$\pi + e$, πe, e^{π}, π^{e}, e^{e}, π^{π}, $e^{\pi e}$, $\pi^{e\pi}$, $\log \pi$ are Sequentially Transcendental Numbers

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Abstract: A number ξ may be algebraic or transcendental.

If a sequence of transcendental numbers ξ_n converges to ξ,

We then say that

$$\xi = \text{Sequentially Transcendental Number.}$$

A sequence of transcendentals need not converge to a transcendental number.

In 2022, we derived[1] an expansion for 1, which we named **The Archimedes Series for 1**

$$1 = \frac{\pi}{4} + \frac{1}{3}\left(\frac{\pi}{4}\right)^3 + \frac{2}{15}\left(\frac{\pi}{4}\right)^5 + \frac{17}{315}\left(\frac{\pi}{4}\right)^7 + \frac{62}{2835}\left(\frac{\pi}{4}\right)^9 +$$

$$+ \frac{(819)(691)}{3^6 5^2 7(11)(91)}\left(\frac{\pi}{4}\right)^{11} + \frac{5461}{3^5 5^2 7(11)(13)}\left(\frac{\pi}{4}\right)^{13} + \ldots. + \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_n \left(\frac{\pi}{4}\right)^{2n-1} + ..$$

where B_n =Bernoulli Numbers

Since π is transcendental, the Partial Sums of the Archimedes Series for 1 are transcendental numbers that converge to the

[1] H. Vic Dannon, "Archimedes Series", Gauge Institute Journal, Vol. 18, No 3. August 2022, pp 1-11

algebraic number 1

And the

Transcendental partial sums 1_n **transform to Algebraic** 1

We shall say that besides being an Algebraic Number,

1 is a Sequentially Transcendental Number

That is, there is a sequence of transcendentals that converges to 1.

In fact, for an infinite hyper-real N we cannot compute the algebraic partial sum with N transcendental terms.

As far as we can ever compute, for any finite n**,**

the partial sum 1_n **is Transcendental.**

We show that

$$\pi + e,\ \pi e,\ e^{\pi},\ \pi^{e},\ e^{e},\ \pi^{\pi},\ e^{\pi e},\ \pi^{\pi e},\ \log \pi$$

are sequentially transcendental numbers.

This **Does Not** mean that **they are** transcendental But that

As far as we can ever compute, for any finite j**,**

the sequences' terms transcendental numbers.

Indeed, a sequence of transcendental numbers need not converge to a transcendental number.

In 2022, we derived[2] an expansion for 1, which we named **The Archimedes Series for 1**

[2] H. Vic Dannon, "Archimedes Series", Gauge Institute Journal, Vol. 18, No 3. August 2022, pp 1-11

$$1 = \frac{\pi}{4} + \frac{1}{3}(\frac{\pi}{4})^3 + \frac{2}{15}(\frac{\pi}{4})^5 + \frac{17}{315}(\frac{\pi}{4})^7 + \frac{62}{2835}(\frac{\pi}{4})^9 +$$

$$+ \frac{(819)(691)}{3^6 5^2 7(11)(91)}(\frac{\pi}{4})^{11} + \frac{5461}{3^5 5^2 7(11)(13)}(\frac{\pi}{4})^{13} + ... + \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_n (\frac{\pi}{4})^{2n-1} + ..$$

where B_n =Bernoulli Numbers

Since π is transcendental, the Partial Sums of the Archimedes Series for 1 are transcendental numbers that converge to the algebraic number 1

And the

Transcendental partial sums 1_n transform to Algebraic 1

We shall say that besides being an Algebraic Number,

1 is a Sequentially Transcendental Number

That is, there is a sequence of transcendentals that converges to 1.

In fact, for an infinite hyper-real N we cannot compute the algebraic partial sum with N transcendental terms.

As far as we can ever compute, for any finite n,

the partial sum 1_n is Transcendental.

Similarly, here,

As far as we can ever compute, for any finite j,

the sequences' terms are transcendental numbers.

This leads us to discuss the meaning of Sequential Transcendence versus Transcendence. And we conclude that

Sequential Transcendence is

a superior characterization of a number.

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Contents

1.

