The Quantum Optics of a Photon Composed of Two Circulating Subelectrons

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Abstract: Assuming that **two circulating subelectrons compose the photon**, it follows that the smaller the photon's radius, the larger its energy. And it explains the wave nature of light.

Then, we explain phenomena of quantum optics with the composite photon.

Assuming further that **three circulating subelectrons compose the electron**, it follows that the smaller the electron's radius, the larger its energy. And the electron's orbit is determined by its radius.

This may explain why an electron moves between two of its orbits

The identification of a spectral line with energy difference between two electron orbits never explained how an electron gains or looses energy before it moves to a different orbit. It is plausible that change in the electron's structure determines the electron's energy and orbit.

The composed electron and photon explain the interaction of gamma photons with electron positron. How radiation becomes matter, and vice versa.

Contents

0.
The Wave-Particle Photon

A black body emits electromagnetic radiation in all possible frequencies. A Radiation Law that suited the measurements for black body radiation was obtained by Planck in 1901[1]. By Planck, The average electromagnetic energy density per unit volume at frequency ν cycles/second emitted by a perfect radiator is

$$u(\nu, T) = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{\frac{h\nu}{kT}} - 1}$$

To obtain that, the radiation energy had to be assumed to be composed of atomic packets, multiples of

$$h\nu\,.$$

Later, the spectrum of the Hydrogen Atom, the Photo-electric effect, and the Compton effect established that as a fact.

However, the emission of a structure-less photon by an electron is enigmatic. Feynman's father asked him[2]

> *"...If the photon comes out of the atom when it goes from the excited to the lower state,*
> *the photon must have been in the atom, in the excited state...*

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[1] H. Vic Dannon, "*Zero Point Energy: Planck's Radiation Law*", Gauge Institute Journal Vol.1 No 3, August 2005.

[2] Richard Feynman, The Physics Teacher, September 1969, 319

..How a photon comes out without it having been there in the excited state "

Indeed, a structure-less photon cannot come out of the atom without having been there in the excited state.

But a photon composed of sub-photons can come out.

A structure-less photon cannot explain diffraction and polarization of light. It defies intuition, and commonsense because there is no particle that is also a wave.

To explain diffraction and polarization, it is common to assume the experimentally unconfirmed, hence, nonphysical Schrödinger's wave equation that replaces a particle with a cloud where the particle is likely to be, and turns quantum optics into wave optics where the frequency ν_{ϕ} is physically unrelated to the photon.

Instead of assuming the impossible, that a particle may be also a wave, **we maintain that**

Each photon generates, and carries along its own electromagnetic field.

That field is attached to the photon,

and propagates with it.

Diffraction and polarization of light can be explained by assuming that

The photon spins around its center

at the frequency of its radiation ν_{photon} cycles/second.

Indeed, this is a fact. We next show that the Bell-shaped Spectral line indicates photon spinning. And that the photon energy is rotational, Hence, due to photon spinning.

1.
The Photon Spins about its Center at Radiation Frequency

1.1 The Spectral Line Indicates Spinning

The transitions of electrons between two atomic orbits produce a bell-shaped spectral line that includes many frequencies that are close to the frequency of the transition $\nu_{2\to 1}$.

At a given instant, the spectral line contains frequencies between

$$\nu_{2\to 1} - \Delta\nu_{2\to 1},$$

and

$$\nu_{2\to 1} + \Delta\nu_{2\to 1}.$$

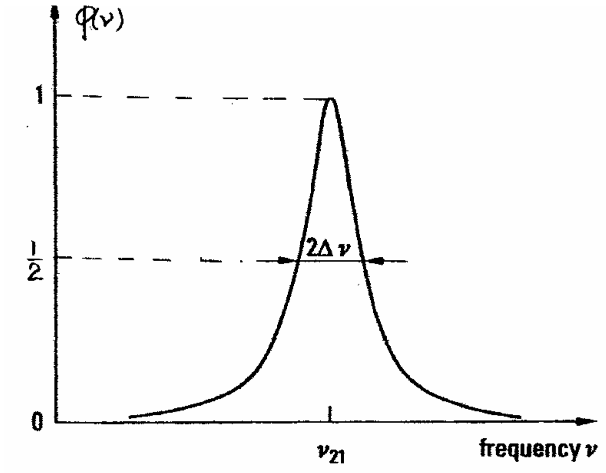
with probability determined by a normal curve

$$\phi(\nu)$$

with width

$$2\Delta\nu_{2\to 1}$$

at $\phi(\nu) = \dfrac{1}{2}$



Each frequency belongs to a different photon at that instant.

But only the radiation frequency is admissible. There is no theory that supports the existence of photons with any frequency except for the electromagnetic radiation frequency.

Therefore, the spectral line represents frequencies of photons at their creation. Each photon evolves from spinning at lower frequency into spinning at higher frequency.

1.2 The Kinetic Energy of the Photon is Rotational

Since light can be slowed to bicycle speed, the kinetic energy of the photon is rotational.

The kinetic energy[3] of the photon is given by

$$E^2 = \underbrace{(mc)^2 c^2}_{rotational} + \underbrace{(m_0 c^2)^2}_{linear}$$

Since $m_0 = 0$,

$$E = mc^2 \text{ is rotational.}$$

[3] https://en.wikipedia.org/wiki/Kinetic_energy

2.
Two Circulating Sub-electrons Compose the Photon

Since masses only turn around other masses, it follows that

The spinning photon is made of at least two masses.

To suggest how a photon is structured, we allude to the composition of the electron from subelectrons.

In 1928, J. J. Thomson presented in [Thomson,p.33], experimental evidence that the electron is a composite particle.

> "*…the properties of the electron recently discovered lead to the view that the electron…has itself a structure, being made up of smaller parts which carry charges of electricity*."

Later, Milliken presented in [Milliken, p. 161] evidence from his experiments for the existence of Subelectrons.

> "*…Ehrenhaft and Zerner even analyze our report on oil droplets and find that these also show in certain instances indications of* ***sub-electrons****, for they yield in these observers' hands too low values of e, whether computed from the Brownian movement or from the law of fall*."

This was copied to claim that the proton, and the neutron are composed of subparticles with fractional charges.

The evidence for sub-electrons had to lead to a planetary model for the electron. To balance the electric forces on them, the sub-electrons must be moving in a closed orbit.

The existence of fractional charges, and sub-electrons, is now well-established. For instance, [Wagoner, pp. 541-5]

It is believed that a proton is composed of three sub-protons

Two u's, each with charge $-\frac{2}{3}e$, and one d with charge $\frac{1}{3}e$.

The opposite sign charges attract each other, and keep the proton together.

Similarly, **we assume that an electron is composed of three subelectrons,**

Two δ_2's, each with charge $\frac{2}{3}e$, and one δ_1 with charge $-\frac{1}{3}e$.

And

we assume that *the photon is composed of two sub-electrons.*

either δ_2 with $\frac{2}{3}e$, and $-\delta_2$ with $-\frac{2}{3}e$,

***or* δ_1 with $\frac{1}{3}e$, and $-\delta_1$ with $-\frac{1}{3}e$.**

In [Dan1][4], we showed that in either case, the radiation energy depends on the frequency ν alone.

It follows that the

the anti-photon is composed of two sub-electrons.

either $-\delta_2$ with $-\frac{2}{3}e$, and δ_2 with $\frac{2}{3}e$,

[4] H. Vic Dannon, The Composite Photon, Gauge Institute Journal, Volume 16, No. 4, November 2020.

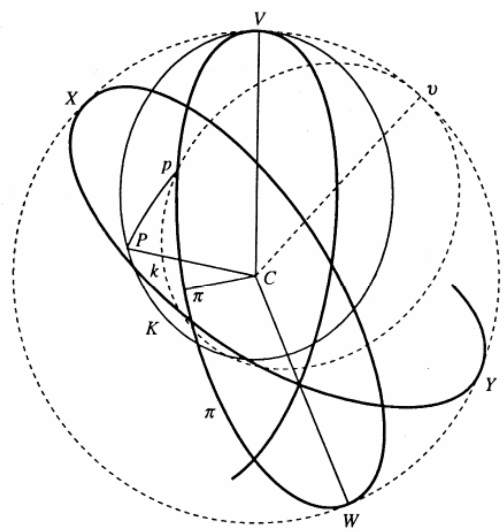
or $-\delta_1$ **with** $-\frac{1}{3}e$**, and** δ_1 **with**$\frac{1}{3}e$**.**

That is, the anti-photon equals the photon. The photon is its own anti-particle.

Then, the centripetal forces of repulsion will balance the electric forces of attraction. And

The sub-electrons will be at fixed distance r_ϕ**.**

Under non-inverse squared law force, the orbit is on a sphere:



[Chandrasekhar, p.195]

But the inverse-squared electric force limits the orbit to the plane, the way the gravitational force limits the planets orbits to the ecliptic plane, and limits the stars orbits to the galactic plane.

Thus,

The Orbit is in the Plane.

The electric force between the subphotons that is proportional to the inverse square of the distance, and the centripetal forces on the subphotons that are proportional to the distance are central forces.

By [Routh,p.274], central forces result in a closed orbit. Thus,

The Orbit is Closed

By [Routh, p.280] Central Force orbits are stable. That is,

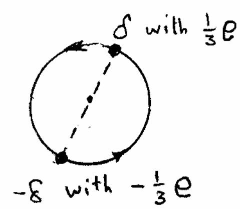
The Orbit is Bounded between Two Circles

Light Polarization suggests that

The Orientation of the Orbit Does Not Change.

Since the distance between the subelectrons r_ϕ is fixed,

The orbit is a Circle.



The center of the circle propagates at light speed c.

Harmonic motion of the subelectrons along the circle at light speed c explains light diffraction. Then,

$$\lambda_\phi \nu_\phi = c = \omega_\phi r_\phi = 2\pi \nu_\phi r_\phi$$

$$\boxed{\lambda_\phi = 2\pi r_\phi}$$

That is, the wavelength of the photon[5], is the circumference of the sub-photons' orbit.

The radius of the orbit is

$$\boxed{r_\phi = \frac{\lambda_\phi}{2\pi} = \frac{c}{2\pi\nu_\phi} = \frac{c}{\omega_\phi}}$$

The sub-photons encircle the center of the photon at frequency

$$\boxed{\nu_\phi = \frac{c}{2\pi r_\phi}}$$

The electromagnetic energy of the photon is

$$\boxed{h\nu_\phi = \frac{hc}{2\pi r_\phi}}$$

That energy equals $m_\phi c^2$, where the mass of the photon is

$$\boxed{m_\phi = \frac{h\nu_\phi}{c^2} = \frac{h}{2\pi c r_\phi}}$$

Then, the linear momentum of the photon is

$$\boxed{m_\phi c = \frac{h\nu_\phi}{c} = \frac{h}{2\pi r_\phi}}$$

Thus,

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[5] known as "Compton Wavelength of the Photon".

Compton Wavelength of the Electron $= \frac{\lambda_e}{2\pi} = \frac{c}{2\pi\nu_e} =$ the Classical Electron Radius $= \frac{1}{4\pi\varepsilon_0}\frac{e^2}{m_e c^2}$

where the electron is a ball of radius r_e with charge density $\frac{e}{\frac{4}{3}\pi r_e^3}$ defying the electric repulsion that will push all charges to the sphere. And defying the subelectrons. And carrying never-detected electromagnetic energy with frequency ν_e.

the photon's mass, energy, and momentum are

proportional to $\frac{1}{r_\phi}$, and greater as r_ϕ is smaller

For two photons, ϕ_1, and ϕ_2,

$$\boxed{\frac{r_{\phi_1}}{r_{\phi_2}} = \frac{\nu_{\phi_2}}{\nu_{\phi_1}} = \frac{m_{\phi_2}}{m_{\phi_1}} = \frac{h\nu_{\phi_2}}{h\nu_{\phi_1}}}$$

For instance, $\lambda_{blue} \approx 4400A^\circ$, $\lambda_{red} \approx 6600A^\circ$,

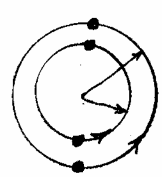
$$\frac{\nu_{red}}{\nu_{blue}} = \frac{r_{blue}}{r_{red}} = \frac{\lambda_{blue}}{\lambda_{red}} \approx \frac{4400A^\circ}{6600A^\circ} = \frac{2}{3}.$$

That is,

A blue photon radius is $\frac{2}{3}$ of a red photon radius,

A blue photon energy is $\frac{3}{2}$ of a red photon energy,

In the blue photon, the subelectrons circulate along the circle with smaller radius.



