Euler's Gamma is Irrational and Sequentially Transcendental Number

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Abstract A number is algebraic if it is a zero of a polynomial in integer coefficients.

The algebraic numbers form a field with respect to addition , and multiplication. The sum, and product of two algebraic numbers is an algebraic number.

A number that is not algebraic, transcends algebraic numbers, and is called transcendental. The following are believed to be transcendental

e (Hermit),

π (Lindemann),

e^{rational} (Hermit),

$e^{\text{algebraic}}$ (Lindemann),

$(\text{algebraic})^{\text{irrational algebraic}}$ (Gelfond-Schneider).

$$e^{\pi} = (e^{i\pi})^{-i} = (-1)^{-i}$$

$$e^{n\pi} = (e^{i\pi})^{-ni} = (-1)^{-ni}$$

Euler defined his Gamma Constant γ by the infinite sum

$$\left(1+\tfrac{1}{2}+\tfrac{1}{3}+...+\tfrac{1}{N}\right)-\left(\log 2+\log\tfrac{3}{2}+\log\tfrac{4}{3}+...+\log\tfrac{N+1}{N}\right)=$$

$$=(1-\log 2)+(\tfrac{1}{2}-\log\tfrac{3}{2})+(\tfrac{1}{3}-\log\tfrac{4}{3})+...+(\tfrac{1}{N}-\log\tfrac{N+1}{N})$$

And computed it with his summation formula.

$$\gamma = 0.577218....$$

Vacca expanded Gamma in infinite series of reciprocals of all the natural numbers.

The series sum has a denominator that includes the product of all the prime numbers. But there is no number that is divided by all the primes. Hence, no number for the denominator of a rational number that equals the series.

Therefore,

Euler's Gamma is Irrational.

The same argument applies to show that π, e, $\zeta(3), \zeta(5), \zeta(7), ...$ are Not rational numbers.

Vacca series expansion of Gamma applies to show that

Gamma is Not a Liouville number.

And we cannot say that Gamma is transcendental on account of its being a Liouville number which it is not.

We show that

Gamma is the limit of a sequence γ_n of transcendentals.

This **Does Not** mean that Gamma is transcendental But that

As far as we can ever compute, for any finite n,

the partial sum γ_n is a transcendental number.

Indeed, a sequence of transcendentals need not converge to a transcendental number.

In 2022, we derived[1] an expansion for 1, which we named **The Archimedes Series for 1**

$$1 = \frac{\pi}{4} + \frac{1}{3}\left(\frac{\pi}{4}\right)^3 + \frac{2}{15}\left(\frac{\pi}{4}\right)^5 + \frac{17}{315}\left(\frac{\pi}{4}\right)^7 + \frac{62}{2835}\left(\frac{\pi}{4}\right)^9 +$$

$$+ \frac{(819)(691)}{3^6 5^2 7(11)(91)}\left(\frac{\pi}{4}\right)^{11} + \frac{5461}{3^5 5^2 7(11)(13)}\left(\frac{\pi}{4}\right)^{13} + ... + \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_n \left(\frac{\pi}{4}\right)^{2n-1} + ..$$

where $B_n =$ Bernoulli Numbers

Since π is transcendental, the Partial Sums of the Archimedes Series for 1 are transcendental numbers that converge to the algebraic number 1

And the

Transcendental partial sums 1_n transform to Algebraic 1

We shall say that besides being an Algebraic Number,

1 is a Sequentially Transcendental Number

That is, there is a sequence of transcendentals that converges to 1.

In fact, for an infinite hyper-real N we cannot compute the algebraic partial sum with N transcendental terms.

As far as we can ever compute, for any finite n,

the partial sum 1_n is Transcendental.

Similarly, for γ

[1] H. Vic Dannon, "Archimedes Series", Gauge Institute Journal, Vol. 18, No 3. August 2022, pp 1-11

As far as we can ever compute, for any finite n,

the partial sum γ_n is a Transcendental number.

This leads us to discuss the meaning of Sequential Transcendence versus Transcendence.

And we conclude that

Sequential Transcendence is

a superior characterization of a number.

Contents

1.

Zeta Series for Gamma

For an infinite Hyper-real N, Euler defined

$$\boxed{\gamma = (1 - \log 2) + \left(\frac{1}{2} - \log\frac{3}{2}\right) + \left(\frac{1}{3} - \log\frac{4}{3}\right) + ... + \left(\frac{1}{N} - \log\frac{N+1}{N}\right)}$$

Proof: the Harmonic Series is

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{N} + \frac{1}{N+1} + ...$$

Euler defined $s(N)$ by

$$ds(N) = \frac{1}{N+1}dN.$$

$$s(N) = \gamma + \log(1+N)$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{N}$$

$$\gamma = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{N} - \log(1+N)$$

$$= \left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{N}\right) - \left(\log 2 + \log\frac{3}{2} + \log\frac{4}{3} + ... + \log\frac{N+1}{N}\right)$$

$$= (1 - \log 2) + \left(\frac{1}{2} - \log\frac{3}{2}\right) + \left(\frac{1}{3} - \log\frac{4}{3}\right) + ... + \left(\frac{1}{N} - \log\frac{N+1}{N}\right).\square$$

From this γ can be expanded in zeta series

$$\boxed{\gamma = \frac{1}{2}\zeta(2) - \frac{1}{3}\zeta(3) + \frac{1}{4}\zeta(4) - \frac{1}{5}\zeta(5) + ... = 0.577218....}$$

$$\zeta(2) = 1.644934066848226436...$$

$$\zeta(3) = 1.202056903159594285...$$

$$\zeta(4) = 1.082323233711138191...$$

$$\zeta(5) = 1.036927755143369926...$$

$$\zeta(6) = 1.017343061984449139...$$

$$\zeta(7) = 1.008349277381922826...$$

$$\zeta(8) = 1.004077356197944339...$$

$$\zeta(9) = 1.002008392826082214...$$

$$\zeta(10) = 1.000994575127818085...$$

$$\zeta(11) = 1.000494188604119464...$$

$$\zeta(12) = 1.000246086553308048...$$

$$\zeta(13) = 1.000122713347578489...$$

$$\zeta(14) = 1.000061248135058704...$$

$$\zeta(15) = 1.000030588236307020...$$

$$\zeta(16) = 1.000015282259408651...$$

....