$\pi + e$ is Irrational, and Sequentially Transcendental Number

Irrationality

$$\pi + e = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \ldots\right) + \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$$

is the sum of infinitely many rational numbers with common denominator that includes the product of all the prime numbers

$$p_1 \cdot p_2 \cdot p_3 \cdot \ldots.$$

There is no finite natural number

$$q$$

that is divided by all the primes.

Thus, there is no rational number $\frac{p}{q}$ that equals $\pi + e$

Therefore, $\pi + e$ is Not a rational number.

That is, $\pi + e$ is Irrational.□

Sequentially Transcendental

For $n = 1, 2, 3, \ldots, N$, $\boxed{\pi + e_n = \text{transcendental}}$

Proof:

$$\pi + e_n = (\text{transcendental}) + (\underbrace{1 + 1 + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{n!}}_{\text{rational}})$$

Therefore,

$$\pi + e_n - \text{algebraic} = \textbf{transcendental.}$$

If $\pi + e_n =$ algebraic, then by the field property of algebraic numbers, $\pi + e_n - \text{algebraic} =$ algebraic.

From the contradiction, it follows that $\pi + e_n =$ transcendental.□

This holds for any $n = 1,2,3,4,...$,

Consequently,

$$\boxed{\pi + e = \text{Sequentially Transcendental Number}}$$

2.

πe is Irrational, and Sequentially Transcendental Number

Irrationality

$$\pi e = 4\left(1 - \frac{1}{3} + \frac{1}{5} - ...\right)\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\right)$$

is the sum of infinitely many rational numbers with common denominator that includes the product of all the prime numbers

$$p_1 \cdot p_2 \cdot p_3 \cdot ....$$

There is no finite natural number

$$q$$

that is divided by all the primes.

Thus, there is no rational number $\frac{p}{q}$ that equals πe

Therefore, πe is Not a rational number.

That is, πe is Irrational.□

Sequentially Transcendental

For $n = 1,2,3,...,N$, $\boxed{\pi e_n = \text{transcendental}}$

Proof:

$$\pi e_n = (\text{transcendental})(\underbrace{1+1+\tfrac{1}{2!}+\tfrac{1}{3!}+...+\tfrac{1}{n!}}_{\text{rational}})$$

Therefore,

$$\frac{\pi e_n}{\text{algebraic}} = \text{transcendental.}$$

If $\pi e_n =$ algebraic, then by the field property of algebraic numbers,

$$\frac{\pi e_n}{\text{algebraic}} = \frac{\text{algebraic}}{\text{algebraic}} = \text{algebraic}.$$

From that contradiction, it follows that $\pi e_n =$ transcendental. $\square$

This holds for any $n = 1, 2, 3, 4, ...,$

Consequently,

$$\boxed{\pi e = \text{Sequentially Transcendental Number}}$$

3.

e^{π} is Irrational, and Sequentially Transcendental Number

Irrationality

$$e^{\pi} = \frac{1}{0!} + \frac{1}{1!}4\left(1 - \frac{1}{3} + \frac{1}{5} - ...\right) + \frac{1}{2!}4^2\left(1 - \frac{1}{3} + \frac{1}{5} - ...\right)^2 + \frac{1}{3!}4^3\left(1 - \frac{1}{3} + \frac{1}{5} - ...\right)^3 + \frac{1}{4!}4^4\left(1 - \frac{1}{3} + \frac{1}{5} - ...\right)^4 + ...$$

is the sum of infinitely many rational numbers with common denominator that includes the product of all the prime numbers

$$p_1 \cdot p_2 \cdot p_3 \cdot ....$$

There is no finite natural number

$$q$$

that is divided by all the primes.

Thus, there is no rational number $\frac{p}{q}$ that equals e^{π}

Therefore, e^{π} is Not a rational number.

That is, e^{π} is Irrational.□

Sequentially Transcendental

For $n = 1, 2, 3, ..., N$, $\boxed{e^{\pi_n} = \text{Transcendental}}$

Proof:

$$e^{\pi_n} = e^{4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots+\frac{(-1)^{n+1}}{2n-1}\right)}$$

By Hermit, $e^{4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots+\frac{(-1)^{n+1}}{2n-1}\right)} = e^{\text{rational}} =$ transcendental

This holds for any $n = 1, 2, 3, 4, \ldots,$

Consequently,

$$\boxed{e^{\pi} = \text{Sequentially Transcendental Number}}$$

4.

