_0)$, *is a Bernoulli Random Variable.*

(2) $E[\dot{P}] = \lambda$,

(3) $\mathrm{Var}[\dot{P}] = \lambda\delta(t_0)$

Proof:

(1) For each $t = t_0$, we need to find a Random Signal $\dot{P}(\zeta, t_0)$, so that for any dt,

$$E\left[\left[\frac{P(\zeta, t_0 + dt) - P(\zeta, t_0)}{dt} - \dot{P}(\zeta, t_0)\right]^2\right] = \text{infinitesimal},$$

Since $P(\zeta, t_0 + dt) - P(\zeta, t_0)$, is a Bernoulli Random Variable P_i,

$$E\left[\left\{\frac{P(\zeta, t + dt) - P(\zeta, t)}{dt} - \dot{P}(\zeta, t)\right\}^2\right] = E\left[\left\{\frac{P_i}{dt} - \dot{P}\right\}^2\right]$$

Therefore, at time $t = t_0$, the Random Variable

$$\frac{1}{dt}P_i,$$

is the derivative of the Random Walk $P(\zeta, t_0)$.□

(2) $$E[\dot{P}] = \tfrac{1}{dt}\underbrace{E[P_i]}_{\lambda dt} = \lambda.\Box$$

(3) $$\mathrm{Var}[\dot{P}] = E[\dot{P}^2] - (\underbrace{E[\dot{P}]}_{\lambda})^2$$

$$= \frac{1}{(dt)^2} \underbrace{E[P_i^2]}_{\lambda dt + \lambda^2 (dt)^2} - \lambda^2$$

$$= \lambda \frac{1}{dt}$$

$$= \lambda \delta(t_0),$$

By [Dan4].□

15.

Integration sums of $f(t)$ with respect to $P(\zeta,t)$

Let $f(t)$ be a hyper-real function on the bounded time interval $[a,b]$. $f(t)$ need not be bounded.

At each $a \le t \le b$, there is a Bernoulli Random Variable

$$dP(\zeta,t) = P(\zeta,t+dt) - P(\zeta,t) = P_i(\zeta,t) = \dot{P}(\zeta,t)dt\,.$$

We form the **Integration Sum**

$$\sum_{t=a}^{t=b} f(t)dP(\zeta,t) = \sum_{t=a}^{t=b} f(t)P_i(\zeta,t) = \sum_{t=a}^{t=b} f(t)\dot{P}(\zeta,t)dt$$

For any dt,

(1) the First Moment of the Integration Sum is

$$E\left[\sum_{t=a}^{t=b} f(t)\dot{P}(\zeta,t)dt\right] = \sum_{t=a}^{t=b} f(t)\underbrace{E[\dot{P}(\zeta,t)]}_{\lambda}dt = \lambda\int_{t=a}^{t=b} f(t)dt\,,$$

assuming $f(t)$ integrable.

(2) the Second Moment of the Integration sum is

$$E\left[\left(\sum_{t=a}^{t=b} f(t)P_i(\zeta,t)\right)^2\right] = E\left[\left(\sum_{t=a}^{t=b} f(t)P_i(\zeta,t)\right)\left(\sum_{\tau=a}^{\tau=b} f(\tau)P_j(\zeta,\tau)\right)\right]$$

$$= \sum_{t=a}^{t=b}\sum_{\tau=a}^{\tau=b} f(t)f(\tau)E[P_j(\zeta,\tau)P_i(\zeta,t)]$$

Since the Bernoulli Random Variables are independent,

$$E[P_j(\zeta,\tau)P_i(\zeta,t)] = E[P_i^2(\zeta,t)] = \lambda dt(\underbrace{1+\lambda dt}_{\approx 1})$$

only for $t = \tau$. Then,

$$E\left[\left(\sum_{t=a}^{t=b} f(t)B_i(\zeta,t)\right)^2\right] = \lambda\sum_{t=a}^{t=b} f^2(t)dt\,,$$

$$= \lambda\int_{t=a}^{t=b} f^2(t)dt\,,$$

assuming $f(t)$ integrable.

Thus, assuming $f(t)$ integrable, for any dt, the Integration Sum is a unique well-defined hyper-real Random Variable $I_P(\zeta)$. We call $I_P(\zeta)$ the integral of $f(t)$, with respect to $P(\zeta,t)$ from $x = a$, to $x = b$, and denote it by

$$\int_{t=a}^{t=b} f(t)dP(\zeta,t).$$

16.

Evolution of Linear Oscillator due to Shot Noise Voltage $\dot{P}(\zeta, t)$

16.1 Shot Noise Voltage

Since the electron charge is $1.6 \times 10^{-19}\mathrm{Coulomb}$, a current of 10^{-3} Ampere has about 6.24×10^{15} electrons.

the number of electrons will fluctuate by millions of electrons per second, and the fluctuations modeled by a Poisson Process $P(\zeta, t)$, will generate a Shot Noise Voltage $\dot{P}(\zeta, t)$.

This noise which is independent of the temperature, and the frequency, will be added to the Thermal Noise that is proportional to the temperature, and to the Flicker Noise which spectral density is inversely proportional to the frequency.

At higher temperatures, shot noise is negligible compared to the Thermal Noise, and at lower frequencies, it is negligible compared to the Flicker noise.

But at low temperatures, and high frequencies, Shot Noise may be the main noise.

16.2 Evolution Equation of Linear Oscillator Process $X_P(\zeta,t)$ driven by Shot Noise Voltage

The balance of voltages in a linear oscillator circuit due to the Shot Noise Voltage Component is

$$dX_P(\zeta,t) = \alpha dP(\zeta,t) - \beta X_P(\zeta,t)dt\,.$$

16.3 Integrating Factor Solution

$$X_P(\zeta,t) = e^{-\beta t}X_P(\zeta,0) + \alpha\int_{\tau=0}^{\tau=t} e^{-\beta(t-\tau)}dP(\zeta,\tau)$$

Proof: following the proof of 7.2.□

We obtain the same solution for the linear oscillator evolution equation by the Variation of Parameters Method.

17.

Linear Oscillator Process $X_P(\zeta, t)$ due to Shot Noise Voltage $\dot{P}(\zeta, t)$

17.1 The Poisson Distribution of $X_P(\zeta, t)$

$$X_P(\zeta, t) = e^{-\beta t} X_P(\zeta, 0) + \alpha \int_{\tau=0}^{\tau=t} e^{-\beta(t-\tau)} dP(\zeta, \tau)$$

is Poisson Distributed with

Mean

$$E[X_P(\zeta, t)] = \lambda \tfrac{\alpha}{\beta} + (E[X_P(\zeta, 0)] - 1) e^{-\beta t},$$

and Variance

$$\mathrm{Var}[X_P(\zeta, t)] = e^{-2\beta t}\, \mathrm{Var}\left[X_P(\zeta, 0) \right] + \lambda \tfrac{\alpha^2}{\beta}(1 - e^{-2\beta t})(\tfrac{1}{2} - \tfrac{\lambda}{\beta}).$$

<u>*Proof*</u>: Since $dP(\zeta, \tau)$ is Poisson Distributed, so is

$I_P(\zeta) = \int_{\tau=0}^{\tau=t} e^{-\beta(t-\tau)} dP(\zeta, \tau)$. Hence, $X_P(\zeta, t)$ is Poisson distributed.

$$E[I_P(\zeta)] = E\left[\sum_{\tau=0}^{\tau=t} e^{\beta\tau} dP(\zeta,\tau)\right],$$

$$= \sum_{\tau=0}^{\tau=t} e^{\beta\tau} \underbrace{E[\underbrace{dP(\zeta,\tau)}_{P_i}]}_{\lambda d\tau}$$

$$= \lambda \int_{\tau=0}^{\tau=t} e^{\beta\tau} d\tau = \tfrac{\lambda}{\beta}(e^{\beta t} - 1).$$

$$E[X_P(\zeta,t)] = \underbrace{E[e^{-\beta t} X_P(\zeta,0)]}_{e^{-\beta t}E[X_P(\zeta,0)]} + \underbrace{E[\alpha e^{-\beta t} I_P(\zeta)]}_{\alpha e^{-\beta t}\underbrace{E[I_P(\zeta)]}_{\frac{\lambda}{\beta}(e^{\beta t}-1)}},$$

$$= e^{-\beta t}E[X_P(\zeta,0)] + \lambda \tfrac{\alpha}{\beta}(1 - e^{-\beta t})$$

$$= \lambda \tfrac{\alpha}{\beta} + (E[X_P(\zeta,0)] - 1)e^{-\beta t}.\square$$

$$\mathrm{Var}[I_P(\zeta)] = E\Big[{I_P}^2(\zeta) \Big] - \underbrace{E\Big[I_P(\zeta) \Big]}_{\frac{\lambda}{\beta}(e^{\beta t}-1)}{}^2,$$

$$= \sum_{\tau=0}^{\tau=t} e^{2\beta\tau} \underbrace{E[P_i^2]}_{\lambda dt} - \tfrac{\lambda^2}{\beta^2}(e^{\beta t} - 1)^2,$$

$$= \lambda \sum_{\tau=0}^{\tau=t} e^{2\beta\tau} d\tau - \tfrac{\lambda^2}{\beta^2}(e^{\beta t} - 1)^2$$

$$= \lambda \tfrac{1}{2\beta}(e^{2\beta t} - 1) - \tfrac{\lambda^2}{\beta^2}(e^{\beta t} - 1)^2$$

$$= \tfrac{\lambda}{\beta}(e^{\beta t} - 1)\Big[\tfrac{e^{\beta t}+1}{2} - \tfrac{\lambda}{\beta}(e^{\beta t} - 1)\Big]$$

$$= \tfrac{\lambda}{\beta}(e^{\beta t} - 1)(e^{\beta t} + 1)(\tfrac{1}{2} - \tfrac{\lambda}{\beta})$$

$$= \tfrac{\lambda}{\beta}(e^{2\beta t} - 1)(\tfrac{1}{2} - \tfrac{\lambda}{\beta}).$$

$$\mathrm{Var}[X_P(\zeta,t)] = \underbrace{\mathrm{Var}\Big[e^{-\beta t}X_P(\zeta,0)\Big]}_{e^{-2\beta t}\,\mathrm{Var}\big[X_P(\zeta,0)\big]} + \underbrace{\mathrm{Var}\Big[\alpha e^{-\beta t}I_P(\zeta)\Big]}_{\alpha^2 e^{-2\beta t}\,\mathrm{Var}\big[I_P(\zeta)\big]}$$

$$= e^{-2\beta t}\,\mathrm{Var}\Big[X_P(\zeta,0)\Big] + \lambda\tfrac{\alpha^2}{\beta}(1 - e^{-2\beta t})(\tfrac{1}{2} - \tfrac{\lambda}{\beta}).\square$$

