Infinitesimal Calculus of Random Signals and White Noise

H. Vic Dannon
vic0@comcast.net
April, 2013

Abstract The Auto-Correlation of Thermal Noise Voltage $\dot{B}(\zeta, t)$ is a Hyper-real Delta Function. To avoid that singularity, the Autocorrelation is Fourier transformed into the Power Spectral Density.

We set up the Infinitesimal Calculus of Random Signals $X(\zeta, t)$, and apply the Hyper-real Delta Function, and the Hyper-real Fourier Transform to compute the Signal's Power.

The writings on Random signals that are based on the Calculus of Limits are inundated with too many errors, and misconceptions to be listed here. For instance, we note that

$$E[X^2(\zeta, t)] \neq \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} S_{XX}(\omega) d\omega .$$

The correct procedure is to Inverse Fourier Transform $S_{XX}(\omega)$, and evaluate it for infinitesimal τ. That is,

$$E[X^2(\zeta,t)] \approx \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} S_{XX}(\omega)e^{i\omega\tau}d\omega\bigg|_{\tau=\text{infinitesimal}}.$$

The following is concerned with such errors.

Keywords: Infinitesimal, Infinite-Hyper-real, Hyper-real, Calculus, Limit, Continuity, Derivative, Integral, Delta Function, Random Variable, Random Process, Random Signal, Stochastic Process, Stochastic Calculus, Probability Distribution, Bernoulli Random Variables, Binomial Distribution, Gaussian, Normal, Expectation, Variance, Random Walk, Auto-Correlation, Signal's Power, Stationary, Ergodic, Characteristic Function, Power Spectral Density, White Noise.

2000 Mathematics Subject Classification 26E35; 26E30; 26E15; 26E20; 26A06; 26A12; 03E10; 03E55; 03E17; 03H15; 46S20; 97I40; 97I30.

Contents

Introduction

0.1 Infinitesimal Calculus

Recently we have shown that when the Real Line is represented as the infinite dimensional space of all the Cauchy sequences of rational numbers, the hyper-reals are spanned by the constant hyper-reals, a family of infinitesimal hyper-reals, and the associated family of infinite hyper-reals.

The infinitesimal hyper-reals are smaller than any real number, yet bigger than zero.

The reciprocals of the infinitesimal hyper-reals are the infinite hyper-reals. They are greater than any real number, yet strictly smaller than infinity.

A neighborhood of infinitesimals separates the zero hyper-real from the reals, and each real number is the center of an interval of hyper-reals, that includes no other real number.

The Hyper-reals are totally ordered, and are lined up on a line, the hyper-real line.

A hyper-real function is a mapping from the hyper-real line into the hyper-real line.

Infinitesimal Calculus is the Calculus of hyper-real functions.

Infinitesimal Calculus is far more effective than the ε, δ Calculus, because being based on almost zero numbers, it allows us to deal with their reciprocals, the almost infinite numbers. We have no use for infinity by itself, but to comprehend the effects of singularities, we have use for the almost infinite.

Such almost infinite numbers are the values of the Delta Function.

0.2 The Delta Function

The Delta Function, the idealization of an impulse signal, is a Hyper-Real function which definition and analysis require Infinitesimal Calculus, and Infinite Hyper-reals.

Engineering Texts define Delta by its sampling property

$$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1,$$

that avoids specifying $\delta(0)$, although the Delta Function is not Riemann integrable in the Calculus of Limits, and is not Lebesgue integrable in Measure Theory.

In fact, in the Calculus of Limits, only the Cauchy Principal Value Integral of the Delta Function exists, and it equals zero.

Only in Infinitesimal Calculus, can the Delta Function be defined, differentiated, and integrated.

The Delta Function enables us to define the Fourier Transform with minimal requirements on the transformed function.

Then, the Fourier Integral Theorem states the sifting property for the Delta Function

$$f(x) = \int_{\xi=-\infty}^{\xi=\infty} f(\xi)\delta(\xi - x)d\xi \, .$$

0.3 Random Signals

Probability Distributions are defined on Random Variables. Random Variables assign numerical vales to outcomes. Thus, maps outcomes into the real line.

Random Signals are Random Variables that evolve in time.

Thermal Noise is the Voltage generated by electrons drifting in a resistor, and colliding with the resistor molecules.

0.4 Auto-Correlation, and Signal Power

The Autocorrelation Function measures the dependence between the Random Signal at two different times.

When the times are infinitesimally close, the Autocorrelation becomes the Mean of the signal Power.

0.5 Power Spectral Density, and White Noise

The Autocorrelation of Thermal Noise Voltage is a Delta Function, and the Mean of the Thermal Noise Power is an infinite hyper-real.

To avoid that singularity, the Autocorrelation of Thermal Noise Voltage is Fourier Transformed to the Power Spectral Density.

Since the Fourier Transform of the Hyper-real Delta Function is 1, the Power Spectral Density of Thermal Noise Voltage is constant for all frequencies, and is called White Noise.

Thus, The Power Spectral Density serves as Signal's Power measure when the Signal's Power is an infinite Hyper-real, and the Signal's Power is preferred when the Power spectral density is an infinite hyper-real.

1.

Hyper-real Line

The minimal domain and range, needed for the definition and analysis of a hyper-real function, is the hyper-real line.

Each real number α can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of infinitesimal hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the infinite hyper-reals.
4. The infinite hyper-reals are greater than any real number, yet strictly smaller than infinity.

5. The infinite hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a non-constant hyper-real.
7. The Hyper-reals are the totality of constant hyper-reals, a family of infinitesimals, a family of infinitesimals with negative sign, a family of infinite hyper-reals, a family of infinite hyper-reals with negative sign, and non-constant hyper-reals.
8. The hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.
9. That line includes the real numbers separated by the non-constant hyper-reals. Each real number is the center of an interval of hyper-reals, that includes no other real number.
10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.
11. Zero is not an infinitesimal, because zero is not strictly greater than zero.

12. We do not add infinity to the hyper-real line.

13. The infinitesimals, the infinitesimals with negative signs, the infinite hyper-reals, and the infinite hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.

14. The hyper-real line is embedded in $\mathbb{R}^{\infty}$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the hyper-real onto the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal hyper-reals, or to the infinite hyper-reals, or to the non-constant hyper-reals.

16. No neighbourhood of a hyper-real is homeomorphic to an $\mathbb{R}^{n}$ ball. Therefore, the hyper-real line is not a manifold.

17. The hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

2.

Hyper-real Function

2.1 Definition of a hyper-real function

$f(x)$ is a hyper-real function, iff it is from the hyper-reals into the hyper-reals.

This means that any number in the domain, or in the range of a hyper-real $f(x)$ is either one of the following

real

real + infinitesimal

real – infinitesimal

infinitesimal

infinitesimal with negative sign

infinite hyper-real

infinite hyper-real with negative sign

Clearly,

2.2 *Every function from the reals into the reals is a hyper-real function.*

3.

Integral of Hyper-real Function

In [Dan3], we defined the integral of a Hyper-real Function.

Let $f(x)$ be a hyper-real function on the interval $[a,b]$.

The interval may not be bounded.

$f(x)$ may take infinite hyper-real values, and need not be bounded.

At each

$$a \le x \le b\,,$$

there is a rectangle with base $[x - \frac{dx}{2}, x + \frac{dx}{2}]$, height $f(x)$, and area

$$f(x)dx\,.$$

We form the **Integration Sum** of all the areas for the x's that start at $x = a$, and end at $x = b$,

$$\sum_{x \in [a,b]} f(x)dx\,.$$

If for any infinitesimal dx, the Integration Sum has the same hyper-real value, then $f(x)$ is integrable over the interval $[a,b]$.

Then, we call the Integration Sum the integral of $f(x)$ from $x = a$, to $x = b$, and denote it by

$$\int_{x=a}^{x=b} f(x)dx .$$

If the hyper-real is infinite, then it is the integral over $[a,b]$,

If the hyper-real is finite,

$$\int_{x=a}^{x=b} f(x)dx = \text{ real part of the hyper-real} . \square$$

3.1 The countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers, $Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty .$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[a,b]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(x)dx$.

The Lower Integral is the Integration Sum where $f(x)$ is replaced

by its lowest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

3.2 $$\sum_{x \in [a,b]} \left(\inf_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

The Upper Integral is the Integration Sum where $f(x)$ is replaced by its largest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

3.3 $$\sum_{x \in [a,b]} \left(\sup_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

If the integral is a finite hyper-real, we have

3.4 *A hyper-real function has a finite integral if and only if its upper integral and its lower integral are finite, and differ by an infinitesimal.*

4.

Delta Function

In [Dan4], we defined the Delta Function, and established its properties

1. The Delta Function is a hyper-real function defined from the hyper-real line into the set of two hyper-reals $\left\{0, \frac{1}{dx}\right\}$. The hyper-real 0 is the sequence $\langle 0, 0, 0, ... \rangle$. The infinite hyper-real $\frac{1}{dx}$ depends on our choice of dx.
2. We will usually choose the family of infinitesimals that is spanned by the sequences $\left\langle \frac{1}{n} \right\rangle, \left\langle \frac{1}{n^2} \right\rangle, \left\langle \frac{1}{n^3} \right\rangle, \ldots$ It is a semigroup with respect to vector addition, and includes all the scalar multiples of the generating sequences that are non-zero. That is, the family includes infinitesimals with negative sign. Therefore, $\frac{1}{dx}$ will mean the sequence $\langle n \rangle$. Alternatively, we may choose

the family spanned by the sequences $\left\langle \frac{1}{2^n} \right\rangle, \left\langle \frac{1}{3^n} \right\rangle, \left\langle \frac{1}{4^n} \right\rangle$,... Then, $\frac{1}{dx}$ will mean the sequence $\left\langle 2^n \right\rangle$. Once we determined the basic infinitesimal dx, we will use it in the Infinite Riemann Sum that defines an Integral in Infinitesimal Calculus.