π^e is Sequentially Transcendental Number

For $n = 1, 2, 3, ..., N$, $\boxed{\pi^{e_n} = \text{Transcendental}}$

Proof:

$$\pi^{e_n} = \pi^{\underbrace{1+1+\frac{1}{2!}+\frac{1}{3!}+...+\frac{1}{n!}}_{rational}}$$

Then, π^{e_n} must be transcendental.

Because

$$\pi^{\underbrace{1+1+\frac{1}{2!}+\frac{1}{3!}+...+\frac{1}{n!}}_{rational}} = \textbf{algebraic} \Rightarrow \pi = (\text{algebraic})^{rational} = \text{algebraic}$$

This holds for any $n = 1, 2, 3, 4, ...$,

Consequently,

$$\boxed{\pi^e = \text{Sequentially Transcendental Number}}$$

5.

π^{π} is Sequentially Transcendental Number

For $n = 1, 2, 3, ..., N$, $\boxed{\pi^{\pi_n} = \text{Transcendental}}$

Proof:

$$\pi^{\pi_n} = \pi^{\underbrace{4\left(1 - \frac{1}{3} + \frac{1}{5} + ... + \frac{(-1)^{n+1}}{2n-1}\right)}_{rational}}$$

Then, π^{π_n} must be transcendental.

Because

$$\pi^{\underbrace{4\left(1 - \frac{1}{3} + \frac{1}{5} + ... + \frac{(-1)^{n+1}}{2n-1}\right)}_{rational}} = \text{algebraic}$$

$$\Rightarrow \pi = (\text{algebraic})^{rational} = \text{algebraic}$$

This holds for any $n = 1, 2, 3, 4, ...,$

Consequently,

$$\boxed{\pi^{\pi} = \text{Sequentially Transcendental Number}}$$

6.

e^e is Irrational, and Sequentially Transcendental Number

Irrationality

$$e^e = \frac{1}{0!} + \frac{1}{1!}\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\right) + \frac{1}{2!}\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\right)^2$$
$$+ \frac{1}{3!}\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\right)^3 + \frac{1}{4!}\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\right)^4 + ...$$

is the sum of infinitely many rational numbers with common denominator that includes the product of all the prime numbers

$$p_1 \cdot p_2 \cdot p_3 \cdot ....$$

There is no finite natural number

$$q$$

that is divided by all the primes.

Thus, there is no rational number $\frac{p}{q}$ that equals e^e

Therefore, e^e is Not a rational number.

That is, e^e is Irrational.□

Sequentially Transcendental

For $n = 1, 2, 3, ..., N$, $\boxed{e^{e_n} = \text{Transcendental}}$

Proof:

$$e^{e_n} = e^{\underbrace{1+1+\frac{1}{2!}+\frac{1}{3!}+...+\frac{1}{n!}}_{rational}}$$

Then, e^{e_n} must be transcendental.

Because

$$e^{\underbrace{1+1+\frac{1}{2!}+\frac{1}{3!}+...+\frac{1}{n!}}_{rational}} = \textbf{algebraic} \Rightarrow e = (\text{algebraic})^{rational} = \text{algebraic}$$

This holds for any $n = 1, 2, 3, 4, ...,$

Consequently,

$$\boxed{e^e = \text{Sequentially Transcendental Number}}$$

7.

$e^{\pi e}$ is Irrational, and Sequentially Transcendental Number

Irrationality

$$e^{\pi e} = \frac{1}{0!} + \frac{1}{1!}\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\right)4\left(1 - \frac{1}{3} + \frac{1}{5} + ...\right)$$

$$+ \frac{1}{2!}\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\right)^2 4^2\left(1 - \frac{1}{3} + \frac{1}{5} + ...\right)^2$$

$$+ \frac{1}{3!}\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\right)^3 4^3\left(1 - \frac{1}{3} + \frac{1}{5} + ...\right)^3 + ...$$

is the sum of infinitely many rational numbers with common denominator that includes the product of all the prime numbers

$$p_1 \cdot p_2 \cdot p_3 \cdot ....$$

There is no finite natural number

$$q$$

that is divided by all the primes.