17.2 The Linear Oscillator Process Steady State

At Steady State, $t =$ infinite hyper-real Θ,

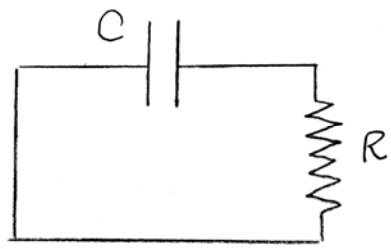
$$E[X_P(\zeta,\Theta)] \approx \lambda\frac{\alpha}{\beta}$$

$$\mathrm{Var}[X_P(\zeta,\Theta)] \approx \lambda\frac{\alpha^2}{\beta}(\tfrac{1}{2} - \tfrac{\lambda}{\beta}).$$

18.

RC Oscillator Process driven by Shot Noise Voltage $\dot{P}(\zeta, t)$

18.1 Evolution Equation of $q_P(\zeta, t)$ in RC Linear Oscillator due to Shot Noise Voltage $\dot{P}(\zeta, t)$



Current fluctuations in the circuit, generate Shot Noise Voltage $\dot{P}(\zeta, t)$ that drives a random current $i_P(\zeta, t) = \dfrac{dq_P(\zeta, t)}{dt}$ through the Circuit.

$q_P(\zeta, t)$ is the random charge on the Capacitor C

The balance of voltages due to the shot noise is

$$\frac{dq_P(\zeta, t)}{dt} R + \frac{q_P(\zeta, t)}{C} = \dot{P}(\zeta, t),$$

$$dq_P(\zeta,t) = \underbrace{\tfrac{1}{R}}_{\alpha}\underbrace{\dot{P}(\zeta,t)dt}_{dP(\zeta,t)} - \underbrace{\tfrac{1}{RC}}_{\beta}q_P(\zeta,t)dt\,.$$

18.2 The Random Charge Process $q_P(\zeta,t)$

$$q_P(\zeta,t) = e^{-\frac{1}{RC}t}q_P(\zeta,0) + \tfrac{1}{R}\int_{\tau=0}^{\tau=t} e^{-\frac{1}{RC}(t-\tau)}dP(\zeta,\tau)$$

Proof: By 16.3,

$$q_P(\zeta,t) = e^{-\beta t}q_P(\zeta,0) + \alpha\int_{\tau=0}^{\tau=t} e^{-\beta(t-\tau)}dP(\zeta,\tau),$$

By 18.1, $\alpha = \frac{1}{R}$, $\beta = \frac{1}{RC}$,

$$= e^{-\frac{1}{RC}t}q_P(\zeta,0) + \tfrac{1}{R}\int_{\tau=0}^{\tau=t} e^{-\frac{1}{RC}(t-\tau)}dP(\zeta,\tau).\square$$

18.3 The Poisson Distribution of the Random Charge

$$q_P(\zeta,t) = e^{-\frac{1}{RC}t}q_P(\zeta,0) + \tfrac{1}{R}\int_{\tau=0}^{\tau=t} e^{-\frac{1}{RC}(t-\tau)}dP(\zeta,\tau)$$

is Poisson Distributed with

Mean

$$E[q(_P\zeta,t)] = \lambda C + \big[E[q_P(\zeta,0)] - 1\big]e^{-\frac{1}{RC}t},$$

and Variance

$$\mathrm{Var}[q_P(\zeta,t)] = e^{-2\frac{1}{RC}t}\,\mathrm{Var}\Big[\,q_P(\zeta,0)\Big] + D\tfrac{C}{R}(1-e^{-2\frac{1}{RC}t})$$

Proof: By 17.1,

$$E[q_P(\zeta,t)] = \lambda\tfrac{\alpha}{\beta} + (E[q_P(\zeta,0)]-1)e^{-\beta t}$$

$$= \lambda C + (E[q_P(\zeta,0)]-1)e^{-\beta t}.\square$$

$$\mathrm{Var}[q_P(\zeta,t)] = e^{-2\beta t}\,\mathrm{Var}\Big[\,q_P(\zeta,0)\Big] + \lambda\tfrac{\alpha^2}{\beta}(1-e^{-2\beta t})(\tfrac{1}{2}-\tfrac{\lambda}{\beta})$$

$$= e^{-2\frac{1}{RC}t}\,\mathrm{Var}_P\Big[\,q(\zeta,0)\Big] + \lambda\tfrac{C}{R}(1-e^{-2\frac{1}{RC}t})(\tfrac{1}{2}-\lambda RC).\square$$

18.4 The Random Charge $q_P(\zeta,t)$ Steady State

At the Steady State, $t =$ infinite hyper-real Θ,

$$E[q_P(\zeta,\Theta)] \approx \lambda C.$$

$$\mathrm{Var}[q_P(\zeta,\Theta)] \approx \lambda\tfrac{C}{R}(\tfrac{1}{2}-\lambda RC).$$

18.5 The Charge Noise Energy at the Steady State

$$\frac{E[q^2(\zeta,\Theta)]}{2C} = \tfrac{\lambda}{4R},$$

where k is Boltzmann Constant,

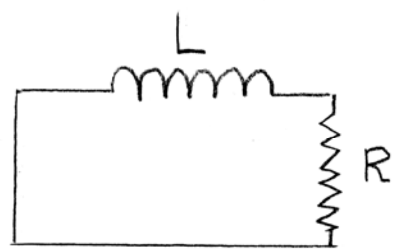
and T is the absolute Temperature.

Proof: $$\tfrac{E[q^2(\zeta,\Theta)]}{2C} = \tfrac{1}{2C}\{\underbrace{\mathrm{Var}[q(\zeta,\Theta)]}_{\lambda\frac{C}{R}(\frac{1}{2}-\lambda RC)} + (\underbrace{E[q(\zeta,\Theta)]}_{\lambda C})^2\} = \tfrac{\lambda}{4R}.\square$$

19.

RL Oscillator Process driven by Shot Noise Voltage $\dot{P}(\zeta, t)$

19.1 Evolution Equation of Current $i_P(\zeta, t)$ in RL Oscillator due to Shot Noise Voltage $\dot{P}(\zeta, t)$



Current fluctuations in the circuit, generate Shot Noise Voltage $\dot{P}(\zeta, t)$ that drives a random current $i_P(\zeta, t)$ through the Circuit. $L\dfrac{di_P(\zeta,t)}{dt}$ is the Random Voltage on L.

The balance of voltages due to the shot noise is

$$L\frac{di_P(\zeta,t)}{dt} + i_P(\zeta,t)R = \dot{P}(\zeta,t),$$

$$di_P(\zeta,t) = \underbrace{\tfrac{1}{L}}_{\alpha}\underbrace{\dot{P}(\zeta,t)dt}_{dP(\zeta,t)} - \underbrace{\tfrac{R}{L}}_{\beta} i_P(\zeta,t)dt\,.$$

19.2 The Current Process $i_P(\zeta, t)$ due to Shot Noise

$$i_P(\zeta,t) = e^{-\frac{R}{L}t} i_P(\zeta,0) + \frac{1}{L}\int_{\tau=0}^{\tau=t} e^{-\frac{R}{L}(t-\tau)} dP(\zeta,\tau)$$

Proof: By 16.3,

$$i_P(\zeta,t) = e^{-\beta t} i_P(\zeta,0) + \alpha\int_{\tau=0}^{\tau=t} e^{-\beta(t-\tau)} dP(\zeta,\tau),$$

By 19.1, $\alpha = \frac{1}{L}$, $\beta = \frac{R}{L}$,

$$= e^{-\frac{R}{L}t} i_P(\zeta,0) + \frac{1}{L}\int_{\tau=0}^{\tau=t} e^{-\frac{R}{L}(t-\tau)} dP(\zeta,\tau).\square$$

19.3 Poisson Distribution of the Shot Noise Current

$$i_P(\zeta,t) = e^{-\frac{R}{L}t} i_P(\zeta,0) + \frac{1}{L}\int_{\tau=0}^{\tau=t} e^{-\frac{R}{L}(t-\tau)} dP(\zeta,\tau)$$

is Poisson Distributed with

Mean

$$E[i_P(\zeta,t)] = \frac{\lambda}{R} + (E[i_P(\zeta,0)] - 1)e^{-\frac{R}{L}t},$$

and Variance

$$\mathrm{Var}[i_P(\zeta,t)] = e^{-2\frac{R}{L}t}\,\mathrm{Var}\big[\,i_P(\zeta,0)\big] + D\frac{1}{RL}(1 - e^{-2\frac{R}{L}t})$$