3. The Delta Function is strictly smaller than ∞

4. We define, $\delta(x) \equiv \frac{1}{dx} \chi_{\left[-\frac{dx}{2}, \frac{dx}{2}\right]}(x)$,

$$\text{where} \quad \chi_{\left[-\frac{dx}{2}, \frac{dx}{2}\right]}(x) = \begin{cases} 1, x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right] \\ 0, \text{otherwise} \end{cases}.$$

5. Hence,
   - for $x < 0$, $\delta(x) = 0$
   - at $x = -\frac{dx}{2}$, $\delta(x)$ jumps from 0 to $\frac{1}{dx}$,
   - for $x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right]$, $\delta(x) = \frac{1}{dx}$.
   - at $x = 0$, $\delta(0) = \frac{1}{dx}$
   - at $x = \frac{dx}{2}$, $\delta(x)$ drops from $\frac{1}{dx}$ to 0.
   - for $x > 0$, $\delta(x) = 0$.

❖ $x\delta(x) = 0$

6. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle \chi_{[-\frac{1}{2},\frac{1}{2}]}(x), 2\chi_{[-\frac{1}{4},\frac{1}{4}]}(x), 3\chi_{[-\frac{1}{6},\frac{1}{6}]}(x) ... \right\rangle$

7. If $dx = \left\langle \frac{2}{n} \right\rangle$, $\delta(x) = \left\langle \frac{1}{2\cosh^2 x}, \frac{2}{2\cosh^2 2x}, \frac{3}{2\cosh^2 3x}, ... \right\rangle$

8. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle e^{-x}\chi_{[0,\infty)}, 2e^{-2x}\chi_{[0,\infty)}, 3e^{-3x}\chi_{[0,\infty)}, ... \right\rangle$

9. $$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1.$$

10. $$\delta(\xi - x) = \frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} e^{-ik(\xi - x)}dk$$

5.

Hyper-real Random Variable

A **Random Variable**

$$X(\zeta)$$

is a real-valued function that maps any event (=outcome) ζ, in the Sample space S, into a real number x, in $\mathbb{R}$.

S includes the non-event ϕ, and $X(\phi) = 0$.

In [Dan5], we presented hyper-real Random Variables

1) Hyper-real $X(\zeta)$

$X(\zeta)$ is Hyper-real Random Variable iff its values may include infinitesimals, and infinite hyper-reals

2) Hyper-real Probability Distribution of $X(\zeta)$

Let $X(\zeta)$ be Hyper-real, and define,

$$dF(x) = \Pr(x - \tfrac{1}{2}dx \leq X(\zeta) \leq x + \tfrac{1}{2}dx).$$

Then,

$$F(x) = \sum_{x = X(\zeta),\ \zeta \in \mathrm{S}} dF(x).$$

is a Hyper-real Probability Distribution of $X(\zeta)$

3) Hyper-real Probability Density of $X(\zeta)$

Let $X(\zeta)$ be Hyper-real. If there is Hyper-real $f(x)$ so that

$$dF(x) = f(x)dx\,,$$

Then

$$f(x) = \frac{dF(x)}{dx}$$

is the Hyper-real Probability Density of $X(\zeta)$.

4) Expectation of Hyper-real $X(\zeta)$

$$E[X(\zeta)] \equiv \sum_{x=X(\zeta),\ \zeta\in \mathrm{S}} x dF(x)\,,$$

is a Hyper-real number.

If $dF(x) = f(x)dx$,

$$E[X(\zeta)] = \sum_{x=X(\zeta),\ \zeta\in \mathrm{S}} x f(x) dx\,.$$

5) 2nd Moment of Hyper-real $X(\zeta)$

$$E[X^2(\zeta)] \equiv \sum_{x=X(\zeta),\ \zeta\in \mathrm{S}} x^2 dF(x)$$

is a Hyper-real number.

6) A Normal Random Variable $N(\zeta)$, with $E[N(\zeta)] = \mu$, and $\mathrm{Var}[N(\zeta)] = \sigma^2$, has a probability density function $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.

The Variance of a Hyper-real $N(\zeta)$ may be an infinitesimal, or an infinite hyper real.

7) Infinite Hyper-real Variance

$$\sigma = \frac{1}{dx} \Rightarrow f(x) = \text{infinitesimal}$$

8) Infinitesimal Variance

$$\sigma = dx \Rightarrow f(x) = \text{Delta Function}$$

6.

Hyper-real Random Signal

A **Random Signal** is a Random Variable that depends also on the time t:

$$X(\zeta, t).$$

Then, the outcome of a Black ball,

$$\zeta = B$$

is identified with the outcome of drawing one Black ball, and one Red ball successively,

$$BR, \text{ and } RB,$$

and with the drawing of one Black ball, and two Red balls successively,

$$BRR,\ RBR,\ RRB,$$

etc.

For a given outcome ζ_0,

$$X(\zeta_0, t) = x_{\zeta_0}(t),$$

is a function of t, a <u>Sample Function</u>, or <u>Process Realization</u>.

In [Dan5], we presented Hyper-real Random Signals

1) Hyper-real $X(\zeta, t)$

A Random Signal is Hyper-real <u>iff</u> the time variable t, and the values of $X(\zeta,t)$ may include infinitesimals, and infinite hyper-reals.

2) Hyper-real Probability Distribution of $X(\zeta,t)$

Let $X(\zeta,t)$ be Hyper-real, fix $t=t_0$, and define,

$$dF(x,t_0)=\Pr(x-\tfrac{1}{2}dx\le X(\zeta,t_0)<x+\tfrac{1}{2}dx).$$

Then,

$$F(x,t_0)=\sum_{x=X(\zeta,t_0),\ \zeta\in S} dF(x,t_0).$$

is a Hyper-real Probability Distribution of $X(\zeta,t_0)$.

3) Hyper-real Probability Density of $X(\zeta,t)$

Let $X(\zeta,t)$ be Hyper-real, and fix $t=t_0$. If there is Hyper-real $f(x,t_0)$ so that

$$dF(x,t_0)=f(x,t_0)dx,$$

Then

$$f(x,t_0)=\frac{dF(x,t_0)}{dx}$$

is the Hyper-real Probability Density of $X(\zeta,t_0)$.

4) Expectation of Hyper-real $X(\zeta,t)$

Let $X(\zeta,t)$ be Hyper-real, fix $t=t_0$, and define

$$E[X(\zeta,t_0)]\equiv\sum_{x=X(\zeta,t_0),\ \zeta\in S} x\,dF(x,t_0),$$

If $dF(x,t_0) = f(x,t_0)dx$,

$$E[X(\zeta,t_0)] = \sum_{x=X(\zeta,t_0),\ \zeta\in S} xf(x,t_0)dx.$$

5) 2[nd] Moment of Hyper-real $X(\zeta,t)$

$$E[X^2(\zeta,t)] \equiv \sum_{x=X(\zeta,t),\ \zeta\in S} x^2 dF(x,t).$$

6) Variance of Hyper-real $X(\zeta,t)$

$$\mathrm{Var}[X(\zeta,t)] \equiv E[X^2(\zeta,t)] - (E[X(\zeta,t)])^2.$$

7) Continuity of $X(\zeta,t)$

Hyper-real $X(\zeta,t)$ is continuous at $t = t_0$ iff for any dt,

$$E\{[X(\zeta,t_0+dt) - X(\zeta,t_0)]^2\} = \text{infinitesimal},$$

$\Leftrightarrow$

$$\sum_{X(\zeta,t_0),\ \zeta\in S} [X(\zeta,t_0+dt) - X(\zeta,t_0)]^2 dF(x,t_0) = \text{infinitesimal}$$

If $dF(x,t_0) = f(x,t_0)dx$,

$\Leftrightarrow$

$$\sum_{X(\zeta,t_0),\ \zeta\in S} [X(\zeta,t_0+dt) - X(\zeta,t_0)]^2 f(x,t_0)dx = \text{infinitesimal}$$

8) $X(\zeta,t)$ *is continuous at* $t = t_0$ $\Rightarrow$ $E[X(\zeta,t_0)]$ *is continuous*

9) Derivative of $X(\zeta,t)$

Hyper-real $X(\zeta, t)$ ***has derivative with respect to*** t ***at*** $t = t_0$ ***<u>iff</u> there is a Random Signal*** $X'(\zeta, t) = \partial_t X(\zeta, t)$, ***so that for any*** dt,

$$E\left[\left[\frac{X(\zeta, t_0 + dt) - X(\zeta, t_0)}{dt} - X'(\zeta, t_0)\right]^2\right] = \text{infinitesimal},$$

$\Leftrightarrow$

$$\sum_{x = X(\zeta, t_0),\ \zeta \in \mathrm{S}} \left[\frac{x(\zeta, t_0 + dt) - x(\zeta, t_0)}{dt} - x'(\zeta, t_0)\right]^2 dF(x, t_0) = \text{infinitesimal}$$

If $dF(x, t_0) = f(x, t_0)dx$,

$\Leftrightarrow$

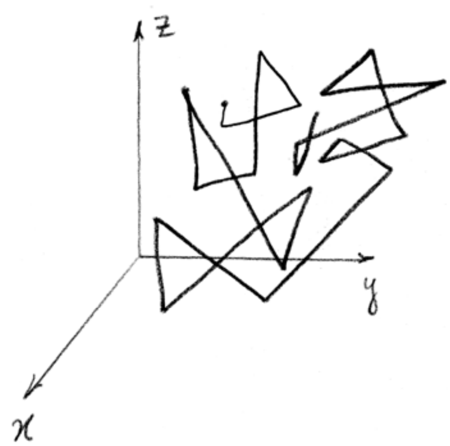
$$\sum_{x = X(\zeta, t_0),\ \zeta \in \mathrm{S}} \left[\frac{x(\zeta, t_0 + dt) - x(\zeta, t_0)}{dt} - x'(\zeta, t_0)\right]^2 f(x, t_0)dx = \text{infinitesimal}$$

7.

