

# Extension of Lorentz Group Representations for Chiral Fermions

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## Abstract

We derive a formulation of the Naimark extension for Dirac spinors. Non-commuting rotation (spin) and boost generators are extended by first grouping operators into left and right handed pairs and then defining ancillary spin-1/2 vacuum meters for the three space dimensions. The result is an explicit example of a recently-proposed theory, in which the extension of the Lorentz group to commuting operators is an underlying structure for three generations of elementary fermions. We suggest that the extension appears at the scale of the weak vector bosons, through the Cabibbo-Kobayashi-Maskawa matrix in the  $W^\pm$  couplings and  $SU(2)_L \times U(1)$ -generated left-right mixing in the  $Z^0$  coupling.

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## INTRODUCTION

The principles of quantum measurement are at the foundation of particle physics. For example, particle spin and momentum assignments are determined by quantum representations of the Lorentz group [1], and quantum electrodynamics as a local  $U(1)$  gauge theory emerges naturally from the phase invariance of quantum observables. However, this close association of particles and quantum measurement appears unconnected in the broader standard model. In particular, the existence of three fermionic generations and non-Abelian gauge theory interactions are independent of quantum representation theory. The specific generational and isospin structures in the standard model are seemingly not constrained by quantum theoretical foundations. In this paper we describe the particle states and decay patterns that arise at the scale of the weak interaction when explicit connections are made between the standard model and the theories of quantum representation and measurement.

In particular, we show that quantum representation theory can be generalized to explain fermionic generations from the requirement that the Lorentz group operators are represented on distinguishable fermionic states. Quark and lepton fermionic generations create representations of the Lorentz group that extend standard particle quantum number assignments. Entanglements of particle quantum states are integral to this extended representation and involve the weak interaction bosons  $\{Z^0, W^\pm\}$ . The  $Z^0$  boson entangles left and right handed fermions within each generation, and the  $W^\pm$  bosons provide entanglements between generations. The  $Z^0$  entanglement is directly through the  $SU(2)_L \times U(1)$  fermionic current, whereas the  $W^\pm$  entanglement mechanism is more subtle. The latter is through the Cabibbo-Kobayashi-Maskawa (CKM) matrix: right handed quark spectators, in a density matrix representation of left handed  $W^\pm$ -mediated decays, lead naturally to CKM quark mixings.

## NAIMARK EXTENSIONS

Extended quantum representations and entanglements require augmenting observable operators through the introduction of vacuum ‘meter’ states, in a procedure known as Naimark extension [2–4]. The basic idea of the extension, applied to a coupled harmonic oscillator, has been first introduced by Arthurs and Kelly [5]. Non-commuting system momentum  $p$

and position  $q$  operators are extended to commuting operators,

$$\begin{aligned}\theta_1 &= p + P \\ \theta_2 &= q - Q,\end{aligned}$$

where  $P$  and  $Q$  are the momentum and position operators of an entirely independent ‘meter’ harmonic oscillator. The meter harmonic oscillator is in the vacuum state  $|0\rangle$  with vanishing expectation values,  $\langle 0|P|0\rangle$  and  $\langle 0|Q|0\rangle$ . The combined, system and meter, state is given by  $|\psi\rangle|0\rangle$ ; and, the fact that  $[\theta_1, \theta_2]$  is zero implies that the expectations,  $\langle \psi|p|\psi\rangle$  and  $\langle \psi|q|\psi\rangle$ , can be obtained simultaneously in this scheme. From the standpoint of the original  $(p, q)$  system, the Naimark extension provides a realization of the phase space picture of quantum mechanics, which can be viewed as equivalent to the Schrödinger or Heisenberg pictures [6]. Theoretical constructions—such as coherent, squeezed coherent, and Bloch states—are more naturally described in the phase space picture [7]. The experimental implementation of squeezed coherent states requires the ancillary Hilbert spaces of a Naimark extension [8, 9].

Based on this more complete description of a particle state, Levine and Dannon [2] have argued that the above extension represents the correct definition of system position and momentum measurement. Examples of simultaneous measurement for non-relativistic position and momentum, as well as spin, have been derived in earlier works by Levine and Tucci [10, 11]. Here we explicitly derive the Naimark extension for Dirac spinors. We show that the extension of spin and boost operators for spin-1/2 particles falls naturally into three generations of left and right handed fermions, the starting point of the standard model for the weak interaction [12, 13]. In addition, we suggest that left-right mixing (revealed in  $Z^0$  couplings) and CKM inter-generational mixing (revealed in  $W^\pm$  couplings) are together the phenomenology of a Lorentz group Naimark extension. The two-quark states used to derive CKM mixings are motivated by constructions in quantum information theory.

## DIRAC SPINORS AND THE LORENTZ GROUP

Consider relativistic fermions described as Dirac 4-spinors. The system is uniquely prescribed by non-commuting spin operators (notation as in Ref. [13]),

$$\Sigma_i = \begin{bmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{bmatrix}, \quad i = 1, 2, 3, \quad (1)$$

[**and**] the generators of infinitesimal spin rotations and boosts[.]

$$S_i(\beta) = \begin{bmatrix} a_+ & a_- \sigma_i \\ a_- \sigma_i & a_+ \end{bmatrix}, \quad i = 1, 2, 3, \quad (2)$$

where  $a_{\pm} = \sqrt{\frac{1}{2}(\gamma \pm 1)}$  with  $\gamma = 1/\sqrt{1 - \beta^2}$  and  $\beta = v/c$ . Infinitesimal boost generators are defined in the limit  $\beta \ll 1$ . To first order in  $\beta$ , we have

$$S_i(\beta) = 1 - \frac{1}{2}\beta \gamma_5 \Sigma_i, \quad (3)$$

where

$$\gamma_5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad (4)$$

The matrices  $\gamma_5 \Sigma_i$  form a boost generator set,

$$\{\gamma_5 \Sigma_1, \gamma_5 \Sigma_2, \gamma_5 \Sigma_3\}.$$

The total set of operators,

$$\{\Sigma_1, \Sigma_2, \Sigma_3, \gamma_5 \Sigma_1, \gamma_5 \Sigma_2, \gamma_5 \Sigma_3\},$$

generate spin rotations and boosts that uniquely prescribe the fermion state. The condition,

$$[\gamma_5, \Sigma_i] = 0,$$

yields commutation relations for the full operator set—given by

$$[\Sigma_i, \Sigma_j] = 2i\epsilon_{ijk}\Sigma_k, \quad (5)$$

$$[\Sigma_i, \gamma_5 \Sigma_j] = 2i\epsilon_{ijk}\gamma_5 \Sigma_k, \quad (6)$$

and

$$[\gamma_5 \Sigma_i, \gamma_5 \Sigma_j] = 2i\epsilon_{ijk}\Sigma_k, \quad (7)$$

for  $i, j, k \in \{1, 2, 3\}$ . The operators and commutation relations in Eqs. (5)–(7) define the classic spin-1/2 Lorentz group.