Proof: By 17.1,

$$E[i_P(\zeta,t)] = \lambda\frac{\alpha}{\beta} + (E[i_P(\zeta,0)] - 1)e^{-\beta t}$$

$$= \frac{\lambda}{R} + (E[i_P(\zeta,0)] - 1)e^{-\frac{R}{L}t}.\square$$

$$\mathrm{Var}[i_P(\zeta,t)] = e^{-2\beta t}\,\mathrm{Var}\Big[i_P(\zeta,0)\Big] + \lambda\frac{\alpha^2}{\beta}(1 - e^{-2\beta t})(\tfrac{1}{2} - \tfrac{\lambda}{\beta})$$

$$= e^{-2\frac{R}{L}t}\,\mathrm{Var}\Big[i_P(i,0)\Big] + \frac{\lambda}{RL}(1 - e^{-2\frac{R}{L}t})(\tfrac{1}{2} - \tfrac{\lambda L}{R}).\square$$

19.4 Shot Noise Current Steady State

At the Steady State, $t =$ infinite hyper-real Θ,

$$E[i_P(\zeta,\Theta)] \approx \frac{\lambda}{R}.$$

$$\mathrm{Var}[i_P(\zeta,\Theta)] \approx \frac{\lambda}{RL}(\tfrac{1}{2} - \tfrac{\lambda L}{R}).$$

19.5 Shot Noise Current Energy at the Steady State

$$\frac{1}{2}E[{i_P}^2(\zeta,\Theta)]L = \frac{\lambda}{4R},$$

where k is Boltzmann Constant,

and T is the absolute Temperature.

Proof:

$$\frac{1}{2}E[{i_P}^2(\zeta,\Theta)]L = \frac{1}{2}\{\underbrace{\mathrm{Var}[i_P(\zeta,\Theta)]}_{\frac{\lambda}{RL}(\frac{1}{2}-\frac{\lambda L}{R})} + (\underbrace{E[i_P(\zeta,\Theta)]}_{\frac{\lambda}{R}})^2\}L = \frac{\lambda}{4R}.\square$$

20.

Evolution of Harmonic Oscillator driven by Shot Noise Voltage $\dot{P}(\zeta,t)$

20.1 Evolution Equation of Harmonic Oscillator driven by Shot Noise $\dot{P}(\zeta,t)$

The Evolution Equation for the Harmonic Oscillator driven by a Poisson Process $\dot{P}(\zeta,t)$ is

$$d\dot{x}(\zeta,t) = \alpha dP(\zeta,t) - \beta\dot{x}(\zeta,t)dt - \omega^2 x(\zeta,t)dt\,,$$

$$\ddot{x}(\zeta,t) = \alpha\dot{P}(\zeta,t) - \beta\dot{x}(\zeta,t) - \omega^2 x(\zeta,t).$$

20.2 Variation of Parameters Solution

for the Harmonic Oscillator Process

$$d\dot{X}_P(\zeta,t) = \alpha dP(\zeta,t) - \beta\dot{X}_P(\zeta,t)dt - \omega^2 X_P(\zeta,t)dt\,,$$

is $$X_P(\zeta,t) = \underbrace{e^{-\frac{1}{2}\beta t}\{A_1(\zeta,0)e^{\frac{1}{2}\kappa t} + A_2(\zeta,0)e^{-\frac{1}{2}\kappa t}\}}_{X_{\text{homogeneuos}}(\zeta,t)} +$$

$$+\underbrace{\frac{\alpha}{\kappa}\int_{\tau=0}^{\tau=t}\{e^{m_1(t-\tau)}-e^{m_2(t-\tau)}\}dP(\zeta,\tau)}_{X_{\text{particular}}(\zeta,t)},$$

provided $\kappa = \sqrt{\beta^2 - 4\omega^2} > 0$,

$$m_1 = -\tfrac{1}{2}\beta + \tfrac{1}{2}\kappa,$$

$$m_2 = -\tfrac{1}{2}\beta - \tfrac{1}{2}\kappa$$

Proof: Following the proof of 11.2.□

20.3 The Time-Rate Random Process $\dot{X}_P(\zeta,t)$

$$\dot{X}_P(\zeta,t) = e^{-\frac{1}{2}\beta t}\{m_1 A_1(\zeta,0)e^{\frac{1}{2}\kappa t} + m_2 A_2(\zeta,0)e^{-\frac{1}{2}\kappa t}\}$$

$$+\frac{\alpha}{\kappa}\int_{\tau=0}^{\tau=t}\{m_1 e^{m_1(t-\tau)} - m_2 e^{m_2(t-\tau)}\}dP(\zeta,\tau),$$

provided $\kappa = \sqrt{\beta^2 - 4\omega^2} > 0$,

$$m_1 = -\tfrac{1}{2}\beta + \tfrac{1}{2}\kappa,$$

$$m_2 = -\tfrac{1}{2}\beta - \tfrac{1}{2}\kappa$$

Proof: Following the proof of 11.3.□

21.

Harmonic Oscillator Process due to Shot Noise Voltage $\dot{P}(\zeta, t)$

21.1 The Poisson Distribution of the Harmonic Oscillator Process $X_P(\zeta, t)$ driven by Shot Noise

$$X_P(\zeta,t) = e^{-\frac{1}{2}\beta t}\{A_1(\zeta,0)e^{\frac{1}{2}\kappa t} + A_2(\zeta,0)e^{-\frac{1}{2}\kappa t}\}$$

$$+\frac{\alpha}{\kappa}\int_{\tau=0}^{\tau=t}\{e^{m_1(t-\tau)} - e^{m_2(t-\tau)}\}dP(\zeta,\tau)$$

is Poisson Distributed with

Mean

$$E[X_P(\zeta,t)] = E[A_1(\zeta,0)]e^{m_1 t} + E[A_2(\zeta,0)]e^{m_2 t}$$

$$+\lambda\frac{\alpha}{\kappa}\left\{\frac{e^{m_1 t}}{m_1} - \frac{e^{m_2 t}}{m_2} + \frac{\kappa}{\omega^2}\right\}$$

and Variance

$$\mathrm{Var}[X_P(\zeta,t)] = e^{-(\beta-\kappa)t}\,\mathrm{Var}\left[A_1(\zeta,0)\right] + e^{-(\beta+\kappa)t}\,\mathrm{Var}\left[A_2(\zeta,0)\right] +$$

$$+\frac{\alpha^2}{\kappa^2}\lambda\left\{-\frac{e^{-(\beta-\kappa)t}}{(\beta-\kappa)} + \frac{e^{-\beta t}}{\beta} - \frac{e^{-(\beta+\kappa)t}}{\beta+\kappa}\right\} + \lambda\frac{\alpha^2}{2\beta\omega^2}\frac{\beta^2-2\omega^2}{\beta^2-4\omega^2} +$$

$$-\lambda^2 \frac{\alpha^2}{\kappa^2}\left[\frac{e^{m_1 t}}{m_1} - \frac{e^{m_2 t}}{m_2} + \frac{\kappa}{\omega^2}\right]^2,$$

$$\textit{provided} \quad \kappa = \sqrt{\beta^2 - 4\omega^2} > 0\,,$$

$$m_1 = -\tfrac{1}{2}\beta + \tfrac{1}{2}\kappa\,,$$

$$m_2 = -\tfrac{1}{2}\beta - \tfrac{1}{2}\kappa\,.$$

Proof: Since $dP(\zeta, \tau)$ is Poisson Distributed, so is

$J_P(\zeta) = \int\limits_{\tau=0}^{\tau=t} \{e^{m_1(t-\tau)} - e^{m_2(t-\tau)}\} dP(\zeta, \tau)$. Hence, $X_P(\zeta, t)$ is Poisson distributed.

$$E[J_P(\zeta)] = E\left[\sum_{\tau=0}^{\tau=t} \{e^{m_1(t-\tau)} - e^{m_2(t-\tau)}\} dP(\zeta, \tau)\right],$$

$$= \sum_{\tau=0}^{\tau=t} \{e^{m_1(t-\tau)} - e^{m_2(t-\tau)}\} \underbrace{E[\underbrace{dP(\zeta, \tau)}_{P_i}]}_{\lambda d\tau}$$

$$= \lambda e^{m_1 t} \int\limits_{\tau=0}^{\tau=t} e^{-m_1 \tau} d\tau - \lambda e^{m_2 t} \int\limits_{\tau=0}^{\tau=t} e^{-m_2 \tau} d\tau\,.$$

$$= \frac{\lambda}{m_1}(e^{m_1 t} - 1) - \frac{\lambda}{m_2}(e^{m_2 t} - 1)$$

$$= \lambda \left\{ \frac{e^{m_1 t}}{m_1} - \frac{e^{m_2 t}}{m_2} \right\} + \lambda \left\{ -\frac{1}{m_1} + \frac{1}{m_2} \right\}$$

$$= \lambda \left\{ \frac{e^{m_1 t}}{m_1} - \frac{e^{m_2 t}}{m_2} \right\} + \underbrace{2\lambda \underbrace{\left\{ \tfrac{1}{\beta - \kappa} - \tfrac{1}{\beta + \kappa} \right\}}_{\frac{\kappa}{2\omega^2}}}_{\lambda \frac{\kappa}{\omega^2}}$$

$$= \lambda \left\{ \frac{e^{m_1 t}}{m_1} - \frac{e^{m_2 t}}{m_2} + \frac{\kappa}{\omega^2} \right\}$$

$$E[X_P(\zeta, t)] = \underbrace{E[A_1(\zeta,0)e^{m_1 t} + A_2(\zeta,0)e^{m_2 t}]}_{\left\{E[A_1(\zeta,0)]e^{m_1 t} + E[A_2(\zeta,0)]e^{m_2 t}\right\}} + \underbrace{E[\tfrac{\alpha}{\kappa} J_P(\zeta)]}_{\frac{\alpha}{\kappa} E[J_P(\zeta)]}$$

$$= E[A_1(\zeta,0)]e^{m_1 t} + E[A_2(\zeta,0)]e^{m_2 t}$$

$$+ \lambda \frac{\alpha}{\kappa} \left\{ \frac{e^{m_1 t}}{m_1} - \frac{e^{m_2 t}}{m_2} + \frac{\kappa}{\omega^2} \right\} . \square$$