Random Walk

The Random Walk of small particles in fluid is named after Brown, who first observed it, Brownian Motion. It models other processes, such as the fluctuations of a stock price.

In a volume of fluid, the path of a particle is in any direction in the volume, and of variable size



7.1 The Bernoulli Random Variables of the Walk

We restrict the Walk here to the line, in uniform infinitesimal size steps dx:

To the left, with probability

$$p = \frac{1}{2},$$

or to the right, with probability

$$q = \tfrac{1}{2}.$$

At fixed time t, after

N infinitesimal time intervals dt,

$N = \frac{t}{dt}$, is a fixed infinite hyper-real,

the particle have made

K infinitesimal steps of size dx to the right,

and

L infinitesimal steps of size dx to the left,

and is at the point

$$x = (\underbrace{K - L}_{M})dx = Mdx.$$

K, L, M, are infinite hyper-reals.

At the ith step we define the Bernoulli Random Variable,

$$B_i(\text{right step}) = dx, \quad \zeta_1 = \text{right step}.$$

$$B_i(\text{left step}) = -dx, \quad \zeta_2 = \text{left step}.$$

where $i = 1, 2, \ldots, N$.

$$\Pr(B_i = dx) = p = \tfrac{1}{2},$$

$$\Pr(B_i = -dx) = q = \tfrac{1}{2},$$

$$E[B_i] = dx \cdot \tfrac{1}{2} + (-dx) \cdot \tfrac{1}{2} = 0,$$

$$E[B_i^{\,2}] = (dx)^2 \cdot \tfrac{1}{2} + (-dx)^2 \cdot \tfrac{1}{2} = (dx)^2$$

$$\text{Var}[B_i] = \underbrace{E[B_i^{\,2}]}_{(dx)^2} - (\underbrace{E[B_i]}_{0})^2 = (dx)^2.$$

7.2 The Distribution of the Random Walk

$$B(\zeta, t) = B_1 + B_2 + ... + B_N$$

is a Random Signal with

$$E[B(\zeta, t)] = 0,$$

$$\text{Var}[B(\zeta, t)] = N(dx)^2.$$

Proof: Since the B_i are independent,

$$E[B(\zeta, t)] = \underbrace{E[B_1]}_{0} + ... + \underbrace{E[B_N]}_{0} = 0$$

$$\text{Var}[B(\zeta, t)] = \underbrace{\text{Var}[B_1]}_{(dx)^2} + ... + \underbrace{\text{Var}[B_N]}_{(dx)^2} = N(dx)^2. \square$$

In [Dan5], we presented the Infinitesimal Calculus of Random Walk

1) The Gaussian Distribution of the Walk

If $(dx)^2 = 2D(dt)$, *where* D *is a constant*

Then, the distribution of $B(\zeta, t)$ *is infinitesimally close to a Gaussian distribution of a Random Signal with*

$$\mu = 0,$$

$$\sigma = \sqrt{t2D} = \sqrt{N}dx.$$

$$f(x,t) \approx \frac{1}{\sqrt{2\pi}\sqrt{t2D}} e^{-\frac{1}{2}\frac{x^2}{t2D}}$$

2) Increments of Random Walk

<u>If</u> $(dx)^2 = 2D(dt)$,

<u>Then</u> **a)** *For any* $\tau > 0$, *the distribution of*

$$B(\zeta, t+\tau) - B(\zeta, t) \text{ is}$$

infinitesimally close to a Gaussian distribution that has

$$\mu = 0,$$

$$\sigma^2 = \tau 2D,$$

and depends only on τ *(Stationary Process).*

b) *For fixed* t, *and any* dt, *the increments*

$$B(\zeta, t) - B(\zeta, t - dt),$$

$$B(\zeta, t - dt) - B(\zeta, t - 2dt),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$B(\zeta, dt) - B(\zeta, 0),$$

are independent, random variables.

3) $(dx)^2 = (2D)dt \Rightarrow$ Random Walk is Continuous

4) $(dx)^2 = (2D)dt \Rightarrow$ ***The Derivative of Random Walk is***

$$\dot{B} = \frac{1}{dt} B_i ,$$

where (a) $B_i = B(\zeta, t_0 + dt) - B(\zeta, t_0)$***, is a Bernoulli Random Variable.***

(b) $E[\dot{B}] = 0$,

(c) $\mathrm{Var}[\dot{B}] = 2D\delta(t_0)$,

5) $E[B(\zeta, t)]$ ***has unbounded Variation in*** $[a, b]$.

8.

$$\int_{t=a}^{t=b} f(t)dB(\zeta,t)$$

While $E[B(\zeta,t)]$ has unbounded Variation in $[a,b]$, integration with respect to $B(\zeta,t)$ is possible.

Let $f(t)$ be a hyper-real function on the bounded time interval $[a,b]$. $f(t)$ need not be bounded.

At each $a \le t \le b$, there is a Bernoulli Random Variable

$$dB(\zeta,t) = B(\zeta,t+dt) - B(\zeta,t) = B_i(\zeta,t) = \dot{B}(\zeta,t)dt.$$

We form the **Integration Sum**

$$\sum_{t=a}^{t=b} f(t)dB(\zeta,t) = \sum_{t=a}^{t=b} f(t)B_i(\zeta,t) = \sum_{t=a}^{t=b} f(t)\dot{B}(\zeta,t)dt$$

For any dt,

(1) the First Moment of the Integration Sum is

$$E\left[\sum_{t=a}^{t=b} f(t)\dot{B}(\zeta,t)dt\right] = \sum_{t=a}^{t=b} f(t)\underbrace{E[\dot{B}(\zeta,t)]}_{0}dt = 0.$$

(2) the Second Moment of the Integration sum is

$$E\left[\left(\sum_{t=a}^{t=b} f(t)B_i(\zeta,t)\right)^2\right] = E\left[\left(\sum_{t=a}^{t=b} f(t)B_i(\zeta,t)\right)\left(\sum_{\tau=a}^{\tau=b} f(\tau)B_j(\zeta,\tau)\right)\right]$$

$$= \sum_{t=a}^{t=b}\sum_{\tau=a}^{\tau=b} f(t)f(\tau)E[B_j(\zeta,\tau)B_i(\zeta,t)]$$

Since the Bernoulli Random Variables are independent,

$$E[B_j(\zeta,\tau)B_i(\zeta,t)] = E[B_i^2(\zeta,t)] = (dx)^2$$

only for $t = \tau$. Then,

$$E\left[\left(\sum_{t=a}^{t=b} f(t)B_i(\zeta,t)\right)^2\right] = \sum_{t=a}^{t=b} f^2(t)\underbrace{(dx)^2}_{(2D)dt},$$

$$= 2D\sum_{t=a}^{t=b} f^2(t)dt,$$

$$= 2D\int_{t=a}^{t=b} f^2(t)dt.$$

assuming $(dx)^2 = (2D)dt$, and $f(t)$ integrable

Thus, for any dt, the Integration Sum is a unique well-defined hyper-real Random Variable $I(\zeta)$.

We call $I(\zeta)$ the integral of $f(t)$, with respect to $B(\zeta,t)$ from $x = a$, to $x = b$, and denote it by $\int_{t=a}^{t=b} f(t)dB(\zeta,t)$.

9.

Hyper-real Auto-Correlation

9.1 The Hyper-real Auto-Correlation Function

between the Hyper-real Random Variables $X(\zeta, t_1)$*, and* $X(\zeta, t_2)$ *is the Mean of the Hyper-real Random Variable* $X(\zeta, t_1)X(\zeta, t_2)$,

$$E[X(\zeta, t_1)X(\zeta, t_2)] \equiv R_{XX}(t_1, t_2).$$

9.2 Example

$$X(\zeta, t) = a\cos(bt + \zeta),$$

where $\zeta(\varsigma)$ *is a Random Variable uniformly distributed in* $[0, 2\pi]$*, has Probability Density Function* $f_X(\varsigma) = \frac{1}{2\pi}$*, and*

$$E[X(\zeta, t_1)X(\zeta, t_2)] = \tfrac{1}{2}a^2\cos b(t_1 - t_2)$$

<u>*Proof*</u>: $E[X(\zeta, t_1)X(\zeta, t_2)] = \int\limits_{\varsigma=0}^{\varsigma=2\pi} a^2\cos(bt_1 + \varsigma)\cos(bt_2 + \varsigma)\frac{1}{2\pi}d\varsigma$

$$= \tfrac{1}{2}\int\limits_{\varsigma=0}^{\varsigma=2\pi} a^2\{\cos(bt_1 - bt_2) + \cos(bt_1 + bt_2 + 2\varsigma)\}\frac{1}{2\pi}d\varsigma$$

$$= \underbrace{\tfrac{1}{2}a^2\{\cos b(t_1 - t_2)}_{R_{XX}(t_1 - t_2)} + \tfrac{1}{4\pi}a^2 \underbrace{\int_{\varsigma=0}^{\varsigma=2\pi} \cos(bt_1 + bt_2 + 2\varsigma)d\varsigma}_{0}.\square$$

9.3 Auto-Correlation of $\dot{B}(\zeta, t)$ is a Delta Function

$$E[\dot{B}(\zeta,t_1)\dot{B}(\zeta,t_2)] = 2D\delta(t_1 - t_2),$$

where $(dx)^2 = 2D(dt)$.