Proof:

$$\log\left(1 + \frac{1}{1}\right) = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - ...$$

$$\Rightarrow \boxed{1 - \log 2 = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - ...}$$

$$\log\left(1+\frac{1}{2}\right)=\frac{1}{2}-\frac{1}{2}\frac{1}{2^2}+\frac{1}{3}\frac{1}{2^3}-\frac{1}{4}\frac{1}{2^4}+\frac{1}{5}\frac{1}{2^5}-...$$

$$\Rightarrow \boxed{\frac{1}{2}-\log\frac{3}{2}=\frac{1}{2}\frac{1}{2^2}-\frac{1}{3}\frac{1}{2^3}+\frac{1}{4}\frac{1}{2^4}-\frac{1}{5}\frac{1}{2^5}+....}$$

$$\log\left(1+\frac{1}{3}\right)=\frac{1}{3}-\frac{1}{2}\frac{1}{3^2}+\frac{1}{3}\frac{1}{3^3}-\frac{1}{4}\frac{1}{3^4}+\frac{1}{5}\frac{1}{3^5}-...$$

$$\Rightarrow \boxed{\frac{1}{3}-\log\frac{4}{3}=\frac{1}{2}\frac{1}{3^2}-\frac{1}{3}\frac{1}{3^3}+\frac{1}{4}\frac{1}{3^4}-\frac{1}{5}\frac{1}{3^5}+...}$$

$$\log\left(1+\frac{1}{4}\right)=\frac{1}{4}-\frac{1}{2}\frac{1}{4^2}+\frac{1}{3}\frac{1}{4^3}-\frac{1}{4}\frac{1}{4^4}+\frac{1}{5}\frac{1}{4^5}-...$$

$$\Rightarrow \boxed{\frac{1}{4}-\log\frac{5}{4}=\frac{1}{2}\frac{1}{4^2}-\frac{1}{3}\frac{1}{4^3}+\frac{1}{4}\frac{1}{4^4}-\frac{1}{5}\frac{1}{4^5}+...}$$

....

$$\log\left(1+\frac{1}{N}\right)=\frac{1}{N}-\frac{1}{2}\frac{1}{N^2}+\frac{1}{3}\frac{1}{N^3}-\frac{1}{4}\frac{1}{N^4}+\frac{1}{5}\frac{1}{N^5}-...$$

$$\Rightarrow \boxed{\frac{1}{N}-\log\frac{N+1}{N}=\frac{1}{2}\frac{1}{N^2}-\frac{1}{3}\frac{1}{N^3}+\frac{1}{4}\frac{1}{N^4}-\frac{1}{5}\frac{1}{N^5}+...}$$

$$\gamma=\left(1-\log 2\right)+\left(\frac{1}{2}-\log\frac{3}{2}\right)+\left(\frac{1}{3}-\log\frac{4}{3}\right)+\cdots+\left(\frac{1}{N}-\log\frac{N+1}{N}\right)$$

$$=\frac{1}{2}\underbrace{\left(1+\frac{1}{2^2}+\frac{1}{3^2}+...\frac{1}{m^2}+...+\frac{1}{N^2}\right)}_{\zeta(2)}$$

$$-\frac{1}{3}\underbrace{\left(1+\frac{1}{2^3}+\frac{1}{3^3}+...+\frac{1}{m^3}+...+\frac{1}{N^3}\right)}_{\zeta(3)}$$

$$+\frac{1}{4}\underbrace{\left(1+\frac{1}{2^4}+\frac{1}{3^4}+...+\frac{1}{m^4}+...+\frac{1}{N^4}\right)}_{\zeta(4)}$$

....

2.

γ is Irrational

Vacca Series[2] for γ

$$\gamma = 1\left(\frac{1}{2}-\frac{1}{3}\right)$$
$$+2\left(\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}\right)$$
$$+3\left(\frac{1}{8}-\frac{1}{9}+\frac{1}{10}-\frac{1}{11}+\frac{1}{12}-\frac{1}{13}+\frac{1}{14}-\frac{1}{15}\right)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$+m\left(\frac{1}{2^m}-\frac{1}{2^m+1}+\dots\dots\dots+\frac{1}{2^{m+1}-2}-\frac{1}{2^{m+1}-1}\right)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

γ is Irrational

Proof:

$$\gamma = 1\frac{1}{2\cdot 3}$$
$$+2\left(\frac{1}{4\cdot 5}+\frac{1}{6\cdot 7}\right)$$
$$+3\left(\frac{1}{8\cdot 9}+\frac{1}{10\cdot 11}+\frac{1}{12\cdot 13}+\frac{1}{14\cdot 15}\right)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$+m\left(\frac{1}{2^m(2^m+1)}+\dots\dots\dots+\frac{1}{(2^{m+1}-2)(2^{m+1}-1)}\right)+\dots$$

[2] Steven R. Finch, "Mathematical Constants", Cambridge U Press, 2003, p.31

$$\gamma = 0.57721\ 56649\ 01532\ .....$$

is the sum of infinitely many rational numbers with common denominator that is the product of all the natural numbers,

$$2 \cdot 3 \cdot 4 \cdot ... = N!$$

which includes the product of all the prime numbers

$$p_1 \cdot p_2 \cdot p_3 \cdot ....$$

There is no finite natural number

$$q$$

that is divided by all the primes.

Thus, there is no rational number $\frac{p}{q}$ that equals Gamma.

Therefore, Gamma is Not a rational number.

That is, Gamma is Irrational.□

3.

γ is Not a Liouville Number

Liouville showed that

<u>If</u>

$$\alpha = \text{the zero of a reduced polynomial}$$

$$P_n(x) \text{ of order } n,$$

<u>and if</u> α is the limit of a sequence of rational numbers,

$$\frac{p_m}{q_m},$$

so that

$$p_m, \text{ and } q_m \text{ are relatively prime,}$$

<u>And if</u> there is a constant $C_m > 0$ so that

$$q_m > C_m$$

<u>Then</u>,

$$\boxed{\left|\alpha - \frac{p_m}{q_m}\right| > \left(\frac{1}{q_m}\right)^{n+1}}$$

The negation of this statement is a criteria for transcendence.