From the Proof of 12.1,

$$E\left[J_P{}^2(\zeta) \right] = \sum_{\tau=0}^{\tau=t} \{ e^{m_1(t-\tau)} - e^{m_2(t-\tau)} \}^2 \underbrace{E[P_i^2]}_{\lambda d\tau}$$

$$= \lambda \int_{\tau=0}^{\tau=t} \{ e^{2m_1(t-\tau)} - e^{-\beta(t-\tau)} + e^{2m_2(t-\tau)} \} d\tau \, .$$

$$= \lambda\left\{\frac{e^{2m_1t}-1}{2m_1} + \frac{e^{-\beta t}-1}{\beta} + \frac{e^{2m_2t}-1}{2m_2}\right\}$$

$$= \left\{\frac{e^{2m_1t}}{2m_1} + \frac{e^{-\beta t}}{\beta} + \frac{e^{2m_2t}}{2m_2}\right\} + \underbrace{\left\{\frac{1}{\beta-\kappa} - \frac{1}{\beta} + \frac{1}{\beta+\kappa}\right\}}_{\frac{\beta^2-2\omega^2}{2\beta\omega^2}}$$

$$\mathrm{Var}[J_P(\zeta)] = E\left[{J_P}^2(\zeta)\right] - (E\left[J_P(\zeta)\right])^2$$

$$= \lambda\left\{\frac{e^{2m_1t}}{2m_1} + \frac{e^{-\beta t}}{\beta} + \frac{e^{2m_2t}}{2m_2}\right\} + \lambda\frac{\beta^2-2\omega^2}{2\beta\omega^2} - \lambda^2\left\{\frac{e^{m_1t}}{m_1} - \frac{e^{m_2t}}{m_2} + \frac{\kappa}{\omega^2}\right\}^2$$

$$= \lambda\left\{\frac{e^{2m_1t}}{2m_1} + \frac{e^{-\beta t}}{\beta} + \frac{e^{2m_2t}}{2m_2} + \frac{\beta^2-2\omega^2}{2\beta\omega^2} - \lambda\left[\frac{e^{m_1t}}{m_1} - \frac{e^{m_2t}}{m_2} + \frac{\kappa}{\omega^2}\right]^2\right\}$$

$$\mathrm{Var}[X(\zeta,t)] = \underbrace{\mathrm{Var}\left[\{A_1(\zeta,0)e^{m_1t} + A_2(\zeta,0)e^{m_2t}\}\right]}_{e^{2m_1t}\,\mathrm{Var}\left[A_1(\zeta,0)\right]+e^{2m_2t}\,\mathrm{Var}\left[A_2(\zeta,0)\right]} + \underbrace{\mathrm{Var}\left[\tfrac{\alpha}{\kappa}J_P(\zeta)\right]}_{\frac{\alpha^2}{\kappa^2}\mathrm{Var}\left[J_P(\zeta)\right]}$$

$$= e^{2m_1t}\,\mathrm{Var}\left[A_1(\zeta,0)\right] + e^{2m_2t}\,\mathrm{Var}\left[A_2(\zeta,0)\right] +$$

$$+\frac{\alpha^2}{\kappa^2}\lambda\left\{\frac{e^{2m_1t}}{2m_1} + \frac{e^{-\beta t}}{\beta} + \frac{e^{2m_2t}}{2m_2} + \frac{\beta^2-2\omega^2}{2\beta\omega^2} - \lambda\left[\frac{e^{m_1t}}{m_1} - \frac{e^{m_2t}}{m_2} + \frac{\kappa}{\omega^2}\right]^2\right\}$$

$$= e^{-(\beta-\kappa)t}\,\mathrm{Var}\left[A_1(\zeta,0)\right] + e^{-(\beta+\kappa)t}\,\mathrm{Var}\left[A_2(\zeta,0)\right] +$$

$$+\frac{\alpha^2}{\kappa^2}\lambda\left\{-\frac{e^{-(\beta-\kappa)t}}{(\beta-\kappa)} + \frac{e^{-\beta t}}{\beta} - \frac{e^{-(\beta+\kappa)t}}{\beta+\kappa}\right\} + \lambda\frac{\alpha^2}{2\beta\omega^2}\frac{\beta^2-2\omega^2}{\beta^2-4\omega^2} +$$

$$-\lambda^2 \frac{\alpha^2}{\kappa^2}\left[\frac{e^{m_1 t}}{m_1} - \frac{e^{m_2 t}}{m_2} + \frac{\kappa}{\omega^2}\right]^2 . \square$$

21.2 The Harmonic Process Equilibrium State

At Equilibrium, $t =$ infinite hyper-real Θ,

$$E[X_P(\zeta,\Theta)] \approx \lambda \frac{\alpha}{\omega^2},$$

$$\mathrm{Var}[X_P(\zeta,\Theta)] \approx \lambda \frac{\alpha^2}{\omega^2}\left\{\frac{1}{2\beta}\frac{\beta^2 - 2\omega^2}{\beta^2 - 4\omega^2} - \lambda \frac{1}{\omega^4}\right\},$$

provided $\beta^2 - 4\omega^2 > 0$.

<u>*Proof*</u>:

$$E[X_P(\zeta,\Theta)] = E[A_1(\zeta,0)]e^{m_1\Theta} + E[A_2(\zeta,0)]e^{m_2\Theta}$$

$$+\lambda \frac{\alpha}{\kappa}\left\{\frac{e^{m_1\Theta}}{m_1} - \frac{e^{m_2\Theta}}{m_2} + \frac{\kappa}{\omega^2}\right\}$$

$$= \left\{E[A_1(\zeta,0)] + \lambda \tfrac{\alpha}{\kappa}\right\} e^{-\frac{1}{2}(\beta - \sqrt{\beta^2 - 4\omega^2})\Theta} +$$

$$+\left\{E[A_2(\zeta,0)] - \lambda \tfrac{\alpha}{\kappa}\right\} e^{-\frac{1}{2}(\beta + \sqrt{\beta^2 - 4\omega^2})\Theta} + \lambda \tfrac{\alpha}{\omega^2}.$$

Since we assume $\beta^2 > 4\omega^2$, then,

$\beta - \sqrt{\beta^2 - 4\omega^2} > 0$, and $e^{-\frac{1}{2}(\beta - \sqrt{\beta^2 - 4\omega^2})\Theta} \approx 0$

$\beta + \sqrt{\beta^2 - 4\omega^2} > 0$, and $e^{-\frac{1}{2}(\beta + \sqrt{\beta^2 - 4\omega^2})\Theta} \approx 0$

Hence, $E[X_P(\zeta,\Theta)] \approx \lambda \frac{\alpha}{\omega^2}.\square$

$$\mathrm{Var}[X_P(\zeta,\Theta)] = \underbrace{e^{-(\beta-\kappa)\Theta}\,\mathrm{Var}\Big[A_1(\zeta,0)\Big]}_{\approx 0,\ \text{since } \beta-\kappa>0} + \underbrace{e^{-(\beta+\kappa)\Theta}\,\mathrm{Var}\Big[A_2(\zeta,0)\Big]}_{\approx 0,\ \text{since } \beta+\kappa>0} +$$

$$+\frac{\alpha^2}{\kappa^2}\lambda \underbrace{\left\{\frac{e^{-(\beta-\kappa)\Theta}}{-(\beta-\kappa)} + \frac{e^{-\beta\Theta}}{\beta} + \frac{e^{-(\beta+\kappa)\Theta}}{-(\beta+\kappa)}\right\}}_{\approx 0} +$$

$$+D\frac{\alpha^2}{\beta\omega^2}\frac{\beta^2-2\omega^2}{\beta^2-4\omega^2}$$

$$\underbrace{-\lambda^2\frac{\alpha^2}{\kappa^2}\left[\frac{e^{m_1 t}}{m_1} - \frac{e^{m_2 t}}{m_2} + \frac{\kappa}{\omega^2}\right]^2}_{\approx \lambda^2\alpha^2\frac{1}{\omega^4}}$$

$$\approx \lambda\frac{\alpha^2}{\omega^2}\left\{\frac{1}{2\beta}\frac{\beta^2-2\omega^2}{\beta^2-4\omega^2} - \lambda\frac{1}{\omega^4}\right\}.\square$$