<u>Proof</u>:

$$E[\dot{B}(\zeta,t_1)\dot{B}(\zeta,t_2)] = E[\tfrac{dB(\zeta,t_1)}{dt}\tfrac{dB(\zeta,t_2)}{dt}]$$

$$= \tfrac{1}{(dt)^2}E[dB(\zeta,t_1)dB(\zeta,t_2)]$$

where $dB(\zeta,t_1)$ is some Bernoulli Random Variable $B_i(\zeta,t_1)$,

and $dB(\zeta,t_2)$ is some Bernoulli Random Variable $B_j(\zeta,t_2)$.

$$= \tfrac{1}{(dt)^2}E[B_i(\zeta,t_1)B_j(\zeta,t_2)].$$

For $t_1 \neq t_2$, $B_i(\zeta,t_1)$, and $B_j(\zeta,t_2)$ are independent Random variables, and we have

$$= \tfrac{1}{(dt)^2}\underbrace{E[B_i(\zeta,t_1)]}_{0}\underbrace{E[B_j(\zeta,t_2)]}_{0} = 0$$

For $t_1 = t_2$, $B_i(\zeta,t_1) = B_j(\zeta,t_2)$, and we have

$$= \frac{1}{(dt)^2} \underbrace{E[B_i^{\,2}(\zeta, t_1)]}_{(dx)^2} = 2D \frac{1}{dt}$$

Therefore,

$$E[\dot{B}(\zeta, t_1)\dot{B}(\zeta, t_2)] = 2D\delta(t_1 - t_2). \square$$

10.

Stationary Hyper-real $X(\zeta, t)$

A Hyper-real Random Signal $X(\zeta, t)$ has a Stationary probability density <u>iff</u> $f_X(x,t)$ is time-independent .

Then, $E[X(\zeta, t)] = \text{constant}$

A Hyper-real Random Signal $X(\zeta, t)$ has a Stationary joint probability density <u>iff</u> $f_{X(\zeta,t_1)X(\zeta,t_2)}(x_1, x_2, t_1, t_2)$ depends on $t_1 - t_2$. Then, $E[X(\zeta, t_1)X(\zeta, t_2)] = R_{XX}(t_1 - t_2)$

10.1 Stationary Hyper-real Random Signal

A Hyper-real Random Signal $X(\zeta, t)$ is Stationary <u>iff</u>

1) $E[X(\zeta, t)] = \text{constant}$

2) $E[X(\zeta, t_1)X(\zeta, t_2)] = R_{XX}(t_1 - t_2)$

10.2 Example 9.2 $X(\zeta, t) = a\cos(bt + \zeta)$ *is stationary*

<u>*Proof*</u>:

1) $E[X(\zeta, t)] = a \int\limits_{\varsigma=0}^{\varsigma=2\pi} \cos(bt + \varsigma)\frac{1}{2\pi}d\varsigma = \frac{a}{2\pi}\sin(bt + \varsigma)\Big|_{\varsigma=0}^{\varsigma=2\pi} = 0.$

2) $E[X(\zeta,t_1)X(\zeta,t_2)] = R_{XX}(t_1 - t_2)$,

is shown in the proof of 9.2.□

11.

Ergodic $X(\zeta,t)$ and the Ergodic Hypothesis

11.1 *Time-Averaging and Mean Operations Commute*

$$\frac{1}{2T}\sum_{t=-T}^{t=T} E[X(\zeta,t)]dt = E\left[\frac{1}{2T}\sum_{t=-T}^{t=T} X(\zeta,t)dt\right]$$

$$\frac{1}{2T}\sum_{t=-T}^{t=T} E[X(\zeta,t)X(\zeta,t+\tau)]dt = E\left[\frac{1}{2T}\sum_{t=-T}^{t=T} X(\zeta,t)X(\zeta,t+\tau)dt\right]$$

11.2 Ergodic Random Signal

A Stationary Hyper-real Random Signal $X(\zeta,t)$ is Ergodic iff for an infinite hyper-real time T, and any outcome η,

I) $E[X(\zeta,t)] = \frac{1}{2T}\sum_{t=-T}^{t=T} X(\eta,t)dt$

II) $E[X(\zeta,t)X(\zeta,t+\tau)] = \frac{1}{2T}\sum_{t=-T}^{t=T} X(\eta,t)X(\eta,t+\tau)dt$.

Since the random signal is stationary,

$E[X(\zeta,t)]$ is a constant C_0, and

$E[X(\zeta,t)X(\zeta,t+\tau)]$ is a function of the time difference $\alpha(\tau)$

Thus, **I)** says that for any outcome η,

$$\frac{1}{2T}\int_{t=-T}^{t=T} X(\eta,t)dt = C_0.$$

And **II)** says that for any outcome η,

$$\frac{1}{2T}\int_{t=-T}^{t=T} X(\eta,t)X(\eta,t+\tau)dt = \alpha(\tau)$$

11.3 Example $X(\zeta,t) = a\cos(bt+\zeta)$ ***is Ergodic***

Proof:

By 10.2, $X(\zeta,t) = a\cos(bt+\zeta)$ is stationary.

I) For any ς, and an infinite Hyper-real time T,

$$\frac{1}{2T}\int_{t=-T}^{t=T} a\cos(bt+\varsigma)dt = \frac{a}{2Tb}\sin(bt+\varsigma)\Big|_{t=-T}^{t=T}$$

$$= \frac{1}{T}\frac{a}{2b}\{\underbrace{\sin(\varsigma+bT)-\sin(\varsigma-bT)}_{2\sin bT\cos\varsigma}\}$$

$$= \frac{1}{T}\underbrace{\frac{a}{b}\cos(\varsigma)\sin(bT)}_{\text{finite Hyper-real}}$$

$$= \text{infinitesimal}.$$

That is,

$$\frac{1}{2T}\int_{t=-T}^{t=T} a\cos(bt+\varsigma)dt \approx 0 = E[X(\zeta,t)].\square$$

II) $$\frac{1}{2T}\int_{t=-T}^{t=T} a\cos(bt+\varsigma)a\cos(bt+b\tau+\varsigma)dt =$$

$$= \frac{1}{2T}\frac{1}{2}a^2\int_{t=-T}^{t=T}\{\cos(b\tau)+\cos(2bt+b\tau+2\varsigma)\}dt$$

$$= \underbrace{\tfrac{1}{2}a^2\cos b\tau}_{R_{XX}(\tau)} + \frac{1}{4T}\frac{a^2}{2b}\{\underbrace{\sin(b\tau+2\varsigma+2bT)-\sin(b\tau+2\varsigma-2bT)}_{2\sin(bT)\cos(b\tau+2\varsigma)}\}$$

$$= R_{XX}(\tau) + \underbrace{\frac{1}{4T}\underbrace{\frac{a^2}{b}\sin(bT)\cos(b\tau+2\varsigma)}_{\text{finite Hyper-real}}}_{\text{infinitesimal}}$$

$$\approx R_{XX}(\tau) = E[X(\zeta,t)X(\zeta,t+\tau)].\square$$

11.4 The Ergodic Hypothesis

is the assumption that a physical Random Signal is Ergodic. If a measured Random Signal has unknown distribution, the Hypothesis is assumed to replace the Mean, and the Auto-Correlation with the time averages of the measured signal.

12.

Hyper-real Characteristic Function $\varphi_X(\xi)$

The Inverse Fourier Transform of a Probability Density Function is used to obtain the Moments of Random Variable.

Let $f_X(x)$ be a Hyper-real Probability Density Function of the Random Variable $X(\zeta)$. Then, [Dan9], the Inverse Fourier Transform of $f_X(x)$

$$\mathcal{F}^{-1}\{f_X\} = \tfrac{1}{2\pi}\int_{x=-\infty}^{x=\infty} e^{i\xi x} f_X(x)dx\,,$$

exists.

The Hyper-real Characteristic Function of $X(\zeta)$ is

$$\varphi_X(\xi) = E[e^{i\xi X}],$$

$$= \int_{x=-\infty}^{x=\infty} e^{i\xi x} f_X(x)dx\,,$$

$$= 2\pi\mathcal{F}^{-1}\{f_X\}\,.$$

12.1 Example of Bernoulli Random Variable $B(\zeta)$ so that

$$\Pr(B = 1) = p\,, \quad \text{and} \quad \Pr(B = 0) = q\,,$$

$$\begin{aligned}\varphi_B(\xi) &= E[e^{i\xi B}] \\ &= e^{i\xi \cdot 1}p + e^{i\xi \cdot 0}q \\ &= e^{i\xi}p + q\,.\end{aligned}$$

12.2 $$f_X(x) = \tfrac{1}{2\pi}\int_{x=-\infty}^{x=\infty} e^{-i\xi x}\varphi_X(\xi)dx = \tfrac{1}{2\pi}\mathcal{F}\{\varphi_X\}.$$

12.3 $$\int_{\xi=-\infty}^{\xi=\infty} \varphi_X(\xi)d\xi = 2\pi f_X(0)$$

Proof:

$$\begin{aligned}\int_{\xi=-\infty}^{\xi=\infty} \varphi_X(\xi)d\xi &= \int_{\xi=-\infty}^{\xi=\infty}\left(\int_{x=-\infty}^{x=\infty} e^{i\xi x}f_X(x)dx\right)d\xi \\ &= \int_{x=-\infty}^{x=\infty}\underbrace{\left(\int_{\xi=-\infty}^{\xi=\infty} e^{i\xi x}d\xi\right)}_{2\pi\delta(x)}f_X(x)dx \\ &= 2\pi f_X(0)\end{aligned}$$

12.4 $f_X(0) \neq 0 \;\; \Rightarrow$

(I) $\int\limits_{\xi=-\infty}^{\xi=\infty} \frac{1}{2\pi f_X(0)} \varphi_X(\xi) d\xi = 1$, *and* $\frac{1}{2\pi f_X(0)} \varphi_X(\xi)$ *is a*

Probability Density Function of $X(\zeta)$.