Thus, there is no rational number $\frac{p}{q}$ that equals $e^{\pi e}$

Therefore, $e^{\pi e}$ is Not a rational number.

That is, $e^{\pi e}$ is Irrational.□

Sequentially Transcendental

For $n = 1,2,3,...,N$, $\boxed{e^{\pi_n e_n} = \text{transcendental}}$

Proof:

$$e^{\pi_n e_n} = e^{\underbrace{4\left(1-\frac{1}{3}+...+\frac{(-1)^{n-1}}{2n-1}\right)\left(\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+...+\frac{1}{n!}\right)}_{rational}} = transcendental$$

This holds for any $n = 1,2,3,4,...,$

Consequently,

$$\boxed{e^{\pi e} = \text{Sequentially Transcendental Number}}$$

8.

$\pi^{\pi e}$ is Sequentially Transcendental Number

For $n = 1, 2, 3, \ldots, N$, $\boxed{\pi^{\pi_n e_n} = \text{transcendental}}$

Proof:

$$\pi^{\pi_n e_n} = \pi^{\underbrace{4\left(1-\frac{1}{3}+\ldots+\frac{(-1)^{n-1}}{2n-1}\right)\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)}_{rational}} = transcendental$$

This holds for any $n = 1, 2, 3, 4, \ldots$,

Consequently,

$$\boxed{\pi^{\pi e} = \text{Sequentially Transcendental Number}}$$

9.

$\log \pi$ is Sequentially Transcendental Number

For $n = 1, 2, 3, ..., N$, $\boxed{\log(\pi_n) = \text{transcendental}}$

Proof:

$$\log(\pi_n) = \log\left[4\left(1 - \frac{1}{3} + ... + \frac{(-1)^{n-1}}{2n-1}\right)\right] = transcendental$$

This holds for any $n = 1, 2, 3, 4, ...,$

Consequently,

$$\boxed{\log \pi = \text{Sequentially Transcendental Number}}$$

10.

Sequential Transcendence versus Transcendence

A number ξ is Transcendental if it is not the root of any n degree polynomial equation with rational coefficients, for any finite natural number n.

This definition excludes any infinite hyper-real number N. Indeed,

10.1

The Transcendental number π is the root of a polynomial equation with rational coefficients of degree N.

Proof One such polynomial equation of degree N with rational coefficients follows from our 2022 derivation[3] of an expansion for 1, which we named **The Archimedes Series for 1**

$$1=\frac{\pi}{4}+\frac{1}{3}\left(\frac{\pi}{4}\right)^3+\frac{2}{15}\left(\frac{\pi}{4}\right)^5+\frac{17}{315}\left(\frac{\pi}{4}\right)^7+\frac{62}{2835}\left(\frac{\pi}{4}\right)^9+$$

$$+\frac{(819)(691)}{3^6 5^2 7(11)(91)}\left(\frac{\pi}{4}\right)^{11}+\frac{5461}{3^5 5^2 7(11)(13)}\left(\frac{\pi}{4}\right)^{13}+....+\frac{2^{2n}(2^{2n}-1)}{(2n)!}B_n\left(\frac{\pi}{4}\right)^{2n-1}+..$$

where B_n =Bernoulli Numbers.□

Similarly, our definition of sequential Transcendence breaks down for $n=N$

[3] H. Vic Dannon, "Archimedes Series", Gauge Institute Journal, Vol. 18, No 3. August 2022, pp 1-11

We defined a number ξ to be sequentially transcendental if for any finite natural number n there a transcendental number as close as we wish to ξ.

This definition excludes any infinite hyper-real number N. Indeed, if we allow $n = N$, then

10.2

For an Algebraic, and Sequentially Transcendental α,

ξ_N must be algebraic

For instance, for the Algebraic number 1, the partial sum

$$1_N = \frac{\pi}{4} + \frac{1}{3}\left(\frac{\pi}{4}\right)^3 + \frac{2}{15}\left(\frac{\pi}{4}\right)^5 + \frac{17}{315}\left(\frac{\pi}{4}\right)^7 + \frac{62}{2835}\left(\frac{\pi}{4}\right)^9 +$$

$$+ \frac{(819)(691)}{3^6 5^2 7(11)(91)}\left(\frac{\pi}{4}\right)^{11} + \frac{5461}{3^5 5^2 7(11)(13)}\left(\frac{\pi}{4}\right)^{13} + ... + \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_N \left(\frac{\pi}{4}\right)^{2n-1}$$