## NAIMARK EXTENSION OF DIRAC SPINORS

The Naimark extension of Eqs. (5)–(7) is constructed by first defining mutually commuting pairs,

$$\left\{ \frac{1}{2}(1 + \gamma_5)\Sigma_i, \frac{1}{2}(1 - \gamma_5)\Sigma_i \right\}, \quad i = 1, 2, 3, \quad (8)$$

of separate left and right handed operators, and then carrying operators with different  $i$ -values on different independent fermions. In Refs. [10] and [11], for non-relativistic position/momentum and spin, a distinction is made between quantum measurements entirely on vacuum meters and measurements in which the original system participates. An example of the latter case that allows simultaneity for relativistic spin and momentum reduces to the problem of finding entangled Hilbert spaces in which non-commuting observables reside on distinguishable quarks. The spin and momentum properties of a single quark require two other vacuum quarks to act as the meters. The final Naimark extension is given by

$$\begin{bmatrix} \frac{1}{2}(1 + \gamma_5)\Sigma_1 & \frac{1}{2}(1 + \gamma_5)\bar{\Sigma}_2 & \frac{1}{2}(1 + \gamma_5)\bar{\bar{\Sigma}}_3 \\ \frac{1}{2}(1 - \gamma_5)\Sigma_1 & \frac{1}{2}(1 - \gamma_5)\bar{\Sigma}_2 & \frac{1}{2}(1 - \gamma_5)\bar{\bar{\Sigma}}_3 \end{bmatrix}, \quad (9)$$

where each component  $i = 1, 2, 3$  is represented in a different fermionic generation. Here, the different generations are denoted by a different number of operator overbars. Note that the expectation values of

$$\frac{1}{2}(1 \pm \gamma_5)\Sigma_i = \frac{1}{2}(1 \pm \gamma_5)\Sigma_i \frac{1}{2}(1 \pm \gamma_5) \quad (10)$$

for  $i = 1, 2, 3$  are equivalent to a two-step process in which the state  $|\psi\rangle$  is projected onto left ( $L$ ) or right ( $R$ ) components,

$$\begin{aligned} |\psi\rangle_L &= \frac{1}{2}(1 - \gamma_5)|\psi\rangle \\ |\psi\rangle_R &= \frac{1}{2}(1 + \gamma_5)|\psi\rangle, \end{aligned}$$

and the expectation value of the spin operator follows.

## FERMIONIC REPRESENTATIONS OF LORENTZ NAIMARK EXTENSION AND ENTANGLEMENTS DUE TO VECTOR BOSONS

The pattern in Eq. (9), a Naimark extension of the Lorentz group of spin and boost operators, suggests an underlying representational structure for known massive fermions

given by

$$\begin{bmatrix} e_R & \mu_R & \tau_R \\ e_L & \mu_L & \tau_L \end{bmatrix}, \quad (11)$$

$$\begin{bmatrix} u_R & c_R & t_R \\ u_L & c_L & t_L \end{bmatrix}, \quad (12)$$

and

$$\begin{bmatrix} d_R & s_R & b_R \\ d_L & s_L & b_L \end{bmatrix}, \quad (13)$$

with a similar pattern for left handed (massless) neutrinos.

In addition to mutual commutativity via ancillary Hilbert spaces, observable simultaneity requires entanglement with vacuum meters. The mixings among the patterns in Eqs. (11)–(13) are an indication that this extended quantum representation appears at the time-space scale of the weak interaction. As described in Ref. [14], entanglement for quarks arises from the diagonalization of the couplings to Higgs particles expressed as rotations,

$$\vec{f}'_R = W_{u(d)} \cdot \vec{f}_R \quad (14)$$

and

$$\vec{f}'_L = U_{u(d)} \cdot \vec{f}_L, \quad (15)$$

where  $u(d)$  corresponds to up(down) quarks and

$$\vec{f} = (f^1, f^2, f^3)^T$$

corresponds to triplets  $(u, c, t)$  and  $(d, s, b)$ . The CKM matrix,  $U_u^\dagger U_d$ , is the only observable mixing across generations in the standard model—as revealed in  $W^\pm$ -mediated left handed processes.

This inter-generational mixing is not observable in neutral currents coupled to  $Z^0$ . However, in the standard model hypercharge provides a mixing of left and right handed fermions that is observable in  $Z^0$  coupling—but not observable in the left-isospin coupled  $W^\pm$  bosons. The  $Z^0$  boson couples to the neutral fermionic current,

$$j_0^\mu = \bar{f} \gamma^\mu (C_v - C_a \gamma^5) f, \quad (16)$$

where  $C_v$  and  $C_a$  are dependent on the fermion type, and  $C_v$  further depends on the weak angle  $\theta_w \approx 28.7^\circ$  (Note that, for left handed neutrinos, there is no left-right mixing via  $C_v$

and  $C_a$ ). The expression in Eq. (16) can be written as a pure vector current  $\bar{F}\gamma^\mu F$ , where

$$F = \eta_1 f_L + \eta_2 f_R \quad (17)$$

with

$$\begin{aligned} \eta_1 &= \frac{1}{X} \sqrt{C_v + C_a}, \\ \eta_2 &= \frac{1}{X} \sqrt{C_v - C_a}, \end{aligned}$$

and

$$X = \sqrt{|C_v + C_a|^2 + |C_v - C_a|^2}.$$

Equation (17) demonstrates that  $Z^0$  coupling involves an entanglement between left and right handed quarks. Accordingly, without  $Z^0$  interaction, right handed quarks exist as spectators to  $W^\pm$ -mediated isospin-changing transitions in a left handed quark Hilbert space. This condition leads to the observed Cabibbo-like entanglement. For example, considering only the first two generations of up-type (up, charm) and down-type (down, strange) quarks, a two-quark input state to  $W^\pm$  and  $Z^0$  decays is given by

$$\begin{aligned} |\psi\rangle &= \beta_d |d_L d'_R 0\rangle + \beta_s |s_L s'_R 0\rangle \\ &+ \beta_u |u_L u'_R 0\rangle + \beta_c |c_L c'_R 0\rangle, \end{aligned} \quad (18)$$

where  $\beta_d^2 + \beta_s^2 + \beta_u^2 + \beta_c^2 = 1$ , and  $|0\rangle$  represents the vacuum in a boson Hilbert space  $E$  given by  $\{|Z^0\rangle, |W^+\rangle, |W^-\rangle, |0\rangle\}$ . If we assume that the  $W^\pm$  decays are described by a unitary operator  $V$ , then the output state is given by