(II) $\Phi_X(\xi) \equiv \int\limits_{\alpha=-\infty}^{\alpha=\xi} \frac{1}{2\pi f_X(0)} \varphi_X(\alpha) d\alpha$ *is a*

Probability Distribution Function of $X(\zeta)$,

so that $d\Phi_X(\xi) = \frac{1}{2\pi f_X(0)} \varphi_X(\xi) d\xi$

12.5 $X_1(\zeta)$, $X_2(\zeta)$ independent $\Rightarrow$ $\varphi_{X_1+X_2}(\xi) = \varphi_{X_1}(\xi)\varphi_{X_2}(\xi)$

<u>*Proof*</u>:

$$\varphi_{X_1+X_2}(\xi) = E[e^{i\xi(X_1+X_2)}] = E[e^{i\xi X_1} e^{i\xi X_2}] = \underbrace{E[e^{i\xi X_1}]}_{\varphi_{X_1}(\xi)} \underbrace{E[e^{i\xi X_2}]}_{\varphi_{X_2}(\xi)}$$

12.6 $$\left.\frac{d\varphi}{dx}\right|_{\xi=0} = iE[X],$$

$$\left.\frac{d^2\varphi}{dx^2}\right|_{\xi=0} = i^2 E[X^2],$$

$$\left.\frac{d^3\varphi}{dx^3}\right|_{\xi=0} = i^3 E[X^3],$$

………………………………

<u>*Proof*</u>:

$$\varphi_X(\xi) = E[e^{i\xi X}]$$

$$= E[1 + i\xi X + \tfrac{1}{2!}(i\xi)^2 X^2 + \tfrac{1}{3!}(i\xi)^3 X^3 + ...]$$

$$= 1 + i\xi E[X] + \tfrac{1}{2!}(i\xi)^2 E[X^2] + \tfrac{1}{3!}(i\xi)^3 E[X^3] + ... \square$$

13.

Power Spectral Density of Stationary $X(\zeta, t)$

13.1 Mean Power of a Random Signal

The Mean Power of a Random Signal $X(\zeta, t)$ is its second Moment

$$E[X^2(\zeta, t)].$$

$X(\zeta, t)$ may be the Noise Voltage over a resistor of one Ohm, or the Noise Current through a resistor of one Ohm.

13.2 Example $X(\zeta, t) = a\cos(bt + \zeta)$,

where $\zeta(\varsigma)$ *is a Random Variable uniformly distributed in* $[0, 2\pi]$*, has Probability Density Function* $f_X(\varsigma) = \frac{1}{2\pi}$*, and*

$$E[X^2(\zeta, t)] = \tfrac{1}{2}a^2$$

<u>*Proof*</u>: By 9.2, $E[X(\zeta, t)X(\zeta, t)] = \frac{1}{2}a^2 \cos b(t - t) = \frac{1}{2}a^2$.□

13.3 Mean Power of Thermal Noise $\dot{B}(\zeta, t)$

$$E[\dot{B}^2(\zeta, t)] = 2D\begin{cases} \frac{1}{dt}, & \text{on } [t - \frac{1}{2}dt, t + \frac{1}{2}dt] \\ 0, & \text{elsewhere} \end{cases}$$

Proof: By 9.3.□

Since the Fourier Transform of the Delta Function equals 1, we prefer to Fourier Transform the Power. But a stationary Signal has,

$$E[X(\zeta, t)X(\zeta, t)] = R_{XX}(t - t),$$

and no time variable to integrate over.

Therefore, we Fourier Transform $E[X(\zeta, t_1)X(\zeta, t_2)]$.

By [Dan9], the Fourier Transform of the Hyper-real $E[X(\zeta, t_1)X(\zeta, t_2)]$ is well-defined.

13.4 Power Spectral Density at $\omega = 2\pi\nu$ of Stationary $X(\zeta, t)$,

$$S_{XX}(\omega) \equiv \mathcal{F}\{E[X(\zeta, t)X(\zeta, t + \tau)]\}$$

$$= \int_{\tau=-\infty}^{\tau=\infty} E[X(\zeta, t)X(\zeta, t + \tau)]e^{-i\omega\tau}d\tau$$

$$= \int_{\tau=-\infty}^{\tau=\infty} E[X(\zeta,t)X(\zeta,t+\tau)]e^{-i2\pi\nu\tau}d\tau = S_{XX}(\nu)$$

Then, $E[X(\zeta,t)X(\zeta,t+\tau)]$ is the Inverse Fourier Transform of the Hyper-real Function $S_{XX}(\omega)$,

13.5 $E[X(\zeta,t)X(\zeta,t+\tau)] = \mathcal{F}^{-1}\{S_{XX}(\omega)\}$

$$= \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} S_{XX}(\omega)e^{i\omega\tau}d\omega$$

$$= \int_{\nu=-\infty}^{\nu=\infty} S_{XX}(\nu)e^{i2\pi\nu\tau}d\nu$$

Therefore,

13.6 $E[X^2(\zeta,t)] \approx \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} S_{XX}(\omega)e^{i\omega\tau}d\omega\Big|_{\tau=\text{infinitesimal}}$,

$$\approx \int_{\nu=-\infty}^{\nu=\infty} S_{XX}(\nu)e^{i2\pi\nu\tau}d\nu\Big|_{\tau=\text{infinitesimal}},$$

$$\approx \left[\mathcal{F}^{-1}\{S_{XX}(\omega)\}\right]_{\tau=\text{infinitesimal}}.$$

Equating $E[X^2(\zeta,t)]$ to $\frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} S_{XX}(\omega)d\omega$ is clearly wrong.

In fact,

13.7 $$\boxed{E[X^2(\zeta,t)] \neq \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} S_{XX}(\omega)d\omega}$$

<u>*Proof*</u>: Let $X(\zeta,t) = \dot{B}(\zeta,t)$. Then,

$$E[\dot{B}(\zeta,t)\dot{B}(\zeta,t+\tau)] = 2D\delta(\tau), \text{ (by 9.3)}$$

$$E[\dot{B}^2(\zeta,t)] = 2D\delta(0)$$

$$S_{XX}(\nu) = \mathcal{F}\{2D\delta(\tau)\} = 2D\underbrace{\mathcal{F}\{\delta(\tau)\}}_{1} = 2D$$

$$\int_{\nu=-\infty}^{\nu=\infty} S_{XX}(\nu)d\nu = 2D\int_{\nu=-\infty}^{\nu=\infty} d\nu$$

$$= \infty > 2D\delta(0) = E[\dot{B}^2(\zeta,t)],$$

because $\delta(0)$ is an infinite hyper-real, and any infinite hyper-real is strictly smaller than ∞.□

13.8 Example $X(\zeta,t) = a\cos(\omega_0 t + \zeta)$,

where $\zeta(\varsigma)$ is a Random Variable uniformly distributed in $[0,2\pi]$, has Probability Density Function $f_X(\varsigma) = \frac{1}{2\pi}$, and

$$S_{XX}(\omega) = \tfrac{1}{2}a^2\pi\{\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\}$$

Proof: $$S_{XX}(\omega) = \int_{\tau=-\infty}^{\tau=\infty} E[X(\zeta,t)X(\zeta,t+\tau)]e^{-i\omega\tau}d\tau$$

By 9.2,

$$E[X(\zeta,t_1)X(\zeta,t_2)] = \tfrac{1}{2}a^2\cos\omega_0(t_1-t_2)$$

Hence,

$$S_{XX}(\omega) = \tfrac{1}{2}a^2\int_{\tau=-\infty}^{\tau=\infty} \underbrace{\cos(\omega_0\tau)}_{\frac{1}{2}\{e^{i\omega_0\tau}+e^{-i\omega_0\tau}\}}\ e^{-i\omega\tau}d\tau$$

$$= \tfrac{1}{2}a^2\int_{\tau=-\infty}^{\tau=\infty} \tfrac{1}{2}\{e^{i\omega_0\tau}+e^{-i\omega_0\tau}\}e^{-i\omega\tau}d\tau$$

$$= \tfrac{1}{4}a^2\underbrace{\int_{\tau=-\infty}^{\tau=\infty} e^{-i(\omega-\omega_0)\tau}d\tau}_{2\pi\delta(\omega-\omega_0)} + \tfrac{1}{4}a^2\underbrace{\int_{\tau=-\infty}^{\tau=\infty} e^{-i(\omega+\omega_0)\tau}d\tau}_{2\pi\delta(\omega+\omega_0)}$$

$$= \tfrac{1}{2}a^2\pi\{\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\}.\square$$

13.9 Power Spectral Density of $\dot{B}(\zeta,t)$

$$S_{\dot{B}\dot{B}}(\omega) = 2D$$

is constant in all frequencies $\omega = 2\pi\nu$, *and the Thermal Noise is called **White Noise**.*