<u>If</u>

$$\tau = \text{limit of a sequence of rational numbers,}$$

$$\frac{p_m}{q_m}, \text{ so that } p_m, \text{ and } q_m \text{ are relatively prime,}$$

<u>and if</u> for each $m = 1, 2, 3, \ldots$,

$$\boxed{\left|\tau - \frac{p_m}{q_m}\right| < \left(\frac{1}{q_m}\right)^{m+1}}$$

Then,

$$\tau = transcendental\,.$$

The partial sums of the Vacca Series are such rationals

$$\begin{aligned}\frac{p_m}{q_m} = 1&\frac{1}{2\cdot 3}\\ &+2\left(\frac{1}{4\cdot 5}+\frac{1}{6\cdot 7}\right)\\ &+3\left(\frac{1}{8\cdot 9}+\frac{1}{10\cdot 11}+\frac{1}{12\cdot 13}+\frac{1}{14\cdot 15}\right)\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &+m\left(\frac{1}{2^m(2^m+1)}+\ldots\ldots\ldots\ldots+\frac{1}{(2^{m+1}-2)(2^{m+1}-1)}\right)\end{aligned}$$

$$\frac{1}{q_m} = \frac{1}{(2^{m+1}-1)!}$$

$$\left|\gamma - \frac{p_m}{q_m}\right| = (m+1)\left(\frac{1}{2^{m+1}(2^{m+1}+1)} + \ldots. + \frac{1}{(2^{m+2}-2)(2^{m+2}-1)}\right) + \ldots.$$

is not bounded by

$$\left(\frac{1}{q_m}\right)^{m+1} = \left(\frac{1}{(2^{m+1}-1)!}\right)^{m+1} \text{ for all } m = 1,2,3,\ldots$$

Therefore, γ is not a Liouville Number.□

4.

γ is Sequentially Transcendental

For $n = 1, 2, 3, 4...$

$$\boxed{\gamma_n \equiv 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \log(n+1) = \text{transcendental}}$$

Proof:

$$\gamma_n \equiv \underbrace{1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}}_{\text{rational}} + \underbrace{\log\frac{1}{n+1}}_{\text{transcendental}}$$

By Lindemann, $e^{\tau} = \text{algebraic} \Rightarrow \tau = \text{transcendental}$

Therefore,

$$e^{\log\frac{1}{N+1}} = \frac{1}{N+1} = \text{algebraic} \Rightarrow \log\frac{1}{N+1} = \text{transcendental}$$

Therefore,

$$\gamma_n - \frac{p_n}{n!} = \text{transcendental.}$$

If $\gamma_n = \text{algebraic}$, then by the field property of algebraic numbers,

$$\gamma_n - \frac{p_n}{n!} = \text{algebraic.}$$

From that contradiction, it follows that $\gamma_n = \text{transcendental.}\square$

For the infinite hyper-real number N,

$$\gamma_N = (1 - \log 2) + \left(\frac{1}{2} - \log\frac{3}{2}\right) + ... + \left(\frac{1}{N} - \log\frac{N+1}{N}\right).$$

$$\gamma_{N+1} = (1 - \log 2) + ... + \left(\frac{1}{N} - \log\frac{N+1}{N}\right) + \left(\frac{1}{N+1} - \log\frac{N+2}{N+1}\right)$$

$$\gamma_{N+1} - \gamma_N = \frac{1}{N+1} - \log\frac{N+2}{N+1}$$

$$= \frac{1}{N+1} - \log\left(1 + \frac{1}{N+1}\right)$$

$$= \frac{1}{N+1} - \left(\frac{1}{N+1} - \frac{1}{2}(\frac{1}{N+1})^2 + \frac{1}{3}(\frac{1}{N+1})^3 - ...\right)$$

$$= \frac{1}{2}(\frac{1}{N+1})^2 - \frac{1}{3}(\frac{1}{N+1})^3 + \frac{1}{4}(\frac{1}{N+1})^4 - ...$$

$$= \text{Order of } (\frac{1}{N+1})^2$$

That is, γ_N is infinitesimally close to γ.

Gamma is the limit of a sequence γ_n of transcendentals.

This **Does Not** mean that Gamma is transcendental But that

As far as we can ever compute, for any finite n

the approximation γ_n is transcendental.

Indeed, a sequence of transcendentals need not converge to a transcendental number.

In 2022, we derived[3] an expansion for 1, which we named **The Archimedes Series for 1**

[3] H. Vic Dannon, "Archimedes Series", Gauge Institute Journal, Vol. 18, No 3. August 2022, pp 1-11

$$1 = \frac{\pi}{4} + \frac{1}{3}\left(\frac{\pi}{4}\right)^3 + \frac{2}{15}\left(\frac{\pi}{4}\right)^5 + \frac{17}{315}\left(\frac{\pi}{4}\right)^7 + \frac{62}{2835}\left(\frac{\pi}{4}\right)^9 +$$

$$+ \frac{(819)(691)}{3^6 5^2 7(11)(91)}\left(\frac{\pi}{4}\right)^{11} + \frac{5461}{3^5 5^2 7(11)(13)}\left(\frac{\pi}{4}\right)^{13} + ... + \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_n \left(\frac{\pi}{4}\right)^{2n-1} + ..$$

where B_n =Bernoulli Numbers

Since π is transcendental, the Partial Sums of the Archimedes Series for 1 are transcendental numbers that converge to the algebraic number 1

And the

Transcendental partial sums 1_n transform to Algebraic 1

We shall say that besides being an Algebraic Number,

1 is a Sequentially Transcendental Number

That is, there is a sequence of transcendental numbers that converges to 1.

In fact, for an infinite hyper-real N we cannot compute the algebraic partial sum with N transcendental terms.

As far as we can ever compute, for any finite n,

the partial sum 1_n is a Transcendental number.

We conclude that

γ is Sequentially Transcendental Number

Meaning that,

As far as we can ever compute, for any finite n,

the partial sum γ_n is a transcendental number.

5.

e^{γ} is Irrational and Sequentially Transcendental Number

Irrationality

Using Vacca expansion for γ

$$e^{\gamma} = \frac{1}{0!} +$$
$$+\frac{1}{1!}\left(1\frac{1}{2\cdot 3} + 2\left(\frac{1}{4\cdot 5} + \frac{1}{6\cdot 7}\right) + 3\left(\frac{1}{8\cdot 9} + \frac{1}{10\cdot 11} + \frac{1}{12\cdot 13} + \frac{1}{14\cdot 15}\right) + ..\right)$$
$$+\frac{1}{2!}\left(1\frac{1}{2\cdot 3} + 2\left(\frac{1}{4\cdot 5} + \frac{1}{6\cdot 7}\right) + 3\left(\frac{1}{8\cdot 9} + \frac{1}{10\cdot 11} + \frac{1}{12\cdot 13} + \frac{1}{14\cdot 15}\right)\right)^2 + ...$$

is the sum of infinitely many rational numbers with common denominator that includes the product of all the prime numbers

$$p_1 \cdot p_2 \cdot p_3 \cdot ....$$

There is no finite natural number

$$q$$

that is divided by all the primes.