$$\begin{aligned} |\psi'\rangle &= V|\psi\rangle \\ &= \beta_d \langle d_L d'_R 0 | V | d_L d'_R 0 \rangle |d_L d'_R 0\rangle + \beta_d \langle d_L d'_R Z^0 | V | d_L d'_R 0 \rangle |d_L d'_R Z^0\rangle \\ &+ \beta_s \langle s_L s'_R 0 | V | s_L s'_R 0 \rangle |s_L s'_R 0\rangle + \beta_s \langle s_L s'_R Z^0 | V | s_L s'_R 0 \rangle |s_L s'_R Z^0\rangle \\ &+ \beta_u \langle u_L u'_R 0 | V | u_L u'_R 0 \rangle |u_L u'_R 0\rangle + \beta_u \langle u_L u'_R Z^0 | V | u_L u'_R 0 \rangle |u_L u'_R Z^0\rangle \\ &+ \beta_c \langle c_L c'_R 0 | V | c_L c'_R 0 \rangle |c_L c'_R 0\rangle + \beta_c \langle c_L c'_R Z^0 | V | c_L c'_R 0 \rangle |c_L c'_R Z^0\rangle \\ &+ \beta_d (\langle u_L W^- | V | d_L 0 \rangle |u_L\rangle + \langle c_L W^- | V | d_L 0 \rangle |c_L\rangle) |d'_R W^-\rangle \\ &+ \beta_s (\langle u_L W^- | V | s_L 0 \rangle |u_L\rangle + \langle c_L W^- | V | s_L 0 \rangle |c_L\rangle) |s'_R W^-\rangle \\ &+ \beta_u (\langle d_L W^+ | V | u_L 0 \rangle |d_L\rangle + \langle s_L W^+ | V | u_L 0 \rangle |s_L\rangle) |u'_R W^+\rangle \\ &+ \beta_c (\langle d_L W^+ | V | c_L 0 \rangle |d_L\rangle + \langle s_L W^+ | V | c_L 0 \rangle |s_L\rangle) |c'_R W^+\rangle, \end{aligned}$$

where the  $Z^0$  boson couples to either  $q_L$  or  $q'_R$  and the  $W^\pm$  couples only to  $q_L$ . For notational clarity we have dropped the  $q'_R$  dependence in  $W^\pm$ -mediated decays—i.e.,  $\langle q_L q'_R W^\pm | V | \tilde{q}_L q'_R 0 \rangle \longrightarrow \langle q_L W^\pm | V | \tilde{q}_L 0 \rangle$ . Unitarity of  $V$  implies the following conditions:

$$\begin{aligned} & |\langle u_L W^- | V | d_L 0 \rangle|^2 + |\langle c_L W^- | V | d_L 0 \rangle|^2 + |\langle d_L d'_R Z^0 | V | d_L d'_R 0 \rangle|^2 + |\langle d_L d'_R 0 | V | d_L d'_R 0 \rangle|^2 = \\ & |\langle u_L W^- | V | s_L 0 \rangle|^2 + |\langle c_L W^- | V | s_L 0 \rangle|^2 + |\langle s_L s'_R Z^0 | V | s_L s'_R 0 \rangle|^2 + |\langle s_L s'_R 0 | V | s_L s'_R 0 \rangle|^2 = \\ & |\langle d_L W^+ | V | u_L 0 \rangle|^2 + |\langle s_L W^+ | V | u_L 0 \rangle|^2 + |\langle u_L u'_R Z^0 | V | u_L u'_R 0 \rangle|^2 + |\langle u_L u'_R 0 | V | u_L u'_R 0 \rangle|^2 = \\ & |\langle d_L W^+ | V | c_L 0 \rangle|^2 + |\langle s_L W^+ | V | c_L 0 \rangle|^2 + |\langle c_L c'_R Z^0 | V | c_L c'_R 0 \rangle|^2 + |\langle c_L c'_R 0 | V | c_L c'_R 0 \rangle|^2 = \end{aligned} \quad 1.0, \quad (19)$$

$$\langle u_L W^- | V | d_L 0 \rangle^* \langle u_L W^- | V | s_L 0 \rangle + \langle c_L W^- | V | d_L 0 \rangle^* \langle c_L W^- | V | s_L 0 \rangle = 0, \quad (20)$$

and

$$\langle d_L W^+ | V | u_L 0 \rangle^* \langle d_L W^+ | V | c_L 0 \rangle + \langle s_L W^+ | V | u_L 0 \rangle^* \langle s_L W^+ | V | c_L 0 \rangle = 0. \quad (21)$$

The output density matrix  $\rho'$  for the left handed system is obtained by taking the trace of the right handed quark and vector boson Hilbert spaces,  $R$  and  $E$ , respectively. This gives the result,

$$\begin{aligned} \rho' &= Tr_{RE} (|\psi'\rangle\langle\psi'|) \\ &= \beta_d^2 A_d^2 |d_L\rangle\langle d_L| + \beta_s^2 A_s^2 |s_L\rangle\langle s_L| \\ &\quad + \beta_u^2 (1 - A_u^2) |\psi_u\rangle\langle\psi_u| + \beta_c^2 (1 - A_c^2) |\psi_c\rangle\langle\psi_c| \\ &\quad + \beta_u^2 A_u^2 |u_L\rangle\langle u_L| + \beta_c^2 A_c^2 |c_L\rangle\langle c_L| \\ &\quad + \beta_d^2 (1 - A_d^2) |\psi_d\rangle\langle\psi_d| + \beta_s^2 (1 - A_s^2) |\psi_s\rangle\langle\psi_s|, \end{aligned} \quad (22)$$

where

$$A_q^2 = |\langle q_L q'_R 0 | V | q_L q'_R 0 \rangle|^2 + |\langle q_L q'_R Z^0 | V | q_L q'_R 0 \rangle|^2, \quad q = d, s, u, c, \quad (24)$$

and

$$|\psi_d\rangle = \frac{1}{\sqrt{1 - A_d^2}} (\langle u_L W^- | V | d_L 0 \rangle |u_L\rangle + \langle c_L W^- | V | d_L 0 \rangle |c_L\rangle), \quad (25)$$



$$|\psi_s\rangle = \frac{1}{\sqrt{1-A_s^2}} (\langle u_L W^- | V | s_L 0 \rangle | u_L \rangle + \langle c_L W^- | V | s_L 0 \rangle | c_L \rangle), \quad (26)$$

$$|\psi_u\rangle = \frac{1}{\sqrt{1-A_u^2}} (\langle d_L W^+ | V | u_L 0 \rangle | d_L \rangle + \langle s_L W^+ | V | u_L 0 \rangle | s_L \rangle), \quad (27)$$

$$|\psi_c\rangle = \frac{1}{\sqrt{1-A_c^2}} (\langle d_L W^+ | V | c_L 0 \rangle | d_L \rangle + \langle s_L W^+ | V | c_L 0 \rangle | s_L \rangle). \quad (28)$$

From the unitarity conditions in Eqs. (20) and (21), we have  $\langle \psi_d | \psi_s \rangle = 0$  and  $\langle \psi_u | \psi_c \rangle = 0$ . By considering the  $W^\pm$ -mediated decays of left handed quarks with right handed quark spectators, orthogonal bases  $\{ |\psi_d\rangle, |\psi_s\rangle \}$  and  $\{ |\psi_u\rangle, |\psi_c\rangle \}$  emerge that are rotated relative to sets  $\{ |u_L\rangle, |c_L\rangle \}$  and  $\{ |d_L\rangle, |s_L\rangle \}$ , respectively.