Proof: $$S_{\dot{B}\dot{B}}(\omega) = \int_{\tau=-\infty}^{\tau=\infty} E[\dot{B}(\zeta,t)\dot{B}(\zeta,t+\tau)]e^{-i\omega\tau}d\tau$$

By 9.3,

$$E[\dot{B}(\zeta, t_1)\dot{B}(\zeta, t_2)] = 2D\delta(t_1 - t_2)$$

Hence,

$$S_{\dot{B}\dot{B}}(\omega) = 2D \int_{\tau=-\infty}^{\tau=\infty} \delta(\tau) e^{-i\omega\tau} d\tau$$

$$= 2De^{-i\omega 0} = 2D. \square$$

13.10 Power of Band-Limited White Noise $S_{\dot{B}\dot{B}}(\nu)$

$S_{\dot{B}\dot{B}}(\nu) = 2D\chi_{[-\nu_0, \nu_0]}(\nu)$ ***is limited to the frequency band*** $[-\nu_0, \nu_0]$

Then $$E[\dot{B}(\zeta, t)\dot{B}(\zeta, t+\tau)] = (2D)(2\nu_0)\frac{\sin(2\pi\nu_0\tau)}{2\pi\nu_0\tau}$$

$$E[\dot{B}^2(\zeta, t)] \approx (2D)(2\nu_0)$$

Proof: $$E[\dot{B}(\zeta, t)\dot{B}(\zeta, t+\tau)] = \int_{\nu=-\infty}^{\nu=\infty} \underbrace{S_{\dot{B}\dot{B}}(\nu)}_{2D\chi_{[-\nu_0,\nu_0]}(\nu)} e^{i2\pi\nu\tau} d\nu$$

$$= 2D \int_{\nu=-\nu_0}^{\nu=\nu_0} e^{i2\pi\nu\tau} d\nu$$

$$= 2D \frac{1}{\pi\tau} \underbrace{\frac{1}{2i}\{e^{i2\pi\nu_0\tau} - e^{-i2\pi\nu_0\tau}\}}_{\sin(2\pi\nu_0\tau)}$$

$$= (2D)(2\nu_0)\frac{\sin(2\pi\nu_0\tau)}{2\pi\nu_0\tau}.\square$$

Hence, for infinitesimal τ,

$$E[\dot{B}^2(\zeta,t)] \approx (2D)(2\nu_0).\square$$

14.

Time-Limited Stationary Signal

Let the Stationary Random Signal $X_L(\zeta,t)$ vanish out of the time-window $[-L,L]$,

$$X_L(\zeta,t)=\begin{cases}X(\zeta,t), & \text{on } [-L,L]\\ 0, & \text{otherwise}\end{cases}.$$

For each ζ, the Fourier Transform of $X_L(\zeta,t)$ is

$$\hat{X}_L(\zeta,\omega)=\mathcal{F}\{X_L(\zeta,t)\}=\int_{t=-L}^{t=L} X_L(\zeta,t)e^{-i\omega t}dt\,.$$

14.1 Example $X_L(\zeta,t)=\begin{cases}a\cos(\omega_0 t+\zeta) & \text{on [-L,L]}\\ 0, & \text{otherwise}\end{cases}$,

where $\zeta(\varsigma)$ is a Random Variable uniformly distributed in $[0,2\pi]$, has Probability Density Function $f_X(\varsigma)=\frac{1}{2\pi}$, and

$$\hat{X}_L(\zeta,\omega)=ae^{i\zeta}\frac{\sin(w_0-w)L}{w_0-w}+ae^{-i\zeta}\frac{\sin(w_0+w)L}{w_0+w}$$

<u>*Proof*</u>: $\hat{X}_L(\zeta,\omega)=a\int_{t=-L}^{t=L}\cos(w_0 t+\zeta)e^{-i\omega t}dt$

$$= a\tfrac{1}{2}e^{i\zeta}\int_{t=-L}^{t=L} e^{i(w_0-w)t}dt + a\tfrac{1}{2}\int_{t=-L}^{t=L} e^{-i(w_0+w)t}dt$$

$$= ae^{i\zeta}\frac{e^{i(w_0-w)L}-e^{-i(w_0-w)L}}{2i(w_0-w)} + ae^{-i\zeta}\frac{e^{i(w_0+w)L}-e^{-i(w_0+w)(-L)}}{-2i(w_0+w)}$$

$$= ae^{i\zeta}\frac{\sin(w_0-w)L}{w_0-w} + ae^{-i\zeta}\frac{\sin(w_0+w)L}{w_0+w}.\square$$

14.2 Power of Time-Limited Stationary Signal

$$E[X_L^2(\zeta,t)] = \int_{\nu=-\infty}^{\nu=\infty} \tfrac{1}{2L}E\left[\left|\hat{X}_L(\zeta,\nu)\right|^2\right]d\nu\,.$$

Proof:

By Parseval's Identity for Fourier Transforms, [Spiegel, p.175],

$$\int_{t=-\infty}^{t=\infty}\left|X_L(\zeta,t)\right|^2 dt = \tfrac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty}\left|\hat{X}_L(\zeta,\omega)\right|^2 d\omega$$

$$= \int_{\nu=-\infty}^{\nu=\infty}\left|\hat{X}_L(\zeta,\nu)\right|^2 d\nu\,.$$

Substituting for $X_L(\zeta,t)$,

$$\int_{t=-L}^{t=L} X_L^2(\zeta,t)dt = \int_{\nu=-\infty}^{\nu=\infty} \left|\hat{X}_L(\zeta,\nu)\right|^2 d\nu$$

Applying the Mean to both sides,

$$\int_{t=-L}^{t=L} E[X_L^2(\zeta,t)]dt = \int_{\nu=-\infty}^{\nu=\infty} E\left[\left|\hat{X}_L(\zeta,\nu)\right|^2\right]d\nu .$$

For a Stationary Signal, $E[X(\zeta,t)X(\zeta,t)] = R_{XX}(t-t) = \text{const}$

Thus, integrating the left hand side,

$$2LE[X_L^2(\zeta,t)] = \int_{\nu=-\infty}^{\nu=\infty} E\left[\left|\hat{X}_L(\zeta,\nu)\right|^2\right]d\nu$$

$$E[X_L^2(\zeta,t)] = \int_{\nu=-\infty}^{\nu=\infty} \tfrac{1}{2L}E\left[\left|\hat{X}_L(\zeta,\nu)\right|^2\right]d\nu .\square$$

14.3 Power Spectral Density of Ergodic Signal $X_L(\zeta,t)$

$$\tfrac{1}{2L}E\left[\left|\hat{X}_L(\zeta,\nu)\right|^2\right] = S_{XX}(\nu)$$

Proof: $\tfrac{1}{2L}E\left[\left|\hat{X}_L(\zeta,\nu)\right|^2\right] = \tfrac{1}{2L}E\left[\hat{X}_L^*(\zeta,\nu)\hat{X}_L(\zeta,\nu)\right]$

$$= \tfrac{1}{2L}E\left[\left(\int_{t=-L}^{t=L} X_L(\zeta,t)e^{-i\omega t}dt\right)^* \int_{t+\tau=-L}^{t+\tau=L} X_L(\zeta,t+\tau)e^{-i\omega(t+\tau)}d\tau\right]$$

$$= \frac{1}{2L} E\left[\int_{t=-L}^{t=L} X_L(\zeta,t)e^{i\omega t}dt \int_{\tau=-L-t}^{\tau=L-t} X_L(\zeta,t+\tau)e^{-i\omega(t+\tau)}d\tau \right]$$

$$= \frac{1}{2L} E\left[\int_{\tau=-L-t}^{\tau=L-t} \left(\int_{t=-L}^{t=L} X_L(\zeta,t)X_L(\zeta,t+\tau)dt \right) e^{-i\omega\tau}d\tau \right]$$

$$= \int_{\tau=-L-t}^{\tau=L-t} \underbrace{\left(\frac{1}{2L} \int_{t=-L}^{t=L} E\big[X_L(\zeta,t)X_L(\zeta,t+\tau) \big] dt \right)}_{=E\left[X_L(\zeta,t)X_L(\zeta,t+\tau)\right],\ \text{since Ergodic}} e^{-i\omega\tau}d\tau$$

$$= \int_{\tau=-L-t}^{\tau=L-t} E\big[X_L(\zeta,t)X_L(\zeta,t+\tau) \big] e^{-i\omega\tau}d\tau$$

$$= \int_{\tau=-\infty}^{\tau=\infty} E\big[X_L(\zeta,t)X_L(\zeta,t+\tau) \big] e^{-i\omega\tau}d\tau$$

$$= S_{XX}(\omega) = S_{XX}(\nu). \square$$

14.4 Example $\quad X_L(\zeta,t) = \begin{cases} a\cos(\omega_0 t + \zeta) & \text{on [-L,L]} \\ 0, & \text{otherwise} \end{cases},$

where $\zeta(\varsigma)$ *is a Random Variable uniformly distributed in* $[0,2\pi]$*, has Probability Density Function* $f_X(\varsigma) = \frac{1}{2\pi}$*, and*

(a) $S_{XX}(\omega) = \frac{1}{2}a^2\pi\left\{\frac{\sin^2 L(\omega-\omega_0)}{\pi L(\omega-\omega_0)^2} + \frac{\sin^2 L(\omega+\omega_0)}{\pi L(\omega+\omega_0)^2}\right\}$

(b) ***For*** $L = \text{infinite hyper-real} = \langle n \rangle$,

$\left\langle \frac{\sin^2 n(\omega-\omega_0)}{\pi n(\omega-\omega_0)^2}\right\rangle$, ***and*** $\left\langle \frac{\sin^2 n(\omega+\omega_0)}{\pi n(\omega+\omega_0)^2}\right\rangle$ ***are the***

Delta Functions of 13.6.

