Zeta(3) is Irrational Number and Sequentially Transcendental Number

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Abstract A number is algebraic if it is a zero of a polynomial in integer coefficients.

The algebraic numbers form a field with respect to addition , and multiplication. The sum, and product of two algebraic numbers is an algebraic number.

A number that is not algebraic, transcends algebraic numbers, and is called transcendental. The following are believed to be transcendental

e (Hermit),

π (Lindemann),

e^{rational} (Hermit),

$e^{\text{algebraic}}$ (Lindemann),

$(\text{algebraic})^{\text{irrational algebraic}}$ (Gelfond-Schneider).

$$e^{\pi} = (e^{i\pi})^{-i} = (-1)^{-i}$$

$$e^{n\pi} = (e^{i\pi})^{-ni} = (-1)^{-ni}$$

Recently, we showed[1] that Euler's Constant γ is a limit of a sequence of Transcendental numbers

Riemann's $\zeta(3)$ is Euler's infinite sum

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\ldots$$

The series sum has a denominator that includes the product of all the prime numbers. But there is no number that is divided by all the primes. Hence, no number for the denominator of a rational number that equals the series. And we conclude that

$\zeta(3)$ **is not a rational number**.

The same argument applies to show that $\zeta(5), \zeta(7), \ldots\ldots$ are Not rational numbers.

Euler computed $\zeta(3)$ with his summation formula.

$$\zeta(3) = 1.202056903159594 2853\ldots$$

Euler sought an answer[2] to the question whether the $\zeta(3)$ Series sums up to a rational number times π^3.

And Nahin, wrote a book[3] dedicated to that question.

Recently, we answered Euler's question in the negative. We showed[4] that $\zeta(3)$ does Not sum up to a rational number times π^3.

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[1] H. Vic Dannon, "Euler's Gamma is Irrational and Sequentially Transcendental Number", Gauge Institute Journal, Vol. 19, No.3, August 2023

[2] Euler's letters to James Stirling in "James Stirling, This about Series and such things" Scottish Academic Press 1988, pp. 142-151.

[3] Paul J. Nahin, "In Pursuit of Zeta-3, The World's Most Mysterious Unsolved Mathematical Problem", Princeton University Press, 2021.

[4] H. Vic Dannon, "All the π^3 series, and the $\zeta(3)$ series". Gauge Institute Journal, Vol. 18, No. 4, November 2022.

We show here that

$\zeta(3)$ **is Not a Liouville number**.

And we cannot say that $\zeta(3)$ is transcendental on account of its being a Liouville number which it is not.

Plouffe[5] gave a formula for $\zeta(3)$ by

$$\zeta(3) = \frac{7}{180}\pi^3 - 2\left[\frac{1}{e^{2\pi}-1} + \frac{1}{2^3}\frac{1}{e^{4\pi}-1} + \frac{1}{3^3}\frac{1}{e^{6\pi}-1} + \frac{1}{4^3}\frac{1}{e^{8\pi}-1} + ...\right]$$

Using this expansion, we show that

$\zeta(3)$ **is the limit of partial sums** $\zeta_j(3)$ **of transcendentals.**

This **Does Not** mean that $\zeta(3)$ is transcendental But that

As far as we can ever compute, for any finite j**,**

the partial sum $\zeta_j(3)$ **is a transcendental number**.

Indeed, a sequence of transcendental numbers need not converge to a transcendental number.

In 2022, we derived[6] an expansion for 1, which we named **The Archimedes Series for 1**

$$1 = \tfrac{\pi}{4} + \tfrac{1}{3}\left(\tfrac{\pi}{4}\right)^3 + \tfrac{2}{15}\left(\tfrac{\pi}{4}\right)^5 + \tfrac{17}{315}\left(\tfrac{\pi}{4}\right)^7 + \tfrac{62}{2835}\left(\tfrac{\pi}{4}\right)^9 +$$

$$+ \tfrac{(819)(691)}{3^6 5^2 7(11)(91)}\left(\tfrac{\pi}{4}\right)^{11} + \tfrac{5461}{3^5 5^2 7(11)(13)}\left(\tfrac{\pi}{4}\right)^{13} + ... + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!}B_n\left(\tfrac{\pi}{4}\right)^{2n-1} + ..$$

where B_n =Bernoulli Numbers

[5] Simon Plouffe, "Identities Inspired by Ramanujan Notebooks" July 1998, https://web.archive.org/web/20090130142844/http://www.lacim.uqam.ca/~plouffe/identities.html

[6] H. Vic Dannon, "Archimedes Series", Gauge Institute Journal, Vol. 18, No 3. August 2022, pp 1-11

Since π is transcendental, the Partial Sums of the Archimedes Series for 1 are transcendental numbers that converge to the algebraic number 1

And the

Transcendental partial sums 1_n transform to Algebraic 1

We shall say that besides being an Algebraic Number,

1 is a Sequentially Transcendental Number

That is, there is a sequence of transcendentals that converges to 1.