Thus, there is no rational number $\frac{p}{q}$ that equals e^{γ}

Therefore, e^{γ} is Not a rational number.

That is, e^{γ} is Irrational.□

Sequentially Transcendental

For $n = 1,2,3,...,N$, $\boxed{e^{\gamma_n} = \text{Transcendental}}$

Proof:

$$e^{\gamma_n} = e^{1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}} e^{-\log 2 - \log\frac{3}{2} - \log\frac{4}{3} - .... - \log\frac{n+1}{n}}$$

By Hermit, $e^{1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}} = e^{\text{rational}} =$ transcendental

$$e^{\gamma_n} = (\text{transcendental})\, e^{\log\frac{1}{2}} e^{\log\frac{2}{3}} e^{\log\frac{3}{4}} ... e^{\log\frac{n}{n+1}}$$

$$= (\text{transcendental}) \underbrace{\frac{1}{\not 2}\frac{\not 2}{\not 3}\frac{\not 3}{4}...\frac{n-1}{\not n}\frac{\not n}{n+1}}_{\frac{1}{n+1} = \text{algebraic}}$$

Therefore,

$$(n+1)e^{\gamma_n} = \text{transcendental.}$$

If $e^{\gamma_n} =$ algebraic, then by the field property of algebraic numbers,

$(n+1)e^{\gamma_n} =$ algebraic.

From that contradiction it follows that $e^{\gamma_n} =$ transcendental.□

This holds for any $n = 1,2,3,4,...$,

Consequently,

$$\boxed{e^{\gamma} = \text{Sequentially Transcendental}}$$

6.

$e^{\pi\gamma}$ is Irrational and Sequentially Transcendental Number

Irrationality

Using Vacca expansion for γ, and Leibniz expansion for π

$$e^{\gamma\pi} = \frac{1}{0!} +$$
$$+\frac{1}{1!}\left(1\frac{1}{2\cdot 3} + 2\left(\frac{1}{4\cdot 5} + \frac{1}{6\cdot 7}\right) + ...\right)4\left(1 - \frac{1}{3} + \frac{1}{5} - ...\right)$$
$$+\frac{1}{2!}\left(1\frac{1}{2\cdot 3} + 2\left(\frac{1}{4\cdot 5} + \frac{1}{6\cdot 7}\right) + ...\right)^2 4^2\left(1 - \frac{1}{3} + \frac{1}{5} - ...\right)^2 + ...$$

is the sum of infinitely many rational numbers with common denominator that includes the product of all the prime numbers

$$p_1 \cdot p_2 \cdot p_3 \cdot ....$$

There is no finite natural number

$$q$$

that is divided by all the primes.

Thus, there is no rational number $\frac{p}{q}$ that equals $e^{\gamma\pi}$

Therefore, $e^{\gamma\pi}$ is Not a rational number.

That is, $e^{\gamma\pi}$ is Irrational.□

Sequentially Transcendental

For $n = 1,2,3,\ldots,N$, $\boxed{e^{\pi_n\gamma_n} = \text{transcendental}}$

Proof:

$$e^{\gamma_n\pi_n} = \underbrace{e^{\underbrace{\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)4\left(1-\frac{1}{3}+\ldots+\frac{(-1)^{n-1}}{2n-1}\right)}_{rational}}}_{Transcendental}$$

$$\times e^{\left[-\log 2-\log\frac{3}{2}-\log\frac{4}{3}-\ldots.-\log\frac{n+1}{n}\right]4\left(1-\frac{1}{3}+\ldots+\frac{(-1)^{n-1}}{2n-1}\right)}$$

$$= (Transcendental)(\underbrace{[e^{\log\frac{1}{2}}][e^{\log\frac{2}{3}}][e^{\log\frac{3}{4}}]\ldots[e^{\log\frac{n}{n+1}}]}_{\frac{1}{n+1}})^{4\left(1-\frac{1}{3}+\ldots+\frac{(-1)^{n-1}}{2n-1}\right)}$$

$$= (\text{transcendental})\left(\frac{1}{n+1}\right)^{4\left(1-\frac{1}{3}+\ldots+\frac{(-1)^{n-1}}{2n-1}\right)}$$

Therefore,

$$\underbrace{(n+1)^{4\left(1-\frac{1}{3}+\ldots+\frac{(-1)^{n-1}}{2n-1}\right)}}_{\text{Algebraic}} e^{\pi_n\gamma_n} = \text{transcendental.}$$

If $e^{\pi_n\gamma_n}$ = algebraic, then by the field property of algebraic numbers, $\underbrace{(n+1)^{4\left(1-\frac{1}{3}+\ldots+\frac{(-1)^{n-1}}{2n-1}\right)}}_{\text{Algebraic}} e^{\pi_n\gamma_n}$ = algebraic.

From that contradiction, it follows that $e^{\pi_n\gamma_n}$ = transcendental.□

This holds for any $n = 1,2,3,4,\ldots,$

Consequently,

$$\boxed{e^{\pi\gamma} = \text{Sequentially Transcendental}}$$

7.

γe is Irrational and Sequentially Transcendental Number

Irrationality

Using Vacca expansion for γ, and Euler's expansion for e

$$\gamma e = \left(1\frac{1}{2\cdot 3} + 2\left(\frac{1}{4\cdot 5} + \frac{1}{6\cdot 7}\right) + ...\right)\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\right)$$

is the sum of infinitely many rational numbers with common denominator that includes the product of all the prime numbers

$$p_1 \cdot p_2 \cdot p_3 \cdot ....$$

There is no finite natural number

$$q$$

that is divided by all the primes.

Thus, there is no rational number $\frac{p}{q}$ that equals $e^{\gamma\pi}$

Therefore, $e^{\gamma\pi}$ is Not a rational number.

