Poincare Conjecture Proof

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Abstract: We prove that a 3-dimensional manifold Σ^3 that is Compact, Simply-Connected, and Borderless is homeomorphic to the 3-dimensional sphere S^3.

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Introduction

A 2-dimensional manifold Σ^2 is a Topological Space with

- Hausdorff separation. That is, distinct points have disjoint neighborhoods,
- Each point x, has a 2-dimensional open neighborhood U_x^2, homeomorphic to a disk D_x^2 in the Euclidean plane E^2.

The homeomorphism $h_x : U_x^2 \to D_x^2$ is a bicontinuous bijection that endows each point x with a local coordinate system. The set of all of these coordinate systems is called an <u>Atlas</u> for the manifold.

It is well-known [4], that a 2-manifold Σ^2 that is Compact, Simply-Connected, and Borderless is Homeomorphic to the 2-dimensional sphere S^2.

For instance, the surface of an Ellipsoid is such a 2-dimensional manifold.

On the other hand, the Torus is not Simply-Connected, and is not homeomorphic to the 2-dimensional sphere.

In 1904, Poincare conjectured [7], that a three-dimensional manifold Σ^3 that is Compact, Simply-Connected, and Borderless, may be homeomorphic to the three-dimensional sphere S^3.

A 3-dimensional manifold Σ^3 is a Topological space with

- Hausdorff separation. That is, distinct points have disjoint neighborhoods,
- Each point x, has a 3-dimensional open neighborhood U_x^3, homeomorphic to a 3-dimensional ball B_x^3 in the Euclidean space E^3.

The homeomorphism $h_x : U_x^3 \to B_x^3$ is a bicontinuous bijection that endows each point x with a local coordinate system. The set of all of the coordinate systems is called an Atlas for the manifold.

Denote the Poincare Manifold, by Σ^3, its topology by $\mathcal{T}$, and an open set in it by G.

We first see that the Poincare manifold has a Metric.

1. Poincare Manifold and its Metric

1.1 $\boxed{\text{The Poincare Manifold is Second Countable}}$

Proof: A Topological Space is Second Countable if its topology has a countable base. The 3-sphere

$$S^3 = \left\{ \left(x_1, x_2, x_3, x_4 \right) \in E^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \right\}$$

is Second Countable as a subspace of the closed 4-ball

$$B^4 = \left\{ \left(x_1, x_2, x_3, x_4 \right) \in E^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 1 \right\}.$$

A countable base for the 3-sphere consists of 3-dimensional balls with rational radius centered at points with rational components.

By [8], any compact manifold has a countable base.

Thus, Poincare Manifold has a countable base for its topology.□

1.2 $\boxed{\text{The Poincare Manifold has a Metric}}$

Proof: By Uryson's Metrization Theorem [3], if a topological space is Second Countable, Hausdorff, and Compact, it is Metrizable.

The semi-metric is a metric because of the Hausdorff separation, $x \neq y \Rightarrow d(x,y) > 0$. Thus, the Poincare Manifold is a metric space. □

2. Loops on the Poincare Manifold

A manifold is Simply-Connected <u>iff</u>

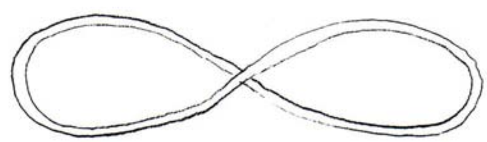
any loop in it is contractible to a point.

For instance, on the 2-dimensional sphere, any loop is contractible to a point.

On the other hand, the 2-dimensional Torus surface has loops that can not be contracted to a point.

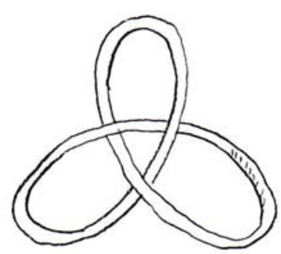
A non-self-intersecting curve homeomorphic to a loop, may not be deformed into a loop, and <u>we do not consider it a loop</u>.

For instance, a non-self-intersecting figure-eight is homeomorphic to a loop, but in the confinement of two dimensions, it cannot be deformed into a loop.