When the subelectrons circulating in the circle of the blue photon transfer to the circle of the red photon, the photon changes its color from blue to red.

Instead of annihilation of a blue photon

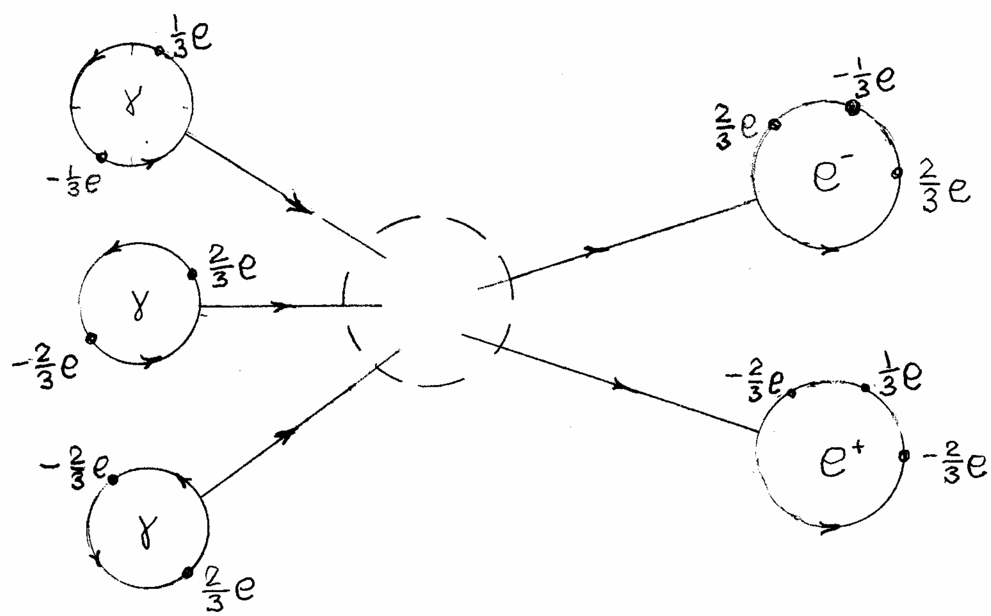
followed by the creation of a red photon,

we observe a change of radius and

a change of color of the same photon.

3. Photon Creation at Matter Annihilation

The composed electron and composed photon explain how an electron-positron pair may transform into 3 gamma photons, and vice versa:



4. Photon's Creation by Electron's Drop to a lower Orbit

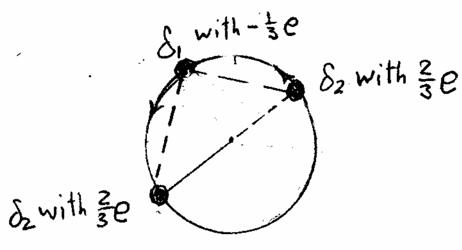
The photon does not exist in the electron before the photon is created.

The energy of the photon is the difference between the electron orbits energies. Upon transition from an orbit with energy ε_2 to an orbit with energy ε_1, the energy $\varepsilon_2 - \varepsilon_1$ transforms into two subelectrons that circulate the center of the photon with frequency

$\dfrac{\varepsilon_2 - \varepsilon_1}{h}$.

We assume that,

The three subelectrons circulate along a stable circle that stays in the same plane.

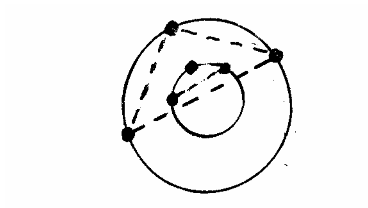


Then,

The larger electron has less energy, and smaller orbit.

The farther the circulating subelectrons are from each other, the lower is the electron's energy, and the lower is the electron's orbit.

The electron with a larger radius

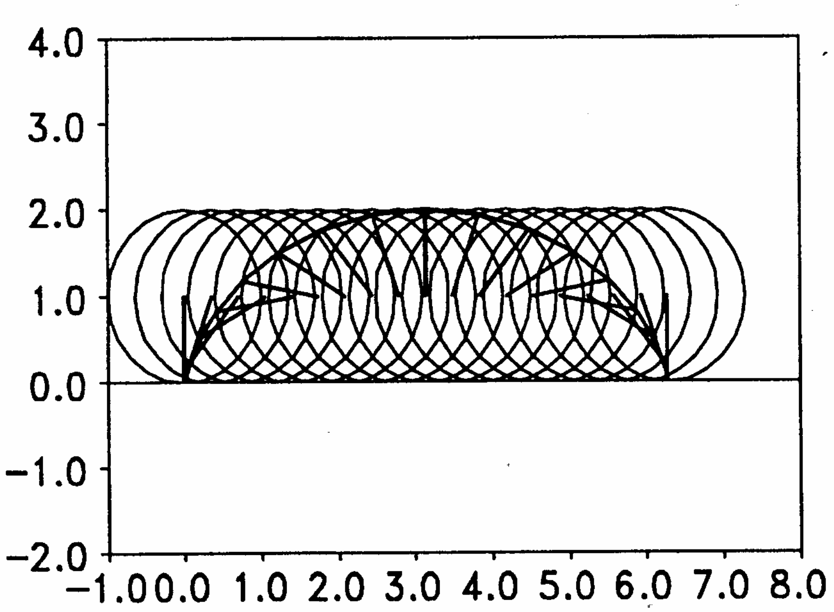


drops into a lower orbit, and creates a photon.

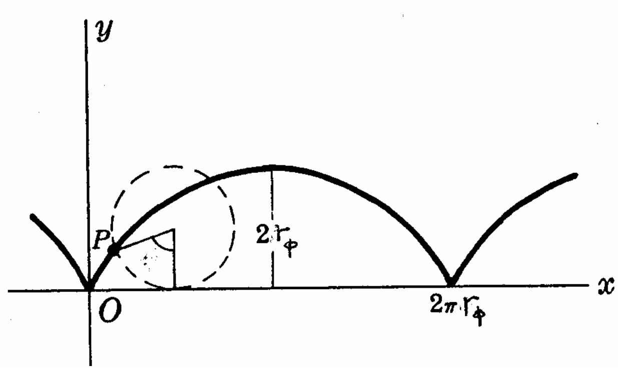
The energy difference $\varepsilon_{\text{higher orbit}} - \varepsilon_{\text{lower orbit}}$, ***transforms into two subphotons that encircle its center at light speed and at frequency*** $\nu_{2\to 1} = \dfrac{\varepsilon_2 - \varepsilon_1}{h}$.

5.
The Sub-photons trace a Cycloid

As a sub-photon encircles the center of the photon, the center moves along a line in the photon direction of propagation.



The combined motion traces a cycloid[6].



[6] https://en.wikipedia.org/wiki/Cycloid

6.
The Photon is NOT an Optical Soliton

A structure-less photon can be perceived as a solitary wave in one dimension: A pulse that propagates in one direction, maintains its profile unchanged, and leaves behind it an undisturbed path.

An Optical Soliton[7] appears in a nonlinear medium such as optical fiber. A photon appears in the vacuum which is a linear medium.

Can an optical soliton represent a photon too?

An Optical Soliton propagating in an optical fiber is an electric field pulse

$$E(x,t) = a(x,t)e^{i(\beta_0 x - \omega_0 t)}$$

where

$$a(x,t) = \text{the envelope of the pulse,}$$

$$\beta_0 = \text{phase constant.}$$

The envelope satisfies a Nonlinear Schrödinger Equation[8]

$$\frac{1}{2}\frac{\partial^2 a}{\partial \tau^2} + i\frac{\partial a}{\partial \zeta} + N^2 \left|a\right|^2 a = 0$$

where

$$\tau = \alpha_1 t - \alpha_2 x\,, \quad \alpha_1, \alpha_2 = \text{constants}$$

$$\zeta = \alpha_3 x\,, \quad \alpha_3 = \text{constant}$$

$$N^2 = \alpha_4 > 0\,, \quad \alpha_4 = \text{constant}$$

The envelope is the solitary wave

[7] https://en.wikipedia.org/wiki/Soliton_(optics)
[8] https://en.wikipedia.org/wiki/Nonlinear_Schrödinger_equation

$$a(\tau,\zeta)=\frac{2}{e^{\tau}+e^{-\tau}}e^{i\frac{1}{2}\zeta}$$

This envelope is not related in any way to the wave equation that is satisfied by the harmonic motion of the subphotons, and explains light properties of diffraction, interference, and polarization.

Therefore, the photon cannot be presented by an optical soliton.

7.
The Photon Energy is Rotational

From $m_{photon}c^2 = h\nu$, the photon's mass is

$$\boxed{m_{photon} = \frac{h\nu}{c^2}}$$

For a green photon, $\lambda_{green} \sim 4000\dot{A} = 4 \cdot 10^{-7}\,\mathrm{m}$,

$$\nu_{green} = \frac{3 \cdot 10^8}{4 \cdot 10^{-7}} = (7.5)10^{14} \ / \ \mathrm{sec}$$

$$\frac{h\nu_{green}}{c^2} = \frac{(6.652)10^{-34}(7.5)10^{14}}{9 \cdot 10^{16}} \sim (5.5) \cdot 10^{-36}kg$$

In comparison, an H_2 molecule has mass

$$(3.32)10^{-27}kg.$$

Over half a Billion times the mass of a green photon.

7.1 The Photon has No Rest-Mass

$$m_0 = m_{photon}\sqrt{1 - \frac{v_{photon}^2}{c^2}} = \frac{h\nu}{c^2}\sqrt{1 - \frac{c^2}{c^2}} = 0\,.$$

The photon ceases to exist at rest.

Since light can be slowed to bicycle speed, the kinetic energy of the photon is rotational.