That is, $e^{\gamma\pi}$ is Irrational.□

Sequentially Transcendental

For $n = 1,2,3,...,N$, $\boxed{\gamma_n e_n = \text{transcendental}}$

Proof:

$$\gamma_n e_n = (\text{transcendental})\underbrace{(1 + 1 + \tfrac{1}{2!} + \tfrac{1}{3!} + ... + \tfrac{1}{n!})}_{\text{rational algebraic}}$$

Therefore,

$$\frac{\gamma_n e_n}{\text{algebraic}} = \textbf{transcendental.}$$

If $\gamma_n e_n =$ algebraic, then by the field property of algebraic numbers, $\frac{\gamma_n e_n}{\text{algebraic}} = \frac{\text{algebraic}}{\text{algebraic}} = \text{algebraic}$.

From that contradiction, it follows that $\gamma_n e_n =$ transcendental.□

This holds for any $n = 1, 2, 3, 4, ...$, including the hyper-real number N. Consequently,

$$\boxed{\gamma e = \text{Sequentially Transcendental Number}}$$

8.

$\gamma\pi$ is Irrational and Sequentially Transcendental Number

Irrationality

Using Vacca expansion for γ, and Leibniz expansion for π

$$\gamma\pi = \left(1\frac{1}{2\cdot 3} + 2\left(\frac{1}{4\cdot 5} + \frac{1}{6\cdot 7}\right) + ...\right)4\left(1 - \frac{1}{3} + \frac{1}{5} - ...\right)$$

is the sum of infinitely many rational numbers with common denominator that includes the product of all the prime numbers

$$p_1 \cdot p_2 \cdot p_3 \cdot ....$$

There is no finite natural number

$$q$$

that is divided by all the primes.

Thus, there is no rational number $\frac{p}{q}$ that equals $\gamma\pi$

Therefore, $\gamma\pi$ is Not a rational number.

That is, $\gamma\pi$ is Irrational.□

Sequentially Transcendental

For $n = 1, 2, 3, ..., N$, $\boxed{\gamma_n\pi_n = \text{transcendental}}$

Proof:

$$\gamma_n \pi_n = (\text{transcendental}) \underbrace{4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... + \frac{(-1)^{n+1}}{2n-1}\right)}_{\text{algebraic}}$$

Therefore, $\dfrac{\gamma_n \pi_n}{\text{algebraic}} =$ transcendental.

If $\gamma_n \pi_n =$ algebraic, then by the field property of algebraic numbers, $\dfrac{\gamma_n \pi_n}{\text{algebraic}} =$ algebraic.

From that contradiction, it follows that $\gamma_n \pi_n =$ transcendental.□

This holds for any $n = 1, 2, 3, 4, ...,$

Consequently,

$$\boxed{\gamma\pi = \text{Sequentially Transcendental Number}}$$

9.

$\gamma + e$ is Irrational and Sequentially Transcendental Number

Irrationality

Using Vacca expansion for γ, and Euler's expansion for e

$$\gamma + e = \left(1\frac{1}{2\cdot 3} + 2\left(\frac{1}{4\cdot 5} + \frac{1}{6\cdot 7}\right) + ...\right) + \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\right)$$

is the sum of infinitely many rational numbers with common denominator that includes the product of all the prime numbers

$$p_1 \cdot p_2 \cdot p_3 \cdot ....$$

There is no finite natural number

$$q$$

that is divided by all the primes.

Thus, there is no rational number $\frac{p}{q}$ that equals $\gamma + e$

Therefore, $\gamma + e$ is Not a rational number.

That is, $\gamma + e$ is Irrational.□

Sequentially Transcendental

For $n = 1, 2, 3, ..., N$, $\boxed{\gamma_n + e_n = \text{transcendental}}$

Proof:

$$\gamma_n + e_n = (\text{transcendental}) + (\underbrace{1 + 1 + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{n!}}_{\text{rational algebraic}})$$

Therefore,

$$\gamma_n + e_n - \text{algebraic} = \textbf{transcendental.}$$

If $\gamma_n + e_n =$ algebraic, then by the field property of algebraic numbers, $\gamma_n + e_n - \text{algebraic} =$ algebraic.

From the contradiction, it follows that $\gamma_n + e_n =$ transcendental.□

This holds for any $n = 1, 2, 3, 4, ...,$

Consequently,

$$\boxed{\gamma + e = \text{Sequentially Transcendental Number}}$$

10.

$\gamma + \pi$ is Irrational and Sequentially Transcendental Number

Irrationality

Using Vacca expansion for γ, and Leibniz expansion for π

$$\gamma + \pi = \left(1\frac{1}{2\cdot 3} + 2\left(\frac{1}{4\cdot 5} + \frac{1}{6\cdot 7}\right) + ...\right) + 4\left(1 - \frac{1}{3} + \frac{1}{5} - ...\right)$$

is the sum of infinitely many rational numbers with common denominator that includes the product of all the prime numbers

$$p_1 \cdot p_2 \cdot p_3 \cdot ....$$

There is no finite natural number

$$q$$

that is divided by all the primes.

Thus, there is no rational number $\frac{p}{q}$ that equals $\gamma + \pi$

Therefore, $\gamma + \pi$ is Not a rational number.

That is, $\gamma + \pi$ is Irrational.□

Sequentially Transcendental

For $n = 1, 2, 3, ..., N$, $\boxed{\gamma_n + \pi_n = \text{transcendental}}$

Proof:

$$\gamma_n + \pi_n = (\text{transcendental}) + \underbrace{4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... + \frac{(-1)^{n+1}}{2n-1}\right)}_{\text{algebraic}}$$

Therefore, $\gamma_n + \pi_n - \text{algebraic} =$ transcendental.

If $\gamma_n + \pi_n =$ algebraic, then by the field property of algebraic numbers, $\gamma_n + \pi_n - \text{algebraic} =$ algebraic.

From the contradiction, it follows that $\gamma_n + \pi_n =$ transcendental.□

This holds for any $n = 1, 2, 3, 4, ...,$

Consequently,

$$\boxed{\gamma + \pi = \text{Sequentially Transcendental Number}}$$

11.