Three dimensions are required to twist, unknot, and deform the figure-eight into a loop.

Similarly, the following knot



is homeomorphic to a loop, but cannot be deformed into a loop in the confinement of two dimensions.

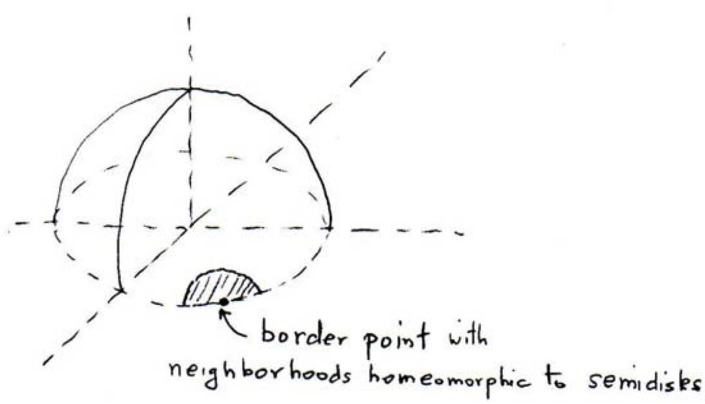
Four dimensions are required to twist, unknot, and deform it into a loop [6], [2].

3. Border of the Poincare Manifold

A 2-manifold is <u>borderless</u> iff it is

- Connected, and
- Its finite triangulation is such that each side of a triangle in the triangulation is glued to one and only one side of another triangle [1].

This guarantees no border points with neighborhoods homeomorphic to semi-disks, as it happens with points on the equator of a half-hemisphere.



The half-hemisphere has a non-closed triangulation, and the equator is its border.

In contrast, the 2-dimensional sphere has a closed triangulation, and is borderless.

Edwin Moise established in [5] that any 3-dimensional manifold has a finite triangulation.

Thus, a 3-manifold is borderless iff it is

- Connected, and
- Its finite triangulation is such that each face of a tetrahedron in the triangulation is glued to one and only one face of another tetrahedron.

4. Pencil Bundle Structure on the Poincare Manifold

We construct the Poincare Homeomorphism between the Poincare Manifold and the 3-sphere, as the union of homeomorphisms from disjoint 2-maniflds that partition the Poincare Manifold, onto respective disjoint 2-spheres that partition the 3-sphere.

We will show that each of the disjoint 2-manifolds is Compact, Simply-Connected, and Borderless.

Then, by the theorem for 2-manifolds, each 2-manifold is homeomorphic to its corresponding 2-sphere.

The Poincare Homeomorphism will be the union of these homeomorphisms.

4.1 Circle Bundle, and Pencil Bundle

The 2-dimensional sphere

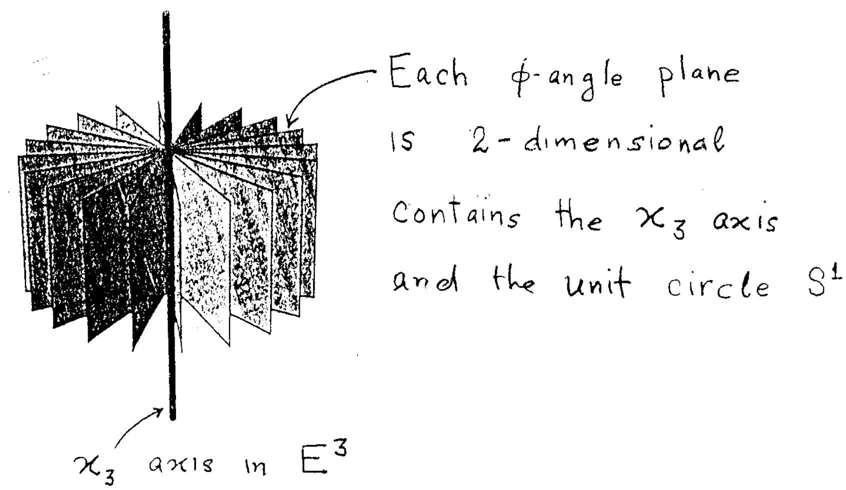
$$S^2 = \left\{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\right\}$$

is a <u>circle bundle</u> [10], a fiber bundle whose fibers are circles, over the 1-dimensional sphere

$$S^1 = \left\{(x_1, x_2) : x_1^2 + x_2^2 = 1\right\}.$$

Each circle in the bundle is parametrized by the angle $\phi \in [0, 2\pi]$.