The kinetic energy[9] of the photon is given by

$$E^2 = \underbrace{(mc)^2 c^2}_{rotational} + \underbrace{(m_0 c^2)^2}_{linear}$$

Since $m_0 = 0$,

$$E = mc^2$$

is rotational. That is,

7.2 The Photon Mass is packed in its Rotational Motion

7.3 The Photon Mass Variation

$$\boxed{\Delta m_\phi = \frac{h}{c^2}\Delta\nu_\phi = \frac{h}{2\pi c}\Delta\frac{1}{r_\phi} = -\frac{h}{2\pi c}\frac{\Delta r_\phi}{r_\phi^2}}$$

[9] https://en.wikipedia.org/wiki/Kinetic_energy

8.
The Photon Momentum

The photon's momentum is the linear momentum of the photon in its rotational motion about its center

$$\boxed{p_\phi = m_\phi c = \frac{h\nu_\phi}{c} = \frac{h}{2\pi c r_\phi}}$$

8.1 The Photon Momentum Variation

$$\boxed{\Delta p_\phi = \frac{h}{c}\Delta\nu_\phi = -\frac{h}{2\pi c}\frac{\Delta r_\phi}{r_\phi^2}}$$

8.2 Atom Recoil at Photon Emission

Sodium (=Natrium) D-line represents the emission of a yellow photon with frequency

$$\nu_{yellow} = (5.091)10^{14} / \sec$$

and recoil momentum

$$\begin{aligned} m_{Natrium} v_{Natrium} &= p_{photon} \\ &= \frac{h\nu_{yellow}}{c} \\ &= \frac{(6.6256)10^{-34}(Joul)(\sec)(5.091)10^{14} / \sec}{(3)10^8 m / \sec} \\ &= (1.12)10^{-27} W / \sec \end{aligned}$$

$m_{Na} = (3.82)10^{-26} kg \quad \Rightarrow$ **The recoil speed of the Sodium Atom is**

$$v_{Na} = \frac{(11.2)10^{-28} W \ / \sec}{(3.82)10^{-26} kg} \approx (2.93)10^{-2} m \ / \sec \approx 3cm \ / \sec$$

9.
The Photon Electromagnetic Field

A rotating photon carries an Electromagnetic field

$$\vec{E}_{photon}$$

that lies in a plane perpendicular to

$$\vec{k} = \text{the photon's propagation direction}$$

And has the energy

$$\frac{1}{2}\varepsilon\vec{E}_{photon}\cdot\vec{E}^{*}_{photon} = h\nu$$

A free photon, has a propagating plane field

$$\vec{E}_{photon} = \sqrt{\frac{2h\nu}{\varepsilon}}e^{i\phi}e^{-i\vec{k}\cdot\vec{r}}e^{i\omega t}\vec{1}_{\vec{k}}$$

A photon locked in a cube of side d, has a standing waves field,

$$\vec{E}_{photon} = \sqrt{\frac{2h\nu}{\varepsilon}}e^{i\phi}\left(\frac{2}{d}\right)^{\frac{3}{2}}\sin\frac{n_x\pi x}{d}\sin\frac{n_y\pi y}{d}\sin\frac{n_z\pi z}{d}e^{2\pi i\nu t}\vec{1}_{\vec{n}}$$

where $\vec{1}_{\vec{n}} =$ polarization vector of the photon.

10.
Photon Field Polarization

A filtered linearly-polarized photon field has a wave vector that oscillates along a line. Then, the field is

$$\vec{E}(\vec{r},t,k,\omega) = \vec{A}e^{-ikz}e^{i\omega t}.$$

A circularly-polarized photon field has a wave vector that traces a circle.

A right-handed circularly-polarized wave of amplitude 1 is

$$\frac{1}{\sqrt{2}}(\vec{1}_x + i\vec{1}_y)e^{-ikz}e^{i\omega t}.$$

Its spin vector is

$$\vec{s} = \hbar\vec{1}_{\vec{k}}.$$

The photon will exert torque on a half-wave plate of quartz.

A left-handed circularly-polarized wave of amplitude 1 is

$$\frac{1}{\sqrt{2}}(\vec{1}_x - i\vec{1}_y)e^{-ikz}e^{i\omega t}.$$

Its spin vector is

$$\vec{s} = -\hbar\vec{1}_{\vec{k}}$$

The photon will exert torque on a half-wave plate of quartz.

The sum of right-handed, and left-handed circularly-polarized waves is a linearly-polarized wave.

The electric field of a plane wave is

$$E(z,t) = \mathrm{Re}\left\{a_0(e^{i\varphi_0} + e^{i(\varphi_0+\frac{\pi}{2})})e^{i\omega(t-\frac{z}{c})}\right\}$$

$$= a_0 \cos[\omega(t - \frac{z}{c}) + \varphi_0] + a_0 \cos[\omega(t - \frac{z}{c}) + \varphi_0 + \frac{\pi}{2}]$$

$$\omega(t - \frac{z}{c}) + \varphi_0 \equiv \alpha_0(z,t)$$

$$= \underbrace{a_0 \cos \alpha_0(z,t)}_{E_x(z,t)} + \underbrace{-a_0 \sin \alpha_0(z,t)}_{E_y(z,t)}$$

Then,

$$\underbrace{\left(\frac{E_x(z,t)}{a_x}\right)}_{\xi}^2 + \underbrace{\left(\frac{E_y(z,t)}{a_y}\right)}_{\eta}^2 = 1$$

is a circle in ξ, and η, centered at the origin with radius

$$a = 1.$$

For fixed z, the electric field rotates <u>clockwise</u> when viewed from the origin in the direction of propagation. And the wave is <u>right-circularly</u> polarized.

If the phase difference between $E_x(z,t)$, and $E_y(z,t)$ is $-\frac{\pi}{2}$, then

$$E_y(z,t) = a_0 \sin \alpha_0(z,t)$$

and the electric field rotates <u>counter-clockwise</u> when viewed from the origin in the direction of propagation. And the wave is <u>left-circularly</u> polarized

11.
The Photon Field Intensity

For fixed instant t the photon's field propagates along z with wave length λ.

The intensity of the field is

$$\frac{1}{(\text{medium impedence})} a_0^2$$

The probability of detecting a photon at point $\vec{r}$ at an area dA normal to its propagation at time dt

$$p(\vec{r})dA\,dt$$

is proportional to the local intensity

$$\left|U(\vec{r})\right|^2 dA\,dt\,.$$

A standing wave photon field in $0 \le z \le d$ with intensity $\sin^2(\pi \frac{z}{d})$ is most likely to be detected at $z = \frac{1}{2}d$.

12.
Interference of Photons' Fields

In Young's interference experiment, a plane wave is filtered through two pinholes that are apart by distance

$$2a\,.$$

A spherical wave propagates from each pinhole. The waves interfere at a screen at distance

$$d$$

from the pinholes filter plane.

The angle subtended by the two pinholes at the screen is

$$\theta = \frac{2a}{d}$$

The width of a dark strip on the screen is

$$\frac{\lambda}{\theta} = \frac{\lambda d}{2a}\,.$$

At the screen, the intensity of each of the interfering waves is

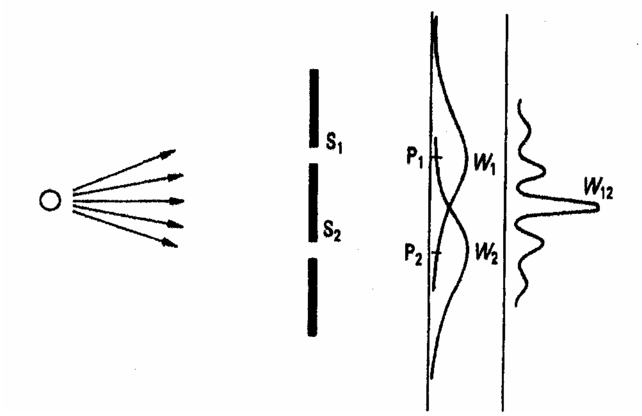
$$I_0\,.$$

In the Fresnel approximation the intensity is

$$I(x) = 2I_0\left(1 + \cos 2\pi \frac{x}{\lambda \,/\, \theta}\right)$$

$$= 2I_0\left(1 + \cos 2\pi \frac{x}{\lambda d \,/\, 2a}\right).$$

In interference, some photons will pass through one slit, and other photons through the other slit.



If S_2 is covered, photons pass only through S_1, and distribute on the screen with

$$W_1 = \left|\Phi_1\right|^2.$$

If S_1 is covered, photons pass only through S_2, and distribute on the screen with

$$W_2 = \left|\Phi_2\right|^2.$$

The interference has

$$W_{12} = \left|\Phi_1 + \Phi_2\right|^2 = \left|\Phi_1\right|^2 + \left|\Phi_2\right|^2 + \Phi_1\Phi_2^* + \Phi_1^*\Phi_2.$$

13.
Photon Field Flux

Photon Flux Density $\frac{I(\vec{r},t)}{h\nu} \equiv \phi(\vec{r},t)$ (photons/$cm^2 \sec$)

Sunlight	10^{14}
10mW He-Ne laser at $\lambda = 6330\dot{A}$ On spot of radius $10\mu m$	10^{22}

Photon Flux over A

$$\int_A \phi(\vec{r},t)dA = \frac{I(\vec{r},t)}{h\nu} = \frac{1}{h\nu} \underbrace{\int_A I(\vec{r},t)dA}_{P(t)=\text{optical power}}$$

photons with $\lambda = 2000\dot{A}$ have

$$\nu = \frac{c}{\lambda} = \frac{3(10^8)}{2(10^{-7})} \sim 1.5(10^{15})Hz$$

$$h\nu \sim (6.6)10^{-34}(1.5)10^{15} \sim 10^{-18}(Joul)(\sec)$$

Optical power $10^{-9}W$, is due to photon flux of

$$\frac{10^{-9}W}{10^{-18}(Joul)(\sec)} = 10^9 \text{ photons/second.}$$

that is, a photon every $\frac{1}{billion}$ second.

14.
The Photon's Schrodinger's Equation

The spinning photon is a harmonic oscillator. Assume that it has restoring force proportional to its displacement x

$$F = -\kappa x\,,$$

oscillating frequency

$$\nu\,,$$

mass

$$m = \frac{h\nu}{c^2}\,,$$

momentum

$$p\,,$$

kinetic energy

$$\frac{1}{2}\frac{p^2}{m}\,,$$

potential energy

$$\int \kappa x dx = \frac{1}{2}\kappa x^2$$

balanced by

$$\int m(\omega^2 x)dx = \frac{1}{2}m\omega^2 x^2$$

angular velocity

$$2\pi\nu = \omega = \sqrt{\frac{\kappa}{m}}\,,$$

total energy

$$E = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}m\omega^2 x^2\,.$$

And assume that it satisfies Schrödinger's wave equation

$$-\left(\frac{h}{2\pi}\right)^2\frac{1}{2m}\psi''(x)+\frac{1}{2}m\omega^2x^2\psi(x)=E\psi(x)$$

The solutions are orthonormal $\psi_n(x)$ for discrete Energies

$$E_n=(n+\tfrac{1}{2})h\nu$$

$$-\left(\frac{h}{2\pi}\right)^2\frac{1}{2m}\psi_n''(x)+\frac{1}{2}m\omega^2x^2\psi_n(x)=(n+\tfrac{1}{2})h\nu\psi_n(x)$$

$$-\left(\frac{h}{2\pi}\right)^2\frac{1}{2\frac{h\nu}{c^2}}\psi''(x)+\frac{1}{2}\frac{h\nu}{c^2}\omega^2x^2\psi(x)=(n+\tfrac{1}{2})h\nu\psi(x)$$

$$-\frac{1}{8\pi^2}\frac{c^2}{\nu^2}\psi''(x)+\frac{1}{c^2}2\pi^2\nu^2x^2\psi(x)=(n+\tfrac{1}{2})\psi(x)$$

$$-\frac{1}{2}\frac{c^2}{\omega^2}\psi''(x)+\frac{1}{2}\frac{\omega^2}{c^2}x^2\psi(x)=(n+\frac{1}{2})\psi(x)$$

The Orthonormal Eigen-functions are

$$\psi_n(x)=\frac{1}{\sqrt{2^n n!}}\sqrt[4]{\frac{2m\omega}{h}}H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)e^{-\frac{m\omega}{2\hbar}x^2}$$

$$=\frac{1}{\sqrt{2^n n!}}\sqrt[4]{\frac{2\frac{h\nu}{c^2}2\pi\nu}{h}}H_n\left(\sqrt{\frac{\frac{h\nu}{c^2}\omega}{\frac{h}{2\pi}}}x\right)e^{-\frac{\frac{h\nu}{c^2}2\pi\nu}{2\frac{h}{2\pi}}x^2}$$

$$=\frac{1}{\sqrt{2^n n!\sqrt[4]{\pi}}}\sqrt{\frac{\omega}{c}}H_n\left(\frac{\omega}{c}x\right)e^{-\frac{1}{2}\frac{\omega^2}{c^2}x^2}$$

where

$$H_n(z)=(-1)^ne^{z^2}\frac{d^n}{dz^n}e^{-z^2},$$

are Hermite polynomials of order n.