<u>*Proof*</u>: **(a)** By 14.1,

$$\hat{X}_L(\zeta,\omega) = ae^{i\zeta}\frac{\sin(w_0-w)L}{w_0-w} + ae^{-i\zeta}\frac{\sin(w_0+w)L}{w_0+w}$$

$$\hat{X}_L{}^*(\zeta,\omega) = ae^{-i\zeta}\frac{\sin(w_0-w)L}{w_0-w} + ae^{i\zeta}\frac{\sin(w_0+w)L}{w_0+w}$$

$$\hat{X}_L(\zeta,t)\hat{X}_L^*(\zeta,t) = a^2\frac{\sin^2(w_0-w)L}{(w_0-w)^2} + a^2\frac{\sin^2(w_0+w)L}{(w_0+w)^2}$$

$$+a^2\underbrace{(e^{2i\zeta}+e^{-2i\zeta})}_{2\cos(2\zeta)}\frac{\sin[(w_0-w)L]}{w_0-w}\frac{\sin[(w_0+w)L]}{w_0+w}$$

$$E\left[\left|\hat{X}_L(\zeta,t)\right|^2\right] = a^2\frac{\sin^2(w_0-w)L}{(w_0-w)^2} + a^2\frac{\sin^2(w_0+w)L}{(w_0+w)^2}$$

$$+a^2\frac{\sin[(w_0-w)L]}{w_0-w}\frac{\sin[(w_0+w)L]}{w_0+w}\underbrace{\int_{\varsigma=0}^{\varsigma=2\pi} 2\cos(2\varsigma)d\varsigma}_{0}$$

$$\underbrace{\tfrac{1}{2L}E\left[\left|\hat{X}_L(\zeta,t)\right|^2\right]}_{S_{XX}(\omega)} = \tfrac{1}{2}a^2\pi\left\{\frac{\sin^2(w_0-w)L}{\pi L(w_0-w)^2} + a^2\frac{\sin^2(w_0+w)L}{\pi L(w_0+w)^2}\right\}$$

$$S_{XX}(\omega) = \tfrac{1}{2}a^2\pi\left\{\frac{\sin^2 L(\omega-\omega_0)}{\pi L(\omega-\omega_0)^2} + \frac{\sin^2 L(\omega+\omega_0)}{\pi L(\omega+\omega_0)^2}\right\}.\square$$

(b) $\delta_n(x) = \dfrac{\sin^2(nx)}{\pi n x^2}$ is a Delta Sequence because

(1) Each of its components has the sifting property

$$\tfrac{1}{\pi}\int_{x=-\infty}^{x=\infty}\frac{\sin^2(nx)}{(nx)^2}d(nx) = \tfrac{2}{\pi}\int_{x=0}^{x=\infty}\frac{\sin^2\xi}{\xi^2}d\xi = 1,$$

by [Spiegel, p.96, (15.36)].

(2) Each component is continuous.

(3) Each component peaks at $nx = 0$, to $\dfrac{n}{\pi}$.

because for $nx = \left\langle\tfrac{1}{n}\right\rangle$, $\quad \delta_n\left(\left\langle\tfrac{1}{n^2}\right\rangle\right) = \dfrac{n}{\pi}\underbrace{\left(\dfrac{\sin\left(\left\langle\frac{1}{n}\right\rangle\right)}{\left\langle\frac{1}{n}\right\rangle}\right)^2}_{\approx 1}$.

Therefore, by Section 4, the sequence represents the Hyper-real Delta Function, and For $L = \text{infinite hyper-real} = \langle n\rangle$,

$$\left\langle\frac{\sin^2 n(\omega-\omega_0)}{\pi n(\omega-\omega_0)^2}\right\rangle, \text{ and } \left\langle\frac{\sin^2 n(\omega+\omega_0)}{\pi n(\omega+\omega_0)^2}\right\rangle$$

are the Delta Functions of 13.6.□

15.

White Noise

15.1 White Noise Current

The Power Spectral Density of Thermal Noise Current $I(\zeta, t)$ through a resistor R is [Engelberg],

$$\mathcal{F}\left\{E[I(\zeta,t)I(\zeta,t+\tau)]\right\} = \frac{2kT}{R},$$

where $k = (1.37)10^{-23}$ is Boltzmann Constant, and T is the absolute Temperature.

15.2 White Noise Voltage

The Power Spectral Density of the Thermal Noise Voltage $V(\zeta,t) = I(\zeta,t)R$ over the resistor R is

$$\mathcal{F}\underbrace{\left\{E[I(\zeta,t)RI(\zeta,t+\tau)R]\right\}}_{R^2 E[I(\zeta,t)I(\zeta,t+\tau)]} = R^2\,\mathcal{F}\underbrace{\left\{E[I(\zeta,t)I(\zeta,t+\tau)]\right\}}_{\frac{2kT}{R}} = 2kTR$$

15.3 $$kTR = D,$$

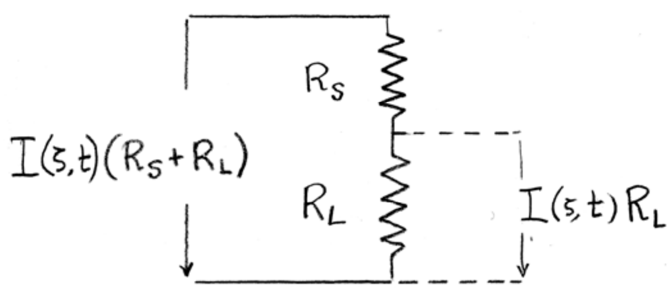
the Drift Coefficient of the Electrons' Random Walk

<u>*Proof*</u>: $2kTR = \mathcal{F}\left\{E[\dot{B}(\zeta,t)\dot{B}(\zeta,t+\tau)]\right\} = 2D\,.\square$

15.4 Spectral Power in a Frequency Band

By Nyquist Formula, [Henry], The Spectral Power in a frequency band $\Delta\nu$ is $4kTR\Delta\nu$.

15.5 White Noise on R_L in R_S-R_L Filter



The Spectral Power Density of the Thermal Noise Voltage generated in the R_S-R_L filter over the resistor R_L is

$$\mathcal{F}\underbrace{\left\{E[I(\zeta,t)R_L I(\zeta,t+\tau)R_L]\right\}}_{R_L^2 E[I(\zeta,t)I(\zeta,t+\tau)]} = R_L^2\,\underbrace{\mathcal{F}\left\{E[I(\zeta,t)I(\zeta,t+\tau)]\right\}}_{\frac{2kT}{R_S+R_L}}$$

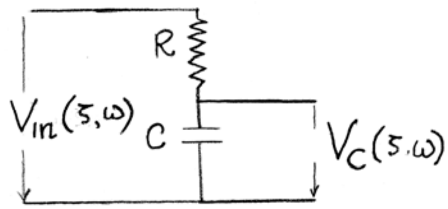
$$= 2kT\frac{R_L}{R_S+R_L}R_L\,.$$

15.6 White and Thermal Noise on C in R-C Filter

The White Noise on C is $E\left[\left|\hat{V}_C(\zeta,\omega)\right|^2\right] = \frac{kT}{C}\frac{2\frac{1}{RC}}{\omega^2+\left(\frac{1}{RC}\right)^2}$.

The Thermal Noise on C is $E[\left|V_C(\zeta,t)\right|^2] \approx \frac{kT}{C}$.

Proof:



The White Noise Voltage on the Capacitor C is

$$\hat{V}_C(\zeta,\omega) = \hat{V}_{in}(\zeta,\omega)\frac{\frac{1}{j\omega C}}{R+\frac{1}{j\omega C}} = \hat{V}_{in}(\zeta,\omega)\frac{\frac{1}{RC}}{j\omega+\frac{1}{RC}}$$

$$\left|\hat{V}_C(\zeta,\omega)\right|^2 = \left|\hat{V}_{in}(\zeta,\omega)\right|^2\left|\frac{\frac{1}{RC}}{j\omega+\frac{1}{RC}}\right|^2$$

$$E\left[\left|\hat{V}_C(\zeta,\omega)\right|^2\right] = E\left[\left|\hat{V}_{in}(\zeta,\omega)\right|^2\right]\frac{1}{2RC}\frac{2\frac{1}{RC}}{\omega^2+\left(\frac{1}{RC}\right)^2}$$