It should be emphasized that  $Z^0$  left-right mixing does not appear in the current  $\bar{F}\gamma_\mu F$ . Consequently, although the right handed spectators and left handed system quarks in Eq. (18) both interact with the  $Z^0$  boson, they do not mix handedness. Furthermore, the  $W^\pm$  channel is isolated to left handed inputs. Even without the  $Z^0$  boson, quark mass is a left-right mixing parameter for the Naimark extension in Eq. (9). However, we are motivated by the idea that the weak bosons represent a dynamical entangling mechanism for a Lorentz group Naimark extension. The left-right mixing in Eq. (17) combine with the CKM matrix for a full Naimark extension of the Lorentz group of spin and boost operators. The mixing involves the weak angle  $\theta_w$  and the CKM matrix parameter set (most prominently the Cabibbo angle  $\theta_C \approx 13.1^\circ$ ).

## QUANTUM INFORMATION

In quantum information theory the state  $|\psi\rangle$  in Eq. (18) is known as a purification of the input density matrix  $\rho = \text{Trace}_{RE}(|\psi\rangle\langle\psi|)$  [15]. The concept of purification is at the center of definitions of entanglement fidelity and entropy exchange for transmission through noisy quantum channels [16–18]. In a follow-on paper we evaluate the entanglement in Eqs. (25)–(28) by considering the  $W^\pm$ -mediated weak decay as a noisy quantum information channel [19]. Weak decays of the four left handed fermions ( $u, d, c, s$ ) are interpreted as a single quantum information channel, which we denote the  $W^\pm$  channel. The noise in the channel results from weak boson interactions. This interpretation of particle decays as information

channels also extends the theory of quantum measurement. A generalization of quantum statistical counting to accessibly distinguishable particles results in an entropic criterion for quark mixing angles equivalent to the GIM mechanism [20].

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# The $W^\pm$ -Mediated Weak Decay as an Information Channel

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## Abstract

This paper defines the channel capacity for  $W^\pm$ -mediated quark decays, and connects the description to Cabibbo rotations for the first two quark generations. Right handed quarks are defined as spectators to the left handed isospin transitions, a role which is analogous to reservoir Hilbert spaces in quantum information theory. Mixed quark states emerge naturally from this two-quark state description. It is suggested that maximum channel capacity is a condition related to distinguishable input state counting at the weak interaction time-space scale. The  $W^\pm$  channel capacity is computed assuming different sets of conditions. In the high energy limit, rotated quark inputs analogous to the GIM mechanism are shown to maximize channel capacity.

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## 1 Introduction

In information theory, inputs that are resolved as distinct messages through a noisy communication channel are termed ‘accessibly distinguishable’. Channel capacity is a measure of the number of resolvable inputs in the limit of a large number of independent channel operations, each accessing the same stochastic input ensemble [1]. In this paper we use the recently developed quantum channel capacity formalism [2, 3] to count distinguishable states through weak quark decays, and suggest that this count is fundamental to the measurement of particle identity through decays triggered by penetrating  $W^\pm$  and  $Z^0$  bosons.

In a recent paper, we showed that the chiral structure of the standard model suggests that left handed  $W^\pm$ -mediated quark decays draw from a density matrix ensemble of input quarks [4]. The density matrix results from a mixed two-quark state in which right handed quark components are spectators to left handed isospin-changing  $W^\pm$ -mediated decays. The two-quark state is known in quantum information theory as a ‘purification’ [5, 6]. The observable decay rate is appropriately realized by repeated application of the weak interaction quantum channel to this density matrix ensemble. From the standpoint of left handed quarks, the  $W^\pm$  and  $Z^0$  bosons represent an interacting ‘near environment’ that defines the channel operation [6]. It is the  $W^\pm$  interactions that change the left handed particle identity, hence the expression  $W^\pm$  *information channel*. The construction of particle decays as quantum information channels was first described in Ref. [7].

In recent years, the physics of distinguishability through a quantum channel with a purifying ‘reservoir’ and noisy environmental interactions has been worked out [8, 9]. In this paper the concept of accessible distinguishability is applied specifically to the  $W^\pm$ -mediated Cabibbo-rotated quark decays in the first two fermionic generations. The four quarks and their  $W^\pm$  decay patterns form, as a single unit, a complex communication channel. For example, with the  $W^-$  channel, the decay products of up-type (up and charm) quarks are added into a system density matrix of input down-type (down and strange) quarks. The density matrix is purified into an orthonormalizing state using a reservoir Hilbert space consisting of

right handed quarks, which are spectators to left handed isospin changing decays. The evolution of this state through the decay channel determines the output density matrix from which channel capacity is computed. While the right handed reservoir quarks and system left handed quarks both undergo  $Z^0$ -mediated transitions, the right handed quarks remain spectators to the  $W^\pm$  channel operation. In Ref. [4], we noted that the output density matrix contains a quark basis rotation involving up-type and down-type quarks. It seems that the distinctive feature of the weak decays that leads to the rotation is the nearness of the weak boson environment — a single particle providing stochastic environmental interaction that both triggers the decay and is the source of channel noise. Typically, decay channel environments are ‘far type’, involving interactions with a sea of particles, photons or gluons, as the channel operation progresses.

This paper is organized as follows. In Section 2, using the reservoir-environment structure discussed above, the output quark density matrix is defined. An output quark basis rotation is demonstrated. Section 3 contains the definition of channel capacity through the  $W^\pm$ -mediated channel and optimum mixing is derived to maximize accessible distinguishability through the channel. Section 4 contains the evaluation of channel capacity for the case of equal  $W^\pm$ -mediated decay rates among input quarks, appropriate in the high energy limit in which all quarks participate. Rotated orthogonal quark input states, equivalent to channel inputs from the Glashow-Illiopolis-Maiani (GIM) mechanism [10, 11], are shown to maximize the  $W^\pm$  channel capacity.

