The Gamma Constant, and Euler's Zeta Functions

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Abstract The Gamma Constant was defined by Euler as the difference between the harmonic series,

$$\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \dots$$

and the logarithm of the infinite hyper-real number

$$\log(N+1).$$

Later, this has been replaced with a shorthand limit notation

$$\lim_{n\to\infty}\left\{1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log(n+1)\right\}$$

Either way, the difference between two infinite hyper-reals is incomprehensible. And it remains absolutely unclear how such difference can be determined exactly.

We maintain that Euler's definition is only preliminary, and should be developed further to obtain a comprehensible expression for the Gamma Constant.

We present here two such series expansions of the Gamma Constant in terms Euler's Zeta Functions, that are stated in [Finch, p. 30]and in [Weisstein, p.1239]

$$\gamma = \left(1 - \log 2\right) + \left(\frac{1}{2} - \log\frac{3}{2}\right) + \left(\frac{1}{3} - \log\frac{4}{3}\right) + \left(\frac{1}{4} - \log\frac{5}{4}\right) + ...$$

$$\frac{1}{n} - \log\frac{n+1}{n} > 0, \text{ is monotonically decreasing to zero}$$

and the series converges absolutely

Developing this further,

$$\gamma = \frac{1}{2}\zeta(2) - \frac{1}{3}\zeta(3) + \frac{1}{4}\zeta(4) - \frac{1}{5}\zeta(5) + ...$$

Finally, we define the Ksi Functions, and in terms of the Zeta, and the Ksi functions we obtain

$$\gamma = \frac{1}{2}\zeta(2)\left(1 + \frac{1}{2}\xi(2)\left[1 + \frac{1}{2}\xi(4)\left\{1 + \frac{1}{2}\xi(8)\left(1 + \frac{1}{2}\xi(16)\left[1 + ...\right]\right)\right\}\right]\right) +$$

$$-\frac{1}{3}\zeta(3)\left(1 - \frac{1}{2}\xi(3)\left[1 + \frac{1}{2}\xi(6)\left\{1 + \frac{1}{2}\xi(12)\left[1 + \frac{1}{2}\xi(24)\left(1 + ...\right)\right]\right\}\right]\right)$$

$$-\frac{1}{5}\zeta(5)\left(1 - \frac{1}{2}\xi(5)\left[1 + \frac{1}{2}\xi(10)\left\{1 + \frac{1}{2}\xi(20)\left(1 + \frac{1}{2}\xi(40)\left[1 + ...\right]\right)\right\}\right]\right)$$

....

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Contents

0.

Euler's Gamma Constant

We denote by N an infinite hyper-real. For instance,

$$N = < 1, 2, 3, ... > .$$

For $k = 1, 2, 3, 4, ...$ Euler's Zeta Function is

$$\zeta(k) = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \frac{1}{5^k} + ...$$

The Gamma Constant was defined by Euler as the difference between the harmonic series,

$$\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + ...$$

and the logarithm of the infinite hyper-real number

$$\log(N + 1).$$

That is, the Gamma constant is the sequence

$$\begin{bmatrix} 1 - \log 2 \\ 1 + \frac{1}{2} - \log 3 \\ 1 + \frac{1}{2} + \frac{1}{3} - \log 4 \\ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \log 5 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \log(n+1) \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{bmatrix}$$

Later, this has been replaced with a shorthand limit notation

$$\lim_{n\to\infty}\left\{1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}-\log(n+1)\right\}$$

Either way, the difference between two infinite hyper-reals is incomprehensible. And it remains absolutely unclear how such difference can be determined exactly.

No other constant that is central in Mathematics, and in Number Theory is defined in such a foggy way.

For instance,

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-....$$

$$e=\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+....$$

$$\log 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-....$$

We maintain that Euler's definition is only preliminary, and should be developed further to obtain a comprehensible expression for the Gamma Constant.

We present here two such series expansions of the Gamma Constant in terms Euler's Zeta Functions

Both are stated in [Finch, p. 30] without any comment.

1.