21.3 The Poisson Distribution of the Time-Rate Random Process

$$\dot{X}_P(\zeta,t) = A_1(\zeta,t)m_1 e^{m_1 t} + A_2(\zeta,t)m_2 e^{m_2 t} +$$

$$+\frac{\alpha}{\kappa}\int_{\tau=0}^{\tau=t}\{m_1 e^{m_1(t-\tau)} - m_2 e^{m_2(t-\tau)}\}dP(\zeta,\tau)$$

is Poisson Distributed with

Mean

$$E[\dot{X}(\zeta,t)] = \left\{m_1 E[A_1(\zeta,0)] + \frac{\alpha}{\kappa}\lambda\right\}e^{m_1 t} + \left\{m_2 E[A_2(\zeta,0)] - \frac{\alpha}{\kappa}\lambda\right\}e^{m_2 t},$$

and Variance

$$\mathrm{Var}[\dot{X}(\zeta,t)] = {m_1}^2 e^{-(\beta-\kappa)t}\,\mathrm{Var}\big[A_1(\zeta,0)\big] +$$

$$+{m_2}^2 e^{-(\beta+\kappa)t}\,\mathrm{Var}\big[A_2(\zeta,0)\big] +$$

$$+\tfrac{\alpha^2}{\kappa^2}\lambda e^{-\beta t}\left\{-\tfrac{1}{4}(\beta-\kappa)e^{\kappa t} + 2\tfrac{\omega^2}{\beta} - \tfrac{1}{4}(\beta+\kappa)e^{-\kappa t}\right\} + \lambda\tfrac{\alpha^2}{2\beta}$$

$$-\tfrac{\alpha^2}{\kappa^2}\lambda^2\Big[e^{-(\beta-\kappa)t} - 2e^{-\beta t} + e^{-(\beta+\kappa)t}\Big],$$

provided $\kappa = \sqrt{\beta^2 - 4\omega^2} > 0\,,$

$$m_1 = -\tfrac{1}{2}\beta + \tfrac{1}{2}\kappa\,,$$

$$m_2 = -\tfrac{1}{2}\beta - \tfrac{1}{2}\kappa\,.$$

<u>Proof</u>: Since $dP(\zeta,\tau)$ is Poisson Distributed, so is

$J_P(\zeta) = \int\limits_{\tau=0}^{\tau=t}\{m_1 e^{m_1(t-\tau)} - m_2 e^{m_2(t-\tau)}\}dP(\zeta,\tau)$. Hence, $\dot{X}_P(\zeta,t)$ is Poisson Distributed.

$$E[J_P(\zeta)] = E\left[\sum_{\tau=0}^{\tau=t}\{m_1 e^{m_1(t-\tau)} - m_2 e^{m_2(t-\tau)}\}dP(\zeta,\tau)\right],$$

$$= \sum_{\tau=0}^{\tau=t} \{m_1 e^{m_1(t-\tau)} - m_2 e^{m_2(t-\tau)}\} \underbrace{E[\underbrace{dP(\zeta,\tau)}_{P_i}]}_{\lambda dt}.$$

$$= \lambda e^{m_1 t} \underbrace{\int_{\tau=0}^{\tau=t} m_1 e^{-m_1\tau} dt}_{1-e^{-m_1 t}} - \lambda e^{m_2 t} \underbrace{\int_{\tau=0}^{\tau=t} m_2 e^{-m_2\tau} dt}_{1-e^{-m_2 t}},$$

$$= \lambda(e^{m_1 t} - e^{m_2 t})$$

$$E[\dot{X}(\zeta,t)] = \underbrace{E[m_1 A_1(\zeta,0)e^{m_1 t} + m_2 A_2(\zeta,0)e^{m_2 t}]}_{\left\{m_1 E[A_1(\zeta,0)]e^{m_1 t} + m_2 E[A_2(\zeta,0)]e^{m_2 t}\right\}} + \underbrace{E[\tfrac{\alpha}{\kappa} J_P(\zeta)]}_{\frac{\alpha}{\kappa}\underbrace{E[J_P(\zeta)]}_{\lambda(e^{m_1 t} - e^{m_2 t})}}$$

$$= \left\{m_1 E[A_1(\zeta,0)] + \frac{\alpha}{\kappa}\lambda\right\} e^{m_1 t} + \left\{m_2 E[A_2(\zeta,0)] - \frac{\alpha}{\kappa}\lambda\right\} e^{m_2 t}. \square$$

$$E\left[{J_P}^2(\zeta)\right] = \sum_{\tau=0}^{\tau=t} \{m_1 e^{m_1(t-\tau)} - m_2 e^{m_2(t-\tau)}\}^2 \underbrace{E[P_i^2]}_{\lambda d\tau}$$

$$= \lambda \int_{\tau=0}^{\tau=t} \{{m_1}^2 e^{2m_1(t-\tau)} - 2m_1 m_2 e^{-\beta(t-\tau)} + {m_2}^2 e^{2m_2(t-\tau)}\} d\tau.$$

$$= \lambda \left\{ m_1 \frac{e^{2m_1 t} - 1}{2} + 2 \underbrace{m_1 m_2}_{\omega^2} \frac{e^{-\beta t} - 1}{\beta} + m_2 \frac{e^{2m_2 t} - 1}{2} \right\}$$

$$= \lambda e^{-\beta t}\left[\tfrac{1}{2}m_1 e^{\kappa t} + 2\tfrac{\omega^2}{\beta} + \tfrac{1}{2}m_2 e^{-\kappa t}\right] + \lambda \underbrace{\{\underbrace{-\tfrac{1}{2}m_1 - \tfrac{1}{2}m_2}_{\frac{1}{2}\beta} - 2\tfrac{\omega^2}{\beta}\}}_{\frac{\kappa^2}{2\beta}}$$

$$= \lambda\left\{e^{-\beta t}\left[\tfrac{1}{2}m_1 e^{\kappa t} + 2\tfrac{\omega^2}{\beta} + \tfrac{1}{2}m_2 e^{-\kappa t}\right] + \tfrac{\kappa^2}{2\beta}\right\}$$

$$\mathrm{Var}[J_P(\zeta)] = E\left[J_P{}^2(\zeta)\right] - (E\left[J_P(\zeta)\right])^2$$

$$= \lambda\left\{e^{-\beta t}\left[\tfrac{1}{2}m_1 e^{\kappa t} + 2\tfrac{\omega^2}{\beta} + \tfrac{1}{2}m_2 e^{-\kappa t}\right] + \tfrac{\kappa^2}{2\beta} - \lambda(e^{m_1 t} - e^{m_2 t})^2\right\}$$

$$\mathrm{Var}[\dot{X}(\zeta,t)] = \underbrace{\mathrm{Var}\left[\{m_1 A_1(\zeta,0)e^{m_1 t} + m_2 A_2(\zeta,0)e^{m_2 t}\}\right]}_{m_1{}^2 e^{2m_1 t}\,\mathrm{Var}\left[A_1(\zeta,0)\right] + m_2{}^2 e^{2m_2 t}\,\mathrm{Var}\left[A_2(\zeta,0)\right]} + \underbrace{\mathrm{Var}\left[\tfrac{\alpha}{\kappa}J_P(\zeta)\right]}_{\frac{\alpha^2}{\kappa^2}\mathrm{Var}\left[J_P(\zeta)\right]}$$

$$= m_1{}^2 e^{2m_1 t}\,\mathrm{Var}\left[A_1(\zeta,0)\right] + m_2{}^2 e^{2m_2 t}\,\mathrm{Var}\left[A_2(\zeta,0)\right] +$$

$$+ \frac{\alpha^2}{\kappa^2}\lambda\left\{e^{-\beta t}\left[\tfrac{1}{2}m_1 e^{\kappa t} + 2\tfrac{\omega^2}{\beta} + \tfrac{1}{2}m_2 e^{-\kappa t}\right] + \tfrac{\kappa^2}{2\beta} - \lambda(e^{m_1 t} - e^{m_2 t})^2\right\}$$

$$= m_1{}^2 e^{-(\beta-\kappa)t}\,\mathrm{Var}\left[A_1(\zeta,0)\right] + m_2{}^2 e^{-(\beta+\kappa)t}\,\mathrm{Var}\left[A_2(\zeta,0)\right] +$$

$$+ \tfrac{\alpha^2}{\kappa^2}\lambda e^{-\beta t}\left\{-\tfrac{1}{4}(\beta-\kappa)e^{\kappa t} + 2\tfrac{\omega^2}{\beta} - \tfrac{1}{4}(\beta+\kappa)e^{-\kappa t}\right\} + \lambda\tfrac{\alpha^2}{2\beta}$$

$$-\tfrac{\alpha^2}{\kappa^2}\lambda^2\left[e^{-(\beta-\kappa)t} - 2e^{-\beta t} + e^{-(\beta+\kappa)t}\right].\square$$

21.4 The Time-Rate Process Equilibrium State

At Equilibrium, $t =$ infinite hyper-real Θ,

$$E[\dot{X}(\zeta,\Theta)] \approx 0\,,$$

$$\mathrm{Var}[\dot{X}(\zeta,\Theta)] \approx \lambda\frac{\alpha^2}{2\beta} > 0\,,$$

provided $\beta^2 - 4\omega^2 > 0\,.$

<u>Proof</u>:

$$E[\dot{X}(\zeta,\Theta)] = \left\{m_1 E[A_1(\zeta,0)] + \tfrac{\alpha}{\kappa}\lambda\right\}e^{m_1\Theta} + \left\{m_2 E[A_2(\zeta,0)] - \tfrac{\alpha}{\kappa}\lambda\right\}e^{m_2\Theta}$$

$$= \left\{m_1 E[A_1(\zeta,0)] + \tfrac{\alpha}{\kappa}\lambda\right\}e^{-\frac{1}{2}(\beta-\sqrt{\beta^2-4\omega^2})\Theta} +$$

$$+\left\{m_2 E[A_1(\zeta,0)] - \tfrac{\alpha}{\kappa}\lambda\right\}e^{-\frac{1}{2}(\beta+\sqrt{\beta^2-4\omega^2})\Theta}.$$