In fact, for an infinite hyper-real N we cannot compute the algebraic partial sum with N transcendental terms.

As far as we can ever compute, for any finite n,

the partial sum 1_n is Transcendental.

Similarly, for $\zeta(3)$

As far as we can ever compute, for any finite j,

the partial sum $\zeta_j(3)$ is a transcendental number.

And Similarly,

$\zeta(5), \zeta(7), \zeta(9), \ldots$**, are sequentially transcendental numbers.**

This leads us to discuss the meaning of Sequential Transcendence versus Transcendence.

And we conclude that

Sequential Transcendence is

a superior characterization of a number.

Contents

1.

$\zeta(2), \zeta(4), \zeta(6), \ldots$ are Transcendental Numbers

$$\zeta(2) = 1.6449340668482264364\ldots = \frac{1}{6}\pi^2$$

$$\zeta(4) = 1.0823232337111381915\ldots = \frac{1}{90}\pi^4$$

$$\zeta(6) = 1.0173430619844491397\ldots = \frac{1}{945}\pi^6$$

$$\zeta(8) = 1.0040773561979443393\ldots = (rational)\pi^8$$

$$\zeta(10) = 1.0009945751278180853\ldots = (rational)\pi^{10}$$

$$\zeta(12) = 1.0002460865533080482\ldots = (rational)\pi^{12}$$

$$\zeta(14) = 1.0000612481350587048\ldots = (rational)\pi^{14}$$

$$\zeta(16) = 1.0000152822594086518\ldots = (rational)\pi^{16}$$

....

$$\zeta(2k) = (rational)\pi^{2k}$$

Any power of π is transcendental. Because

$$\pi^n = \text{algebraic} \Rightarrow \pi = \sqrt[n]{\text{algebraic}} = \text{algebraic}$$

2.

$\zeta(3), \zeta(5), \zeta(7), ...$ are Irrational

$$\boxed{\zeta(3) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \text{...is irrational}}$$

<u>*Proof*</u>: For $n = 1, 2, 3, ...$

$$\zeta_n(3) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + ... + \frac{1}{n^3}$$

sums up to the quotient

$$\frac{p_n}{(n!)^3}.$$

For an infinite hyper-real N,

$\zeta_N(3)$ has a common denominator $(N!)^3$

where $N =$ infinite hyper-real.

That is, a common denominator that is the product of all the natural numbers,

$$2 \cdot 3 \cdot 4 \cdot ... = N!$$

being cubed.

This denominator includes the product of all the prime numbers

$$p_1 \cdot p_2 \cdot p_3 \cdot ....$$

There is no finite natural number q that is divided by all the primes.

Thus, there is no rational number $\frac{p}{q}$ that equals $\zeta(3)$.

Therefore,

$\zeta(3)$ is Not a rational number.

That is,

$\zeta(3)$ is Irrational.□

Similarly, for any $k = 1, 2, 3, \dots$,

$$\boxed{\zeta(2k+1) = 1 + \frac{1}{2^{2k+1}} + \frac{1}{3^{2k+1}} + \frac{1}{4^{2k+1}} + \dots = \text{irrational number}}$$

$\zeta(5) =$ 1.0369277551433699263... $=$ irrational number

$\zeta(7) =$ 1.0083492773819228268... $=$ irrational number

$\zeta(9) =$ 1.0020083928260822144... $=$ irrational number

$\zeta(11) =$ 1.0004941886041194645... $=$ irrational number

$\zeta(13) =$ 1.0001227133475784891... $=$ irrational number

$\zeta(15) =$ 1.0000305882363070204... $=$ irrational number

....

3.

$\zeta(3)$ is Not a Liouville Number

Liouville showed that If

α is the zero of a reduced polynomial

$P_n(x)$ of order n,

and if α is the limit of a sequence of rational numbers,

$$\frac{p_m}{q_m},$$

so that p_m, and q_m are relatively prime,

And there is a constant $C_m > 0$ so that for

$$q_m > C_m$$

Then,

$$\boxed{\left|\alpha - \frac{p_m}{q_m}\right| > \left(\frac{1}{q_m}\right)^{n+1}}$$

The negation of this statement is a criteria for transcendence.

If

τ is the limit of a sequence of rational numbers,

$$\frac{p_m}{q_m},$$

so that

p_m, and q_m are relatively prime,

and if for each $m = 1, 2, 3, \ldots$,

$$\boxed{\left|\tau - \frac{p_m}{q_m}\right| < \left(\frac{1}{q_m}\right)^{m+1}}$$

Then, $\tau = transcendental$

The partial sums of Euler's expansion of $\zeta(3)$ are such rationals

$$\frac{p_m}{q_m} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + ... + \frac{1}{m^3} = \frac{p_m}{(m!)^3}$$

$$\frac{1}{q_m} = \frac{1}{(m!)^3}$$

$$\left|\zeta(3) - \frac{p_m}{q_m}\right| = \frac{1}{(m+1)^3} + \frac{1}{(m+2)^3} + ...$$

is not bounded by $\left(\frac{1}{q_m}\right)^{m+1} = \frac{1}{(m!)^{3(m+1)}}$ for all $m = 1, 2, 3, ...$

Therefore, $\zeta(3)$ is not a Liouville Number.