where $B_n =$ Bernoulli Numbers for $n = 1, 2, ...N$

is infinitesimally close to 1,

$$1 = 1_N + \text{infinitesimal}$$

Therefore,

$$\underbrace{1}_{\text{algebraic}} = \{\text{the standard part of } 1\} = 1_N$$

That is, for any finite n,

$$1_n = \text{transcendental}$$

But for an infinite hyper-real N

$$1_N = \text{algebraic}$$

It follows that

Our definitions of Transcendental, and Sequentially Transcendental apply only to finite n

In any event, we cannot compute with any infinite n.

But if we are limited to finite n, then the transcendental π is actually the Leibniz rational partial sum

$$\pi_n = 4\left\{1 - \frac{1}{3} + \frac{1}{5} - .... + \frac{(-1)^{n+1}}{2n-1}\right\}$$

that can be made as close as we can compute to π

And γ is actually the transcendental

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + ...\frac{1}{n} - \log(n+1)$$

that can be made as close as we can compute to γ

In other words, since transcendence breaks down at the forever incomprehensible infinity

Sequential Transcendence is way more informative then Transcendence

In 2022, we derived[4] an expansion for π, which we named **The Archimedes Series for π**

$$\pi = \tfrac{1}{2}\{\alpha(2\pi) + \tfrac{1}{3}\alpha^3(2\pi) + \tfrac{2}{15}\alpha^5(2\pi) + \tfrac{17}{315}\alpha^7(2\pi) + \tfrac{62}{2835}\alpha^9(2\pi) +$$

$$+\tfrac{(819)(691)}{3^6 5^2 7(11)(91)}\alpha^{11}(2\pi) + \tfrac{5461}{3^5 5^2 7(11)(13)}\alpha^{13}(2\pi) + ... + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!}B_n\alpha^{2n-1}(2\pi) + ...\}$$

$$\text{where } \alpha(2\pi) = \arctan(2\pi) \approx 1.412965137..$$

[4] H. Vic Dannon, "Archimedes Series", Gauge Institute Journal, Vol. 18, No 3. August 2022, pp 1-11

and $B_n =$ Bernoulli Numbers.

The Transcendental partial sums

$$\pi_n = \tfrac{1}{2}\{\alpha(2\pi) + \tfrac{1}{3}\alpha^3(2\pi) + \tfrac{2}{15}\alpha^5(2\pi) + \tfrac{17}{315}\alpha^7(2\pi) + \tfrac{62}{2835}\alpha^9(2\pi) +$$

$$+ \tfrac{(819)(691)}{3^6 5^2 7(11)(91)}\alpha^{11}(2\pi) + \tfrac{5461}{3^5 5^2 7(11)(13)}\alpha^{13}(2\pi) + ... + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!}B_n\alpha^{2n-1}(2\pi)\}$$

represent π better than any inapplicable statement about its not being a root of a polynomial equation.

We conclude that

Sequential Transcendence is

a superior characterization of a number.

Appendix
Transcendental Numbers

Liouville $\boxed{\frac{1}{10}+\frac{1}{10^{2!}}+\frac{1}{10^{3!}}+\frac{1}{10^{4!}}+....=transcendental}$

Hermit $\boxed{e^{\text{rational}}=transcendental}$

Lindemann $\boxed{e^{\text{algebraic}}=transcendental}$

$$\boxed{e^{\tau}=\text{algebraic}\Rightarrow\tau\text{=tanscendental}}$$

$$e^{i\pi}=-1\Rightarrow i\pi=\text{trans}\ \Rightarrow\pi=-i\cdot\text{trans}=\text{trans}$$

α_1,α_2 =**Algebraically Dependent over** $\mathbb{Q}$

iff $P(\alpha_1,\alpha_2)=0$ where $P(x,y)$ has coefficients from $\mathbb{Q}$

$\sqrt{\pi},\pi=$ algebraically dependent over $\mathbb{Q}$ with $P(x,y)=x^2-y$.