Since we assume $\beta^2 > 4\omega^2$, then,

$\beta - \sqrt{\beta^2 - 4\omega^2} > 0$, and $e^{-\frac{1}{2}(\beta-\sqrt{\beta^2-4\omega^2})\Theta} \approx 0$

$\beta + \sqrt{\beta^2 - 4\omega^2} > 0$, and $e^{-\frac{1}{2}(\beta+\sqrt{\beta^2-4\omega^2})\Theta} \approx 0$

Hence, $E[\dot{X}(\zeta,\Theta)] \approx 0\,.\square$

$$\mathrm{Var}[\dot{X}(\zeta,\Theta)] = \underbrace{{m_1}^2 e^{-(\beta-\kappa)\Theta}\,\mathrm{Var}\big[A_1(\zeta,0)\big]}_{\approx 0,\text{ since }\beta-\kappa>0} +$$

$$+\underbrace{{m_2}^2 e^{-(\beta+\kappa)\Theta}\,\mathrm{Var}\big[A_2(\zeta,0)\big]}_{\approx 0,\text{ since }\beta+\kappa>0} +$$

$$+\tfrac{\alpha^2}{\kappa^2}\lambda\underbrace{\left\{-\tfrac{1}{4}(\beta-\kappa)e^{-(\beta-\kappa)\Theta} + 2\tfrac{\omega^2}{\beta}e^{-\beta\Theta} - \tfrac{1}{4}(\beta+\kappa)e^{-(\beta+\kappa)\Theta}\right\}}_{\approx 0} +$$

$$+\lambda \tfrac{\alpha^2}{2\beta}$$

$$-\tfrac{\alpha^2}{\kappa^2}\lambda^2\left[e^{-(\beta-\kappa)\Theta} - 2e^{-\beta\Theta} + e^{-(\beta+\kappa)\Theta}\right]$$

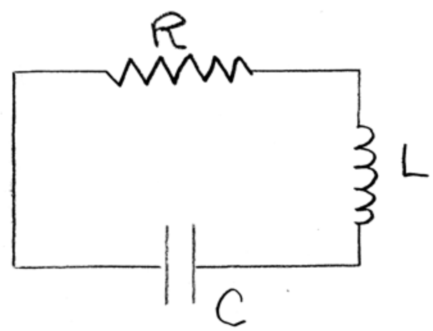
$$\approx \lambda\frac{\alpha^2}{2\beta},$$

since $\beta > 0$.$\square$

22.

RLC Harmonic Oscillator due to Shot Noise Voltage $\dot{P}(\zeta, t)$

22.1 The Evolution Equation of RLC Harmonic Process driven by Shot Noise Voltage $\dot{P}(\zeta, t)$



Current fluctuations in the circuit, generate Shot Noise Voltage $\dot{P}(\zeta, t)$ that drives a random current $i_P(\zeta, t) = \dfrac{dq_P(\zeta, t)}{dt}$ through the Circuit.

$i_P(\zeta, t)R = \dot{q}_P(\zeta, t)R$ is the Random Voltage on R.

$L\dfrac{di_P(\zeta, t)}{dt}$ is the Random Voltage on L.

$\dfrac{q_P(\zeta, t)}{C}$ is the Random Voltage on C.

The balance of voltages due to the shot noise is

$$L\ddot{q}_P(\zeta,t) + R\dot{q}_P(\zeta,t) + \tfrac{1}{C}q_P(\zeta,T) = \dot{P}(\zeta,t),$$

$$\ddot{q}_P(\zeta,t) = \underbrace{\tfrac{1}{L}}_{\alpha}\dot{P}(\zeta,t) - \underbrace{\tfrac{R}{L}}_{\beta}\dot{q}_P(\zeta,t) - \underbrace{\tfrac{1}{LC}}_{\omega^2}q_P(\zeta,t).$$

22.2 The Shot Noise Charge $q_P(\zeta,t)$

$$q_P(\zeta,t) = e^{-\frac{R}{2L}t}\{A_1(\zeta,0)e^{\frac{\sqrt{R^2C^2-4LC}}{2LC}t} + A_2(\zeta,0)e^{-\frac{\sqrt{R^2C^2-4LC}}{2LC}t}\} +$$

$$+\frac{C}{\sqrt{R^2C^2-4LC}}\int_{\tau=0}^{\tau=t}\{e^{-\left\{\frac{R}{2L}-\frac{\sqrt{R^2C^2-4LC}}{2LC}\right\}(t-\tau)} - e^{-\left\{\frac{R}{2L}+\frac{\sqrt{R^2C^2-4LC}}{2LC}\right\}(t-\tau)}\}dP(\zeta,\tau)$$

Proof: By 20.2, the Harmonic Oscillator evolution equation

$$d\dot{X}_P(\zeta,t) = \alpha dP(\zeta,t) - \beta\dot{X}_P(\zeta,t)dt - \omega^2 X_P(\zeta,t)dt,$$

is solved by the Process

$$X_P(\zeta,t) = e^{-\frac{1}{2}\beta t}\{A_1(\zeta,0)e^{\frac{1}{2}\kappa t} + A_2(\zeta,0)e^{-\frac{1}{2}\kappa t}\} +$$

$$+\frac{\alpha}{\kappa}\int_{\tau=0}^{\tau=t}\{e^{m_1(t-\tau)} - e^{m_2(t-\tau)}\}dP(\zeta,\tau),$$

provided $\kappa = \sqrt{\beta^2 - 4\omega^2} > 0$,

$$m_1 = -\tfrac{1}{2}\beta + \tfrac{1}{2}\kappa,$$

$$m_2 = -\tfrac{1}{2}\beta - \tfrac{1}{2}\kappa.$$

Substituting

$$\alpha = \frac{1}{L}, \qquad \beta = \frac{R}{L}, \qquad \omega^2 = \frac{1}{LC},$$

then,

$$\kappa = \sqrt{\frac{R^2}{L^2} - 4\frac{1}{LC}} = \frac{1}{LC}\sqrt{R^2C^2 - 4LC},$$

$$m_1 = -\frac{R}{2L} + \frac{\sqrt{R^2C^2-4LC}}{2LC},$$

$$m_1 = -\frac{R}{2L} - \frac{\sqrt{R^2C^2-4LC}}{2LC},$$

$$q_P(\zeta,t) = e^{-\frac{R}{2L}t}\left\{A_1(\zeta,0)e^{\frac{\sqrt{R^2C^2-4LC}}{2LC}t} + A_2(\zeta,0)e^{-\frac{\sqrt{R^2C^2-4LC}}{2LC}t}\right\} +$$

$$+\frac{C}{\sqrt{R^2C^2-4LC}}\int_{\tau=0}^{\tau=t}\{e^{-\left\{\frac{R}{2L}-\frac{\sqrt{R^2C^2-4LC}}{2LC}\right\}(t-\tau)} - e^{-\left\{\frac{R}{2L}+\frac{\sqrt{R^2C^2-4LC}}{2LC}\right\}(t-\tau)}\}dP(\zeta,\tau).\square$$

22.3 The Poisson Distribution of $q_P(\zeta,t)$

$$q_P(\zeta,t) = e^{-\frac{R}{2L}t}\left\{A_1(\zeta,0)e^{\frac{\sqrt{R^2C^2-4LC}}{2LC}t} + A_2(\zeta,0)e^{-\frac{\sqrt{R^2C^2-4LC}}{2LC}t}\right\} +$$

$$+\frac{C}{\sqrt{R^2C^2-4LC}}\int_{\tau=0}^{\tau=t}\{e^{-\left\{\frac{R}{2L}-\frac{\sqrt{R^2C^2-4LC}}{2LC}\right\}(t-\tau)} - e^{-\left\{\frac{R}{2L}+\frac{\sqrt{R^2C^2-4LC}}{2LC}\right\}(t-\tau)}\}dP(\zeta,\tau)$$

is Poisson Distributed with

Mean

$$E[q_P(\zeta,t)] = e^{-\frac{R}{2L}t}\{E[A_1(\zeta,0)]e^{\frac{\sqrt{R^2C^2-4}}{2LC}t} + E[A_2(\zeta,0)]e^{-\frac{\sqrt{R^2C^2-4}}{2LC}t}\} +$$

$$+\lambda\frac{4LC^2}{\sqrt{R^2C^2-4LC}}e^{-\frac{R}{2L}t}\left(\frac{e^{\frac{\sqrt{R^2C^2-4LC}}{2LC}t}}{-RC+\sqrt{R^2C^2-4LC}}+\frac{e^{-\frac{\sqrt{R^2C^2-4LC}}{2LC}t}}{RC+\sqrt{R^2C^2-4LC}}\right)+\lambda C$$

and Variance

$$\mathrm{Var}[q_P(\zeta,t)] = e^{-\frac{R}{L}t}\{\mathrm{Var}[A_1(\zeta,0)]e^{\frac{\sqrt{R^2C^2-4LC}}{LC}t} + \mathrm{Var}[A_2(\zeta,0)]e^{-\frac{\sqrt{R^2C^2-4LC}}{LC}t}\}$$