Then,

$$\psi(x) = c_0\psi_0(x) + c_1\psi_1(x) + c_2\psi_2(x) + \ldots,$$

$$\Pr\{E = nh\nu\} = \left|c_n\right|^2,$$

$$\Pr\{h\nu \text{ is at } x\} = \left|\psi(x)\right|^2,$$

$$\Pr\{\text{momentum} = p\} = \text{proportional to } \left|\left(\mathcal{F}^{-1}\psi\right)(p)\right|^2,$$

$$\left(\mathcal{F}^{-1}\psi\right)(p) = \frac{1}{\sqrt{h}}\int_{x=-\infty}^{x=\infty}\psi(x)e^{2\pi\frac{p}{h}ix}dx$$

$$\sigma_x\sigma_p \geq \frac{1}{2}\frac{h}{2\pi}$$

So many assumptions to further assume that a physical particle can evaporate into a probability cloud. Thus, applying the Schrödinger wave equation to the photon particle accomplishes nothing.

15. The Photo-Electric Effect: Electron Absorbs a Photon

Only the energy of the absorbed photon is necessary to explain the Photo-Electric effect:

An electron bound to a cathode absorbs a photon with energy $h\nu$.

If $h\nu$ is greater than the stopping potential $q_{electron}U$ between the cathode and the anode, the electron is emitted with kinetic energy

$$\frac{1}{2}m_{electron}v_{electron}^2 = h\nu - q_{electron}U,$$

that is proportional to ν.

Below a threshold frequency ν_0, no photoelectrons are emitted.

At ν_0, the energy balance is

$$0 = h\nu_0 - q_{electron}U$$

Then,

$$\frac{1}{2}m_{electron}v_{electron}^2 = h(\nu - \nu_0)$$

$$\underbrace{m_{electron}^2 v_{electron}^2}_{p_{electron}^2} = 2h(\nu - \nu_0)m_{electron}$$

$$\frac{p_{electron}^2}{p_{photon}^2} = \frac{2h(\nu - \nu_0)m_{electron}}{(\frac{h\nu}{c})^2}$$

$$\boxed{\frac{p_{electron}}{p_{photon}} = \sqrt{\frac{m_{electron}c^2}{h\nu}}\sqrt{2\left(1 - \frac{\nu_0}{\nu}\right)}}$$

Approximating with

$$\underline{\lambda \approx 1\mu m} \Rightarrow \nu = \frac{c}{\lambda} \approx \frac{3 \cdot 10^8}{10^{-6}} = 3 \cdot 10^{14}$$

$$\underline{\nu_0 \approx (1.5)10^{14} \ / \sec} \Rightarrow \sqrt{2\left(1 - \frac{\nu_0}{\nu}\right)} \approx 1$$

$$\sqrt{\frac{m_{electron}c^2}{h\nu}} \approx \sqrt{\frac{9 \cdot 10^{-31}3^2 10^{16}}{(6.6)10^{-34}3 \cdot 10^{14}}} \approx \sqrt{2 \cdot 10^5} \approx 450$$

$$\Rightarrow \frac{p_{electron}}{p_{photon}} \approx 450.$$

$p_{photon} \approx p_{electron} \Rightarrow$

$$\nu \approx \frac{m_{electron}c^2}{h} \approx \frac{9 \cdot 10^{-31}3^2 10^{16}}{(6.6)10^{-34}} \approx (1.2)10^{20} \ / \sec$$

$$\lambda = \frac{c}{\nu} \approx \frac{h}{m_{electron}c} \approx \frac{(6.6)10^{-34}}{9 \cdot 10^{-31}3 \cdot 10^8} \approx (2.4)10^{-12} m$$

16.
Compton Scattering of a Photon off an Electron

Explaining Compton's Scattering requires the energy, and the momentum of a spinning photon.

In Compton's scattering, a photon spinning at ν_0 cycles/second scatters in-elastically from an orbiting electron. The electron's speed in its orbit is relatively small compared with the photon's speed of light, and the electron is considered to be at rest.

The photon spinning slows down to ν_1 cycles/second. That is, the photon changes its color.

The energy $h(\nu_1 - \nu_0)$ transfers to the electron, knocking it of its orbit at speed v, and increasing its mass by

$$\frac{h(\nu_1 - \nu_0)}{c^2}$$

from its rest mass, m_e to its relativistic mass at v,

$$m = \frac{m_e}{\sqrt{1 - \frac{v^2}{c^2}}} \approx m_e(1 + \tfrac{1}{2}\frac{v^2}{c^2})$$

That is,

$$\frac{h(\nu_0 - \nu_1)}{c^2} = m - m_e,$$

$$h(\nu_0 - \nu_1) + m_e c^2 = mc^2,$$

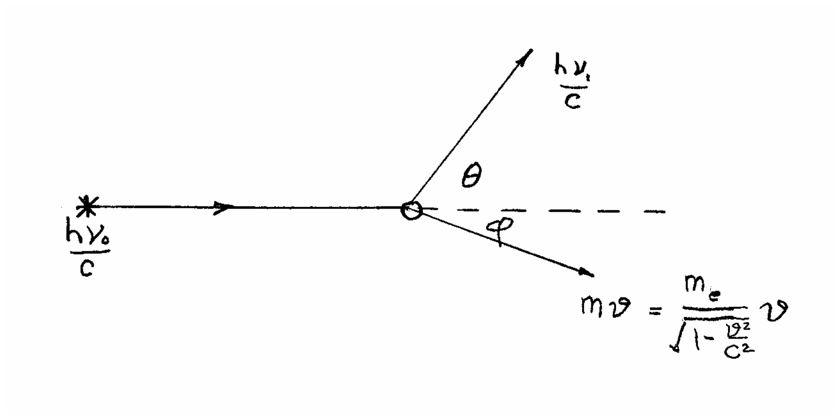
$$\left(h(\nu_0 - \nu_1) + m_e c^2\right)^2 = m^2 c^4,$$

$$h^2(\nu_0 - \nu_1)^2 + 2h(\nu_0 - \nu_1)m_e c^2 + m_e^2 c^4 = m^2 c^4$$

$$h^2(\nu_0 - \nu_1)^2 + 2h(\nu_0 - \nu_1)m_e c^2 = (m^2 - \underbrace{m_e^2}_{m^2(1-\frac{v^2}{c^2})})c^4.$$

Thus, by energy conservation,

$$h^2(\nu_0 - \nu_1)^2 + 2h(\nu_0 - \nu_1)m_e c^2 = m^2 c^2 v^2$$



By momentum conservation along the x axis,

$$\frac{h\nu_0}{c} = \frac{h\nu_1}{c}\cos\theta + mv\cos\varphi,$$

$$h(\nu_0 - \nu_1\cos\theta) = mcv\cos\varphi,$$

$$h^2(\nu_0^2 - 2\nu_0\nu_1\cos\theta + \nu_1^2\cos^2\theta) = m^2c^2v^2\cos^2\varphi.$$

By momentum conservation along the y axis,

$$0 = \frac{h\nu_1}{c}\sin\theta + mv\sin\varphi,$$

$$h^2\nu_1^2\sin^2\theta = m^2c^2v^2\sin^2\varphi.$$

Adding the two momentum conservation equations,

$$h^2(\nu_0^2 - 2\nu_0\nu_1 \cos\theta + \nu_1^2 \cos^2\theta) + h^2\nu_1^2 \sin^2\theta = m^2c^2v^2$$

Thus, by momentum conservation,

$$h^2(\nu_0^2 - 2\nu_0\nu_1 \cos\theta + \nu_1^2) = m^2c^2v^2.$$

Thus, by both momentum and energy conservation,

$$h^2(\nu_0^2 - 2\nu_0\nu_1 \cos\theta + \nu_1^2) = h^2(\nu_0 - \nu_1)^2 + 2h(\nu_0 - \nu_1)m_e c^2,$$

$$-2h^2\nu_0\nu_1 \cos\theta = -2h^2\nu_0\nu_1 + 2h(\nu_0 - \nu_1)m_e c^2,$$

$$h\nu_0\nu_1(1 - \cos\theta) = (\nu_0 - \nu_1)m_e c^2$$

$$h\frac{1 - \cos\theta}{m_e c^2} = \frac{\nu_0 - \nu_1}{\nu_0\nu_1}$$

$$2\frac{h}{m_e c^2}\sin^2 \tfrac{1}{2}\theta = \frac{1}{\nu_1} - \frac{1}{\nu_0}$$

$$\frac{1}{\nu_1} = \frac{1}{\nu_0} + \frac{2h}{m_e c^2}\sin^2 \tfrac{1}{2}\theta$$

$$\boxed{\nu_1 = \frac{\nu_0}{1 + \frac{2h}{m_e c^2}\nu_0 \sin^2 \tfrac{1}{2}\theta}}$$

For $h \sim 6.6 \cdot 10^{-34} Js$,

$m_e \sim 9 \cdot 10^{-31} kg$,

$c \sim 3 \cdot 10^8 m / s$,

$$\frac{\nu_0}{\nu_1} = 1 + \frac{2h}{m_e c^2}\nu_0 \sin^2 \tfrac{1}{2}\theta$$

$$\sim 1 + (6.6)10^{-34}\nu_0 \frac{2\sin^2 \tfrac{1}{2}\theta}{(9)10^{-31}(81)10^{16}}$$

$$\sim 1 + (2)10^{-21}\nu_0 \sin^2 \tfrac{1}{2}\theta$$

Since ν_0 is between $3 \cdot 10^{16}$, and $3 \cdot 10^{19}$, $\dfrac{\nu_0}{\nu_1}$ is between

$$\sim 1 + (0.00006)\sin^2 \tfrac{1}{2}\theta, \quad \text{and} \quad \sim 1 + (0.06)\sin^2 \tfrac{1}{2}\theta$$

Since θ is between 0, and π, $\dfrac{\nu_0}{\nu_1}$ is between

$$\sim 1, \text{ and } \sim 1.06$$

The kinetic energy of the electron is

$$\begin{aligned}
\tfrac{1}{2} m_e v^2 &= h\nu_0 - h\nu_1, \\
&= h\nu_0 - \frac{h\nu_0}{1 + \dfrac{2h}{m_e c^2}\nu_0 \sin^2 \tfrac{1}{2}\theta}, \\
&= h\nu_0 \left(1 - \frac{1}{1 + \dfrac{2h}{m_e c^2}\nu_0 \sin^2 \tfrac{1}{2}\theta} \right) \\
&= h\nu_0 \frac{\dfrac{2h}{m_e c^2}\nu_0 \sin^2 \tfrac{1}{2}\theta}{1 + \dfrac{2h}{m_e c^2}\nu_0 \sin^2 \tfrac{1}{2}\theta} \\
&= h\nu_0 \frac{1}{\dfrac{1}{\dfrac{2h}{m_e c^2}\nu_0 \sin^2 \tfrac{1}{2}\theta} + 1},
\end{aligned}$$

Therefore,

$$\boxed{v = \sqrt{\frac{2h}{m_e}} \sqrt{\frac{\nu_0}{1 + \frac{1}{\frac{2h}{m_e c^2} \nu_0 \sin^2 \frac{1}{2}\theta}}}}$$

For $h \sim 6.6 \cdot 10^{-34} Js$,

$m_e \sim 9 \cdot 10^{-31} kg$,

$c \sim 3 \cdot 10^{8} m / s$,

$$v \sim \sqrt{\frac{2(6.6)10^{-34}}{9 \cdot 10^{-31}}} \sqrt{\frac{\nu_0}{1 + \frac{1}{\frac{2(6.6)10^{-34}}{9 \cdot 10^{-31} 9 \cdot 10^{16}} \nu_0 \sin^2 \frac{1}{2}\theta}}}$$

$$\sim (3.83)10^{-2} \sqrt{\frac{\nu_0}{1 + \frac{(6)10^{19}}{\nu_0 \sin^2 \frac{1}{2}\theta}}}$$