4.

Ramanujan-Plouffe $\zeta(3), \zeta(5), \zeta(7), \ldots$

Inspired by Ramanujan Notebook II, p.293, formulas 25.1, 25.2, 25.3, Plouffe[7] expanded $\zeta(3), \zeta(5), \zeta(7), \ldots$ by

$$\zeta(3) = \frac{7}{180}\pi^3 - 2\left[\frac{1}{e^{2\pi}-1} + \frac{1}{2^3}\frac{1}{e^{4\pi}-1} + \frac{1}{3^3}\frac{1}{e^{6\pi}-1} + \frac{1}{4^3}\frac{1}{e^{8\pi}-1} + \ldots\right]$$

$$\zeta(5) = \frac{5}{1470}\pi^5 - \frac{3024}{1470}\left[\frac{1}{e^{2\pi}-1} + \frac{1}{2^5}\frac{1}{e^{4\pi}-1} + \frac{1}{3^5}\frac{1}{e^{6\pi}-1} + \ldots\right] + \frac{84}{1470}\left[\frac{1}{e^{2\pi}+1} + \frac{1}{2^5}\frac{1}{e^{4\pi}+1} + \frac{1}{3^5}\frac{1}{e^{6\pi}+1} + \ldots\right]$$

$$\zeta(7) = \frac{19}{56700}\pi^5 - \frac{113400}{1470}\left[\frac{1}{e^{2\pi}-1} + \frac{1}{2^7}\frac{1}{e^{4\pi}-1} + \frac{1}{3^7}\frac{1}{e^{6\pi}-1} + \ldots\right]$$

....

Plouffe expansions have the form

$$\zeta(2j+1) = \frac{l_{2j+1}}{k_{2j+1}}\pi^{2j+1} - \frac{m_{2j+1}}{k_{2j+1}}\left[\frac{1}{e^{2\pi}-1} + \frac{1}{2^{2j+1}}\frac{1}{e^{4\pi}-1} + \frac{1}{3^{2j+1}}\frac{1}{e^{6\pi}-1} + \ldots\right] + \frac{n_{2j+1}}{k_{2j+1}}\left[\frac{1}{e^{2\pi}+1} + \frac{1}{2^{2j+1}}\frac{1}{e^{4\pi}+1} + \frac{1}{3^{2j+1}}\frac{1}{e^{6\pi}+1} + \ldots\right]$$

accompanied by the following table[8]

$2j+1$	k_{2j+1}	l_{2j+1}	m_{2j+1}	n_{2j+1}

[7] https://en.wikipedia.org/wiki/Particular_values_of_the_Riemann_zeta_function
[8] The wikipedia table goes up to $2j+1=21$

3	180	7	360	
5	1470	5	3024	84
7	56700	19	113400	
9	18523890	625	3712264	74844
11	425675250	1453	851350500	

In 1998, $\zeta(3)$ was known to 32 million digits. And the formulas were confirmed to 50,000 digits.

We will apply the computer confirmed formula for $\zeta(3)$ to show that $\zeta(3)$ is Sequentially Transcendental number.

5.

$\zeta(3)$ is Sequentially Transcendental Number

By Plouffe,

$$\zeta(3)=\frac{7}{180}\pi^3 - 2\left[\frac{1}{e^{2\pi}-1}+\frac{1}{2^3}\frac{1}{e^{4\pi}-1}+\frac{1}{3^3}\frac{1}{e^{6\pi}-1}+\frac{1}{4^3}\frac{1}{e^{8\pi}-1}+...\right]$$

By Euler,

$$\frac{\pi^3}{32}=1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+....$$

$$\frac{7}{180}\pi^3=\frac{7}{180}32\left[1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+....\right]$$

For $j=1,2,3,...$, define the sequence of rationals

$$p_{3,j}=\frac{7}{180}32\left[1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+....+\frac{(-1)^{j+1}}{(2j+1)^3}\right]\to\frac{7}{180}\pi^3$$

The fraction

$$\frac{1}{e^{2\pi}-1}$$

is a transcendental number, that expands into the infinite series

$$\frac{1}{e^{2\pi}-1}=\frac{1}{e^{2\pi}}\frac{1}{1-\frac{1}{e^{2\pi}}}=\frac{1}{e^{2\pi}}+\frac{1}{e^{4\pi}}+\frac{1}{e^{6\pi}}+...$$

Each partial sum of the infinite series is transcendental too.