In addition to providing a measure of particle distinguishability, the density matrix formulation of the  $W^\pm$  channel provides a quantitative measure of the reversibility of the decay. An entropic quantity determines the extent to which quark inputs can be reconstructed from the decay outputs [6]. The  $W^\pm$  channel reversibility measure is computed in the appendix.

## 2 Density Matrices for the $W^\pm$ Channel

In this section we consider the combined  $W^+$  and  $W^-$  channel as acting on the four quarks in the first two generations. Recall that the Cabibbo rotation matrix results from the product  $U_u^\dagger U_d$ , where  $U_u$  is the mixing matrix of left handed up and charm quarks, and  $U_d$  is the mixing matrix of left handed down and strange quarks [12, 13]. This involvement of both up-type and down-type quarks suggests a joint channel in which the input density matrix is

$$\rho = \xi_1 |d \rangle \langle d| + \xi_2 |s \rangle \langle s| + \xi_3 |u \rangle \langle u| + \xi_4 |c \rangle \langle c|, \quad (1)$$

where  $|q \rangle$  is the *left handed* state of the quark  $q = d, s, u, c$ , and positive  $\xi_i$  satisfy

$$\xi_1 + \xi_2 + \xi_3 + \xi_4 = 1. \quad (2)$$

The set  $\{|q \rangle\}$  defines states in the system Hilbert space  $Q$ . For notational clarity, in the following we denote *right handed* quark states by  $|q' \rangle$ .

Right handed quarks, as spectators in the  $W^\pm$ -mediated decays, fulfill the role of ‘reservoirs’  $R$  in the orthonormalization of the density matrix in Eq. (1). Including an ancillary boson ‘environment’ Hilbert space  $E$ , initially in the vacuum state  $|0 \rangle$ , the two-quark input state to the  $W^\pm$  channel is given by

$$|\psi \rangle = \sqrt{\xi_1} |dd'0 \rangle + \sqrt{\xi_2} |ss'0 \rangle + \sqrt{\xi_3} |uu'0 \rangle + \sqrt{\xi_4} |cc'0 \rangle, \quad (3)$$

which appropriately projects to the system  $Q$  as

$$\rho = \text{Trace}_{RE}(|\psi \rangle \langle \psi|). \quad (4)$$

The so-called *QRE* representation in Eq. (3) was discussed for particle decays in Ref. [7], and for general quantum channels in Refs. [6] and [8].

The operation of the  $W^\pm$  channel is represented by an evolution operator  $V$  acting on the combined Hilbert spaces of the system  $\{|q \rangle\}$  and the environment  $\{|0 \rangle, |Z^0 \rangle, |W^+ \rangle, |W^- \rangle\}$ . The



output state is given by

$$\begin{aligned}
|\psi'\rangle &= V|\psi\rangle = \\
&\sqrt{\xi_1}(\langle dd'0|V|dd'0\rangle|dd'0\rangle + \langle dd'Z^0|V|dd'0\rangle|dd'Z^0\rangle) + \\
&\sqrt{\xi_2}(\langle ss'0|V|ss'0\rangle|ss'0\rangle + \langle ss'Z^0|V|ss'0\rangle|ss'Z^0\rangle) + \\
&\sqrt{\xi_3}(\langle uu'0|V|uu'0\rangle|uu'0\rangle + \langle uu'Z^0|V|uu'0\rangle|uu'Z^0\rangle) + \\
&\sqrt{\xi_4}(\langle cc'0|V|cc'0\rangle|cc'0\rangle + \langle cc'Z^0|V|cc'0\rangle|cc'Z^0\rangle) + \\
&\sqrt{\xi_1}\{\langle uW^-|V|d0\rangle|u\rangle + \langle cW^-|V|d0\rangle|c\rangle\}|d'W^-\rangle + \\
&\sqrt{\xi_2}\{\langle uW^-|V|s0\rangle|u\rangle + \langle cW^-|V|s0\rangle|c\rangle\}|s'W^-\rangle + \\
&\sqrt{\xi_3}\{\langle dW^+|V|u0\rangle|d\rangle + \langle sW^+|V|u0\rangle|s\rangle\}|u'W^+\rangle + \\
&\sqrt{\xi_4}\{\langle dW^+|V|c0\rangle|d\rangle + \langle sW^+|V|c0\rangle|s\rangle\}|c'W^+\rangle,
\end{aligned} \tag{5}$$

where by unitarity of  $V$  we have the conditions

$$\begin{aligned}
&|\langle uW^-|V|d0\rangle|^2 + |\langle cW^-|V|d0\rangle|^2 + |\langle dd'0|V|dd'0\rangle|^2 + \\
&|\langle dd'Z^0|V|dd'0\rangle|^2 = \\
&|\langle uW^-|V|s0\rangle|^2 + |\langle cW^-|V|s0\rangle|^2 + |\langle ss'0|V|ss'0\rangle|^2 + \\
&|\langle ss'Z^0|V|ss'0\rangle|^2 = \\
&|\langle dW^+|V|u0\rangle|^2 + |\langle sW^+|V|u0\rangle|^2 + |\langle uu'0|V|uu'0\rangle|^2 + \\
&|\langle uu'Z^0|V|uu'0\rangle|^2 = \\
&|\langle dW^+|V|c0\rangle|^2 + |\langle sW^+|V|c0\rangle|^2 + |\langle cc'0|V|cc'0\rangle|^2 + \\
&|\langle cc'Z^0|V|cc'0\rangle|^2 = 1.0,
\end{aligned} \tag{6}$$

$$\langle uW^-|V|d0\rangle^* \langle uW^-|V|s0\rangle + \langle cW^-|V|d0\rangle^* \langle cW^-|V|s0\rangle = 0, \tag{7}$$

and

$$\langle dW^+|V|u0\rangle^* \langle dW^+|V|c0\rangle + \langle sW^+|V|u0\rangle^* \langle sW^+|V|c0\rangle = 0. \tag{8}$$

For notational clarity, because the channel operation with  $W^\pm$  output does not involve right handed quarks, the  $q'$  state is dropped from the matrix element;  $\langle qq'W^\pm|V|\tilde{q}q'0\rangle \longrightarrow \langle qW^\pm|V|\tilde{q}0\rangle$ . The conditions in Eqs. (6)-(8) are equivalent to the  $W^\pm$  information channel having ‘trace preserving operator elements’ [14]. Equations (7) and (8) insure that the ratio of Cabibbo-allowed to Cabibbo-suppressed channel amplitudes is the same for each of the up-type and down-type quarks (although not necessarily that the up-type and down-type ratios are equal). This fact further motivates the trace preserving channel representation for these decays.