Since ν_0 **is between** $3 \cdot 10^{16}$, **and** $3 \cdot 10^{19}$, v **is between**

$$\sim (3.83)10^{-2} \sqrt{\frac{(3)10^{16}}{1 + \frac{(6)10^{19}}{(3)10^{16} \sin^2 \frac{1}{2}\theta}}},$$

$$\sim (6.6)10^{6} \sqrt{\frac{1}{1 + 2\frac{10^{3}}{\sin^2 \frac{1}{2}\theta}}}$$

and

$$\sim (3.83)10^{-2}\sqrt{\frac{(3)10^{19}}{1+\frac{(6)10^{19}}{(3)10^{19}\sin^2\frac{1}{2}\theta}}},$$

$$\sim (2.1)10^{8}\sqrt{\frac{1}{1+2\frac{1}{\sin^2\frac{1}{2}\theta}}}$$

Since θ **is between** 0**, and** π**,** v **is between**

~ 0**, and** $\sim (1.2)10^8$

θ **and** φ **are related by**

$$\frac{\sin\varphi}{\cos\varphi}=\frac{mv\sin\varphi}{mv\cos\varphi}$$

$$=\frac{-\frac{h\nu_1}{c}\sin\theta}{\frac{h\nu_0}{c}-\frac{h\nu_1}{c}\cos\theta}$$

$$=\frac{\sin\theta}{-\frac{\nu_0}{\nu_1}+\cos\theta}$$

$$\boxed{\tan\varphi=\frac{\sin\theta}{-\frac{\nu_0}{\nu_1}+\cos\theta}}$$

$$(-\frac{\nu_0}{\nu_1}+\cos\theta)\sin\varphi=\cos\varphi\sin\theta$$

$$\underbrace{\cos\theta\sin\varphi-\cos\varphi\sin\theta}_{\sin(\varphi-\theta)}=\frac{\nu_0}{\nu_1}\sin\varphi$$

$$\boxed{\frac{\sin(\varphi-\theta)}{\sin\varphi}=\frac{\nu_0}{\nu_1}=1+\frac{2h}{m_e c^2}\nu_0\sin^2\tfrac{1}{2}\theta}$$

$$\underline{\theta=0}\Big| \Rightarrow \nu_0=\nu_1 \Rightarrow \tan\varphi=\frac{\sin\theta}{-\dfrac{\nu_0}{\nu_1}+\cos\theta},\ \text{only for } \theta>0$$

$$\underline{\theta=\tfrac{1}{2}\pi}\Big| \Rightarrow \frac{\nu_0}{\nu_1}=1+\frac{h}{m_e c^2}\nu_0$$

Since ν_0 **is between** $3\cdot 10^{16}$**, and** $3\cdot 10^{19}$**,** $\dfrac{\nu_0}{\nu_1}$ **is between**

$$1+\frac{(6.6)10^{-34}}{9\cdot 10^{-31}9\cdot 10^{16}}3\cdot 10^{16}\sim 1.00024$$

and

$$1+\frac{(6.6)10^{-34}}{9\cdot 10^{-31}9\cdot 10^{16}}3\cdot 10^{19}\sim 1.24$$

$$\underline{\theta=\pi}\Big| \Rightarrow \frac{\nu_0}{\nu_1}=1+\frac{2h}{m_e c^2}\nu_0$$

Then, $\dfrac{\nu_0}{\nu_1}$ **is between** ~ 1.0005**, and** ~ 1.5

17. Raman Scattering of Photons off Molecules

In Raman Scattering, a stream of photons with frequency

$$\nu_0$$

scatters from molecules at rest. Each molecule has its system of orbiting electrons.

We assume that the electrons of a particular molecule will respond to the photons by oscillations with frequency that is specific to the molecules.

$$\nu_{Molecular}.$$

The oscillations of the electrons of the molecule generate electric dipole moment

$$\vec{m} = \varepsilon_0[\alpha_0 + \alpha_1 \sin(2\pi\nu_M t)]E_0 \sin(2\pi\nu_0 t)$$

$$= \varepsilon_0\alpha_0 E_0 \sin(2\pi\nu_0 t) + \varepsilon_0\alpha_1 E_0 \sin(2\pi\nu_M t)\sin(2\pi\nu_0 t)$$

$$= \varepsilon_0\alpha_0 E_0 \sin(2\pi\nu_0 t)$$

$$+\varepsilon_0\alpha_1 E_0 \tfrac{1}{2}\cos[2\pi(\nu_0 - \nu_M)t]$$

$$-\varepsilon_0\alpha_1 E_0 \tfrac{1}{2}\cos[2\pi(\nu_0 + \nu_M)t]$$

That is, the electric dipole radiates three frequencies

Some photons will remain unaffected by the electrons and will keep their frequency ν_0 unchanged.

Some photons will part some energy with the electrons of the particular molecule, and rotate slower at frequency

$$\nu_{stokes} = \nu_0 - \nu_M$$

Some photons will absorb energy from the electrons of the particular molecule, and rotate faster at frequency

$$\nu_{Anti-stokes} = \nu_0 + \nu_M$$

The molecules vibrate at frequencies

$$2\nu_M, \quad 3\nu_M, \quad 4\nu_M, \quad 5\nu_M, \ldots$$

The photons of the Raman Scattering are

$$\nu_0 - 2\nu_M, \quad \nu_0 + 2\nu_M$$

$$\nu_0 - 3\nu_M, \quad \nu_0 + 3\nu_M$$

$$\nu_0 - 4\nu_M, \quad \nu_0 + 4\nu_M$$

$$\nu_0 - 5\nu_M, \quad \nu_0 + 5\nu_M$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

18.
The Photon Pressure

An X-ray photon with $\nu_X = 10^{-10}$, hence, $h\nu_X = (6.6)10^{-34}10^{-10}$, is reflected by a mirror with $m_{mirror} = \frac{2}{10^5}\mathrm{kg}$ on a string of length $l = 0.1\mathrm{m}$. By Momentum conservation,

$$h\nu_X = -h\nu_X + m_{mirror}v_{mirror}$$

$$m_{mirror}v_{mirror} = 2h\nu_X$$

And the mirror acquires the kinetic energy

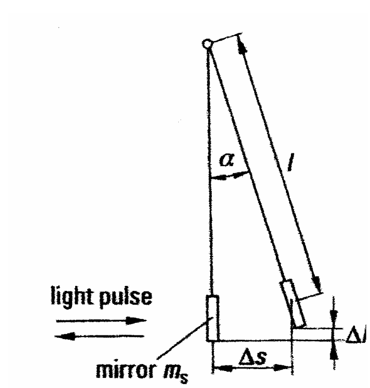
$$\frac{1}{2}m_{mirror}v_{mirror}^2 = \frac{2}{m_{mirror}}\left(h\nu_X\right)^2$$

The mirror deflects by an angle α. Then,

$$\Delta s \approx l\alpha,$$

and

$$\Delta l \approx (\Delta s)\alpha \approx l\alpha^2.$$



The kinetic energy $\frac{2}{m_{mirror}}\left(h\nu_X\right)^2$ converts into the potential energy $m_{mirror}g\Delta l \approx m_{mirror}gl\alpha^2$.

$$\frac{2}{m_{mirror}}\left(h\nu_X\right)^2 \approx m_{mirror} g l \alpha^2$$

$$\boxed{\alpha \approx \sqrt{\frac{2}{gl}}\frac{1}{m_{mirror}} h\nu_X}$$

$$\alpha \approx \sqrt{\frac{2}{(9.8)0.1}}\frac{1}{\frac{2}{10^5}}(6.6)10^{-34}10^{-10} \approx (4.7)10^{-39}$$

$$\Delta s \approx l\alpha \approx (0.1)(4.7)10^{-39} = (4.7)10^{-40}$$

19.
Photon Arrival-Time Uncertainty

$$\Delta E_{photon} = h\Delta\nu$$

Substituting in

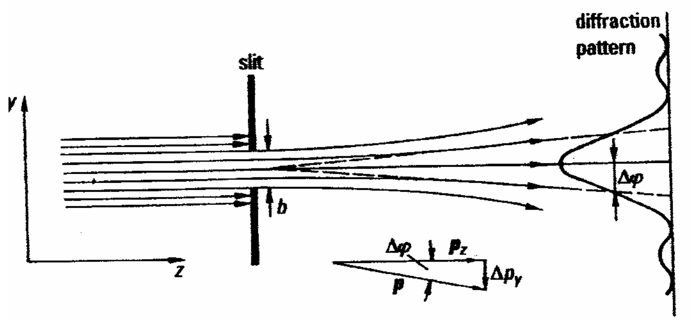
$$\Delta t \Delta E \geq \frac{h}{2\pi}$$

$$(\Delta t)\cancel{h}\Delta\nu \geq \frac{\cancel{h}}{2\pi}$$

$$\Delta t \geq \frac{1}{2\pi\Delta\nu}$$

20.
The Photon-Location Uncertainty

For diffraction at a slit



$$\Delta p_y = \left|p\right|\sin(\Delta\varphi) = \frac{h\nu}{c}\sin(\Delta\varphi) = \frac{h}{\lambda}\sin(\Delta\varphi)$$

Substituting in

$$\Delta y \Delta p_y \geq \frac{h}{2\pi}$$

$$\Delta y \frac{\cancel{h}}{\lambda}\sin(\Delta\varphi) \geq \frac{\cancel{h}}{2\pi}$$

$$\boxed{\Delta y \geq \frac{\lambda}{2\pi\sin(\Delta\varphi)}}$$

21.
Red Shift under Earth Gravitation

$$\Phi_{gravitational} = G\frac{m_{photon}M_{earth}}{R_{earth}} \Rightarrow$$

$$\boxed{\underbrace{\Delta\Phi}_{work} = G\frac{M_{earth}}{R_{earth}}\frac{h}{c^2}\underbrace{\Delta\nu}_{red\ \text{shift}}}$$

22.
The Poisson Distributed Number of Photons, Each Detected with Constant infinitesimal Probability

Let N be an infinite hyper-real number. Assume that N photons are detected within time T, and assume that at each time-interval, $\frac{T}{N}$, at most one photon may be detected.

The chance to detect that one photon is the infinitesimal

$$\boxed{p = \frac{\lambda}{N}},$$

and the chance to detect no photons is

$$q = 1 - \frac{\lambda}{N}$$

The number of detected photons is a random variable X that is <u>Binomially distributed</u>, and the probability to detect k photons is

$$p(k) = \Pr(X = k) = \frac{N!}{k!(N-k)!}\left(\frac{\lambda}{N}\right)^k \left(1 - \frac{\lambda}{N}\right)^{N-k}$$

$$= \frac{1}{(N-k)!} \underbrace{\frac{N!}{N^k}}_{\approx (N-k)!} \frac{\lambda^k}{k!} \underbrace{\left(1 - \frac{\lambda}{N}\right)^{N-k}}_{\approx \left(1-\frac{\lambda}{N}\right)^N = e^{-\lambda}} \approx \frac{\lambda^k}{k!} e^{-\lambda}.$$

$$\boxed{p(k) \approx \frac{\lambda^k}{k!} e^{-\lambda}}$$

A Poisson Distribution, that is, the sum of N Bernoulli Trials where each detection is a Bernoulli Trial. Then,

$$\sum_{k=0}^{k=N} p(k) = \sum_{k=0}^{k=N} \frac{N!}{k!(N-k)!}\left(\frac{\lambda}{N}\right)^k \left(1-\frac{\lambda}{N}\right)^{N-k}$$

$$= \left(\frac{\lambda}{N} + 1 - \frac{\lambda}{N}\right)^N = 1.$$

$$\sum_{k=0}^{k=N} p(k) \approx \sum_{k=0}^{k=N} \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{k=N} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = 1.$$

Its Mean is

$$E(X) = Np = N\frac{\lambda}{N} = \lambda$$

And its Variance is

$$Var(X) = Npq = N\frac{\lambda}{N}\underbrace{\left(1-\frac{\lambda}{N}\right)}_{\approx 1} \approx \lambda.$$