The fraction

$$\frac{1}{e^{4\pi}-1}$$

is a transcendental number, that expands into the infinite series

$$\frac{1}{e^{4\pi}-1}=\frac{1}{e^{4\pi}}\frac{1}{1-\dfrac{1}{e^{4\pi}}}=\frac{1}{e^{4\pi}}+\frac{1}{e^{8\pi}}+\frac{1}{e^{12\pi}}+...$$

That is included in the series of

$$\frac{1}{e^{2\pi}-1}$$

The fraction

$$\frac{1}{e^{6\pi}-1}$$

is a transcendental number, that expands into the infinite series

$$\frac{1}{e^{6\pi}-1}=\frac{1}{e^{6\pi}}\frac{1}{1-\dfrac{1}{e^{6\pi}}}=\frac{1}{e^{6\pi}}+\frac{1}{e^{12\pi}}+\frac{1}{e^{18\pi}}+...$$

That is included in the series of

$$\frac{1}{e^{2\pi}-1}$$

Summing up the infinite series of the powers of $\frac{1}{e^{2\pi}}$,

the coefficient of $\frac{1}{e^{2\pi}}$ is $1=r_1$

the coefficient of $\frac{1}{e^{4\pi}}$ is $1+\frac{1}{2^3}=r_2$

the coefficient of $\frac{1}{e^{6\pi}}$ is $1+\frac{1}{3^3}=r_3$

....

the coefficient of $\frac{1}{e^{2j\pi}}$ is the rational number r_j

Therefore,

$$\left[\frac{1}{e^{2\pi}-1}+\frac{1}{2^3}\frac{1}{e^{4\pi}-1}+\frac{1}{3^3}\frac{1}{e^{6\pi}-1}+\frac{1}{4^3}\frac{1}{e^{8\pi}-1}+...\right]$$

has partial sums

$$r_1\frac{1}{e^{2\pi}}+r_2\frac{1}{e^{4\pi}}+r_3\frac{1}{e^{6\pi}}+r_4\frac{1}{e^{8\pi}}+...r_j\frac{1}{e^{2j\pi}}.$$

These sums must be transcendental. Because for any $l = 1,2,3,...,j$, the powers $e^{-2l\pi}$ are linearly independent over the Rationals. Otherwise, if

$$r_1e^{-2\pi}+r_2e^{-4\pi}+r_3e^{-6\pi}+r_4e^{-8\pi}+...r_je^{-2j\pi}=\alpha \text{ algebraic},$$

$$-\alpha \text{ algebraic}+r_1e^{-2\pi}+r_2e^{-4\pi}+r_3e^{-6\pi}+r_4e^{-8\pi}+...r_je^{-2j\pi}=0,$$

and the transcendental $e^{-2\pi}$ is a zero of a polynomial of degree j.

Therefore, for $j = 1,2,3,4...$, the partial sums

$$\zeta_j(3)=p_{3,j}-2\left[r_1\frac{1}{e^{2\pi}}+r_2\frac{1}{e^{4\pi}}+r_3\frac{1}{e^{6\pi}}+r_4\frac{1}{e^{8\pi}}+...r_j\frac{1}{e^{2j\pi}}\right]\to\zeta(3)$$

are transcendental numbers.

Because if $\zeta_j(3)=$algebraic, then by the field property of algebraic numbers, $\zeta_j(3)-p_{3,j}=$algebraic. And

$$\underbrace{\zeta_j(3)-p_{3,j}}_{\text{algebraic}}=\underbrace{-2\left[r_1\frac{1}{e^{2\pi}}+r_2\frac{1}{e^{4\pi}}+r_3\frac{1}{e^{6\pi}}+r_4\frac{1}{e^{8\pi}}+...r_j\frac{1}{e^{2j\pi}}\right]}_{\text{transcendental}}$$

From that contradiction, it follows that $\zeta_j(3) =$transcendental.

This holds for any $j = 1, 2, 3, 4, \ldots$, including the infinite hyper-real number N. Consequently,

$$\zeta_N(3) = \text{transcendental}$$

$$\zeta_{N+1}(3) = \text{transcendental}$$

$$\zeta_N(3) = p_{3,N} - 2\left[r_1 \frac{1}{e^{2\pi}} + r_2 \frac{1}{e^{4\pi}} + r_3 \frac{1}{e^{6\pi}} + \ldots r_N \frac{1}{e^{2N\pi}}\right].$$

$$\zeta_{N+1}(3) = p_{3,N+1} - 2\left[r_1 \frac{1}{e^{2\pi}} + r_2 \frac{1}{e^{4\pi}} + \ldots + r_N \frac{1}{e^{2N\pi}} + r_{N+1} \frac{1}{e^{2(N+1)\pi}}\right]$$

$$\left|\zeta_{N+1}(3) - \zeta_N(3)\right| \leq \frac{1}{(2N+1)^3} + r_{N+1} \frac{1}{e^{2(N+1)\pi}}$$

$$\leq \text{Order of } \left(\frac{1}{2N+1}\right)^2$$

That is, $\zeta_N(3)$ is infinitesimally close to $\zeta(3)$.