As ϕ varies, the circle follows through a pencil [10] of 1-dimensional spheres S^1.



The figure depicts a <u>pencil</u> of 2-dimensional ϕ-angle planes.

Each is a Euclidean space E^2 that contains a 1-dimensional sphere S^1, and the x_3 axis.

The pencil is generated by turning the 1-sphere S^1 about the axis x_3 that lies on the sphere's diameter in the 3rd dimension.

4.2 $\boxed{\text{The Pencil Bundle Structure of } S^3 \text{ is induced on } \Sigma^3 \text{ and on } E^4}$

Proof: As a compact set in a metric space, the Poincare Manifold is bounded by a 3-dimensional sphere of radius a.

We'll assume that $a = 1$, and that the sphere is centered at the origin of the 4-dimensional space that contains it. That is, the sphere is S^3. The four coordinate axes are $x_1, x_2, x_3,$ and x_4.

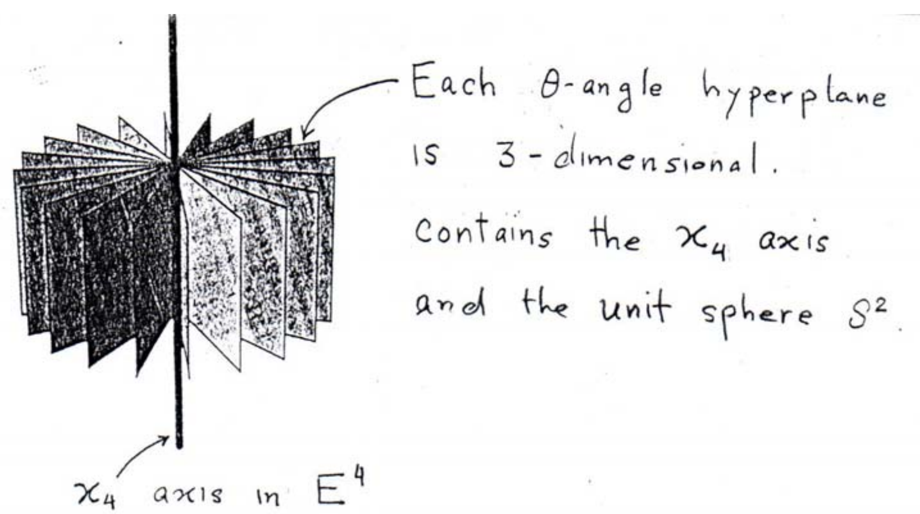
The 3-dimensional sphere S^3 is a circle bundle [10], over the 2-dimensional sphere S^2.

Each circle is parametrized by the angle $\theta \in [0, 2\pi]$.

As θ varies, the circle follows through a pencil [10] of 2-dimensional spheres S^2.

The figure depicts a <u>pencil</u> of 3-dimensional θ-angle hyperplanes.

Each is a Euclidean space E^3 that contains a 2-dimensional sphere S^2, and the x_4 axis.



The pencil is generated by turning the 2-sphere S^2 about the axis x_4 that lies on the sphere's diameter in the 4th dimension. Denote

$$S_\theta^2 = S^3 \bigcap \left(\textit{the } \theta\textit{-angle hyperplane that contains the } x_4\textit{-axis} \right).$$

Then, $\left\{ S_\theta^2 : \theta \in [0, 2\pi] \right\}$ is a partition of S^3 since

$$S_{\theta_1}^2 \bigcap S_{\theta_2}^2 = \varnothing \quad \text{for} \quad \theta_1 \neq \theta_2, \text{ and } S^3 = \bigcup_{\theta \in [0,2\pi]} S_\theta^2 .$$