For input  $\rho$  in Eq. (1), the output system  $Q$  density matrix is given by

$$\mathcal{T}(\rho) = \text{Trace}_{RE}(|\psi' \rangle \langle \psi'|) \quad (9)$$

or

$$\begin{aligned} \mathcal{T}(\rho) = & \xi_1 A_d^2 |d \rangle \langle d| + \xi_2 A_s^2 |s \rangle \langle s| + \xi_3 (1 - A_u^2) |\psi_u \rangle \langle \psi_u| + \\ & \xi_4 (1 - A_c^2) |\psi_c \rangle \langle \psi_c| + \xi_3 A_u^2 |u \rangle \langle u| + \xi_4 A_c^2 |c \rangle \langle c| + \\ & \xi_1 (1 - A_d^2) |\psi_d \rangle \langle \psi_d| + \xi_2 (1 - A_s^2) |\psi_s \rangle \langle \psi_s|, \end{aligned} \quad (10)$$

where

$$A_q^2 = | \langle qq'0 | V | qq'0 \rangle |^2 + | \langle qq'Z^0 | V | qq'0 \rangle |^2, \quad (11)$$

for  $q = d, s, u, c$ , and

$$|\psi_d \rangle = \frac{1}{\sqrt{1 - A_d^2}} \{ \langle uW^- | V | d0 \rangle |u \rangle + \langle cW^- | V | d0 \rangle |c \rangle \}, \quad (12)$$

$$|\psi_s \rangle = \frac{1}{\sqrt{1 - A_s^2}} \{ \langle uW^- | V | s0 \rangle |u \rangle + \langle cW^- | V | s0 \rangle |c \rangle \}, \quad (13)$$

$$|\psi_u \rangle = \frac{1}{\sqrt{1 - A_u^2}} \{ \langle dW^+ | V | u0 \rangle |d \rangle + \langle sW^+ | V | u0 \rangle |s \rangle \}, \quad (14)$$

and

$$|\psi_c \rangle = \frac{1}{\sqrt{1 - A_c^2}} \{ \langle dW^+ | V | c0 \rangle |d \rangle + \langle sW^+ | V | c0 \rangle |s \rangle \}. \quad (15)$$

Note from Eqs. (7) and (8) we have

$$\langle \psi_d | \psi_s \rangle = 0, \quad (16)$$

and

$$\langle \psi_u | \psi_c \rangle = 0. \quad (17)$$

Equations (10)-(17) suggest that the effect of the channel is to introduce orthogonal basis sets  $\{|\psi_u \rangle, |\psi_c \rangle\}$  and  $\{|\psi_d \rangle, |\psi_s \rangle\}$  rotated relative to the sets  $\{|d \rangle, |s \rangle\}$  and  $\{|u \rangle, |c \rangle\}$ , respectively. The rotated bases emerge from the system  $Q$ , reservoir  $R$ , and environment  $E$  representation in Eq. (3) in which right handed (reservoir)

quarks are spectators to the  $W^\pm$  channel operation. The degree of rotation is proportional to decay amplitudes  $\langle qW^\pm|V|\tilde{q}0\rangle$  for a weak interaction evolution operator  $V$  describing  $W^\pm$  and  $Z^0$  mediated spontaneous decays.

### 3 $W^\pm$ Channel Capacity

In quantum information theory channel capacity is derived by maximizing the number of accessibly distinguishable orthogonal input states. Relating this quantity to entropy requires consideration of ‘typical’ output states which dominate the populations of outputs in the limit of many decays [2, 3, 14, 15]. Two typicality subspaces [14] are considered for  $n$  independent decays described by  $\rho$  in Eq. (1): 1) the full output space with an ensemble generated by the density matrix  $(\mathcal{T}(\rho))^{\otimes n}$ , and 2) typical states surrounding a single typical output from a density matrix ensemble of the form

$$\begin{aligned} & (\mathcal{T}(|d\rangle\langle d|))^{\otimes n\xi_1} \otimes (\mathcal{T}(|s\rangle\langle s|))^{\otimes n\xi_2} \otimes (\mathcal{T}(|u\rangle\langle u|))^{\otimes n\xi_3} \otimes \\ & (\mathcal{T}(|c\rangle\langle c|))^{\otimes n\xi_4}. \end{aligned} \quad (18)$$

The latter density matrix is due to the quantum channel noise that surrounds each output, and provides a limit on left handed quark distinguishability due to the  $W^\pm$  and  $Z^0$  boson interactions.

For the  $W^\pm$  channel defined in Section 2, the channel capacity is defined by

$$\begin{aligned} \mathcal{C}(\mathcal{T}) = \max_{\xi_i} \{ & S(\mathcal{T}(\rho)) - \xi_1 S(\mathcal{T}(|d\rangle\langle d|)) - \xi_2 S(\mathcal{T}(|s\rangle\langle s|)) - \\ & \xi_3 S(\mathcal{T}(|u\rangle\langle u|)) - \xi_4 S(\mathcal{T}(|c\rangle\langle c|)) \}, \end{aligned} \quad (19)$$

where  $S(\rho)$  is the quantum entropy of the density matrix  $\rho$  given by

$$S(\rho) = -\text{Trace}(\rho \ln(\rho)) = -\sum_i \lambda_i \ln(\lambda_i) \quad (20)$$

for  $\rho$  eigenvalues  $\{\lambda_i, i = 1, \dots, 4\}$  [14]. Note that the output density matrix  $\mathcal{T}(\rho)$  in Eq. (10) splits into the  $\{|s\rangle, |d\rangle\}$  and  $\{|u\rangle, |c\rangle\}$  subspaces. In order to simplify the diagonalization and evaluation of Eq. (19), we write

$$\mathcal{T}(\rho) = \mathcal{T}_{sd}(\rho) + \mathcal{T}_{uc}(\rho) \quad (21)$$

with down-type

$$\begin{aligned} \mathcal{T}_{ds}(\rho) = & \xi_1 A_d^2 |d \rangle \langle d| + \xi_2 A_s^2 |s \rangle \langle s| + \\ & \xi_3 (1 - A_u^2) (\cos \alpha |d \rangle + \sin \alpha |s \rangle) (\cos \alpha \langle d| + \sin \alpha \langle s|) + \\ & \xi_4 (1 - A_c^2) (-\sin \alpha |d \rangle + \cos \alpha |s \rangle) (-\sin \alpha \langle d| + \cos \alpha \langle s|), \end{aligned} \quad (22)$$

and a similar expression for up-type  $\mathcal{T}_{uc}(\rho)$  given by,

$$\begin{aligned} \mathcal{T}_{uc}(\rho) = & \xi_3 A_u^2 |u \rangle \langle u| + \xi_4 A_c^2 |c \rangle \langle c| + \\ & \xi_1 (1 - A_d^2) (\cos \beta |u \rangle + \sin \beta |c \rangle) (\cos \beta \langle u| + \sin \beta \langle c|) + \\ & \xi_2 (1 - A_s^2) (-\sin \beta |u \rangle + \cos \beta |c \rangle) (-\sin \beta \langle u| + \cos \beta \langle c|). \end{aligned} \quad (23)$$