$\zeta(3) =$ **is the limit of a sequence** $\zeta_j(3)$ **of transcendentals.**

This **Does Not** mean that $\zeta(3)$ is transcendental But that

As far as we can ever compute, for any finite j**,**

the partial sum $\zeta_j(3)$ **is a transcendental number**.

Indeed, a sequence of transcendental numbers need not converge to a transcendental number.

In 2022, we derived[9] an expansion for 1, which we named **The Archimedes Series for 1**

[9] H. Vic Dannon, "Archimedes Series", Gauge Institute Journal, Vol. 18, No 3. August 2022, pp 1-11

$$1 = \frac{\pi}{4} + \frac{1}{3}(\frac{\pi}{4})^3 + \frac{2}{15}(\frac{\pi}{4})^5 + \frac{17}{315}(\frac{\pi}{4})^7 + \frac{62}{2835}(\frac{\pi}{4})^9 +$$

$$+ \frac{(819)(691)}{3^6 5^2 7(11)(91)}(\frac{\pi}{4})^{11} + \frac{5461}{3^5 5^2 7(11)(13)}(\frac{\pi}{4})^{13} + ... + \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_n (\frac{\pi}{4})^{2n-1} + ..$$

where $B_n =$ Bernoulli Numbers

if π is transcendental, the Partial Sums of the Archimedes Series for 1 are transcendental numbers that converge to the algebraic number 1

And the

Transcendental partial sums 1_n transform to Algebraic 1

We shall say that besides being an Algebraic Number,

1 is a Sequentially Transcendental Number

That is, there is a sequence of transcendentals that converges to 1. In fact, for an infinite hyper-real N we cannot compute the algebraic partial sum with N transcendental terms.

As far as we can ever compute, for any finite n,

the partial sum 1_n is Transcendental.

Thus

$\zeta(3)$ is Sequentially Transcendental Number

Meaning that,

As far as we can ever compute, for any finite j,

the partial sum $\zeta_j(3)$ is a transcendental number.

Similarly,

$\zeta(5), \zeta(7), \zeta(9), ...$**, are sequentially transcendental numbers.**

6.

$\zeta(3)e$ is Irrational and Sequentially Transcendental Number

Irrationality

$$\zeta(3)e = \left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + ...\right)\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\right)$$

is the sum of infinitely many rational numbers with common denominator that includes the product of all the prime numbers

$$p_1 \cdot p_2 \cdot p_3 \cdot ....$$

There is no finite natural number

$$q$$

that is divided by all the primes.

Thus, there is no rational number $\frac{p}{q}$ that equals $\zeta(3)e$

Therefore, $\zeta(3)e$ is Not a rational number.

That is, $\zeta(3)e$ is Irrational.□

Sequentially Transcendental

For $j = 1,2,3,...,N$, $\boxed{\zeta_j(3)e_j = \text{transcendental}}$

Proof:

$$\zeta_j(3)e_j = (\text{transcendental})\underbrace{(1 + 1 + \tfrac{1}{2!} + \tfrac{1}{3!} + ... + \tfrac{1}{j!})}_{\frac{p}{j!} = \text{algebraic}}$$

Therefore, $\dfrac{\zeta_j(3)e_j}{\text{algebraic}} =$ transcendental.

If $\zeta_j(3)e_j =$ algebraic, then by the field property of algebraic numbers, $\dfrac{\zeta_j(3)e_j}{\text{algebraic}} =$ algebraic.

From that contradiction, it follows that $\zeta_j(3)e_j =$ transcendental.□

This holds for any $j = 1, 2, 3, 4, \ldots$,

Consequently,

$$\boxed{\zeta(3)e = \text{Sequentially Transcendental Number}}$$

7.

$\zeta(3)\pi$ is Irrational and Sequentially Transcendental Number

Irrationality

$$\zeta(3)\pi = \left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + ...\right)4\left(1 - \frac{1}{3} + \frac{1}{5} - ...\right)$$

is the sum of infinitely many rational numbers with common denominator that includes the product of all the prime numbers

$$p_1 \cdot p_2 \cdot p_3 \cdot ....$$

There is no finite natural number

$$q$$

that is divided by all the primes.

Thus, there is no rational number $\frac{p}{q}$ that equals $\pi + e$

Therefore, $\pi + e$ is Not a rational number.

That is, $\pi + e$ is Irrational.□

Sequentially Transcendental

For $j = 1, 2, 3, ..., N$, $\boxed{\zeta_j(3)\pi_j = \text{transcendental}}$

Proof:

$$\zeta_j(3)\pi_j = (\text{transcendental})\underbrace{4(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + ... + \tfrac{(-1)^{j+1}}{2j-1})}_{\text{algebraic}}$$

Therefore, $\dfrac{\zeta_j(3)\pi_j}{\text{algebraic}} =$transcendental.