Each S_θ^2 corresponds to

$$\Sigma_\theta^2 \equiv \Sigma^3 \bigcap \left(\textit{the } \theta\textit{-angle hyperplane that contains the } x_4\textit{-axis} \right),$$

and $\left\{ \Sigma_\theta^2 : \theta \in [0, 2\pi] \right\}$ is a partition of Σ^3 since

$$\Sigma_{\theta_1} \bigcap \Sigma_{\theta_2} = \varnothing \quad \text{for} \quad \theta_1 \neq \theta_2, \quad \text{and} \quad \Sigma^3 = \bigcup_{\theta \in [0,2\pi]} \Sigma_\theta^2 .$$

Each Σ_θ^2 is embedded in the 3-dimensional Euclidean space

$$E_\theta^3 \equiv E^4 \bigcap \left(\textit{the } \theta\textit{-angle hyperplane that contains the } x_4\textit{-axis} \right),$$

and $\left\{ E_\theta^3 : \theta \in [0, 2\pi] \right\}$ is a partition of E^4 since

$$E_{\theta_1}^3 \bigcap E_{\theta_2}^3 = \varnothing \quad \text{for} \quad \theta_1 \neq \theta_2, \quad \text{and} \quad E^4 = \bigcup_{\theta \in [0,2\pi]} E_\theta^3 . \square$$

4.3 $\boxed{\begin{array}{r} \mathcal{T}_\theta \equiv \mathcal{T} \cap \left(\textit{the } \theta\textit{-angle hyperplane} \right) \text{ is a topology on } \Sigma_\theta^2 \\ \text{with open sets } \mathrm{G}_\theta \equiv G \cap \left(\textit{the } \theta\textit{-angle hyperplane} \right) \end{array}}$

Proof: $\mathcal{T}_\theta$ is the relative topology of $\mathcal{T}$ on Σ_θ^2. $\square$

4.4 $\boxed{\text{Each } \Sigma_\theta^2 \text{ is Hausdorff}}$

Proof: Every subspace of Hausdorff is Hausdorff. □

4.5 $\boxed{\text{Each } \Sigma_\theta^2 \text{ is a 2-manifold with the topology } \mathcal{T}_\theta}$

Proof: Each point $x \in \Sigma_\theta^2$, has a 2-dimensional open neighborhood $U_x^2 = U_x^3 \bigcap \left(\textit{the } \theta\textit{-angle hyperplane}\right)$, homeomorphic to a 2-dimensional disk $D_x^2 = B_x^3 \bigcap \left(\textit{the } \theta\textit{-angle hyperplane}\right)$.□

4.6 $\boxed{\text{Each } \Sigma_\theta^2 \text{ is Compact}}$

Proof: Σ_θ^2 is compact, as a subset of the compact Σ^3.□

4.7 $\boxed{\text{Each } \Sigma_\theta^2 \text{ is Connected}}$

Proof: If Σ_θ^2 is disconnected, then Σ^3 is disconnected.□

4.8 $\boxed{\text{Each } \Sigma_\theta^2 \text{ has a Closed Finite Triangulation}}$

Proof: Σ_θ^2 inherits a finite triangulation from Σ^3.

Each side of a triangle in the induced triangulation on Σ_θ^2, is on the face of some tetrahedron from the triangulation of Σ^3.

The tetrahedron shares its face with one and only one other tetrahedron.

Consequently, the triangle side is glued to one and only one side of another triangle, and the triangulation of Σ_{θ}^{2} is closed.□

4.9 $\boxed{\text{Each } \Sigma_{\theta}^{2} \text{ is Borderless}}$

Proof: By 4.7 and 4.8, each Σ_{θ}^{2} is connected, and has a closed finite triangulation. Therefore, Σ_{θ}^{2} has no boundary.□

4.10 $\boxed{\text{Each } \Sigma_{\theta}^{2} \text{ is Simply-Connected}}$

Proof: If $\Sigma_{\theta_0}^{2}$ is not simply-connected, there is a loop $\ell_{\theta_0}^{2}$ in $\Sigma_{\theta_0}^{2}$ that is prevented from contracting by a 2-disk $D_{\theta_0}^{2}$ that does not belong to $\Sigma_{\theta_0}^{2}$.