The Signal Power is

$$E^2(X) = \lambda^2$$

The Noise Power is

$$\sigma^2(X) = Var(X) = \lambda$$

The Signal to Noise Ratio is

$$\frac{E^2(X)}{\sigma^2(X)} = \lambda.$$

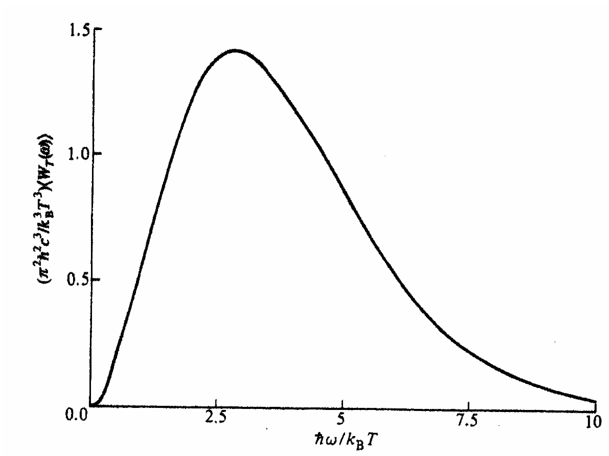
23. The Geometrically-Distributed Number of Detected Photons in Thermal Equilibrium

In 1905, Planck obtained the Radiation Law for electromagnetic energy. Planck showed that there are two constants[10],

The Entropy Constant k[11], and The Energy Constant h

so that at Absolute Temperature T, electromagnetic energy of frequency ν must be emitted in packets of $h\nu$. And the emitted power per unit area per second is

$$\rho(\nu,T) = 2\frac{\nu^2}{c^2}\frac{h\nu}{e^{\frac{h\nu}{kT}} - 1}.$$



[10] We explain how Planck determined that $h = 6.62607015 \times 10^{-34}\ \mathrm{J \cdot Hz^{-1}}$ in H. Vic Dannon, The Entropy Constant k, The Radiation Constant h, The Quantized 2nd Law, The Quantized Temperature, & The Temperature Quantum
Gauge Institute Journal, Volume 15, No 1, February 2019.

[11] Planck determined the Entropy Constant $k = (1.380649)10^{-23}\mathrm{J/K}$, and named it after Boltzmann .

Planck's derivation of his Radiation Law mandates that electromagnetic energy of any frequency ν is emitted in packets of energy $h\nu$. The so called "Quantum Hypothesis" is actually a fact. And the essence of the Quantum Theory.

<u>In Thermal Equilibrium at temperature T</u>, the probability to detect a photon is

$$\boxed{p = 1 - e^{-\frac{h\nu}{k_B T}}},$$

and the probability to detect no photons is

$$\boxed{q = 1 - p = e^{-\frac{h\nu}{k_B T}}}$$

At room Temperature,

$$T_{room} = 27^\circ C = 300^\circ K$$

For a Green photon

$$\lambda_{green} \sim 5500\dot{A} = (5.5)10^{-7}\,\mathrm{m}$$

$$\nu_{green} = \frac{c}{\lambda_{green}} \sim \frac{3 \cdot 10^8}{(5.5)10^{-7}} \sim (5.45454)10^{14} \ / \sec$$

$$\frac{h\nu_{green}}{k_B T_{room}} \sim \frac{(6.626)10^{-34}(5.45454)10^{14}}{(1.38)10^{-23}(3)10^2} \sim 87.3$$

$$p_{green} \sim 1 - e^{-87.3} \sim 1 - \frac{1}{10^{38}}$$

$$q_{green} \sim e^{-87.3} \sim \frac{1}{10^{38}}$$

The probability to detect one photon after $n = 0,1,2,\ldots$ failures is a random variable X Geometrically Distributed[12],

$$\boxed{p(n) = q^n p = e^{-\frac{nh\nu}{k_B T}}(1 - e^{-\frac{h\nu}{k_B T}})}$$

Then,

$$\sum_{n=0}^{n=N} p(n) = \underbrace{\left(1 + \frac{1}{e^{\frac{h\nu}{k_B T}}} + \frac{1}{e^{\frac{2h\nu}{k_B T}}} + \ldots + \frac{1}{e^{\frac{Nh\nu}{k_B T}}}\right)}_{\frac{1}{1-\frac{1}{e^{\frac{h\nu}{k_B T}}}}}\left(1 - \frac{1}{e^{\frac{h\nu}{k_B T}}}\right) = 1$$

Its Mean is

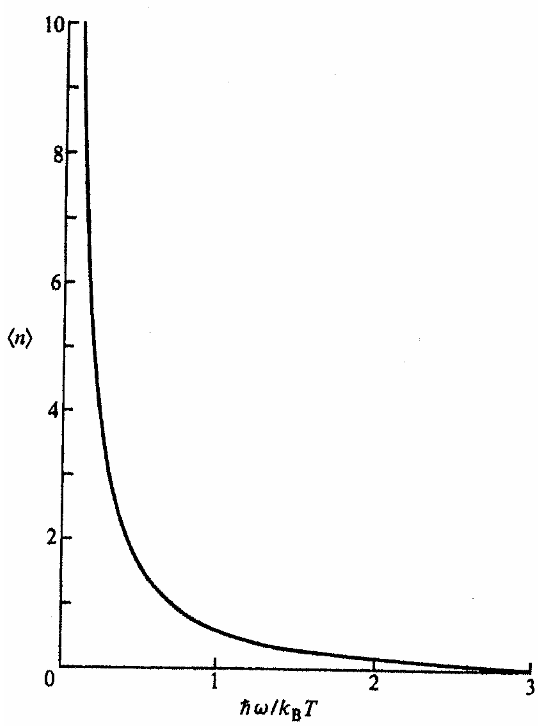
$$\boxed{\mu = \frac{q}{p} = \frac{e^{-\frac{h\nu}{kT}}}{1 - e^{-\frac{h\nu}{kT}}} = \frac{1}{e^{\frac{h\nu}{kT}} - 1} = \frac{\overline{u}}{h\nu} = \overline{n}}$$

which is the average number of photons.

For the green photon, at room temperature

$$\mu_{green} \sim \frac{\frac{1}{10^{38}}}{1 - \frac{1}{10^{38}}} \sim \frac{1}{10^{38}}$$

[12] https://en.wikipedia.org/wiki/Geometric_distribution



Or, from its definition,

$$\mu = 0(1-q) + 1(1-q)q + 2(1-q)q^2 + 3(1-q)q^3 ... + N(1-q)q^N$$

$$= q - q^2 + 2q^2 - 2q^3 + 3q^3 - 3q^4 + ... + Nq^N - Nq^{N+1}$$

$$= q(\underbrace{1 + q + q^2 + q^3 + ...}_{\frac{1}{1-q}})$$

$$= \frac{1}{\frac{1}{q} - 1}$$

Hence,

$$\frac{1}{q} = \frac{1}{\mu} + 1$$

$$\boxed{q = \frac{\mu}{\mu + 1}}$$

$$\boxed{p = \frac{1}{\mu + 1}}$$

$$\frac{1}{p} = \mu + 1$$

$$\boxed{p(n) = \frac{1}{\mu + 1}\left(\frac{\mu}{\mu + 1}\right)^n}$$

The Variance is

$$\boxed{Var[X]} = \frac{q}{p^2} = \underbrace{\frac{q}{p}}_{\mu} \underbrace{\frac{1}{p}}_{\mu+1} = \boxed{\mu^2 + \mu}$$

The Signal to Noise is

$$\frac{\mu^2}{\sigma^2} = \frac{\mu^2}{\mu^2 + \mu} = \frac{\mu}{\mu + 1}.$$

24.
Boltzmann Distributed Molecular Internal Energies Maximize the Entropy

At temperature T, a gas molecule has internal energy ε_i with the Boltzmann Probability

$$\frac{N_i}{N} = b_i = \frac{e^{-\frac{\varepsilon_i}{kT}}}{e^{-\frac{\varepsilon_1}{kT}} + e^{-\frac{\varepsilon_2}{kT}} + \ldots + e^{-\frac{\varepsilon_M}{kT}}}$$

For a molecule with two possible energies, ε_1, ε_2, the distribution

$$\frac{N_1}{N} = p_1 = \frac{1}{\underbrace{e^{-\frac{\varepsilon_1}{kT}} + e^{-\frac{\varepsilon_2}{kT}}}_{Q}} e^{-\frac{\varepsilon_1}{kT}} = Qe^{-\frac{\varepsilon_1}{kT}}$$

$$\frac{N_2}{N} = p_2 = \frac{1}{e^{-\frac{\varepsilon_1}{kT}} + e^{-\frac{\varepsilon_2}{kT}}} e^{-\frac{\varepsilon_2}{kT}} = Qe^{-\frac{\varepsilon_2}{kT}}$$

maximizes the entropy

$$s = -kp_1 \log p_1 - kp_2 \log p_2$$

so that

$$p_1 + p_2 = 1$$

$$p_1\varepsilon_1 + p_2\varepsilon_2 = \varepsilon$$

The Lagrange function to be maximized is

$$G = -kp_1 \log p_1 - kp_2 \log p_2 - \lambda_1(p_1 + p_2 - 1) - \lambda_2(p_1\varepsilon_1 + p_2\varepsilon_2 - \varepsilon)$$

$$\frac{\partial G}{\partial p_1} = -k\log p_1 - k - \lambda_1 - \lambda_2\varepsilon_1 \Rightarrow -k\log p_1 = k + \lambda_1 + \lambda_2\varepsilon_1$$

$$\frac{\partial G}{\partial p_2} = -k\log p_2 - k - \lambda_1 - \lambda_2\varepsilon_2 \Rightarrow -k\log p_2 = k + \lambda_1 + \lambda_2\varepsilon_2$$

$$\Rightarrow \underbrace{-k\log \underbrace{\frac{p_2}{p_1}}_{e^{-\frac{\varepsilon_2-\varepsilon_1}{kT}}}}_{\frac{\varepsilon_2-\varepsilon_1}{T}} = \lambda_2(\varepsilon_2 - \varepsilon_1) \Rightarrow \boxed{\lambda_2 = \frac{1}{T}}$$

$$-k\log p_1 = k + \lambda_1 + \frac{\varepsilon_1}{T} \Rightarrow -k\varepsilon_2(\log Q - \underbrace{\frac{\varepsilon_1}{kT}}) = k\varepsilon_2 + \lambda_1\varepsilon_2 + \underbrace{\frac{\varepsilon_1\varepsilon_2}{T}}$$

$$\Rightarrow -k\log Q = k + \lambda_1$$

$$\Rightarrow \boxed{\lambda_1 = -k(1 + \log Q)}$$

25.
Absorbed Photons

At Temperature T, consider gas molecules, N_1 of them have electron orbits with energy ε_1, and N_2 have electron orbits with energy $\varepsilon_2 = \varepsilon_1 + h\nu$, in an electromagnetic field with radiation power density $\rho(\nu,T)$ per unit volume, and per second.