Series Expansion of the Gamma Constant

$$\gamma = \left(1 - \log 2\right) + \left(\frac{1}{2} - \log\frac{3}{2}\right) + \left(\frac{1}{3} - \log\frac{4}{3}\right) + \left(\frac{1}{4} - \log\frac{5}{4}\right) + \dots$$

$\frac{1}{n} - \log\frac{n+1}{n} > 0$, is monotonically decreasing to zero

and the series converges absolutely

Proof:

We can replace the defining sequence of the gamma Constant with an infinite series:

Since

$$n+1 = 2\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdot\dots\cdot\frac{n}{n-1}\cdot\frac{n+1}{n},$$

$$\log(n+1) = \log 2 + \log\frac{3}{2} + \log\frac{4}{3} + \log\frac{5}{4} + \dots + \log\frac{n}{n-1} + \log\frac{n+1}{n},$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} - \log(n+1) = 1 - \log 2$$

$$+\frac{1}{2} - \log\frac{3}{2}$$

$$+\frac{1}{3} - \log\frac{4}{3}$$

$$+\frac{1}{4} - \log\frac{5}{4}$$

....

$$+\frac{1}{n}-\log\frac{n+1}{n}$$

Therefore, we obtain the infinite series expansion

$$\boxed{\gamma=\left(1-\log 2\right)+\left(\frac{1}{2}-\log\frac{3}{2}\right)+\left(\frac{1}{3}-\log\frac{4}{3}\right)+\left(\frac{1}{4}-\log\frac{5}{4}\right)+...}$$

The terms of the series

$$\underline{\frac{1}{n}-\log\frac{n+1}{n}}$$

are <u>all positive</u> because

$$1+\frac{1}{n}<1+\frac{1}{n}+\frac{1}{2!}\frac{1}{n^2}+\frac{1}{3!}\frac{1}{n^3}+....=e^{\frac{1}{n}},$$

Hence,

$$\log\left(1+\frac{1}{n}\right)<\frac{1}{n},$$

$$\frac{1}{n}-\log\frac{n+1}{n}>0.\square$$

They <u>converge to zero</u> because

$$\frac{1}{n}-\log\frac{n+1}{n}\to 0-\log 1=0.\square$$

And they <u>strictly monotonically decrease to zero</u> because

$$\frac{1}{n+1}<\frac{1}{n}$$

$$0<\frac{1}{n}-\frac{1}{n+1}$$

$$0<\frac{1}{n+1}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$1+\frac{1}{n}<1+\frac{1}{n}+\frac{1}{n+1}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$1+\frac{1}{n}<\left(1+\frac{1}{n+1}\right)\left(1+\frac{1}{n}-\frac{1}{n+1}\right)$$

Since

$$e^{\left(\frac{1}{n}-\frac{1}{n+1}\right)}=\underbrace{1+\left(\frac{1}{n}-\frac{1}{n+1}\right)}+\frac{1}{2!}\left(\frac{1}{n}-\frac{1}{n+1}\right)^2+\frac{1}{3!}\left(\frac{1}{n}-\frac{1}{n+1}\right)^3+\ldots,$$

we have

$$1+\frac{1}{n}<\left(1+\frac{1}{n+1}\right)e^{\left(\frac{1}{n}-\frac{1}{n+1}\right)}$$

$$\frac{1+\frac{1}{n}}{1+\frac{1}{n+1}}<e^{\left(\frac{1}{n}-\frac{1}{n+1}\right)}$$

$$\log\frac{1+\frac{1}{n}}{1+\frac{1}{n+1}}<\frac{1}{n}-\frac{1}{n+1}$$

$$\log\frac{\frac{n+1}{n}}{\frac{n+2}{n+1}}<\frac{1}{n}-\frac{1}{n+1}$$

$$\log\frac{n+1}{n}-\log\frac{n+2}{n+1}<\frac{1}{n}-\frac{1}{n+1}$$

$$\frac{1}{n+1}-\log\frac{n+2}{n+1}<\frac{1}{n}-\log\frac{n+1}{n}.\square$$

Having the positive terms $\frac{1}{n}$ that are strictly monotonically decreasing to zero, characterizes also the harmonic series which diverges.

But the subtraction of $\log\frac{n+1}{n}$ from $\frac{1}{n}$ in the Gamma series, tempers the harmonic series, and converges the resulting Gamma series.