But $\Sigma_{\theta_0}^{2}$ has no boundary, and it cannot contain the disk boundary.

Therefore, the loop $\ell_{\theta_0}^{2}$ must be slipping along the surface of a 2-Torus $T_{\theta_0}^{2}$.

The Torus surface has no boundary and can be part of $\Sigma^2_{\theta_0}$, but the interior of the Torus does not belong to $\Sigma^2_{\theta_0}$.

The interior lies in the 3-dimensional Euclidean space $E^3_{\theta_0}$ that contains $\Sigma^2_{\theta_0}$.

Since the E^3_θ are disjoint from each other, the non-contracting loop $\ell^2_{\theta_0}$, and the Torus $T^2_{\theta_0}$ are embedded only in $E^3_{\theta_0}$, and in none of the other E^3_θ.

Therefore, $\ell^2_{\theta_0}$, does not contract in any other E^3_θ, and in none of the Σ^2_θ. Hence, $\ell^2_{\theta_0}$ does not contract in $\bigcup_{\theta\in[0,2\pi]} (\Sigma^2_\theta, \theta) = \Sigma^3$.

Consequently, the Poincare Manifold has a loop that is not contractible, and Σ^3 is not Simply-Connected.

This says that if Σ^3 is Simply-Connected, each Σ^2_θ is simply-connected.$\square$

4.11 $\boxed{\text{Each } \Sigma_\theta^2 \text{ is homeomorphic to } S_\theta^2 \text{ with } f_\theta : \Sigma_\theta^2 \to S_\theta^2}$

Proof: By 4.5, 4.6, 4.9, and 4.10, each Σ_θ^2 is a 2-dimensional manifold, which is Compact, Borderless, and Simply-Connected.

By the theorem for 2-dimensional manifolds [4], such manifold is homeomorphic to the 2-dimensional sphere.□

5. Constructing the Poincare Homeomorphism

We proceed to construct the Poincare Homeomorphism as the union of the homeomorphisms f_θ.

To that end, we will apply a theorem from [9].

5.1 ❶ *$\mathcal{S}$ is a subbase for a topology $\mathcal{T}$ on X*

❷ *$\widetilde{\mathcal{S}}$ is a subbase for a topology $\widetilde{\mathcal{T}}$ on $\widetilde{X}$*

❸ *$f : X \to \widetilde{X}$ is*

① *one-one function which induces a*

② *one-one correspondence between the elements of*

$\mathcal{S}$, and the elements of $\widetilde{\mathcal{S}}$

Then, f is a homeomorphism between X, and $\widetilde{X}$

Let τ be a topology on S^1, with open sets Θ.

We proceed to write $\mathcal{T}$ in terms of $\mathcal{T}_\theta$.

5.2 $\boxed{\mathcal{T} = \bigcup_{\theta\in[0,2\pi]} \mathcal{T}_\theta \quad \text{has open sets } G = \bigcup_{\theta\in\Theta} G_\theta}$

Proof:

$(\supseteq)$ For each $\theta \in [0,2\pi]$, $\mathcal{T} \supseteq \mathcal{T}_\theta$.□

$(\subseteq)$ If $x \in G \in \mathcal{T}$, then x belongs to some θ-angle hyperplane for some $\theta \in [0,2\pi]$. Hence, $x \in G\bigcap\big(\textit{the } \theta\textit{-angle hyperplane}\big) = G_\theta \in \mathcal{T}_\theta$ by 4.3. □

5.3 *For each* θ, $f_\theta \mathcal{T}_\theta$ *is a topology on* S_θ^2, *with open sets* $f_\theta G_\theta$

Proof: For each $\theta \in [0,2\pi]$, f_θ is homeomorphism.□

5.4 $\boxed{\widetilde{\mathcal{T}} = \bigcup_{\theta\in[0,2\pi]} f_\theta \mathcal{T}_\theta \text{ is a topology on } S^3 \text{ with open sets } \bigcup_{\theta\in\Theta\in\tau} f_\theta G_\theta}$.