The probability for a photon $h\nu$ from $\rho(\nu,T)$ to be absorbed by an electron at an ε_1 orbit over time dt is proportional to $\rho(\nu,T)$, to dt, and to a factor $B_{1\to 2}$. It is

$$\rho(\nu,T)B_{1\to 2}dt$$

The Boltzmann Probability that at temperature T an electron orbits a gas molecule with internal energy ε_1 is

$$p_1 = \frac{N_1}{N} = g_1 e^{-\frac{\varepsilon_1}{kT}}.$$

Thus, the probability for a photon $h\nu$ from $\rho(\nu,T)$ to be absorbed by an ε_1 orbit electron, and move the electron to the ε_2 orbit is

$$\underbrace{\Pr\{h\nu \text{ of } \rho(\nu,T) \text{ absorbed over time } dt/\text{electron orbits with } \varepsilon_1\}}_{\rho(\nu,T)B_{1\to 2}dt} \times$$

$$\times \underbrace{\Pr\{\text{electron orbits with } \varepsilon_1\}}_{p_1 = g_1 e^{-\frac{\varepsilon_1}{kT}}} =$$

$$= \boxed{[\rho(\nu,T)B_{1\to 2}dt][g_1 e^{-\frac{\varepsilon_1}{kT}}]}$$

The width of the spectral line at $\nu_{2\to 1}$ allows absorption of photons with frequencies between

$$\nu_{2\to 1} - \Delta\nu_{2\to 1}, \quad \text{and} \quad \nu_{2\to 1} + \Delta\nu_{2\to 1}.$$

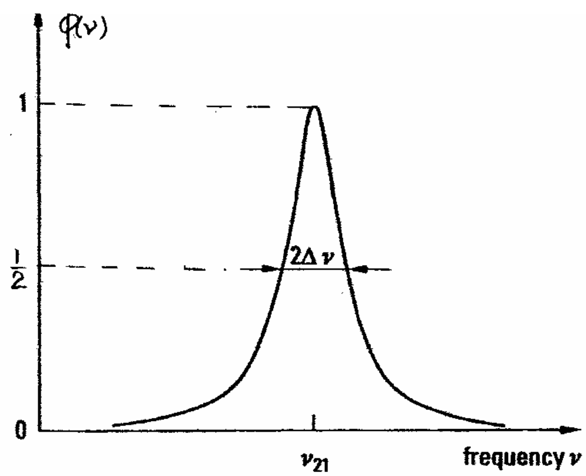
with lesser probability determined by a normal curve

$$\phi(\nu)$$

with width

$$2\Delta\nu_{2\to 1}$$

at $\phi(\nu) = \dfrac{1}{2}$



Since N_1 decreases,

$$dp_1 = -B_{1\to 2} p_1 \rho(\nu, T)\phi(\nu)dt$$

$$\frac{dp_1}{p_1} = -B_{1\to 2}\rho(\nu, T)\phi(\nu)dt$$

The rate of change of density of molecules with energy ε_1 is

$$\frac{dp_1}{dt} = -B_{1\to 2} p_1 \rho(\nu, T)\phi(\nu)$$

26.
Stimulated Photons

At Temperature T, consider gas molecules, N_1 of them have electron orbits with energy ε_1, and N_2 have electron orbits with energy $\varepsilon_2 = \varepsilon_1 + h\nu$, in an electromagnetic field with radiation power density $\rho(\nu, T)$ per unit volume, and per second.

The probability that a photon $h\nu$ from the radiation field will scatter off an electron in an ε_2 orbit, and push it to a lower orbit with ε_1 so that over time dt it releases a photon with the same energy $h\nu$ is proportional to $\rho(\nu, T)$, to dt, and a factor $B_{2\to 1}$. It is

$$\rho(\nu, T)B_{2\to 1}dt$$

The Boltzmann Probability that at temperature T an electron is orbiting a gas molecule with internal energy ε_2 is

$$p_2 = \frac{N_2}{N} = g_2 e^{-\frac{\varepsilon_2}{kT}}.$$

Thus, the probability that $h\nu$ from $\rho(\nu, T)$ will stimulate the release of $h\nu$ from an electron in an ε_2 orbit over time dt is

$$\underbrace{\Pr\{h\nu \text{ of } \rho(\nu,T) \text{ to move electron from } \varepsilon_2 \text{ to } \varepsilon_1 \text{ /electron is in } \varepsilon_2 \text{ orbit}\}}_{\rho(\nu,T)B_{2\to 1}dt}$$

$$\times \underbrace{\Pr\{\text{electron is in } \varepsilon_2 \text{ orbit}\}}_{p_2 = g_2 e^{-\frac{\varepsilon_2}{kT}}} =$$

$$= \boxed{\rho(\nu, T)(B_{2\to 1}dt)(g_2 e^{-\frac{\varepsilon_2}{kT}})}$$

The emitted photon has the same direction, the same momentum and the same energy of the stimulating photon. And the stimulating radiation is amplified.

The width of the spectral line at $\nu_{2\to 1}$ allows emission of photons with frequencies between

$$\nu_{2\to 1} - \Delta\nu_{2\to 1}, \quad \text{and} \quad \nu_{2\to 1} + \Delta\nu_{2\to 1}.$$

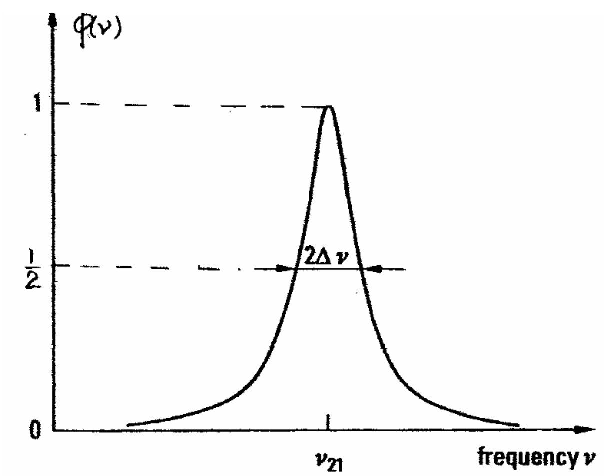
with lesser probability determined by a normal curve

$$\phi(\nu)$$

with width

$$2\Delta\nu_{2\to 1}$$

at $\phi(\nu) = \frac{1}{2}$



Since N_2 decreases,

$$dp_2 = -B_{2\to 1} p_2 \rho(\nu, T)\phi(\nu)dt$$

The rate of change of density of molecules with energy ε_2 is

$$\frac{dp_2}{dt} = -B_{2\to 1} p_2 \rho(\nu, T)\phi(\nu)$$

27.
Spontaneously Emitted Photons

The probability for a molecule with an electron in an ε_2 orbit to <u>spontaneously emit</u> over time dt a photon $h\nu$ is

$$A_{2\to 1}dt$$

At temperature T there are N_2 molecules with electrons in ε_2 orbits.

The Boltzmann Probability that at temperature T a gas molecule has internal energy ε_2 is

$$p_2 = \frac{N_2}{N} = g_2 e^{-\frac{\varepsilon_2}{kT}}$$

The probability for a molecule to <u>spontaneously emit</u> $h\nu$ over dt is

$$\underbrace{\Pr\{\text{emit } h\nu \text{ over } dt \ / \text{ molecule has electron in } \varepsilon_2 \text{ orbit}\}}_{(A_{2\to 1}dt)}$$

$$\times \underbrace{\Pr\{\text{molecule has electron in } \varepsilon_2 \text{ orbit}\}}_{p_2 = g_2 e^{-\frac{\varepsilon_2}{kT}}} =$$

$$= \boxed{(A_{2\to 1}dt) \times (g_2 e^{-\frac{\varepsilon_2}{kT}})}.$$

The spontaneously emitted photon has a random direction.

Since N_2 decreases,

$$dp_2 = -A_{2\to 1}p_2 dt$$

The rate of change of density of molecules with energy ε_2 is

$$\frac{dp_2}{dt} = -A_{2\to1}p_2$$

$$\frac{dp_2}{p_2} = -A_{2\to1}dt$$

$$\log p_2 = -A_{2\to1}t$$

$$p_2(t) = p(0)e^{-A_{2\to1}t}$$

For optical transitions,

$A_{2\to1}$ is between $\frac{1}{10^8}$sec. and $\frac{1}{10^9}$sec.

$$p_2(\frac{1}{A_{2\to1}}) = \frac{p_2(0)}{e}$$

$\Delta t = \frac{1}{A_{2\to1}}$ is the average life-time of the energy state ε_2.

From the uncertainty relation

$$(\Delta h\nu)(\Delta t) \geq \frac{h}{2\pi},$$

$$(\Delta\nu)\frac{1}{A_{2\to1}} \geq \frac{1}{2\pi}$$

$$\boxed{2\pi\Delta\nu \geq A_{2\to1}}$$

28. Stimulated Photons at Thermal Equilibrium

In 1917, Einstein interpreted Planck's Radiation formula in terms of stimulated photons at thermal equilibrium.

<u>**Spontaneously Emitted Photons**</u>,

The probability for a molecule to <u>spontaneously emit</u> over a second a photon $h\nu$ on condition that the molecule has energy ε_2 is

$$A_{2\to 1}$$

The probability for a molecule to <u>spontaneously emit</u> over time dt a photon $h\nu$ on condition that the molecule has energy ε_2 is

$$A_{2\to 1}dt$$

At temperature T there are N_2 molecules with internal energy ε_2.

The Boltzmann Probability that at temperature T a gas molecule has internal energy ε_2 is

$$\frac{N_2}{N} = g_2 e^{-\frac{\varepsilon_2}{kT}}$$

The probability for a molecule to <u>spontaneously emit</u> $h\nu$ over dt is

$$\underbrace{\Pr\{\text{emit } h\nu \text{ over } dt \text{ / molecule has } \varepsilon_2\}}_{(A_{2\to 1}dt)} \times \underbrace{\Pr\{\text{molecule has } \varepsilon_2\}}_{(g_2 e^{-\frac{\varepsilon_2}{kT}})} =$$

$$= (A_{2\to 1}dt) \times (g_2 e^{-\frac{\varepsilon_2}{kT}}).$$

The spontaneously emitted photon has a random direction.

Stimulated Photons

The probability to stimulate a molecule to emit over a second a photon $h\nu$ on condition that the molecule has energy ε_2 is

$$B_{2\to 1}$$

The probability to stimulate a molecule to emit over time dt a photon $h\nu$ on condition that the molecule has energy ε_2 is

$$B_{2\to 1}dt$$

The Boltzmann Probability that at temperature T a gas molecule has internal energy ε_2 is

$$\frac{N_2}{N} = g_2 e^{-\frac{\varepsilon_2}{kT}}$$

The probability for a photon $h\nu$ from $\rho(\nu,T)$ to stimulate a molecule to emit $h\nu$ over time dt is

$$\underbrace{\Pr\{h\nu \text{ of } \rho(\nu,T) \text{ stimulate } h\nu/\text{molecule has } \varepsilon_2\}}_{\rho(\nu,T)B_{2\to 1}dt} \times \underbrace{\Pr\{\text{molecule has } \varepsilon_2\}}_{g_2 e^{-\frac{\varepsilon_2}{kT}}} =$$

$$= \rho(\nu,T)(B_{2\to 1}dt)(g_2 e^{-\frac{\varepsilon_2}{kT}})$$

The emitted photon has the same direction of the stimulating photon. And the stimulating radiation is amplified.