Bernoulli Series for γ

$$\boxed{\gamma = \tfrac{1}{2}B_2 + \tfrac{1}{4}B_4 + \tfrac{1}{6}B_6 + \tfrac{1}{8}B_8 + \tfrac{1}{10}B_{10} + \tfrac{1}{12}B_{12} + \tfrac{1}{14}B_{14} + \tfrac{1}{16}B_{16} + ...}$$

The Bernoulli Numbers are all rational numbers:

$$\tfrac{1}{2}B_2 = \tfrac{1}{2}\tfrac{1}{6} \approx \left[8.(3)\right]10^{-2}$$

$$\tfrac{1}{4}B_4 = \tfrac{1}{4}\tfrac{-1}{30} \approx -\left[8.(3)\right]10^{-3}$$

$$\tfrac{1}{6}B_6 = \tfrac{1}{6}\tfrac{1}{42} \approx \left[4\right]10^{-3}$$

$$\tfrac{1}{8}B_8 = \tfrac{1}{8}\tfrac{-1}{30} \approx -\left[4.1(6)\right]10^{-3}$$

$$\tfrac{1}{10}B_{10} = \tfrac{1}{10}\tfrac{5}{66} \approx \left[7.(57)\right]10^{-3}$$

$$\tfrac{1}{12}B_{12} = \tfrac{1}{12}\tfrac{-691}{2730} \approx -\left[2.1\right]10^{-2}$$

$$\tfrac{1}{14}B_{14} = \tfrac{1}{14}\tfrac{7}{6} \approx \left[8.(3)\right]10^{-2}$$

$$\tfrac{1}{16}B_{16} = \tfrac{1}{16}\tfrac{-3617}{510} \approx -\left[4.432\right]\times 10^{-1}$$

$$\tfrac{1}{18}B_{18} = \tfrac{1}{18}\tfrac{43867}{798} \approx 3.053954330$$

$$\tfrac{1}{20}B_{20} = \tfrac{1}{20}\tfrac{-174611}{330} \approx -26.4562(12)$$

....

<u>*Proof*</u>:

By Euler-Maclaurin Summation,

$$\tfrac{1}{1} + .. + \tfrac{1}{N} - \int_{x=1}^{x=N} \tfrac{1}{x}\,dx =$$

$$= \frac{1}{2}\underbrace{(\frac{1}{1} + \frac{1}{N})}_{1} + \frac{1}{2!}\frac{1}{6}\underbrace{[-x^{-2}]_{x=1}^{x=N}}_{1} + \frac{1}{4!}\frac{-1}{30}\underbrace{[-3!x^{-4}]_{x=1}^{x=N}}_{3!} + \frac{1}{6!}\frac{1}{42}\underbrace{[-5!x^{-6}]_{x=1}^{x=N}}_{5!} +$$

$$+ \frac{1}{8!}\frac{-1}{30}\underbrace{[-7!x^{-8}]_{x=1}^{x=N}}_{7!} + \frac{1}{10!}\frac{5}{66}\underbrace{[-9!x^{-10}]_{x=1}^{x=N}}_{9!} + ...$$

$$= \frac{1}{2}(1 + \frac{1}{6} + \frac{1}{2}\frac{-1}{30} + \frac{1}{3}\frac{1}{42} + \frac{1}{4}\frac{-1}{30} + \frac{1}{5}\frac{5}{66} + ...)$$

12.

Fast-Converging Bernoulli Series

$$\gamma\Big|_{n=10} = 1 + .. + \frac{1}{10} - \log 10 + \frac{1}{2}\frac{1}{10} +$$
$$+ \frac{1}{2}\underbrace{B_2}_{\frac{1}{6}}\frac{1}{100} + \frac{1}{4}\underbrace{B_4}_{\frac{-1}{30}}\frac{1}{10^4} + \frac{1}{6}\underbrace{B_6}_{\frac{1}{42}}\frac{1}{10^6} + \frac{1}{8}\underbrace{B_8}_{\frac{-1}{30}}\frac{1}{10^8} + \frac{1}{10}\underbrace{B_{10}}_{\frac{5}{66}}\frac{1}{10^{10}} + ...$$

$$\frac{1}{2}B_2 = \frac{1}{2}\frac{1}{6} \approx \left[8.(3)\right]10^{-2} \quad \Rightarrow \times 10^{-2} \approx \left[8.(3)\right]10^{-4}$$

$$\frac{1}{4}B_4 = \frac{1}{4}\frac{-1}{30} \approx -\left[8.(3)\right]10^{-3} \Rightarrow \times 10^{-4} \approx -\left[8.(3)\right]10^{-7}$$

$$\frac{1}{6}B_6 = \frac{1}{6}\frac{1}{42} \approx \left[4\right]10^{-3} \quad \Rightarrow \times 10^{-6} \approx \left[4\right]10^{-9}$$

$$\frac{1}{8}B_8 = \frac{1}{8}\frac{-1}{30} \approx -\left[4.1(6)\right]10^{-3} \quad \Rightarrow \times 10^{-8} \approx -\left[4.1(6)\right]10^{-11}$$

$$\frac{1}{10}B_{10} = \frac{1}{10}\frac{5}{66} \approx \left[7.(57)\right]10^{-3} \quad \Rightarrow \times 10^{-10} = \left[7.(57)\right]10^{-13}$$

$$\frac{1}{12}B_{12} = \frac{1}{12}\frac{-691}{2730} \approx -\left[2.1\right]10^{-2} \Rightarrow \times 10^{-12} \approx -\left[2.1\right]10^{-14}$$

$$\frac{1}{14}B_{14} = \frac{1}{14}\frac{7}{6} \approx \left[8.(3)\right]10^{-2} \qquad \Rightarrow \times 10^{-14} = \left[8.(3)\right]\times 10^{-16}$$

$$\frac{1}{16}B_{16} = \frac{1}{16}\frac{-3617}{510} \approx -\left[4.432\right]\times 10^{-1} \Rightarrow \times 10^{-16} \approx -\left[4.432\right]\times 10^{-17}$$

$$\frac{1}{18}B_{18} = \frac{1}{18}\frac{43867}{798} \approx 3.053954330 \qquad \Rightarrow \times 10^{-18} \approx \left[3.05395433\right]10^{-18}$$

$$\frac{1}{20}B_{20} = \frac{1}{20}\frac{-174611}{330} \approx -26.4562(12) \quad \Rightarrow \times 10^{-20} \approx \left[2.64562(12)\right]10^{-19}$$

Proof:

$$1 + .. + \frac{1}{N} - \int_{x=1}^{x=N}\frac{1}{x}dx = 1 + .. + \frac{1}{n} - \underbrace{\int_{x=1}^{x=n}\frac{1}{x}dx}_{\log n} + \left\{\frac{1}{n+1} + .. + \frac{1}{N} - \int_{x=n}^{x=N}\frac{1}{x}dx\right\}$$