$$+\frac{2DC^2LCe^{-\frac{R}{L}t}}{R^2C^2-4LC}\left\{\frac{e^{\frac{\sqrt{R^2C^2-4LC}}{LC}t}}{-RC+\sqrt{R^2C^2-4LC}}+\frac{1}{RC}-\frac{e^{-\frac{\sqrt{R^2C^2-4LC}}{LC}t}}{RC+\sqrt{R^2C^2-4LC}}\right\}$$

$$+D\frac{C}{R}\frac{R^2C-2L}{R^2C-4L}.$$

Proof: By 21.1,

$$E[q_P(\zeta,t)] = E[A_1(\zeta,0)]e^{m_1t} + E[A_2(\zeta,0)]e^{m_2t}$$

$$+\lambda\frac{\alpha}{\kappa}\left(\frac{e^{m_1t}}{m_1}-\frac{e^{m_2t}}{m_2}\right)+\lambda\frac{\alpha}{\omega^2}$$

$$= e^{-\frac{R}{2L}t}\{E[A_1(\zeta,0)]e^{\frac{\sqrt{R^2C^2-4LC}}{2LC}t} + E[A_2(\zeta,0)]e^{-\frac{\sqrt{R^2C^2-4LC}}{2LC}t}\}$$

$$+\lambda\frac{4LC^2}{\sqrt{R^2C^2-4LC}}e^{-\frac{R}{2L}t}\left(\frac{e^{\frac{\sqrt{R^2C^2-4LC}}{2LC}t}}{-RC+\sqrt{R^2C^2-4LC}}+\frac{e^{-\frac{\sqrt{R^2C^2-4LC}}{2LC}t}}{RC+\sqrt{R^2C^2-4LC}}\right)+\lambda C.\square$$

$$\mathrm{Var}[q_P(\zeta,t)] = e^{-(\beta-\kappa)t}\,\mathrm{Var}\big[A_1(\zeta,0)\big]+e^{-(\beta+\kappa)t}\,\mathrm{Var}\big[A_2(\zeta,0)\big]+$$

$$+\lambda\frac{\alpha^2}{\kappa^2}e^{-\beta t}\left\{-\frac{e^{\kappa t}}{\beta-\kappa}+\frac{1}{\beta}-\frac{e^{-\kappa t}}{\beta+\kappa}\right\}+$$

$$+\lambda\frac{\alpha^2}{2\beta\omega^2}\frac{\beta^2-2\omega^2}{\beta^2-4\omega^2}$$

$$-\lambda^2\frac{\alpha^2}{\kappa^2}\left[\frac{e^{m_1 t}}{m_1}-\frac{e^{m_2 t}}{m_2}+\frac{\kappa}{\omega^2}\right]^2$$

$$= e^{-\frac{R}{L}t}\{\mathrm{Var}[A_1(\zeta,0)]e^{\frac{\sqrt{R^2C^2-4LC}}{LC}t}+\mathrm{Var}[A_2(\zeta,0)]e^{-\frac{\sqrt{R^2C^2-4LC}}{LC}t}\}+$$

$$+\frac{4\lambda C^2LCe^{-\frac{R}{L}t}}{R^2C^2-4LC}\left\{\frac{e^{\frac{\sqrt{R^2C^2-4LC}}{LC}t}}{-RC+\sqrt{R^2C^2-4LC}}+\frac{1}{RC}-\frac{e^{-\frac{\sqrt{R^2C^2-4LC}}{LC}t}}{RC+\sqrt{R^2C^2-4LC}}\right\}$$

$$+\lambda\frac{C}{2R}\frac{R^2C-2L}{R^2C-4L}$$

$$-\frac{\lambda^2 4C^4L^2}{R^2C^2-4LC}\left[\frac{2e^{-\frac{R}{L}t}e^{\frac{\sqrt{R^2C^2-4LC}}{LC}t}}{-RC+\sqrt{R^2C^2-4LC}}-\frac{2e^{-\frac{R}{L}t}e^{-\frac{\sqrt{R^2C^2-4LC}}{LC}t}}{RC+\sqrt{R^2C^2-4LC}}+\frac{\sqrt{R^2C^2-4LC}}{2LC}\right]^2.\square$$

22.4 The Shot Noise Charge Steady State

At the Steady State, $t =$ infinite hyper-real Θ,

$$E[q_P(\zeta,\Theta)] \approx \lambda C.$$

$$\mathrm{Var}[q_P(\zeta,\Theta)] \approx \lambda\left\{\frac{C}{2R}\frac{R^2C-2L}{R^2C-4L}-\lambda LC^3\right\}.$$

Proof: By 21.2, $E[q_P(\zeta,\Theta)] \approx \lambda\frac{\alpha}{\omega^2} = \lambda\frac{\frac{1}{L}}{\frac{1}{LC}} = \lambda C.\square$

$$\mathrm{Var}[q_P(\zeta,\Theta)] \approx \lambda \frac{\alpha^2}{\omega^2}\left\{\frac{1}{2\beta}\frac{\beta^2-2\omega^2}{\beta^2-4\omega^2} - \lambda\frac{1}{\omega^4}\right\}$$

$$= \lambda \frac{\frac{1}{L^2}}{\frac{1}{LC}}\left\{\frac{1}{2\frac{R}{L}}\frac{\frac{R^2}{L^2}-2\frac{1}{LC}}{\frac{R^2}{L^2}-4\frac{1}{LC}} - \lambda\frac{1}{\frac{1}{L^2C^2}}\right\}$$

$$= \frac{\lambda C}{2R}\frac{R^2C-2L}{R^2C-4L} - \lambda^2 LC^3 . \square$$

22.5 Shot Noise Energy of $q_P(\zeta,t)$ at the Steady State

$$\tfrac{1}{2C}E[{q_P}^2(\zeta,\Theta)] \approx \frac{\lambda}{4R}\frac{R^2C-2L}{R^2C-4L} - \tfrac{1}{2}\lambda^2LC^2 + \tfrac{1}{2}\lambda^2 C,$$

where k is Boltzmann Constant,

and T is the absolute Temperature.

Proof:

$$\tfrac{1}{2C}E[{q_P}^2(\zeta,\Theta)] = \tfrac{1}{2C}\{\underbrace{\mathrm{Var}[q_P(\zeta,\Theta)]}_{\frac{\lambda C}{2R}\frac{R^2C-2L}{R^2C-4L}-\lambda^2LC^3} + (\underbrace{E[q_P(\zeta,\Theta)]}_{\lambda C})^2\}$$

$$= \frac{\lambda}{4R}\frac{R^2C-2L}{R^2C-4L} - \tfrac{1}{2}\lambda^2LC^2 + \tfrac{1}{2}\lambda^2 C . \square$$

22.6 The Shot Noise Current $i_P(\zeta,t)$

$$\dot{q}_P(\zeta,t) = e^{-\frac{R}{2L}t}\left(-\tfrac{R}{2L} + \tfrac{\sqrt{R^2C^2-4LC}}{2LC}\right)A_1(\zeta,0)e^{\frac{\sqrt{R^2C^2-4LC}}{2LC}t} +$$

$$-e^{-\frac{R}{2L}t}\left(\frac{R}{2L}+\frac{\sqrt{R^2C^2-4LC}}{2LC}\right)A_2(\zeta,0)e^{-\frac{\sqrt{R^2C^2-4LC}}{2LC}t}+$$

$$-\frac{2C}{\sqrt{R^2C^2-4LC}}\left(\frac{R}{2L}-\frac{\sqrt{R^2C^2-4LC}}{2LC}\right)\int_{\tau=0}^{\tau=t} e^{-\left[\frac{R}{2L}-\frac{\sqrt{R^2C^2-4LC}}{2LC}\right](t-\tau)}dP(\zeta,\tau)+$$

$$-\frac{2C}{\sqrt{R^2C^2-4LC}}\left(\frac{R}{2L}+\frac{\sqrt{R^2C^2-4LC}}{2LC}\right)\int_{\tau=0}^{\tau=t} e^{-\left[\frac{R}{2L}+\frac{\sqrt{R^2C^2-4LC}}{2LC}\right](t-\tau)}dP(\zeta,\tau).$$

Proof: By 20.3,

$$\dot{q}_P(\zeta,t)=e^{-\frac{1}{2}\beta t}\{m_1A_1(\zeta,0)e^{\frac{1}{2}\kappa t}+m_2A_2(\zeta,0)e^{-\frac{1}{2}\kappa t}\}$$

$$+\frac{\alpha}{\kappa}\int_{\tau=0}^{\tau=t}\{m_1e^{m_1(t-\tau)}-m_2e^{m_2(t-\tau)}\}dP(\zeta,\tau),$$

$$=e^{-\frac{R}{2L}t}\left(-\frac{R}{2L}+\frac{\sqrt{R^2C^2-4LC}}{2LC}\right)A_1(\zeta,0)e^{\frac{\sqrt{R^2C^2-4LC}}{2LC}t}+$$

$$-e^{-\frac{R}{2L}t}\left(\frac{R}{2L}+\frac{\sqrt{R^2C^2-4LC}}{2LC}\right)A_2(\zeta,0)e^{-\frac{\sqrt{R^2C^2-4LC}}{2LC}t}+$$

$$-\frac{2C}{\sqrt{R^2C^2-4LC}}\left(\frac{R}{2L}-\frac{\sqrt{R^2C^2-4LC}}{2LC}\right)\int_{\tau=0}^{\tau=t} e^{-\left[\frac{R}{2L}-\frac{\sqrt{R^2C^2-4LC}}{2LC}\right](t-\tau)}dP(\zeta,\tau)+$$

$$+\frac{2C}{\sqrt{R^2C^2-4LC}}\left(\frac{R}{2L}+\frac{\sqrt{R^2C^2-4LC}}{2LC}\right)\int_{\tau=0}^{\tau=t} e^{-\left[\frac{R}{2L}+\frac{\sqrt{R^2C^2-4LC}}{2LC}\right](t-\tau)}dP(\zeta,\tau)$$