Absorbed Photons

The Boltzmann Probability that at temperature T a gas molecule has internal energy ε_1 is

$$\frac{N_1}{N} = g_1 e^{-\frac{\varepsilon_1}{kT}}$$

The probability for a molecule with energy ε_1 to <u>absorb</u> a photon $h\nu$ from radiation with power $\rho(\nu,T)$ per unit area, and per second, over time dt is

$$\underbrace{\Pr\{h\nu \text{ of } \rho(\nu,T) \text{ absorbed/molecule has } \varepsilon_1\}}_{\rho(\nu,T)B_{1\to 2}dt} \times \underbrace{\Pr\{\text{molecule has } \varepsilon_1\}}_{g_1 e^{-\frac{\varepsilon_1}{kT}}} =$$

$$= (\rho(\nu,T)B_{1\to 2}dt)(g_1 e^{-\frac{\varepsilon_1}{kT}})$$

At equilibrium at temperature T,

$$g_1 e^{-\frac{\varepsilon_1}{kT}} B_{1\to 2}\rho(\nu,T)dt = g_2 e^{-\frac{\varepsilon_2}{kT}} B_{2\to 1}\rho(\nu,T)dt + g_2 e^{-\frac{\varepsilon_2}{kT}} A_{2\to 1}dt$$

$$e^{\frac{\varepsilon_2-\varepsilon_1}{kT}} g_1 B_{1\to 2}\rho(\nu,T) = g_2 B_{2\to 1}\rho(\nu,T) + g_2 A_{2\to 1}$$

$$e^{\frac{h\nu}{kT}} g_1 B_{1\to 2}\rho(\nu,T) = g_2 B_{2\to 1}\rho(\nu,T) + g_2 A_{2\to 1}$$

$$\underbrace{e^{\frac{h\nu}{kT}}}_{\to 1, T\to\infty} g_1 B_{1\to 2} = g_2 B_{2\to 1} + \frac{1}{\underbrace{\rho(\nu,T)}_{\to 0, T\to\infty}} g_2 A_{2\to 1}$$

$$\boxed{g_1 B_{1\to 2} = g_2 B_{2\to 1}}$$

Substituting in $e^{\frac{h\nu}{kT}} g_1 B_{1\to 2}\rho(\nu,T) = g_2 B_{2\to 1}\rho(\nu,T) + g_2 A_{2\to 1}$

$$\underbrace{\frac{g_1 B_{1\to 2}}{g_2 B_{2\to 1}}}_{1} e^{\frac{h\nu}{kT}}\rho(\nu,T) = \rho(\nu,T) + \frac{A_{2\to 1}}{B_{2\to 1}}$$

$$\rho(\nu, T) = \frac{\dfrac{A_{2\to 1}}{B_{2\to 1}}}{e^{\frac{h\nu}{kT}} - 1}$$

In Thermal Equilibrium, radiation with power $\rho(\nu, T)$ per unit area, and per second is given by Planck's Law,

$$\rho(\nu, T) = 2\frac{h}{c^2}\nu^3 \frac{1}{e^{\frac{h\nu}{kT}} - 1}.$$

Therefore,

$$\boxed{\frac{A_{2\to 1}}{B_{2\to 1}} = 2\frac{h}{c^2}\nu^3}$$

29.
Stimulated Photons at Thermal Non-Equilibrium

At Thermal Equilibrium, by Planck's Radiation Law,

$$\rho(\nu,T) = 2\frac{h}{c^2}\nu^3 \frac{1}{e^{\frac{h\nu}{kT}}-1}$$

Then,

$$e^{\frac{h\nu}{kT}} > 1,$$

$$e^{\frac{\varepsilon_2-\varepsilon_1}{kT}} > 1$$

$$e^{\frac{-\varepsilon_1}{kT}} > e^{\frac{-\varepsilon_2}{kT}}$$

multiplying both sides by $g_1 B_{1\to 2}\rho(\nu,T) = g_2 B_{2\to 1}\rho(\nu,T)$

$$\underbrace{\underbrace{g_1 e^{-\frac{\varepsilon_1}{kT}}}_{\frac{N_1}{N}} B_{1\to 2}\rho(\nu,T)}_{\text{rate of absorbed photons}} > \underbrace{\underbrace{g_2 e^{-\frac{\varepsilon_2}{kT}}}_{\frac{N_2}{N}} B_{2\to 1}\rho(\nu,T)}_{\text{rate of stimulated photons}}$$

$$\boxed{N_1 B_{1\to 2} > N_2 B_{2\to 1}}$$

is the condition for thermal equilibrium,

Thus, the condition for non-equilibrium, and optical gain is

$$\boxed{N_2 B_{2\to 1} > N_1 B_{1\to 2}}$$

Then,

$$\underbrace{\underbrace{g_2 e^{-\frac{\varepsilon_2}{kT}}}_{\frac{N_2}{N}} B_{2\to 1}\rho(\nu,T)}_{\text{rate of stimulated photons}} > \underbrace{\underbrace{g_1 e^{-\frac{\varepsilon_1}{kT}}}_{\frac{N_1}{N}} B_{1\to 2}\rho(\nu,T)}_{\text{rate of absorbed photons}}$$

$$g_1 e^{-\frac{\varepsilon_1}{kT}} B_{1\to 2}\rho(\nu,T) < g_2 e^{-\frac{\varepsilon_2}{kT}} B_{2\to 1}\rho(\nu,T) + g_2 e^{-\frac{\varepsilon_2}{kT}} A_{2\to 1}$$

$$g_1 B_{1\to 2} e^{\frac{h\nu}{kT}}\rho(\nu,T) < g_2 B_{2\to 1}\rho(\nu,T) + g_2 A_{2\to 1}$$

$$\boxed{\rho(\nu,T) < \frac{\dfrac{A_{2\to 1}}{B_{2\to 1}}}{\dfrac{g_1 B_{1\to 2}}{g_2 B_{2\to 1}} e^{\frac{h\nu}{kT}} - 1}}$$

is the Stimulated Photons Law at Non-equilibrium

Appendix 1:

Elliptical Polarization

The electric field of a plane wave is

$$E(z,t) = \mathrm{Re}\left\{(a_x e^{i\varphi_x} + a_y e^{i\varphi_y})e^{i\omega(t-\frac{z}{c})}\right\}$$

$$= a_x \cos[\omega(t-\frac{z}{c}) + \varphi_x] + a_y \cos[\omega(t-\frac{z}{c}) + \varphi_y]$$

$$\omega(t-\frac{z}{c}) + \varphi_x \equiv \alpha_x(z,t)$$

$$\omega(t-\frac{z}{c}) + \varphi_y \equiv \alpha_y(z,t)$$

$$\alpha_x(z,t) - \alpha_y(z,t) = \varphi_x - \varphi_y \equiv \varphi$$

$$= \underbrace{a_x \cos\alpha_x(z,t)}_{E_x(z,t)} + \underbrace{a_y \cos\alpha_y(z,t)}_{E_y(z,t)}$$

$$2\cos\varphi \frac{E_x(z,t)}{a_x}\frac{E_y(z,t)}{a_y} + \sin^2\varphi =$$

$$= 2\cos(\alpha_x - \alpha_y)\cos\alpha_x\cos\alpha_y + 1 - \cos^2(\alpha_x - \alpha_y)$$

$$= \cos(\alpha_x - \alpha_y)\{2\cos\alpha_x\cos\alpha_y - \underbrace{\cos(\alpha_x - \alpha_y)}_{\cos\alpha_x\cos\alpha_y + \sin\alpha_x\sin\alpha_y}\} + 1$$

$$= \underbrace{\cos(\alpha_x - \alpha_y)}_{\cos\alpha_x\cos\alpha_y + \sin\alpha_x\sin\alpha_y}\{\cos\alpha_x\cos\alpha_y - \sin\alpha_x\sin\alpha_y\} + 1$$

$$= \cos^2\alpha_x\cos^2\alpha_y - \sin^2\alpha_x\sin^2\alpha_y + 1$$

$$= \cos^2\alpha_x\cos^2\alpha_y - (1-\cos^2\alpha_x)(1-\cos^2\alpha_y) + 1$$

$$= \cos^2\alpha_x + \cos^2\alpha_y$$

$$= \underbrace{\left(\frac{E_x(z,t)}{a_x}\right)}_{\xi}^2 + \underbrace{\left(\frac{E_y(z,t)}{a_y}\right)}_{\eta}^2$$

To see that the curve

$$\xi^2 + \eta^2 - 2\xi\eta\cos\varphi - \sin^2\varphi = 0$$

is an ellipse in ξ, and η, we rotate it by an angle $\frac{\pi}{4}$ with

$$\begin{bmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

into a curve in λ, and μ.

$$\begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}.$$

Then,

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}\lambda - \frac{1}{\sqrt{2}}\mu \\ \frac{1}{\sqrt{2}}\lambda + \frac{1}{\sqrt{2}}\mu \end{bmatrix}$$

$$0 = \xi^2 + \eta^2 - 2\xi\eta\cos\varphi - \sin^2\varphi$$

$$= (\tfrac{1}{2}\lambda^2 + \tfrac{1}{2}\mu^2 - \lambda\mu) + (\tfrac{1}{2}\lambda^2 + \tfrac{1}{2}\mu^2 + \lambda\mu)$$

$$-2(\frac{1}{\sqrt{2}}\lambda - \frac{1}{\sqrt{2}}\mu)(\frac{1}{\sqrt{2}}\lambda + \frac{1}{\sqrt{2}}\mu)\cos\varphi - \sin^2\varphi$$

$$\lambda^2 + \mu^2 + (\mu^2 - \lambda^2)\cos\varphi = \sin^2\varphi$$

$$\lambda^2(1-\cos\varphi)+\mu^2(1+\cos\varphi)=(1+\cos\varphi)(1-\cos\varphi)$$

$$\frac{\lambda^2}{1+\cos\varphi}+\frac{\mu^2}{1-\cos\varphi}=1$$

$$\frac{(\xi+\eta)^2}{2(1+\cos\varphi)}+\frac{(\xi-\eta)^2}{2(1-\cos\varphi)}=1$$

$$\frac{\left(\cos[\omega(t-\frac{z}{c})+\varphi_x]+\cos[\omega(t-\frac{z}{c})+\varphi_y]\right)^2}{2(1+\cos\varphi)}+$$

$$+\frac{\left(\cos[\omega(t-\frac{z}{c})+\varphi_x]-\cos[\omega(t-\frac{z}{c})+\varphi_y]\right)^2}{2(1-\cos\varphi)}=1,$$

$$\frac{\left[2\cos\left(\omega(t-\frac{z}{c})+\frac{\varphi_x+\varphi_y}{2}\right)\cos\frac{\varphi_y-\varphi_x}{2}\right]^2}{2(1+\cos\varphi)}+$$

$$+\frac{\left[2\sin\left(\omega(t-\frac{z}{c})+\frac{\varphi_x+\varphi_y}{2}\right)\sin\frac{\varphi_y-\varphi_x}{2}\right]^2}{2(1-\cos\varphi)}=1$$

An ellipse in

$$\cos[\omega(t-\frac{z}{c})+\varphi_x]+\cos[\omega(t-\frac{z}{c})+\varphi_y]=$$

$$=2\cos\left(\omega(t-\frac{z}{c})+\frac{\varphi_x+\varphi_y}{2}\right)\cos\frac{\varphi_y-\varphi_x}{2}$$

and in,

$$\cos[\omega(t-\frac{z}{c})+\varphi_x]-\cos[\omega(t-\frac{z}{c})+\varphi_y]=$$

$$=2\sin\left(\omega(t-\frac{z}{c})+\frac{\varphi_x+\varphi_y}{2}\right)\sin\frac{\varphi_y-\varphi_x}{2}$$

centered at the origin with long axis

$$a=\sqrt{2(1+\cos\varphi)},$$

and short axis

$$b=\sqrt{2(1-\cos\varphi)}$$

In polar coordinates, $(\rho(t),\theta(t))$, the ellipse is

$$\rho^2(t)=\frac{a^2b^2}{a^2\sin^2\theta(t)+b^2\cos^2\theta(t)}$$

$$=\frac{4\sin^2\varphi}{2(1+\cos\varphi)\sin^2\theta(t)+2(1-\cos\varphi)\cos^2\theta(t)}$$

The distance from the ellipse center to its focus is

$$c=\sqrt{2(1+\cos\varphi)-2(1-\cos\varphi)}=2\sqrt{\cos\varphi}$$

The eccentricity of the ellipse is

$$e=\frac{c}{a}=\sqrt{\frac{2\cos\varphi}{1+\cos\varphi}}$$

For fixed z, the electric field rotates along the ellipse, and for fixed instant t it propagates along z with wave length λ.

The intensity of the field is

$$\frac{1}{2(\text{medium impedence})}(a_x^2+a_y^2)=$$

$$= \frac{1}{2(\text{medium impedence})} \left\{ \left(\frac{E_x(z,t)}{\cos \alpha_x} \right)^2 + \left(\frac{E_y(z,t)}{\cos \alpha_y} \right)^2 \right\}$$

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