By Euler-Maclaurin summation

$$\frac{1}{n+1}+..+\frac{1}{N}-\int_{x=n}^{x=N}\frac{1}{x}dx=$$

$$=\frac{1}{2}\underbrace{\left(\frac{1}{n}+\frac{1}{N}\right)}_{n^{-1}}+\frac{1}{2!}\frac{1}{6}\underbrace{\left[-x^{-2}\right]_{x=n}^{x=N}}_{n^{-2}}+\frac{1}{4!}\frac{-1}{30}\underbrace{\left[-3!x^{-4}\right]_{x=n}^{x=N}}_{3!n^{-4}}+\frac{1}{6!}\frac{1}{42}\underbrace{\left[-5!x^{-6}\right]_{x=n}^{x=N}}_{5!n^{-6}}+$$

$$+\frac{1}{8!}\frac{-1}{30}\underbrace{\left[-7!x^{-8}\right]_{x=n}^{x=N}}_{7!n^{-8}}+\frac{1}{10!}\frac{5}{66}\underbrace{\left[-9!x^{-10}\right]_{x=n}^{x=N}}_{9!n^{-10}}+...$$

$$=\frac{1}{2}\frac{1}{n}+\frac{1}{2}\frac{1}{6}\frac{1}{n^2}+\frac{1}{4}\frac{-1}{30}\frac{1}{n^4}+\frac{1}{6}\frac{1}{42}\frac{1}{n^6}+\frac{1}{8}\frac{-1}{30}\frac{1}{n^8}+\frac{1}{10}\frac{5}{66}\frac{1}{n^{10}}+...$$

$$\gamma\Big|_{n=10}=1+..+\frac{1}{10}-\log 10+$$

$$+\frac{1}{2}\frac{1}{10}+\frac{1}{2}\frac{1}{6}\frac{1}{100}-\frac{1}{4}\frac{1}{30}\frac{1}{10^4}+\frac{1}{6}\frac{1}{42}\frac{1}{10^6}-\frac{1}{8}\frac{1}{30}\frac{1}{10^8}+\frac{1}{10}\frac{5}{66}\frac{1}{10^{10}}+...$$

13.

Sequential Transcendence versus Transcendence

A number ξ is Transcendental if it is not the root of any n degree polynomial equation with rational coefficients, for any finite natural number n.

This definition excludes any infinite hyper-real number N. Indeed,

13.1

The Transcendental number π is the root of a polynomial equation with rational coefficients of degree N.

Proof One such polynomial equation of degree N with rational coefficients follows from our 2022 derivation[4] of an expansion for 1, which we named **The Archimedes Series for 1**

$$1 = \frac{\pi}{4} + \frac{1}{3}\left(\frac{\pi}{4}\right)^3 + \frac{2}{15}\left(\frac{\pi}{4}\right)^5 + \frac{17}{315}\left(\frac{\pi}{4}\right)^7 + \frac{62}{2835}\left(\frac{\pi}{4}\right)^9 +$$

$$+ \frac{(819)(691)}{3^6 5^2 7(11)(91)}\left(\frac{\pi}{4}\right)^{11} + \frac{5461}{3^5 5^2 7(11)(13)}\left(\frac{\pi}{4}\right)^{13} + ... + \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_n \left(\frac{\pi}{4}\right)^{2n-1} + ..$$

where B_n =Bernoulli Numbers.□

Similarly, our definition of sequential Transcendence breaks down for $n = N$

[4] H. Vic Dannon, "Archimedes Series", Gauge Institute Journal, Vol. 18, No 3. August 2022, pp 1-11

We defined a number ξ to be sequentially transcendental if for any finite natural number n there a transcendental number as close as we wish to ξ.

This definition excludes any infinite hyper-real number N. Indeed, if we allow $n = N$, then

13.2

For an Algebraic, and Sequentially Transcendental α,

ξ_N must be algebraic

For instance, for the Algebraic number 1, the partial sum

$$1_N = \frac{\pi}{4} + \frac{1}{3}\left(\frac{\pi}{4}\right)^3 + \frac{2}{15}\left(\frac{\pi}{4}\right)^5 + \frac{17}{315}\left(\frac{\pi}{4}\right)^7 + \frac{62}{2835}\left(\frac{\pi}{4}\right)^9 +$$

$$+ \frac{(819)(691)}{3^6 5^2 7(11)(91)}\left(\frac{\pi}{4}\right)^{11} + \frac{5461}{3^5 5^2 7(11)(13)}\left(\frac{\pi}{4}\right)^{13} + ... + \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_N \left(\frac{\pi}{4}\right)^{2n-1}$$

where B_n =Bernoulli Numbers for $n = 1,2,...N$

is infinitesimally close to 1,

$$1 = 1_N + \text{infinitesimal}$$

Therefore,

$$\underbrace{1}_{\text{algebraic}} = \{\text{the standard part of } 1\} = 1_N$$

That is, for any finite n,

$$1_n = \text{transcendental}$$

But for an infinite hyper-real N

$$1_N = \text{algebraic}$$

It follows that

Our definitions of Transcendental, and Sequentially Transcendental apply only to finite n

In any event, we cannot compute with any infinite n.

But if we are limited to finite n, then the transcendental π is actually the Leibniz rational partial sum

$$\pi_n = 4\left\{1 - \frac{1}{3} + \frac{1}{5} - .... + \frac{(-1)^{n+1}}{2n-1}\right\}$$

that can be made as close as we can compute to π

And γ is actually the transcendental

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + ...\frac{1}{n} - \log(n+1)$$

that can be made as close as we can compute to γ

In other words, since transcendence breaks down at the forever incomprehensible infinity

Sequential Transcendence is way more informative then Transcendence

In 2022, we derived[5] an expansion for π, which we named **The Archimedes Series for** π

$$\pi = \tfrac{1}{2}\{\alpha(2\pi) + \tfrac{1}{3}\alpha^3(2\pi) + \tfrac{2}{15}\alpha^5(2\pi) + \tfrac{17}{315}\alpha^7(2\pi) + \tfrac{62}{2835}\alpha^9(2\pi) +$$

$$+\tfrac{(819)(691)}{3^6 5^2 7(11)(91)}\alpha^{11}(2\pi) + \tfrac{5461}{3^5 5^2 7(11)(13)}\alpha^{13}(2\pi) + ... + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!}B_n\alpha^{2n-1}(2\pi) + ...\}$$

where $\alpha(2\pi) = \arctan(2\pi) \approx 1.412965137..$

[5] H. Vic Dannon, "Archimedes Series", Gauge Institute Journal, Vol. 18, No 3. August 2022, pp 1-11

and $B_n =$ Bernoulli Numbers.