Lindemann-Weierstrass

$\alpha_1,\alpha_2,\alpha_3=$ algebraic

$\beta_1,\beta_2,\beta_3=$ algebraic, Linearly independent over $\mathbb{Q}$

$$\boxed{\Rightarrow\beta_1e^{\alpha_1}+\beta_2e^{\alpha_2}+\beta_3e^{\alpha_3}=\text{transcendental}}$$

α algebraic $\neq 0 \ \Rightarrow\ \cos\alpha,\ \sin\alpha,\ \tan\alpha=$ transcendental,

α algebraic $\neq 0,1\Rightarrow\ \log\alpha=$transcendental

$\alpha_1,\alpha_2,\alpha_3$ =algebraic, linearly independent over $\mathbb{Q}$

$\Rightarrow e^{\alpha_1}, e^{\alpha_2}, e^{\alpha_2} =$ algebraically independent over $\mathbb{Q}$

$\Rightarrow r_1 e^{\alpha_1} + r_2 e^{\alpha_2} + r_3 e^{\alpha_3} = transcendental$ for any $r_1, r_2, r_3 \in \mathbb{Q}$

Baker $\alpha_1 \neq \alpha_2 \neq \alpha_3$ algebraic

$\Rightarrow e^{\alpha_1}, e^{\alpha_2}, e^{\alpha_3} =$ linearly independent over $\mathbb{A}$

$\Rightarrow \beta_1 e^{\alpha_1} + \beta_2 e^{\alpha_2} + \beta_3 e^{\alpha_3} = transcendental$ for any $\beta_1, \beta_2 \in \mathbb{A}$

Gelfond

$\alpha \neq 0,1$ algebraic , $\beta =$ irrational algebraic $\boxed{\Rightarrow \alpha^{\beta} = \text{transcendental}}$

$$2^{\sqrt{2}},$$

$$\sqrt{2}^{\sqrt{2}},$$

$$e^{\pi} = (e^{i\pi})^{-i} = (-1)^{-i},$$

$$e^{-\frac{1}{2}\pi} = (e^{i\frac{1}{2}\pi})^{i} = (i)^{i}.$$

Gelfond-Schneider

$\alpha_1, \alpha_2 =$ algebraic $\neq 0,1$

$\beta_1, \beta_2 =$ algebraic,

$1, \beta_1, \beta_2 =$ Linearly independent over $\mathbb{Q}$

$$\boxed{\Rightarrow \alpha_1^{\beta_1} \alpha_2^{\beta_2} = transcendental}$$

Baker

$\alpha_1 \neq \alpha_2$ algebraic $\neq 0,1$

$\beta_1, \beta_2 =$ irrational algebraic,

$1, \beta_1, \beta_2 =$ linearly independent over $\mathbb{Q}$

$$\boxed{\Rightarrow \alpha_1^{\beta_1}\alpha_2^{\beta_2} = transcendental}$$

Gelfond-Schneider

$\alpha_1, \alpha_2 =$ algebraic, $\neq 0,1$

$\beta_1, \beta_2 =$ algebraic,

$\log \alpha_1, \log \alpha_2 =$ Linearly independent over $\mathbb{Q}$

$$\boxed{\Rightarrow \beta_1 \log \alpha_1 + \beta_2 \log \alpha_2 = transcendental}$$

Gelfond-Schneider-Baker

$\alpha_1, \alpha_2, \alpha_3 =$ algebraic, $\neq 0,1$

$\beta_0, \beta_1, \beta_2, \beta_3 =$ algebraic,

$\beta_0, \beta_1, \beta_2, \beta_3 =$ Linearly independent over $\mathbb{Q}$

$$\boxed{\Rightarrow e^{\beta_0}\alpha_1^{\beta_1}\alpha_2^{\beta_2}\alpha_3^{\beta_3} = \text{transcendental}}$$

Gelfond-Schneider-Baker

$\alpha_1, \alpha_2, \alpha_3 =$ algebraic, $\neq 0,1$

$\beta_0 \neq 0, \beta_1, \beta_2, \beta_3 =$ algebraic,

$\log \alpha_1, \log \alpha_2, \log \alpha_3 =$ Linearly independent over $\mathbb{Q}$

$$\boxed{\Rightarrow \beta_0 + \beta_1 \log \alpha_1 + \beta_2 \log \alpha_2 + \beta_3 \log \alpha_3 = \text{transcendental}}$$

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