If $\zeta_j(3)\pi_j =$algebraic, then by the field property of algebraic numbers, $\dfrac{\zeta_j(3)\pi_j}{\text{algebraic}} =$algebraic.

From that contradiction, it follows that $\zeta_j(3)\pi_j =$transcendental.□

This holds for any $n = 1, 2, 3, 4, ...,$

Consequently,

$$\boxed{\zeta(3)\pi = \text{Sequentially Transcendental Number}}$$

8.

$\zeta(3) + e$ is Irrational and Sequentially Transcendental Number

Irrationality

$$\zeta(3) + e = \left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + ...\right) + \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\right)$$

is the sum of infinitely many rational numbers with common denominator that includes the product of all the prime numbers

$$p_1 \cdot p_2 \cdot p_3 \cdot ....$$

There is no finite natural number

$$q$$

that is divided by all the primes.

Thus, there is no rational number $\frac{p}{q}$ that equals $\zeta(3) + e$

Therefore, $\zeta(3) + e$ is Not a rational number.

That is, $\zeta(3) + e$ is Irrational.□

Sequentially Transcendental

For $j = 1, 2, 3, ..., N$, $\boxed{\zeta_j(3) + e_j \; = \text{transcendental}}$

Proof:

$$\zeta_j(3) + e_j \; = (\text{transcendental}) + (\underbrace{1 + 1 + \tfrac{1}{2!} + \tfrac{1}{3!} + ... + \tfrac{1}{j!}}_{\frac{p}{j!} = \text{algebraic}})$$

Therefore, $\zeta_j(3) + e_j - \text{algebraic} =$ **transcendental.**

If $\zeta_j(3) + e_j =$ **algebraic, then by the field property of algebraic numbers,** $\zeta_j(3) + e_j - \text{algebraic} =$ **algebraic.**

From the contradiction, it follows that $\zeta_j(3) + e_j =$ **transcendental.**$\square$

This holds for any $j = 1, 2, 3, 4, \dots$ **,**

Consequently,

$$\boxed{\zeta(3) + e = \text{Sequentially Transcendental Number}}$$

9.

$\zeta(3)+\pi$ is Irrational and Sequentially Transcendental Number

Irrationality

$$\zeta(3)+\pi=\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+...\right)+4\left(1-\frac{1}{3}+\frac{1}{5}-...\right)$$

is the sum of infinitely many rational numbers with common denominator that includes the product of all the prime numbers

$$p_1 \cdot p_2 \cdot p_3 \cdot ....$$

There is no finite natural number

$$q$$

that is divided by all the primes.

Thus, there is no rational number $\frac{p}{q}$ that equals $\zeta(3)+\pi$

Therefore, $\zeta(3)+\pi$ is Not a rational number.

That is, $\zeta(3)+\pi$ is Irrational.□

Sequentially Transcendental

For $j = 1, 2, 3, ..., N$, $\boxed{\zeta_j(3) + \pi_j = \text{transcendental}}$

Proof:

$$\zeta_j(3) + \pi_j = (\text{transcendental}) + \underbrace{4(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + ... + \tfrac{(-1)^{j+1}}{2j-1})}_{\text{algebraic}}$$

Therefore, $\zeta_j(3) + \pi_j - \text{algebraic} =$ transcendental.

If $\zeta_j(3) + \pi_j =$ algebraic, then by the field property of algebraic numbers, $\zeta_j(3) + \pi_j - \text{algebraic} =$ algebraic.

From the contradiction, it follows that

$\zeta_j(3) + \pi_j =$ transcendental. $\square$

This holds for any $j = 1, 2, 3, 4, \dots$,

Consequently,

$$\boxed{\zeta(3) + \pi = \text{Sequentially Transcendental Number}}$$

10.

Sequential Transcendence versus Transcendence

A number ξ is Transcendental if it is not the root of any n degree polynomial equation with rational coefficients, for any finite natural number n.

This definition excludes any infinite hyper-real number N. Indeed,

10.1

The Transcendental number π is the root of a polynomial equation with rational coefficients of degree N.

Proof One such polynomial equation of degree N with rational coefficients follows from our 2022 derivation[10] of an expansion for 1, which we named **The Archimedes Series for 1**

$$1 = \frac{\pi}{4} + \frac{1}{3}\left(\frac{\pi}{4}\right)^3 + \frac{2}{15}\left(\frac{\pi}{4}\right)^5 + \frac{17}{315}\left(\frac{\pi}{4}\right)^7 + \frac{62}{2835}\left(\frac{\pi}{4}\right)^9 +$$

$$+ \frac{(819)(691)}{3^6 5^2 7(11)(91)}\left(\frac{\pi}{4}\right)^{11} + \frac{5461}{3^5 5^2 7(11)(13)}\left(\frac{\pi}{4}\right)^{13} + ... + \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_n \left(\frac{\pi}{4}\right)^{2n-1} + ..$$

where B_n =Bernoulli Numbers.□

Similarly, our definition of sequential Transcendence breaks down for $n = N$

[10] H. Vic Dannon, "Archimedes Series", Gauge Institute Journal, Vol. 18, No 3. August 2022, pp 1-11

We defined a number ξ to be sequentially transcendental if for any finite natural number n there a transcendental number as close as we wish to ξ.