22.7 The Poisson Distribution of $i_P(\zeta,t)$

$i_P(\zeta,t) = \dot{q}_P(\zeta,t)$ ***is Poisson Distributed with***

Mean

$$E[i_P(\zeta,t)] = e^{-\frac{R}{2L}t}\left(-\frac{R}{2L} + \frac{\sqrt{R^2C^2-4LC}}{2LC}\right)E[A_1(\zeta,0)]e^{\frac{\sqrt{R^2C^2-4LC}}{2LC}t} +$$

$$-e^{-\frac{R}{2L}t}\left(\frac{R}{2L} + \frac{\sqrt{R^2C^2-4LC}}{2LC}\right)E[A_2(\zeta,0)]e^{-\frac{\sqrt{R^2C^2-4LC}}{2LC}t},$$

and Variance

$$\mathrm{Var}[i_P(\zeta,t)] = \left(-\frac{R}{2L} + \frac{\sqrt{R^2C^2-4LC}}{2LC}\right)^2 e^{-\left\{\frac{R}{L}-\frac{\sqrt{R^2C^2-4LC}}{LC}\right\}t}\,\mathrm{Var}\big[A_1(\zeta,0)\big]$$

$$+\left(\frac{R}{2L} + \frac{\sqrt{R^2C^2-4LC}}{2LC}\right)^2 e^{-\left\{\frac{R}{L}+\frac{\sqrt{R^2C^2-4LC}}{LC}\right\}t}\,\mathrm{Var}\big[A_2(\zeta,0)\big]$$

$$+\frac{1}{4}\frac{1}{R^2C-4L}\lambda\frac{-RC+\sqrt{R^2C^2-4LC}}{L}\,e^{-\left\{\frac{R}{L}-\frac{\sqrt{R^2C^2-4LC}}{LC}\right\}t}$$

$$+2\frac{1}{R^2C-4L}\lambda e^{-\frac{R}{L}t}\,\frac{1}{R}$$

$$-\frac{1}{4}\frac{1}{R^2C-4L}\lambda\frac{RC+\sqrt{R^2C^2-4LC}}{L}\,e^{-\left\{\frac{R}{L}+\frac{\sqrt{R^2C^2-4LC}}{LC}\right\}t}$$

$$+\lambda\frac{1}{2LR}$$

$$-\frac{C^2}{R^2C^2-4LC}\lambda^2\left(e^{-\{\frac{R}{L}-\frac{\sqrt{R^2C^2-4LC}}{LC}\}t} - 2e^{-\frac{R}{L}t} + e^{-\{\frac{R}{L}+\frac{\sqrt{R^2C^2-4LC}}{LC}\}t}\right)$$

Proof: By 21.3,

$$\dot{q}_P(\zeta,t) = A_1(\zeta,t)m_1e^{m_1t} + A_2(\zeta,t)m_2e^{m_2t} +$$

$$+\frac{\alpha}{\kappa}\int_{\tau=0}^{\tau=t}\{m_1 e^{m_1(t-\tau)} - m_2 e^{m_2(t-\tau)}\}dP(\zeta,\tau)$$

is Poisson Distributed with

Mean

$$E[\dot{X}(\zeta,t)] = \left\{m_1 E[A_1(\zeta,0)] + \frac{\alpha}{\kappa}\lambda\right\}e^{m_1 t} + \left\{m_2 E[A_2(\zeta,0)] - \frac{\alpha}{\kappa}\lambda\right\}e^{m_2 t}$$

$$= e^{-\frac{R}{2L}t}\left\{(-\tfrac{R}{2L} + \tfrac{\sqrt{R^2C^2-4LC}}{2LC})E[A_1(\zeta,0)] + \tfrac{2C}{\sqrt{R^2C^2-4LC}}\lambda\right\}e^{\frac{\sqrt{R^2C^2-4LC}}{2LC}t}$$

$$-e^{-\frac{R}{2L}t}\left\{(\tfrac{R}{2L} + \tfrac{\sqrt{R^2C^2-4LC}}{2LC})E[A_2(\zeta,0)] + \tfrac{2C}{\sqrt{R^2C^2-4LC}}\lambda\right\}e^{-\frac{\sqrt{R^2C^2-4LC}}{2LC}t}$$

and Variance

$$\mathrm{Var}[\dot{q}_P(\zeta,t)] = {m_1}^2 e^{-(\beta-\kappa)t}\,\mathrm{Var}\big[A_1(\zeta,0)\big] +$$

$$+{m_2}^2 e^{-(\beta+\kappa)t}\,\mathrm{Var}\big[A_2(\zeta,0)\big] +$$

$$+\tfrac{\alpha^2}{\kappa^2}\lambda e^{-\beta t}\left\{-\tfrac{1}{4}(\beta-\kappa)e^{\kappa t} + 2\tfrac{\omega^2}{\beta} - \tfrac{1}{4}(\beta+\kappa)e^{-\kappa t}\right\} + \lambda\tfrac{\alpha^2}{2\beta}$$

$$-\tfrac{\alpha^2}{\kappa^2}\lambda^2\left[e^{-(\beta-\kappa)t} - 2e^{-\beta t} + e^{-(\beta+\kappa)t}\right],$$

$$= (-\tfrac{R}{2L} + \tfrac{\sqrt{R^2C^2-4LC}}{2LC})^2 e^{-\left\{\frac{R}{L} - \frac{\sqrt{R^2C^2-4LC}}{LC}\right\}t}\,\mathrm{Var}\big[A_1(\zeta,0)\big] +$$

$$+(\tfrac{R}{2L} + \tfrac{\sqrt{R^2C^2-4LC}}{2LC})^2 e^{-\left\{\frac{R}{L} + \frac{\sqrt{R^2C^2-4LC}}{LC}\right\}t}\,\mathrm{Var}\big[A_2(\zeta,0)\big] +$$

$$+\tfrac{1}{4}\tfrac{1}{R^2C-4L}\lambda\tfrac{-RC+\sqrt{R^2C^2-4LC}}{L}\,e^{-\left\{\frac{R}{L} - \frac{\sqrt{R^2C^2-4LC}}{LC}\right\}t}$$

$$+2\tfrac{1}{R^2C-4L}\lambda e^{-\frac{R}{L}t}\tfrac{1}{R}$$

$$-\tfrac{1}{4}\tfrac{1}{R^2C-4L}\lambda\tfrac{RC+\sqrt{R^2C^2-4LC}}{L}e^{-\left\{\frac{R}{L}+\frac{\sqrt{R^2C^2-4LC}}{LC}\right\}t}$$

$$+\lambda\tfrac{1}{2LR}$$

$$-\tfrac{C^2}{R^2C^2-4LC}\lambda^2(e^{-\{\frac{R}{L}-\frac{\sqrt{R^2C^2-4LC}}{LC}\}t}-2e^{-\frac{R}{L}t}+e^{-\{\frac{R}{L}+\frac{\sqrt{R^2C^2-4LC}}{LC}\}t})$$

22.8 The Shot Noise Current Steady State

At Equilibrium, $t =$ infinite hyper-real Θ,

$$E[i_P(\zeta,\Theta)] \approx 0\,,$$

$$\mathrm{Var}[i_P(\zeta,\Theta)] \approx \lambda\frac{1}{2RL}$$

<u>*Proof*</u>: By 21.4,

At Equilibrium, $t =$ infinite hyper-real Θ,

$$E[\dot{q}_P(\zeta,\Theta)] \approx 0\,,$$

$$\mathrm{Var}[\dot{q}_P(\zeta,\Theta)] \approx \lambda\frac{\alpha^2}{2\beta} = \lambda\frac{1}{2RL}.\square$$

22.9 Shot Noise Energy of $i_P(\zeta,t)$ at the Steady State

$$\tfrac{1}{2}E[{i_P}^2(\zeta,\Theta)]L \approx \frac{\lambda}{4R},$$

where k *is Boltzmann Constant,*

and T *is the absolute Temperature.*

Proof:

$$\tfrac{1}{2}E[{i_P}^2(\zeta,\Theta)]L = \tfrac{1}{2}\{\underbrace{\mathrm{Var}[i_P(\zeta,\Theta)]}_{\approx\frac{\lambda}{2RL}} + (\underbrace{E[i_P(\zeta,\Theta)]}_{\approx 0})^2\}L \approx \frac{\lambda}{4R}.\square$$

22.10 RLC Harmonic Oscillator Shot Noise Energy

$$\frac{\lambda}{2R}\frac{R^2C-3L}{R^2C-4L} - \tfrac{1}{2}\lambda^2LC^2 + \tfrac{1}{2}\lambda^2C$$

Proof :

$$\underbrace{\frac{\lambda}{4R}}_{\text{current Shot Noise Energy}} + \underbrace{\frac{\lambda}{4R}\frac{R^2C-2L}{R^2C-4L} - \tfrac{1}{2}\lambda^2LC^2 + \tfrac{1}{2}\lambda^2C}_{\text{Charge Shot Noise Energy}} =$$

$$= \frac{\lambda}{2R}\frac{R^2C-3L}{R^2C-4L} - \tfrac{1}{2}\lambda^2LC^2 + \tfrac{1}{2}\lambda^2C\,.\square$$

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