Proof:

Empty set $\quad \varnothing \in \mathcal{T}_\theta \Rightarrow \varnothing \in f_\theta \mathcal{T}_\theta \Rightarrow \varnothing \in \bigcup_{\theta\in[0,2\pi]} f_\theta \mathcal{T}_\theta = \widetilde{\mathcal{T}}$.□

The Space $\quad \Sigma_\theta^2 \in \mathcal{T}_\theta \Rightarrow \quad S^3 = \bigcup_{\theta\in[0,2\pi]} (S_\theta^2, \theta) \in \bigcup_{\theta\in[0,2\pi]} f_\theta \mathcal{T}_\theta = \widetilde{\mathcal{T}}$.□

Infinite Union

For any $i \in I$, let $\tilde{G}^i \in \widetilde{\mathcal{T}}$. We'll show that $\bigcup_{i\in I} \tilde{G}^i \in \widetilde{\mathcal{T}}$.

For each $i \in I$, there is $\Theta^i \in \tau$, so that for all $\theta^i \in \Theta^i$, there are $G_{\theta^i}^i \in \mathcal{T}_{\theta^i}$ so that $\tilde{G}^i = \bigcup_{\theta^i\in\Theta^i} f_{\theta^i} G_{\theta^i}^i$. Hence,

$$\bigcup_{i \in I} \tilde{G}^i = \bigcup_{i \in I} \bigcup_{\theta^i \in \Theta^i} f_{\theta^i} G^i_{\theta^i} .$$

Taking the union first over all $i \in I$, then over all $\theta \in \bigcup_{i \in I} \Theta^i$,

$$\bigcup_{i \in I} \tilde{G}^i = \bigcup_{\theta \in \bigcup_{i \in I} \Theta^i} \bigcup_{i \in I} f_\theta G^i_\theta$$

Since $\bigcup_{i \in I} \Theta^i \in \tau$, and $\bigcup_{i \in I} f_\theta G^i_\theta \in f_\theta \mathcal{T}_\theta$, we have $\bigcup_{i \in I} \tilde{G}^i \in \widetilde{\mathcal{T}}$.□

Finite intersection

Let $\tilde{G}^1, \tilde{G}^2 \in \tilde{\mathcal{T}}$. We'll show that $\tilde{G}^1 \bigcap \tilde{G}^2 \in \tilde{\mathcal{T}}$.

Since $\tilde{G}^1, \tilde{G}^2 \in \tilde{\mathcal{T}}$, there are $\Theta^1, \Theta^2 \in \tau$ so that for all $\theta^1 \in \Theta^1$, and for all $\theta^2 \in \Theta^2$, there are $G^1_{\theta^1} \in \mathcal{T}_{\theta^1}$, $G^2_{\theta^2} \in \mathcal{T}_{\theta^2}$ so that

$$\tilde{G}^1 = \bigcup_{\theta^1 \in \Theta^1} f_{\theta^1} G^1_{\theta^1}, \text{ and } \tilde{G}^2 = \bigcup_{\theta^2 \in \Theta^2} f_{\theta^2} G^2_{\theta^2} .$$

Hence,

$$\tilde{G}^1 \bigcap \tilde{G}^2 = \bigcup_{\theta^1 \in \Theta^1} f_{\theta^1} G^1_{\theta^1} \bigcap \bigcup_{\theta^2 \in \Theta^2} f_{\theta^2} G^2_{\theta^2}$$

$$= \bigcup_{\theta \in \Theta^1 \bigcap \Theta^2} f_\theta G^1_\theta \bigcap f_\theta G^2_\theta$$

Since $\Theta^1 \bigcap \Theta^2 \in \tau$, and $f_\theta G^1_\theta \bigcap f_\theta G^2_\theta \in f_\theta \mathcal{T}_\theta$, we have $\tilde{G}^1 \bigcap \tilde{G}^2 \in \widetilde{\mathcal{T}}$.□

5.5 $\boxed{\mathcal{T} \textit{ is } \text{Sup Topology on } \Sigma^3, \text{ and } \{\mathcal{T}_\theta : \theta \in [0, 2\pi]\} \text{ partitions } \mathcal{T}}$

Proof: $\mathcal{T}$ is the smallest topology on Σ^3 which is larger than each $\mathcal{T}_\theta$. Such topology is called sup topology [11].