Indeed, since to first order in n

$$\log\left(1+\frac{1}{n}\right) \approx \frac{1}{n},$$

and since

$$\frac{1}{n} > \frac{1}{n+1},$$

then

$$\log\left(1+\frac{1}{n}\right) \geq \frac{1}{n+1},$$

$$-\log\left(1+\frac{1}{n}\right) \leq -\frac{1}{n+1},$$

$$\frac{1}{n} - \log\left(1+\frac{1}{n}\right) \leq \frac{1}{n} - \frac{1}{n+1}$$

$$= \frac{1}{n(n+1)} < \frac{1}{n^2}.$$

Therefore, the Gamma Series is bounded by the $\zeta(2)$ series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ... = \frac{\pi^2}{6}.$$

That is the Gamma Series is absolutely convergent.

Consequently, its terms may be ordered in any way, without changing the fact that is converges.

It is formally correct to write

$$\gamma = \zeta(1) - \log(N+1).$$

But a comprehensible definition of Gamma is by the absolutely convergent series of positive terms

$$\gamma = \left(1 - \log 2\right) + \left(\frac{1}{2} - \log\frac{3}{2}\right) + \left(\frac{1}{3} - \log\frac{4}{3}\right) + \left(\frac{1}{4} - \log\frac{5}{4}\right) + ..$$

In fact, since infinite hyper-reals, and their reciprocals, the infinitesimals, are a crucial extension of the real numbers, we would write

$$\gamma = \left(1 - \log 2\right) + \left(\frac{1}{2} - \log\frac{3}{2}\right) + \left(\frac{1}{3} - \log\frac{4}{3}\right) + ... + \left(\frac{1}{N} - \log\frac{N+1}{N}\right),$$

where N is an infinite hyper-real, and its reciprocal $\frac{1}{N}$, as well as

$\log\frac{N+1}{N}$ are infinitesimals.

2.

Gamma Expanded in Zeta Functions

$$\boxed{\gamma = \frac{1}{2}\zeta(2) - \frac{1}{3}\zeta(3) + \frac{1}{4}\zeta(4) - \frac{1}{5}\zeta(5) + ...}$$

Proof:

By Euler,

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - ...,$$

Hence,

$$\frac{1}{n} - \log\left(1 + \frac{1}{n}\right) = \frac{1}{2}\frac{1}{n^2} - \frac{1}{3}\frac{1}{n^3} + \frac{1}{4}\frac{1}{n^4} + \frac{1}{5}\frac{1}{n^5} - ...$$

Thus,

$$\frac{1}{1} - \log\left(1 + \frac{1}{1}\right) = \frac{1}{2}\frac{1}{1^2} - \frac{1}{3}\frac{1}{1^3} + \frac{1}{4}\frac{1}{1^4} + \frac{1}{5}\frac{1}{1^5} - ...$$

$$\frac{1}{2} - \log\left(1 + \frac{1}{2}\right) = \frac{1}{2}\frac{1}{2^2} - \frac{1}{3}\frac{1}{2^3} + \frac{1}{4}\frac{1}{2^4} + \frac{1}{5}\frac{1}{2^5} - ...$$

$$\frac{1}{3} - \log\left(1 + \frac{1}{3}\right) = \frac{1}{2}\frac{1}{3^2} - \frac{1}{3}\frac{1}{3^3} + \frac{1}{4}\frac{1}{3^4} + \frac{1}{5}\frac{1}{3^5} - ...$$

$$\frac{1}{4} - \log\left(1 + \frac{1}{4}\right) = \frac{1}{2}\frac{1}{4^2} - \frac{1}{3}\frac{1}{4^3} + \frac{1}{4}\frac{1}{4^4} + \frac{1}{5}\frac{1}{4^5} - ...$$

$$\frac{1}{5} - \log\left(1 + \frac{1}{5}\right) = \frac{1}{2}\frac{1}{5^2} - \frac{1}{3}\frac{1}{5^3} + \frac{1}{4}\frac{1}{5^4} + \frac{1}{5}\frac{1}{5^5} - ...$$