In order to connect the parameters in Eqs. (22) and (23) to observables, in addition to Eq. (11), we define

$$\cos \alpha = \langle dW^+ | V | u0 \rangle / \sqrt{1 - A_u^2}, \quad (24)$$

$$\sin \alpha = \langle sW^+ | V | u0 \rangle / \sqrt{1 - A_u^2}, \quad (25)$$

$$\cos \beta = \langle uW^- | V | d0 \rangle / \sqrt{1 - A_d^2}, \quad (26)$$

and

$$\sin \beta = \langle cW^- | V | d0 \rangle / \sqrt{1 - A_d^2}, \quad (27)$$

and note that through Eqs. (7) and (8)  $\alpha$  and  $\beta$  are also connected, respectively, to charm and strange decay amplitudes.

By straight-forward diagonalization of Eqs. (22) and (23), we compute eigenvalues,

$$\lambda_{\pm} = (\omega \pm \sqrt{\Omega})/2 \quad (28)$$

where

$$\omega = \xi_1 A_d^2 + \xi_2 A_s^2 + \xi_3 (1 - A_u^2) + \xi_4 (1 - A_c^2) \quad (29)$$

and

$$\begin{aligned} \Omega = & (\xi_1 A_d^2 - \xi_2 A_s^2)^2 + (\xi_3 (1 - A_u^2) - \xi_4 (1 - A_c^2))^2 + \\ & 2(\xi_1 A_d^2 - \xi_2 A_s^2)(\xi_3 (1 - A_u^2) - \xi_4 (1 - A_c^2)) \cos 2\alpha. \end{aligned} \quad (30)$$

for  $\mathcal{T}_{ds}$ ; and

$$\kappa_{\pm} = (\theta \pm \sqrt{\Theta})/2 \quad (31)$$

where

$$\theta = \xi_3 A_u^2 + \xi_4 A_c^2 + \xi_1(1 - A_d^2) + \xi_2(1 - A_s^2) \quad (32)$$

and

$$\Theta = \frac{(\xi_3 A_u^2 - \xi_4 A_c^2)^2 + (\xi_1(1 - A_d^2) - \xi_2(1 - A_s^2))^2 + 2(\xi_3 A_u^2 - \xi_4 A_c^2)(\xi_1(1 - A_d^2) - \xi_2(1 - A_s^2)) \cos 2\beta}{2} \quad (33)$$

for  $\mathcal{T}_{uc}(\rho)$ .

Using Eqs. (7), (8), (11), and (24) - (27), we note that

$$\cos 2\alpha = (1 - b)/(1 + b), \quad (34)$$

and

$$\cos 2\beta = (1 - \tilde{b})/(1 + \tilde{b}) \quad (35)$$

where

$$b = \frac{|\langle sW^+|V|u0\rangle|^2}{|\langle dW^+|V|u0\rangle|^2} = \frac{|\langle dW^+|V|c0\rangle|^2}{|\langle sW^+|V|c0\rangle|^2} \quad (36)$$

and

$$\tilde{b} = \frac{|\langle cW^-|V|d0\rangle|^2}{|\langle uW^-|V|d0\rangle|^2} = \frac{|\langle uW^-|V|s0\rangle|^2}{|\langle cW^-|V|s0\rangle|^2} \quad (37)$$

are the branching ratios for  $|u\rangle$ ,  $|c\rangle$  and  $|d\rangle$ ,  $|s\rangle$  Cabibbo-suppressed-to-Cabibbo-allowed decays, respectively. Furthermore, from Eq. (6) we can identify

$$p_q = 1 - A_q^2 = 1 - |\langle qq'0|V|qq'0\rangle|^2 - |\langle qq'Z^0|V|qq'0\rangle|^2 \quad (38)$$

as the total squared amplitude for  $|q\rangle$  decay by  $W^\pm$  bosons. The parameters  $b$ ,  $\tilde{b}$ , and  $p_q$  are measured quantities that determine the range of channel capacities achievable through the input mixings  $\{\xi_i, i = 1, 2, 3, 4\}$  in Eq. (1). The Cabibbo hypothesis corresponds to  $b = \tilde{b}$  or  $\cos 2\alpha = \cos 2\beta$ .

The evaluation of the channel capacity follows from the maximization over  $\{\xi_i, i = 1, 2, 3\}$  (with the constraint in Eq. (2) fixing the fourth mixing parameter  $\xi_4$ ) of the function

$$\mathcal{C}(\mathcal{T}) = \frac{\mathcal{H}_4(\lambda_+, \lambda_-, \kappa_+, \kappa_-) - \xi_1 \mathcal{H}_2(A_d^2) - \xi_2 \mathcal{H}_2(A_s^2) - \xi_3 \mathcal{H}_2(A_u^2) - \xi_4 \mathcal{H}_2(A_c^2)}{2} \quad (39)$$

where

$$\mathcal{H}_4(a_1, a_2, a_3, a_4) = - \sum_{i=1}^4 a_i \ln a_i \quad (40)$$

for  $a_1 + a_2 + a_3 + a_4 = 1$ , and

$$\mathcal{H}_2(p) = -p \ln p - (1-p) \ln(1-p). \quad (41)$$

## 4 Channel Capacity in the High Energy Limit

It is interesting to consider a special case of Eq. (39) in which all quarks have the same  $W^\pm$ -mediated decay probability

$$A_u = A_c = A_d = A_s = A, \quad (42)$$

as prescribed by the standard model in the high energy limit in which mass differences are negligible. Substitution of Eq. (42) into Eq. (39) yields

$$\begin{aligned} \mathcal{C}(\rho, \mathcal{T}) = & -(1/2)(\omega + \sqrt{\Omega}) \ln(\omega + \sqrt{\Omega}) - (1/2)(\omega - \sqrt{\Omega}) \ln(\omega - \sqrt{\Omega}) \\ & -(1/2)(\theta + \sqrt{\Theta}) \ln(\theta + \sqrt{\Theta}) - (1/2)(\theta - \sqrt{\Theta}) \ln(\theta - \sqrt{\Theta}) \\ & - \mathcal{H}_2(A^2), \end{aligned} \quad (43)$$

where

$$\omega = s_{12}A^2 + (1 - s_{12})(1 - A^2), \quad (44)$$

$$\theta = (1 - s_{12})A^2 + s_{12}(1 - A^2), \quad (45)$$

$$\Omega = d_{34}^2A^4 + d_{12}^2(1 - A^2)^2 + 2d_{34}d_{12}A^2(1 - A^2) \cos 2\beta, \quad (46)$$

$$\Theta = d_{12}^2A^4 + d_{34}^2(1 - A^2)^2 + 2d_{12}d_{34}A^2(1 - A^2) \cos 2\alpha, \quad (47)$$

with

$$s_{12} = \xi_1 + \xi_2 = 1 - \xi_3 - \xi_4 \quad (48)$$

and

$$d_{12} = \xi_1 - \xi_2, \quad (49)$$

$$d_{34} = \xi_3 - \xi_4. \quad (50)$$

Note that the quantities  $\theta$  and  $\omega$  depend only on  $s_{12}$ , the sum of coefficients in Eq. (1), whereas the quantities  $\Omega$  and  $\Theta$  depend on