The Transcendental partial sums

$$\pi_n = \tfrac{1}{2}\{\alpha(2\pi) + \tfrac{1}{3}\alpha^3(2\pi) + \tfrac{2}{15}\alpha^5(2\pi) + \tfrac{17}{315}\alpha^7(2\pi) + \tfrac{62}{2835}\alpha^9(2\pi) +$$

$$+ \tfrac{(819)(691)}{3^6 5^2 7(11)(91)}\alpha^{11}(2\pi) + \tfrac{5461}{3^5 5^2 7(11)(13)}\alpha^{13}(2\pi) + ... + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!}B_n\alpha^{2n-1}(2\pi)\}$$

represent π better than any inapplicable statement about its not being a root of a polynomial equation.

We conclude that

Sequential Transcendence is

a superior characterization of a number.

Appendix
Transcendental Numbers

Liouville $$\boxed{\frac{1}{10}+\frac{1}{10^{2!}}+\frac{1}{10^{3!}}+\frac{1}{10^{4!}}+....=transcendental}$$

Hermit $$\boxed{e^{\text{rational}}=transcendental}$$

Lindemann $$\boxed{e^{\text{algebraic}}=transcendental}$$

$$\boxed{e^{\tau}=\text{algebraic}\Rightarrow\tau\text{=tanscendental}}$$

$$e^{i\pi}=-1\Rightarrow i\pi=\text{trans}\ \Rightarrow\pi=-i\cdot\text{trans}=\text{trans}$$

α_1,α_2 =**Algebraically Dependent over** $\mathbb{Q}$

<u>iff</u> $P(\alpha_1,\alpha_2)=0$ where $P(x,y)$ has coefficients from $\mathbb{Q}$

$\sqrt{\pi},\pi=$ algebraically dependent over $\mathbb{Q}$ with $P(x,y)=x^2-y$.

Lindemann-Weierstrass

$\alpha_1,\alpha_2,\alpha_3=$ algebraic

$\beta_1,\beta_2,\beta_3=$ algebraic, Linearly independent over $\mathbb{Q}$

$$\boxed{\Rightarrow\beta_1e^{\alpha_1}+\beta_2e^{\alpha_2}+\beta_3e^{\alpha_3}=\text{transcendental}}$$

α algebraic $\neq 0\ \Rightarrow\ \cos\alpha,\ \sin\alpha,\ \tan\alpha=$ transcendental,

α algebraic $\neq 0,1\Rightarrow\ \log\alpha=$transcendental

$\alpha_1,\alpha_2,\alpha_3$ =algebraic, linearly independent over $\mathbb{Q}$

$$\Rightarrow e^{\alpha_1},e^{\alpha_2},e^{\alpha_2}=\text{ algebraically independent over }\mathbb{Q}$$

$\Rightarrow r_1 e^{\alpha_1} + r_2 e^{\alpha_2} + r_3 e^{\alpha_3} = transcendental$ for any $r_1, r_2, r_3 \in \mathbb{Q}$

Baker $\alpha_1 \neq \alpha_2 \neq \alpha_3$ algebraic

$\Rightarrow e^{\alpha_1}, e^{\alpha_2}, e^{\alpha_3} =$ linearly independent over $\mathbb{A}$

$\Rightarrow \beta_1 e^{\alpha_1} + \beta_2 e^{\alpha_2} + \beta_3 e^{\alpha_3} = transcendental$ for any $\beta_1, \beta_2 \in \mathbb{A}$

Gelfond

$\alpha \neq 0,1$ algebraic , $\beta =$ irrational algebraic $\boxed{\Rightarrow \alpha^{\beta} = \text{transcendental}}$

$$2^{\sqrt{2}},$$

$$\sqrt{2}^{\sqrt{2}},$$

$$e^{\pi} = (e^{i\pi})^{-i} = (-1)^{-i},$$

$$e^{-\frac{1}{2}\pi} = (e^{i\frac{1}{2}\pi})^{i} = (i)^{i}.$$

Gelfond-Schneider

$\alpha_1, \alpha_2 =$ algebraic $\neq 0,1$

$\beta_1, \beta_2 =$ algebraic,

$1, \beta_1, \beta_2 =$ Linearly independent over $\mathbb{Q}$

$$\boxed{\Rightarrow \alpha_1^{\beta_1} \alpha_2^{\beta_2} = transcendental}$$

Baker

$\alpha_1 \neq \alpha_2$ algebraic $\neq 0,1$

$\beta_1, \beta_2 =$ irrational algebraic,

$1, \beta_1, \beta_2 =$ linearly independent over $\mathbb{Q}$

$$\boxed{\Rightarrow \alpha_1^{\beta_1}\alpha_2^{\beta_2} = transcendental}$$

Gelfond-Schneider

$\alpha_1, \alpha_2 =$ algebraic, $\neq 0,1$

$\beta_1, \beta_2 =$ algebraic,

$\log\alpha_1, \log\alpha_2 =$ Linearly independent over $\mathbb{Q}$

$$\boxed{\Rightarrow \beta_1 \log\alpha_1 + \beta_2 \log\alpha_2 = transcendental}$$

Gelfond-Schneider-Baker

$\alpha_1, \alpha_2, \alpha_3 =$ algebraic, $\neq 0,1$

$\beta_0, \beta_1, \beta_2, \beta_3 =$ algebraic,

$\beta_0, \beta_1, \beta_2, \beta_3 =$ Linearly independent over $\mathbb{Q}$

$$\boxed{\Rightarrow e^{\beta_0}\alpha_1^{\beta_1}\alpha_2^{\beta_2}\alpha_3^{\beta_3} = \text{transcendental}}$$

Gelfond-Schneider-Baker

$\alpha_1, \alpha_2, \alpha_3 =$ algebraic, $\neq 0,1$

$\beta_0 \neq 0, \beta_1, \beta_2, \beta_3 =$ algebraic,

$\log\alpha_1, \log\alpha_2, \log\alpha_3 =$ Linearly independent over $\mathbb{Q}$

$$\boxed{\Rightarrow \beta_0 + \beta_1 \log\alpha_1 + \beta_2 \log\alpha_2 + \beta_3 \log\alpha_3 = \text{transcendental}}$$

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