Therefore,

$$\gamma = \left(1 - \log 2\right) + \left(\frac{1}{2} - \log\frac{3}{2}\right) + \left(\frac{1}{3} - \log\frac{4}{3}\right) + \left(\frac{1}{4} - \log\frac{5}{4}\right) + ...$$

$$= \frac{1}{2}\underbrace{\left\{1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + ...\right\}}_{\zeta(2)}$$

$$-\frac{1}{3}\underbrace{\left\{1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + ...\right\}}_{\zeta(3)}$$

$$+\frac{1}{4}\underbrace{\left\{1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + ...\right\}}_{\zeta(4)}$$

$$-\frac{1}{5}\underbrace{\left\{1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} + ...\right\}}_{\zeta(5)}$$

$$+ ....$$

$$\gamma = \frac{1}{2}\zeta(2) - \frac{1}{3}\zeta(3) + \frac{1}{4}\zeta(4) - \frac{1}{5}\zeta(5) + ... \square$$

Euler[1] does not use the Zeta notation for his Zeta functions. Having already computed their values, he uses those values to obtain for γ,

$$C = 0.577\ 218\,.$$

Later[2], he obtains

$$C = 0.577\ 215\ 664\ 901\ 532\ 9\,.$$

To us, the expansion of γ in Euler's Zeta functions is more meaningful than the likely infinitely many digits value of the Gamma Constant.

We shall expand this form further with the aid of our Xi functions

[1] Leonhardi Euleri Opera Omnia, Series Prima, Opera Mathematica, Volumen 14, p. 94.

[2] same volume, p.118

3.

Ksi Functions

For $k = 1, 2, 3, 4, \ldots,$

p = prime,

n = natural number,

Euler showed[3]

$$\zeta(k) = 1 + \frac{1}{2^k} + \frac{1}{3^k} + .. + \frac{1}{n^k} + ...$$
$$= \frac{2^k}{2^k - 1} \cdot \frac{3^k}{3^k - 1} \cdot \frac{5^k}{5^k - 1} \cdot .. \cdot \frac{p^k}{p^k - 1} \cdot ..$$

We define the Ksi functions

$$\xi(k) = \frac{2^k}{2^k + 1} \cdot \frac{3^k}{3^k + 1} \cdot \frac{5^k}{5^k + 1} \cdot .. \cdot \frac{p^k}{p^k - 1} \cdot ...$$

Then,

$$\zeta(k)\xi(k) = \zeta(2k)$$

Therefore,

$$\zeta(k) = \frac{\zeta(2k)}{\xi(k)} = \zeta(2k)\frac{2^k + 1}{2^k} \cdot \frac{3^k + 1}{3^k} \cdot \frac{5^k + 1}{5^k} \cdot \frac{7^k + 1}{7^k} \cdot \frac{11^k + 1}{11^k} \cdot ....$$

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[3] [3] Leonhardi Euleri Opera Omnia, Series Prima, Opera Mathematica, Volumen 8, p. 288.

$$\boxed{\xi(k)=\frac{\zeta(2k)}{\zeta(k)}=\zeta(2k)\frac{2^k-1}{2^k}\cdot\frac{3^k-1}{3^k}\cdot\frac{5^k-1}{5^k}\cdot\frac{7^k-1}{7^k}\cdot\frac{11^k-1}{11^k}\cdot....}$$

$$\boxed{\frac{\xi(k)}{\zeta(k)}=\frac{2^k-1}{2^k+1}\cdot\frac{3^k-1}{3^k+1}\cdot\frac{5^k-1}{5^k+1}\cdot\frac{7^k-1}{7^k+1}\cdot\frac{11^k-1}{11^k+1}\cdot....}$$

4.

$\zeta(2), \xi(2)$

$$\boxed{\zeta(2)\xi(2) = \zeta(4) = \frac{\pi^4}{90}}$$

$$\boxed{\zeta(2) = \frac{2^2}{2^2-1}\cdot\frac{3^2}{3^2-1}\cdot\frac{5^2}{5^2-1}\cdot .... = 1 + \frac{1}{2^2} + \frac{1}{3^2} + ... = \frac{\pi^2}{6}}$$

$$\boxed{\frac{2^2-1}{2^2}\cdot\frac{3^2-1}{3^2}\cdot\frac{5^2-1}{5^2}\cdot\frac{7^2-1}{7^2}\cdot\frac{11^2-1}{11^2}\cdot ... = \frac{6}{\pi^2}}$$

$$\boxed{\xi(2) = \frac{2^2}{2^2+1}\cdot\frac{3^2}{3^2+1}\cdot\frac{5^2}{5^2+1}\cdot ..... = \frac{\zeta(4)}{\zeta(2)} = \frac{\pi^2}{15}}$$