Since $V_{in}(\zeta,t)$ is a Random Walk Signal,

$E\left[\left|\hat{V}_{in}(\zeta,\omega)\right|^2\right] = 2kTR$, and the White Noise on C is

$$E\left[\left|\hat{V}_C(\zeta,\omega)\right|^2\right] = \frac{kT}{C}\frac{2\frac{1}{RC}}{\omega^2+\left(\frac{1}{RC}\right)^2}.\square$$

$$\underbrace{\mathcal{F}^{-1}\left\{E\left[\left|\hat{V}_C(\zeta,\omega)\right|^2\right]\right\}}_{E[V_C(\zeta,t)V_C(\zeta,t+\tau)]} = \frac{1}{2RC}\underbrace{\mathcal{F}^{-1}\left\{2kTR\right\}}_{2kTR\delta(\tau)} * \underbrace{\mathcal{F}^{-1}\left\{\frac{2\frac{1}{RC}}{\omega^2+\left(\frac{1}{RC}\right)^2}\right\}}_{e^{-\frac{1}{RC}|\tau|}},$$

$$E[V_C(\zeta,t)V_C(\zeta,t+\tau)] = \tfrac{1}{C}kT\delta(\tau) * e^{-\frac{1}{RC}|\tau|}$$

$$= \tfrac{kT}{C}e^{-\frac{1}{RC}|\tau|}.$$

Thus, for infinitesimal τ, the Thermal Noise Power generated in the R-C filter over the capacitor C is

$$E[|V_C(\zeta,t)|^2] \approx \tfrac{kT}{C}.\square$$

15.7 White and Thermal Noise on R in C-R Filter

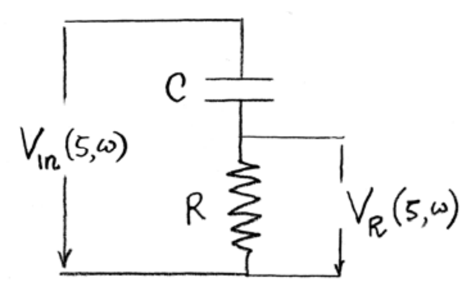
(a) *The White Noise on R is*

$$E\left[\left|\hat{V}_R(\zeta,\omega)\right|^2\right] = 2kTR\left\{1 - \frac{1}{2RC}\frac{2\frac{1}{RC}}{\omega^2 + \left(\frac{1}{RC}\right)^2}\right\}.$$

(b) *The Thermal Noise on R is*

$$E\left[\left|V_R(\zeta,t)\right|^2\right] \approx 2kTR\delta(0) - \tfrac{kT}{C}.$$

<u>*Proof*</u>:



(a) The White Noise Voltage on the Resistor R is

$$\hat{V}_R(\zeta,\omega) = \hat{V}_{in}(\zeta,\omega)\frac{R}{\frac{1}{j\omega C} + R} = \hat{V}_{in}(\zeta,\omega)\frac{j\omega}{\frac{1}{RC} + j\omega}$$

$$\left|\hat{V}_R(\zeta,\omega)\right|^2 = \left|\hat{V}_{in}(\zeta,\omega)\right|^2 \left|\frac{j\omega}{j\omega + \frac{1}{RC}}\right|^2$$

$$E\left[\left|\hat{V}_R(\zeta,\omega)\right|^2\right] = E\left[\left|\hat{V}_{in}(\zeta,\omega)\right|^2\right] \frac{\omega^2}{\omega^2 + \left(\frac{1}{RC}\right)^2}$$

Since $V_{in}(\zeta,t)$ is a Random Walk Signal,

$E\left[\left|\hat{V}_{in}(\zeta,\omega)\right|^2\right] = 2kTR$, and the White Noise on R is

$$E\left[\left|\hat{V}_R(\zeta,\omega)\right|^2\right] = 2kTR\left\{1 - \frac{1}{2RC}\frac{2\frac{1}{RC}}{\omega^2 + \left(\frac{1}{RC}\right)^2}\right\}.\square$$

(b)

$$\underbrace{\mathcal{F}^{-1}\left\{E\left[\left|\hat{V}_R(\zeta,\omega)\right|^2\right]\right\}}_{E[V_C(\zeta,t)V_C(\zeta,t+\tau)]} = \underbrace{\mathcal{F}^{-1}\left\{2kTR\right\}}_{2kTR\delta(\tau)} * \underbrace{\mathcal{F}^{-1}\left\{1 - \frac{1}{2RC}\frac{2\frac{1}{RC}}{\omega^2 + \left(\frac{1}{RC}\right)^2}\right\}}_{\delta(\tau) - \frac{1}{2RC}e^{-\frac{1}{RC}|\tau|}\ \text{[Komo, p.291]}},$$

$$E[V_C(\zeta,t)V_C(\zeta,t+\tau)] = 2kTR\delta(\tau) * \left\{\delta(\tau) - \frac{1}{2RC}e^{-\frac{1}{RC}|\tau|}\right\},$$

$$= 2kTR\delta(\tau) - \frac{D}{RC}e^{-\frac{1}{RC}|\tau|}.$$

Thus, for infinitesimal τ, the Thermal Noise Power generated in the R-C filter over the capacitor C is

$$E\left[\left|V_R(\zeta,t)\right|^2\right] \approx 2kTR\delta(0) - \frac{kT}{C}.\square$$

References

[Benoit] Eric Benoit "*Random Walks and Stochastic Differential Equations*" in "*Nonstandard Analysis in Practice*" edited by Francine Diener, and Marc Diener, Springer, 1995.

[Brook/Wynne] D. Brook and R. J. Wynne, "*Signal processing Principles and Applications*", Edward Arnold, 1988.

[Chandrasekhar] S. Chandrasekhar, "*Stochastic Problems in Physics and Astronomy*" Reviews of Modern Physics, Volume 15, Number1, January 1943.

Reprinted in "*Selected Papers on Noise and Stochastic Processes*" edited by Nelson Wax, Dover, 1954

[Dan1] Dannon, H. Vic, "*Well-Ordering of the Reals, Equality of all Infinities, and the Continuum Hypothesis*" in Gauge Institute Journal Vol.6 No 2, May 2010;

[Dan2] Dannon, H. Vic, "*Infinitesimals*" in Gauge Institute Journal Vol.6 No 4, November 2010;

[Dan3] Dannon, H. Vic, "*Infinitesimal Calculus*" in Gauge Institute Journal Vol.7 No 4, November 2011;

[Dan4] Dannon, H. Vic, "*The Delta Function*" in Gauge Institute Journal Vol.8 No 1, February 2012;

[Dan5] Dannon, H. Vic, "*Infinitesimal Calculus of Random Walk and Poisson Processes*" posted to www.gauge-institute.org

[Dan6] Dannon, H. Vic, "*Einstein's Diffusion and Probability-Wave Equations*" posted to www.gauge-institute.org

[Dan7] Dannon, H. Vic, "*Ito's Integral*" posted to www.gauge-institute.org

[Dan8] Dannon, H. Vic, "*Evolution Equations of Random Walk and Poisson Processes*" posted to www.gauge-institute.org

[Dan9] Dannon, H. Vic, "*Delta Function, the Fourier Transform, and Fourier Integral Theorem*" in Gauge Institute Journal Vol.8 No 1, February 2012;

[Engelberg] Shlomo Engelberg, "*Random Signals and Noise: A Mathematical Introduction*" CRC Press, 2007.

[Gard] Thomas Gard, "*Introduction to Stochastic Differential Equations*" Dekker, 1988

[Gardiner] Crispin Gardiner "*Stochastic Methods*" fourth Edition, Springer, 2009.

[Gnedenko] B. V. Gnedenko, "*The Theory of Probability*", Second Edition, Chelsea, 1963.

[Gradshteyn/Ryzhik] I. S. Gradshteyn, I. M. Ryzhik, Allan Jeffrey, and Daniel Zwillinger, "*Tables of Integrals, Series, and Products*" Seventh Edition, Academic Press, 2007

[Grimmett/Welsh] Geoffrey Grimmett and Dominic Welsh, "*Probability, an introduction*", Oxford, 1986.

[Henry] R. W. Henry, "*Random Walk Model of Thermal Noise for Students in Elementary Physics*" American Journal of Physics, Vol. 41, pp. 1361-1363, 1973.

[Hoel/Port/Stone] Paul Hoel, Sidney Port, Charles Stone, "*Introduction to Stochastic Processes*" Houghton Mifflin, 1972.

[Hsu] Hwei Hsu, "*Probability, Random Variables, & Random Processes*", Schaum's Outlines, McGraw-Hill, 1997.

[Karlin/Taylor] Howard Taylor, Samuel Karlin, "*An Introduction to Stochastic Modeling*", Academic Press, 1984.

[Komo] John J. Komo "*Random Signal Analysis in Engineering Systems*" Academic Press, 1987.

[Kuo] Hui Hsiung Kuo, "*Introduction to Stochastic Integration*", Springer, 2006.

[Larson/Shubert] Harold Larson, Bruno Shubert, "*Probabilistic Models in Engineering Sciences*", Volume II, Random Noise, Signals, and Dynamic Systems", Wiley, 1979.

[Lemons] Don Lemons, "*An Introduction to Stochastic Processes in Physics*", John Hopkins, 2002.

[Middleton] David Middleton, "*An Introduction to Statistical Communication Theory*" McGraw-Hill, 1960.

[Miller/Childers] Scott L. Miller, Donald G. Childers, "*Probability and Random Processes with Applications to Signal Processing and Communications*" Elsevier, 2004.

[Oksendal] Bernt Oksendal, "*Stochastic Differential Equations, An Introduction with Applications*", Fourth Corrected Printing of the Sixth Edition, Springer, 2007.

[Peebles] Peyton Z. Peebles, "*Probability, Random Variables, and Random Signal Principles*" Third Edition, McGraw Hill, 1993.

[Pursley] Michael B. Pursley, "*Random Processes in Linear Systems*", Prentice Hall, 2002.

[Spiegel] Murray Spiegel, "Mathematical Handbook of Formulas and Tables", Schaum's Outlines Series, McGraw-Hill, 1968.

http://en.wikipedia.org/wiki/Shot_noise

http://en.wikipedia.org/wiki/Johnson%E2%80%93Nyquist_noise

http://en.wikipedia.org/wiki/Flicker_noise

http://en.wikipedia.org/wiki/White_noise

http://en.wikipedia.org/wiki/Brownian_noise

http://en.wikipedia.org/wiki/Stochastic_differential_equations

http://en.wikipedia.org/wiki/Wiener_process

http://en.wikipedia.org/wiki/Ornstein%E2%80%93Uhlenbeck_process