This definition excludes any infinite hyper-real number N. Indeed, if we allow $n = N$, then

10.2

For an Algebraic, and Sequentially Transcendental α,

ξ_N must be algebraic

For instance, for the Algebraic number 1, the partial sum

$$1_N = \frac{\pi}{4} + \frac{1}{3}\left(\frac{\pi}{4}\right)^3 + \frac{2}{15}\left(\frac{\pi}{4}\right)^5 + \frac{17}{315}\left(\frac{\pi}{4}\right)^7 + \frac{62}{2835}\left(\frac{\pi}{4}\right)^9 +$$

$$+ \frac{(819)(691)}{3^6 5^2 7(11)(91)}\left(\frac{\pi}{4}\right)^{11} + \frac{5461}{3^5 5^2 7(11)(13)}\left(\frac{\pi}{4}\right)^{13} + ... + \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_N \left(\frac{\pi}{4}\right)^{2n-1}$$

where $B_n =$ Bernoulli Numbers for $n = 1, 2, ...N$

is infinitesimally close to 1,

$$1 = 1_N + \text{infinitesimal}$$

Therefore,

$$\underbrace{1}_{\text{algebraic}} = \{\text{the standard part of } 1\} = 1_N$$

That is, for any finite n,

$$1_n = \text{transcendental}$$

But for an infinite hyper-real N

$$1_N = \text{algebraic}$$

It follows that

Our definitions of Transcendental, and Sequentially Transcendental apply only to finite n

In any event, we cannot compute with any infinite n.

But if we are limited to finite n, then the transcendental π is actually the Leibniz rational partial sum

$$\pi_n = 4\left\{1 - \frac{1}{3} + \frac{1}{5} - .... + \frac{(-1)^{n+1}}{2n-1}\right\}$$

that can be made as close as we can compute to π

And γ is actually the transcendental

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + ...\frac{1}{n} - \log(n+1)$$

that can be made as close as we can compute to γ

In other words, since transcendence breaks down at the forever incomprehensible infinity

Sequential Transcendence is way more informative then Transcendence

In 2022, we derived[11] an expansion for π, which we named **The Archimedes Series for π**

$$\pi = \tfrac{1}{2}\{\alpha(2\pi) + \tfrac{1}{3}\alpha^3(2\pi) + \tfrac{2}{15}\alpha^5(2\pi) + \tfrac{17}{315}\alpha^7(2\pi) + \tfrac{62}{2835}\alpha^9(2\pi) +$$

$$+ \tfrac{(819)(691)}{3^6 5^2 7(11)(91)}\alpha^{11}(2\pi) + \tfrac{5461}{3^5 5^2 7(11)(13)}\alpha^{13}(2\pi) + ... + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!}B_n\alpha^{2n-1}(2\pi) + ...\}$$

where $\alpha(2\pi) = \arctan(2\pi) \approx 1.412965137..$

[11] H. Vic Dannon, "Archimedes Series", Gauge Institute Journal, Vol. 18, No 3. August 2022, pp 1-11

and $B_n =$ Bernoulli Numbers.

The Transcendental partial sums

$$\pi_n = \frac{1}{2}\{\alpha(2\pi) + \frac{1}{3}\alpha^3(2\pi) + \frac{2}{15}\alpha^5(2\pi) + \frac{17}{315}\alpha^7(2\pi) + \frac{62}{2835}\alpha^9(2\pi) +$$

$$+\frac{(819)(691)}{3^6 5^2 7(11)(91)}\alpha^{11}(2\pi)+\frac{5461}{3^5 5^2 7(11)(13)}\alpha^{13}(2\pi)+.... + \frac{2^{2n}(2^{2n}-1)}{(2n)!}B_n\alpha^{2n-1}(2\pi)\}$$

represent π better than any inapplicable statement about its not being a root of a polynomial equation.

We conclude that

Sequential Transcendence is

a superior characterization of a number.

Appendix
Transcendental Numbers

Liouville $\boxed{\frac{1}{10}+\frac{1}{10^{2!}}+\frac{1}{10^{3!}}+\frac{1}{10^{4!}}+....=transcendental}$

Hermit $\boxed{e^{\text{rational}}=transcendental}$

Lindemann $\boxed{e^{\text{algebraic}}=transcendental}$

$$\boxed{e^{\tau}=\text{algebraic}\Rightarrow\tau\text{=tanscendental}}$$

$$e^{i\pi}=-1\Rightarrow i\pi=\text{trans}\ \Rightarrow\pi=-i\cdot\text{trans}=\text{trans}$$

α_1,α_2 =**Algebraically Dependent over** $\mathbb{Q}$

<u>iff</u> $P(\alpha_1,\alpha_2)=0$ where $P(x,y)$ has coefficients from $\mathbb{Q}$

$\sqrt{\pi},\pi=$ algebraically dependent over $\mathbb{Q}$ with $P(x,y)=x^2-y$.