Proof: $\xi(2) = \dfrac{\zeta(4)}{\zeta(2)} = \dfrac{\frac{\pi^4}{90}}{\frac{\pi^2}{6}} = \dfrac{\pi^2}{15}$

$$\boxed{\frac{2^2+1}{2^2}\cdot\frac{3^2+1}{3^2}\cdot\frac{5^2+1}{5^2}\cdot\frac{7^2+1}{7^2}\cdot\frac{11^2+1}{11^2}\cdot ... = \frac{15}{\pi^2}}$$

$$\boxed{\frac{\xi(2)}{\zeta(2)} = \frac{2^2-1}{2^2+1}\cdot\frac{3^2-1}{3^2+1}\cdot\frac{5^2-1}{5^2+1}\cdot\frac{7^2-1}{7^2+1}\cdot\frac{11^2-1}{11^2+1}\cdot ... = \frac{2}{5} = 0.4}$$

Proof: $\dfrac{\xi(2)}{\zeta(2)} = \dfrac{\frac{\pi^2}{15}}{\frac{\pi^2}{6}} = \dfrac{2}{5}$

5.

$\zeta(3), \xi(3)$

$$\boxed{\zeta(3) = \frac{2^3}{2^3-1}\cdot\frac{3^3}{3^3-1}\cdot\frac{5^3}{5^3-1}\cdot .... = 1 + \frac{1}{2^3} + \frac{1}{3^3} + ...}$$

$$\boxed{\xi(3) = \frac{2^3}{2^3+1}\cdot\frac{3^3}{3^3+1}\cdot\frac{5^3}{5^3+1}\cdot .....}$$

$$\boxed{\frac{\xi(3)}{\zeta(3)} = \frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}\cdot\frac{5^3-1}{5^3+1}\cdot\frac{7^3-1}{7^3+1}\cdot\frac{11^3-1}{11^3+1}\cdot ...}$$

$$\boxed{\zeta(3)\xi(3) = \zeta(6) = \frac{\pi^6}{945}}$$

$$\boxed{\zeta(3) = \frac{\pi^6}{945}\cdot\frac{2^3+1}{2^3}\cdot\frac{3^3+1}{3^3}\cdot\frac{5^3+1}{5^3}\cdot\frac{7^3+1}{7^3}\cdot\frac{11^3+1}{11^3}\cdot ...}$$

$$\boxed{\xi(3) = \frac{\pi^6}{945}\cdot\frac{2^3-1}{2^3}\cdot\frac{3^3-1}{3^3}\cdot\frac{5^3-1}{5^3}\cdot\frac{7^3-1}{7^3}\cdot\frac{11^3-1}{11^3}\cdot ...}$$

6.

$\zeta(4), \xi(4)$

$$\boxed{\zeta(4)\xi(4) = \zeta(8) = \frac{\pi^8}{9450}}$$

$$\boxed{\zeta(4) = \frac{2^4}{2^4-1}\cdot\frac{3^4}{3^4-1}\cdot\frac{5^4}{5^4-1}\cdot\frac{7^4}{7^4-1}\cdot\frac{11^4}{11^4-1}\cdot .. = 1 + \frac{1}{2^4} + \frac{1}{3^4} + .. = \frac{\pi^4}{90}}$$

$$\boxed{\frac{2^4-1}{2^4}\cdot\frac{3^4-1}{3^4}\cdot\frac{5^4-1}{5^4}\cdot\frac{7^4-1}{7^4}\cdot\frac{11^4-1}{11^4}\cdot ... = \frac{90}{\pi^4}}$$

$$\boxed{\xi(4) = \frac{2^4}{2^4+1}\cdot\frac{3^4}{3^4+1}\cdot\frac{5^4}{5^4+1}\cdot\frac{7^4}{7^4+1}\cdot\frac{11^4}{11^4+1}\cdot ..... = \frac{\zeta(8)}{\zeta(4)} = \frac{\pi^4}{105}}$$