In [11], each of the topologies is defined on the same space X, and contains it. Here, for each θ, $\mathcal{T}_\theta \subseteq \Sigma_\theta^2 \subseteq E_\theta^3$, and all the $\mathcal{T}_\theta$ are disjoint from each other. $\left\{\mathcal{T}_\theta : \theta \in [0,2\pi]\right\}$ partitions $\mathcal{T}$ since

$\mathcal{T}_{\theta_1} \bigcap \mathcal{T}_{\theta_2} = \varnothing$ for $\theta_1 \neq \theta_2$, and $\mathcal{T} = \bigcup_{\theta\in[0,2\pi]} \mathcal{T}_\theta$. $\square$

5.6 $\boxed{\widetilde{\mathcal{T}} \text{ is Sup Topology on } S^3\text{, and } \left\{f_\theta \mathcal{T}_\theta : \theta \in [0,2\pi]\right\} \text{ partitions } \widetilde{\mathcal{T}}}$

Proof: $f_{\theta_1}\mathcal{T}_{\theta_1} \bigcap f_{\theta_2}\mathcal{T}_{\theta_2} = \varnothing$ for $\theta_1 \neq \theta_2$, and $\widetilde{\mathcal{T}} = \bigcup_{\theta\in[0,2\pi]} f_\theta \mathcal{T}_\theta$. $\square$

5.7 *For each θ, let $\mathcal{S}_\theta$ be a Subbase for $\mathcal{T}_\theta$ on Σ_θ^2, with sets H_θ.*

Then, $\boxed{\bigcup_{\theta\in[0,2\pi]} \mathcal{S}_\theta \text{ is a subbase for } \mathcal{T}, \textit{with} \text{ subbase sets } \bigcup_{\theta\in\Theta\in\tau} H_\theta}$

Proof: If $G \in \mathcal{T}$, there is $\Theta \in \tau$, and there are $G_\theta \in \mathcal{T}_\theta$, so that

$$G = \bigcup_{\theta\in\Theta} G_\theta .$$

For each G_θ, there are base sets B_θ^j, $j \in J$, so that

$$G_\theta = \bigcup_{j\in J} B_\theta^j$$

For each B_θ^j, there are subbase sets $H_\theta^{j,k} \in \mathcal{S}_\theta$, $k = 1...n$ so that

$$B_\theta^j = \bigcap_{k=1}^{n} H_\theta^{j,k}$$

Therefore,

$$G = \bigcup_{\theta\in\Theta} \bigcup_{j\in J} \bigcap_{k=1}^{n} H_\theta^{j,k}$$

$$= \bigcup_{j\in J} \bigcap_{k=1}^{n} \bigcup_{\theta\in\Theta} H_\theta^{j,k}.$$

Thus, $\bigcup_{\theta\in[0,2\pi]} \mathcal{S}_\theta$ is a subbase for $\mathcal{T}$, with subbase sets $\bigcup_{\theta\in\Theta\in\tau} H_\theta$.□

5.8 *For each* θ, $f_\theta\mathcal{S}_\theta$ *is a subbase for* $f_\theta\mathcal{T}_\theta$ *with subbase sets* $f_\theta H_\theta$

Proof: For each $\theta \in [0,2\pi]$, f_θ is homeomorphism.□

5.9 $\boxed{\bigcup_{\theta\in[0,2\pi]} f_\theta\mathcal{S}_\theta \textit{ is a subbase for } \widetilde{\mathcal{T}} \text{ on } S^3 \textit{ with subbase sets } \bigcup_{\theta\in\Theta\in\tau} f_\theta H_\theta}$