$d_{12}$  and  $d_{34}$ , the difference of the coefficients. Optimizing  $\mathcal{C}(\rho, \mathcal{T})$  with respect to  $s_{12}$ , we have  $\partial\mathcal{C}/\partial s_{12} = 0$  leading to the condition

$$\omega^2 - \Omega = \theta^2 - \Theta \quad (51)$$

or

$$\theta = (1/2)(1 + \Omega - \Theta). \quad (52)$$

We are interested in embedded rotations in Eq. (1) involving separately the down-type and up-type quarks. This condition is equivalent to

$$\xi_1 = (\cos^2 \delta)/2, \xi_2 = (\sin^2 \delta)/2, \quad (53)$$

and

$$\xi_3 = (\cos^2 \tilde{\delta})/2, \xi_4 = (\sin^2 \tilde{\delta})/2, \quad (54)$$

which is most generally expressed by  $s_{12} = s_{34} = 1/2$ . Imposing this condition in the channel capacity extrema criteria in Eq. (52), yields  $\theta = 1/2$ , or  $\Theta = \Omega$ . Substitution of Eqs. (46) and (47) into  $\Theta = \Omega$  yields

$$(d_{12}^2 - d_{34}^2)(2A^2 - 1) = 2d_{12}d_{34}A^2(1 - A^2)(\cos 2\beta - \cos 2\alpha). \quad (55)$$

This is the desired result. Equal Cabibbo allowed-to-suppressed branching ratios for up and down quarks, the Cabibbo hypothesis, is equivalent to  $d_{12} = \pm d_{34}$ ; which is satisfied with a rotated orthogonal basis in Eq. (3) given by

$$|\psi \rangle = (1/\sqrt{2})(\cos \delta |dd'0 \rangle + \sin \delta |ss'0 \rangle - \sin \delta |uu'0 \rangle + \cos \delta |cc'0 \rangle), \quad (56)$$

with  $\tilde{\delta} = \delta + \pi/2$ . The result in Eq. (56), in which the full quark set (left- and right-handed and up and down) form a rotated basis, is equivalent to maximum channel capacity for  $W^\pm$  mediated decays. The rotated input quarks of the GIM mechanism, Eq. (56) with Cabibbo angle  $\theta_C = \delta$ , optimize the channel operation.

## 5 Conclusions

In our view, maximizing accessible distinguishability is roughly analogous to maximizing the number of microscopic configurations associated with a macroscopic state in statistical mechanics. Distinguishability, imposed and limited by quantum mechanics, is central

to correct Boltzmann counting even though the microscopic states of a gas are not directly measured. Accessibility to these states is determined by the observable macroscopic state.

For a given quantum transformation, which in this paper is spontaneous decay, there are unique particle mixings that maximize the number of distinguishable inputs to the decay channel. Distinguishability is established by repeated access to a stochastic input ensemble — a process that defines observables related to decays in particular, and more generally to scattering cross sections. Accessible distinguishability of the decay inputs is determined by the output particle identities.

The application of quantum information theory to weak decay phenomenology is very preliminary at this point in our ongoing research. Output left handed quark mixing proportional to decay amplitudes emerges from the *QRE* representation, and GIM mechanism-rotated quarks arise from maximum accessible distinguishability through the  $W^\pm$  channel. This suggests a source of quark mixing that is more fundamental than simply connecting parameterized mixing to S-matrix elements — namely, a mixing due to a chiral fermionic structure and quantum distinguishability.



## A Reversibility of the $W^\pm$ Decay Channel

The observation of weak decay products yields incomplete information on the quark input states. The degree to which these states can be reconstructed from the outputs depends on the amount of entropy deposited into the weak boson environment  $E$ . This quantity is known as exchange entropy which is defined in terms of the density matrix [6]

$$\rho'_E = \text{Trace}_{RQ}(|\psi' \rangle \langle \psi'|) \quad (57)$$

where  $|\psi' \rangle$  is given in Eq. (5). Evaluation of the trace over right handed reservoir  $R$  and left handed system  $Q$  Hilbert spaces yields

$$\begin{aligned} \rho'_E = & \xi_1(|B_d \rangle \langle B_d| + (1 - A_d^2)|W^- \rangle \langle W^-|) + \\ & \xi_2(|B_s \rangle \langle B_s| + (1 - A_s^2)|W^- \rangle \langle W^-|) + \\ & \xi_3(|B_u \rangle \langle B_u| + (1 - A_u^2)|W^+ \rangle \langle W^+|) + \\ & \xi_4(|B_c \rangle \langle B_c| + (1 - A_c^2)|W^+ \rangle \langle W^+|), \end{aligned} \quad (58)$$

where  $A_q$ ,  $q = d, s, u, c$ , is given in Eq. (11) and  $|B_q \rangle$ ,  $q = d, s, u, c$ , are states in the  $\{|0 \rangle, |Z^0 \rangle\}$  - space given by

$$|B_q \rangle = \langle qq'0|V|qq'0 \rangle |0 \rangle + \langle qq'Z^0|V|qq'0 \rangle |Z^0 \rangle. \quad (59)$$

Note that the amplitude  $\langle qq'Z^0|V|qq'0 \rangle$  involves  $Z^0$ -mediated decays  $q \rightarrow q$  of the left handed system quarks and  $q' \rightarrow q'$  of the right handed reservoir quarks. The overall irreversibility of the  $W^\pm$  channel can be assessed by considering the function

$$\text{Irr}(\rho, \mathcal{T}) = S(\rho) + S(\rho'_E) - S(\mathcal{T}(\rho)), \quad (60)$$

where  $\rho$  and  $\mathcal{T}(\rho)$  are given in Eqs. (1) and (21)-(23), respectively. The quantity  $\text{Irr}(\rho, \mathcal{T})$  vanishes if the decay channel is reversible [14].

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