Proof: $\xi(4) = \dfrac{\zeta(8)}{\zeta(4)} = \dfrac{\frac{\pi^8}{9450}}{\frac{\pi^4}{90}} = \dfrac{\pi^4}{105}$

$$\boxed{\frac{2^4+1}{2^4}\cdot\frac{3^4+1}{3^4}\cdot\frac{5^4+1}{5^4}\cdot\frac{7^4+1}{7^4}\cdot\frac{11^4+1}{11^4}\cdot ... = \frac{105}{\pi^4}}$$

$$\boxed{\frac{\xi(4)}{\zeta(4)} = \frac{2^4 - 1}{2^4 + 1} \cdot \frac{3^4 - 1}{3^4 + 1} \cdot \frac{5^4 - 1}{5^4 + 1} \cdot \frac{7^4 - 1}{7^4 + 1} \cdot \frac{11^4 - 1}{11^4 + 1} \cdot ... = \frac{6}{7} \approx 0.857}$$

Proof: $$\frac{\xi(4)}{\zeta(4)} = \frac{\frac{\pi^4}{105}}{\frac{\pi^4}{90}} = \frac{6}{7}$$

7.

$\zeta(5), \xi(5)$

$$\zeta(5) = \frac{2^5}{2^5 - 1} \cdot \frac{3^5}{3^5 - 1} \cdot \frac{5^5}{5^5 - 1} \cdot \frac{7^5}{7^5 - 1} \cdot \frac{11^5}{11^5 - 1} \cdot .... = 1 + \frac{1}{2^5} + \frac{1}{3^5} + ...$$

$$\xi(5) = \frac{2^5}{2^5 + 1} \cdot \frac{3^5}{3^5 + 1} \cdot \frac{5^5}{5^5 + 1} \cdot \frac{7^5}{7^5 + 1} \cdot \frac{11^5}{11^5 + 1} \cdot .....$$

$$\frac{\xi(5)}{\zeta(5)} = \frac{2^5 - 1}{2^5 + 1} \cdot \frac{3^5 - 1}{3^5 + 1} \cdot \frac{5^5 - 1}{5^5 + 1} \cdot \frac{7^5 - 1}{7^5 + 1} \cdot \frac{11^5 - 1}{11^5 + 1} \cdot ...$$

$$\zeta(5)\xi(5) = \zeta(10) = \frac{\pi^{10}}{93555}$$

$$\zeta(5) = \frac{\zeta(10)}{\xi(5)} = \frac{\pi^{10}}{93555} \cdot \frac{2^5 + 1}{2^5} \cdot \frac{3^5 + 1}{3^5} \cdot \frac{5^5 + 1}{5^5} \cdot \frac{7^5 + 1}{7^5} \cdot \frac{11^5 + 1}{11^5} \cdot ...$$

$$\xi(5) = \frac{\zeta(10)}{\zeta(5)} = \frac{\pi^{10}}{93555} \cdot \frac{2^5 - 1}{2^5} \cdot \frac{3^5 - 1}{3^5} \cdot \frac{5^5 - 1}{5^5} \cdot \frac{7^5 - 1}{7^5} \cdot \frac{11^5 - 1}{11^5} \cdot ...$$

8.

$\zeta(6), \xi(6)$

$$\boxed{\zeta(6) = \frac{2^6}{2^6-1}\cdot\frac{3^6}{3^6-1}\cdot\frac{5^6}{5^6-1}\cdot\frac{7^6}{7^6-1}\cdot\frac{11^6}{11^6-1}\cdot .... = 1+\frac{1}{2^6}+\frac{1}{3^6}+.... = \frac{\pi^6}{945}}$$

$$\boxed{\frac{2^6-1}{2^6}\cdot\frac{3^6-1}{3^6}\cdot\frac{5^6-1}{5^6}\cdot\frac{7^6-1}{7^6}\cdot\frac{11^6-1}{11^6}\cdot .... = \frac{945}{\pi^6}}$$

$$\boxed{\xi(6) = \frac{2^6}{2^6+1}\cdot\frac{3^6}{3^6+1}\cdot\frac{5^6}{5^6+1}\cdot ..... = \frac{\zeta(12)}{\zeta(6)} = \frac{691}{5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 15}\pi^6}$$