Proof: If $\tilde{G} \in \tilde{\mathcal{T}}$, there is $\Theta \in \tau$, and there are $f_\theta G_\theta \in f_\theta\mathcal{T}_\theta$, so that

$$\tilde{G} = \bigcup_{\theta\in\Theta} f_\theta G_\theta .$$

For each G_θ, there are base sets B_θ^j , $j \in J$, so that

$$f_\theta G_\theta = \bigcup_{j\in J} f_\theta B_\theta^j$$

For each B_θ^j, there are subbase sets $H_\theta^{j,k} \in \mathcal{S}_\theta$, $k = 1...n$ so that

$$f_\theta B_\theta^j = \bigcap_{k=1}^{n} f_\theta H_\theta^{j,k}$$

Therefore,

$$\tilde{G} = \bigcup_{\theta\in\Theta}\bigcup_{j\in J}\bigcap_{k=1}^{n} f_\theta H_\theta^{j,k}$$

$$= \bigcup_{j\in J}\bigcap_{k=1}^{n}\bigcup_{\theta\in\Theta} f_\theta H_\theta^{j,k}\,.$$

Thus, $\bigcup\limits_{\theta\in[0,2\pi]} f_\theta \mathcal{S}_\theta$ is a subbase for $\widetilde{\mathcal{T}}$, with subbase sets $\bigcup\limits_{\theta\in\Theta\in\mathcal{T}} f_\theta H_\theta$.□

To obtain the Poincare Homeomorphism by 5.1, we need to define a function $f:\Sigma^3 \to S^3$ which is

① one-one

② one-one correspondence between a subbase set for $\mathcal{T}$,

$\bigcup\limits_{\theta\in\Omega} H_\theta$, and a subbase set for $\widetilde{\mathcal{T}}$, $\bigcup\limits_{\theta\in\Omega} f_\theta H_\theta$.

5.10 Definition of f

For $x\in\Sigma^3$, there are unique $\theta\in[0,2\pi]$, and $x_\theta\in\Sigma_\theta^2$ so that

$$x = (x_\theta,\theta)\,.$$

We define the mapping f from Σ^3 onto S^3 by $\boxed{f(x_\theta,\theta) = (f_\theta x_\theta,\theta)}$.

5.11 $\boxed{f \text{ is one-one}}$

Proof: Let

$$x = (x_\theta, \theta), \quad \text{and} \quad y = (y_\phi, \phi).$$

Assume that

$$(f_\theta x_\theta, \theta) = (f_\phi y_\phi, \phi).$$

Since Σ_θ^2 are disjoint,

$$\theta = \phi, \quad \text{and} \quad f_\theta x_\theta = f_\theta y_\theta.$$

Since f_θ is a homeomorphism,

$$x_\theta = y_\theta.$$

Hence,

$$x = y.\square$$

5.12 $\boxed{\begin{array}{l} f \text{ is one-one between the subbase set } \bigcup_{\theta \in \Theta} H_\theta \text{ in } S^3 \\ \text{and the subbase set } \bigcup_{\theta \in \Theta} f_\theta H_\theta \text{ in } \Sigma^3 \end{array}}$

Proof: Let $H = \bigcup_{\theta \in \Theta} H_\theta$, and $K = \bigcup_{\phi \in \Phi} K_\phi$.

Assume that,

$$\left\{(f_\theta H_\theta, \theta) : \theta \in \Theta, H_\theta \subseteq H\right\} = \left\{(f_\phi K_\phi, \phi) : \phi \in \Phi, K_\phi \subseteq K\right\}.$$

Since the Σ_θ^2 are disjoint,

$$\Theta = \Phi, \quad \text{and} \quad f_\theta H_\theta = f_\theta K_\theta, \text{ for any } \theta \in \Theta.$$

Since f_θ is a homeomorphism, for each $\theta \in \Theta$,

$$H_\theta = K_\theta.$$

Hence, $H = K.\square$

5.13 $\boxed{f \text{ is the Poincare Homeomorphism}}$

Proof: By 5.1, 5.2, 5.4, 5.7, 5.9, 5.11, and 5.12, f is a homeomorphism. It is the Poincare Homeomorphism from Σ^3 onto S^3 .□

5.14 The Poincare Homeomorphism restriction to Σ_θ^2 is f_θ.

$$f\Big|_{\Sigma_\theta^2} = f_\theta .$$

5.15 The Poincare Homeomorphism is the union of all the f_θ.

$$f = \left\{ f_\theta : \theta \in [0,2\pi] \right\} = \bigcup_{\theta \in [0,2\pi]} f_\theta .$$

References

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