Lindemann-Weierstrass

$\alpha_1,\alpha_2,\alpha_3=$ algebraic

$\beta_1,\beta_2,\beta_3=$ algebraic, Linearly independent over $\mathbb{Q}$

$$\boxed{\Rightarrow\beta_1e^{\alpha_1}+\beta_2e^{\alpha_2}+\beta_3e^{\alpha_3}=\text{transcendental}}$$

α algebraic $\neq 0\ \Rightarrow\ \cos\alpha,\ \sin\alpha,\ \tan\alpha=$ **transcendental,**

α algebraic $\neq 0,1\Rightarrow\ \log\alpha$ =transcendental

$\alpha_1,\alpha_2,\alpha_3$ =algebraic, linearly independent over $\mathbb{Q}$

$$\Rightarrow e^{\alpha_1},e^{\alpha_2},e^{\alpha_2}=\text{ algebraically independent over }\mathbb{Q}$$

$$\Rightarrow r_1 e^{\alpha_1} + r_2 e^{\alpha_2} + r_3 e^{\alpha_3} = transcendental \text{ for any } r_1, r_2, r_3 \in \mathbb{Q}$$

Baker $\alpha_1 \neq \alpha_2 \neq \alpha_3$ algebraic

$$\Rightarrow e^{\alpha_1}, e^{\alpha_2}, e^{\alpha_3} = \text{linearly independent over } \mathbb{A}$$

$$\Rightarrow \beta_1 e^{\alpha_1} + \beta_2 e^{\alpha_2} + \beta_3 e^{\alpha_3} = transcendental \text{ for any } \beta_1, \beta_2 \in \mathbb{A}$$

Gelfond

$\alpha \neq 0,1$ algebraic , $\beta =$irrational algebraic $\boxed{\Rightarrow \alpha^{\beta} = \text{transcendental}}$

$$2^{\sqrt{2}},$$

$$\sqrt{2}^{\sqrt{2}},$$

$$e^{\pi} = (e^{i\pi})^{-i} = (-1)^{-i},$$

$$e^{-\frac{1}{2}\pi} = (e^{i\frac{1}{2}\pi})^{i} = (i)^{i}.$$

Gelfond-Schneider

$\alpha_1, \alpha_2 =$ algebraic $\neq 0,1$

$\beta_1, \beta_2 =$ algebraic,

$1, \beta_1, \beta_2 =$Linearly independent over $\mathbb{Q}$

$$\boxed{\Rightarrow \alpha_1^{\beta_1} \alpha_2^{\beta_2} = transcendental}$$

Baker

$\alpha_1 \neq \alpha_2$ algebraic $\neq 0,1$

$\beta_1, \beta_2 =$ irrational algebraic,

$1, \beta_1, \beta_2 =$ linearly independent over $\mathbb{Q}$

$$\boxed{\Rightarrow \alpha_1^{\beta_1}\alpha_2^{\beta_2} = transcendental}$$

Gelfond-Schneider

$\alpha_1, \alpha_2 =$ algebraic, $\neq 0,1$

$\beta_1, \beta_2 =$ algebraic,

$\log\alpha_1, \log\alpha_2 =$ Linearly independent over $\mathbb{Q}$

$$\boxed{\Rightarrow \beta_1 \log\alpha_1 + \beta_2 \log\alpha_2 = transcendental}$$

Gelfond-Schneider-Baker

$\alpha_1, \alpha_2, \alpha_3 =$ algebraic, $\neq 0,1$

$\beta_0, \beta_1, \beta_2, \beta_3 =$ algebraic,

$\beta_0, \beta_1, \beta_2, \beta_3 =$ Linearly independent over $\mathbb{Q}$

$$\boxed{\Rightarrow e^{\beta_0}\alpha_1^{\beta_1}\alpha_2^{\beta_2}\alpha_3^{\beta_3} = \text{transcendental}}$$

Gelfond-Schneider-Baker

$\alpha_1, \alpha_2, \alpha_3 =$ algebraic, $\neq 0,1$

$\beta_0 \neq 0, \beta_1, \beta_2, \beta_3 =$ algebraic,

$\log\alpha_1, \log\alpha_2, \log\alpha_3 =$ Linearly independent over $\mathbb{Q}$

$$\boxed{\Rightarrow \beta_0 + \beta_1 \log\alpha_1 + \beta_2 \log\alpha_2 + \beta_3 \log\alpha_3 = \text{transcendental}}$$

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