Proof: $\xi(6) = \dfrac{\zeta(12)}{\zeta(6)} = \dfrac{\dfrac{2^{10}}{13!}\dfrac{691}{105}\pi^{12}}{\dfrac{2^4}{7!}\dfrac{1}{3}\pi^6} = \dfrac{691}{5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 15}\pi^6$

$$\boxed{\frac{2^6+1}{2^6}\cdot\frac{3^6+1}{3^6}\cdot\frac{5^6+1}{5^6}\cdot ..... = \frac{5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 15}{691\cdot\pi^6}}$$

$$\boxed{\frac{\xi(6)}{\zeta(6)} = \frac{2^6-1}{2^6+1}\cdot\frac{3^6-1}{3^6+1}\cdot\frac{5^6-1}{5^6+1}\cdot\frac{7^6-1}{5^6+1}\cdot\frac{11^6-1}{5^6+1}\cdot ... = \frac{691}{715} \approx 0.966}$$

Proof: $$\frac{\xi(6)}{\zeta(6)} = \frac{\dfrac{691}{5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 15}\pi^6}{\dfrac{\pi^6}{945}} = \frac{691}{715}$$

$$\boxed{\zeta(6)\xi(6) = \zeta(12) = \frac{2^{10}}{13!}\frac{691}{105}\pi^{12}}$$

9.

Gamma Expanded in Zeta, and Ksi Functions

The absolute convergence of the Gamma Series allows us to reorder the series

$$\gamma = \frac{1}{2}\zeta(2) - \frac{1}{3}\zeta(3) + \frac{1}{4}\zeta(4) - \frac{1}{5}\zeta(5) + ...$$

$$= \frac{1}{2}\zeta(2)\left(1 + \frac{1}{2}\xi(2)\left[1 + \frac{1}{2}\xi(4)\left\{1 + \frac{1}{2}\xi(8)\left(1 + \frac{1}{2}\xi(16)\left[1 + ...\right]\right)\right\}\right]\right) +$$

$$-\frac{1}{3}\zeta(3)\left(1 - \frac{1}{2}\xi(3)\left[1 + \frac{1}{2}\xi(6)\left\{1 + \frac{1}{2}\xi(12)\left[1 + \frac{1}{2}\xi(24)\left(1 + ...\right)\right]\right\}\right]\right)$$

$$-\frac{1}{5}\zeta(5)\left(1 - \frac{1}{2}\xi(5)\left[1 + \frac{1}{2}\xi(10)\left\{1 + \frac{1}{2}\xi(20)\left(1 + \frac{1}{2}\xi(40)\left[1 + ...\right]\right)\right\}\right]\right)$$

....

Euler did not derive this expansion.

It is not as simple as the expansion in the zeta functions. But it gives an idea about the Ksi functions.

Euler did not use the notation $\zeta(5)$ for

$$1 + \frac{1}{2^5} + \frac{1}{3^5} + ... = \frac{2^5}{2^5 - 1} \cdot \frac{3^5}{3^5 - 1} \cdot \frac{5^5}{5^5 - 1} \cdot \frac{7^5}{7^5 - 1} \cdot \frac{11^5}{11^5 - 1} \cdot ....$$

And did not use the notation $\xi(5)$ for

$$\frac{2^5}{2^5 + 1} \cdot \frac{3^5}{3^5 + 1} \cdot \frac{5^5}{5^5 + 1} \cdot \frac{7^5}{7^5 + 1} \cdot \frac{11^5}{11^5 + 1} \cdot ....$$

He did show its equality to

$$1 - \frac{1}{2^5} - \frac{1}{3^5} + \frac{1}{4^5} - \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} + \frac{1}{10^5} - \frac{1}{11^5} - \frac{1}{12^5} - \frac{1}{13^5} + \dots,$$

where any term with even number of primes at the denominator is positive.

Euler did not investigate the Ksi functions further.

Investigation of the Ksi functions yields many more properties than the few observations that we have presented here.

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