The Geometry of Exponentials. And Euler's Gamma Expanded in Logarithms

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Abstract We explore the geometry of $w = e^z$, $z = \log w$, $\alpha^z = (\rho e^{i\phi})^z$, $e^{i\theta}$, and i .

We expand Euler's Constant Gamma γ in logarithms.

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References

1.

$y = e^x$

For an infinite hyper-real N, $\varepsilon = \frac{1}{N} =$ infinitesimal

$$y = e^x = \left(1 + \frac{x}{N}\right)^N = \left(1 + \varepsilon x\right)^N$$

$$= \left(1 + \frac{\log y}{N}\right)^N = \left(1 + \varepsilon \log y\right)^N$$

Thus, the exponential function looks like $(1+0)^\infty$.

Then,

$$\frac{d}{dx}\left(1 + \varepsilon x\right)^N = N\left(1 + \varepsilon x\right)^{N-1}\varepsilon = \left(1 + \varepsilon x\right)^{N-1} = \left(1 + \varepsilon x\right)^N$$

$$\frac{dy}{dx} = y.$$

Applying the Binomial Theorem with the infinite hyper-real N, we expand the infinite product into an infinite series in powers of x,

$$\left(1 + \varepsilon x\right)^N = 1 + \underbrace{N\varepsilon x}_{x} + \frac{1}{2!}\underbrace{N(N-1)}_{N^2}\varepsilon^2 x^2 + \frac{1}{3!}\underbrace{N(N-1)(N-2)}_{N^3}\varepsilon^3 x^3 + \ldots.$$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots.$$

2.

$x = \log y$

For the Logarithmic function,

$$x = \log y = N\left(y^{\frac{1}{N}} - 1\right) = \frac{y^{\varepsilon} - 1}{\varepsilon}$$

$$= N\left(e^{\frac{x}{N}} - 1\right) = \frac{e^{\varepsilon x} - 1}{\varepsilon}$$

Thus, the logarithmic function looks like $\frac{0}{0}$

$$\frac{d}{dy}\log y = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\frac{d}{dx}e^{x}} = \frac{1}{e^{x}} = \frac{1}{y}$$

Applying the Binomial Theorem with the infinite hyper-real N, we expand the infinite product into an infinite series in powers of y,

$$\log(1+y) = N\left((1+y)^{\frac{1}{N}} - 1\right)$$

$$= \underbrace{N\frac{1}{N}}_{1} y + \frac{1}{2!}N\frac{1}{N}\underbrace{(\frac{1}{N}-1)}_{-1}y^2 + \frac{1}{3!}N\frac{1}{N}\underbrace{(\frac{1}{N}-1)(\frac{1}{N}-2)}_{2}y^3 +$$

$$+\frac{1}{4!}N\frac{1}{N}\underbrace{(\frac{1}{N}-1)(\frac{1}{N}-2)(\frac{1}{N}-3)}_{-2\cdot 3}y^4 + \ldots$$

$$\log(1+y) = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \ldots$$

3.

$w = e^z$

For an infinite hyper-real N, $\varepsilon = \frac{1}{N} =$ infinitesimal

$$w = e^z = \left(1 + \frac{z}{N}\right)^N = \left(1 + \varepsilon z\right)^N$$

$$= \left(1 + \frac{\log w}{N}\right)^N = \left(1 + \varepsilon \log w\right)^N$$

Thus, the exponential function looks like $(1+0)^\infty$.

Then,

$$\frac{d}{dz}e^z = \frac{d}{dz}\left(1+\varepsilon z\right)^N = N\left(1+\varepsilon z\right)^{N-1}\varepsilon = \left(1+\varepsilon z\right)^{N-1} = \left(1+\varepsilon z\right)^N = e^z$$

$$\frac{dw}{dz} = w\,.$$

Applying the Binomial Theorem with the infinite hyper-real N, we expand the infinite product into an infinite series in powers of z,

$$\left(1+\varepsilon z\right)^N = 1 + \underbrace{N\varepsilon}_{1} z + \frac{1}{2!}\underbrace{N(N-1)}_{N^2}\varepsilon^2 z^2 + \frac{1}{3!}\underbrace{N(N-1)(N-2)}_{N^3}\varepsilon^3 z^3 + ...$$

That is,

$$e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + ...$$

$$e^{iy} = 1 + iy - \frac{1}{2!}y^2 - i\frac{1}{3!}y^3 + \frac{1}{4!}y^4 + i\frac{1}{5!}y^5 + ...$$

This series is absolutely convergent, and can be reordered

$$e^{iy} = \underbrace{\left(1 - \frac{1}{2!}y^2 + \frac{1}{4!}y^4 + ...\right)}_{\cos y} + i\underbrace{\left(y - \frac{1}{3!}y^3 + \frac{1}{5!}z^5 - ....\right)}_{\sin y}$$

$$e^{iy} = \cos y + i \sin y$$

$$e^{iy} \text{ is periodic with period} = 2\pi$$

$$\left|e^{iy}\right| = \sqrt{\cos^2 y + \sin^2 y} = 1$$

$$e^z = e^{x+iy} = e^x e^{iy} = e^x \cos y + ie^x \sin y$$

$$e^z = e^{re^{i\theta}} = e^{r\cos\theta} e^{i\sin\theta} = e^{r\cos\theta}\cos(\underbrace{\sin\theta}_{y}) + ie^{r\cos\theta}\sin(\underbrace{\sin\theta}_{y})$$

4.

The $e^{i\theta}$ Rotation

$$e^{i\theta} = \cos\theta + i\sin\theta = \text{point on the unit circle in } \mathbb{C},$$

represented by $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ = rotation by θ

$$e^{-i\theta} = \cos\theta - i\sin\theta \text{ is represented by } \underbrace{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}_{\text{rotation by } -\theta}.$$

Applied to another rotation,

$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)} \leftrightarrow \underbrace{\begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}}_{\text{rotation by } \theta_1+\theta_2}$$

Applied to a point $z = re^{i\phi}$,

$$e^{i\theta}z = e^{i\theta}re^{i\phi} = re^{i(\theta+\phi)} \leftrightarrow r\underbrace{\begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}}_{\text{rotation by } \theta+\phi}$$

$$e^{i\theta_1} + e^{i\theta_2} \leftrightarrow \begin{bmatrix} \cos\theta_1 + \cos\theta_2 & -\sin\theta_1 - \sin\theta_2 \\ \sin\theta_1 + \sin\theta_2 & \cos\theta_1 + \cos\theta_2 \end{bmatrix}$$

$$= 2\cos\tfrac{1}{2}(\theta_1 - \theta_2)\underbrace{\begin{bmatrix} \cos\frac{1}{2}(\theta_1 + \theta_2) & -\sin\frac{1}{2}(\theta_1 + \theta_2) \\ \sin\frac{1}{2}(\theta_1 + \theta_2) & \cos\frac{1}{2}(\theta_1 + \theta_2) \end{bmatrix}}_{\text{rotation by } \frac{1}{2}(\theta_1 + \theta_2)}$$

5.

$z = \log w$

For the Logarithmic function,

$$z = \log w = N\left(w^{\frac{1}{N}} - 1\right) = \frac{w^{\varepsilon} - 1}{\varepsilon}$$

$$= N\left(e^{\frac{z}{N}} - 1\right) = \frac{e^{\varepsilon z} - 1}{\varepsilon}$$

Thus, the logarithmic function looks like $\frac{0}{0}$

$$\frac{d}{dw}\log w = \frac{dz}{dw} = \frac{1}{\frac{dw}{dz}} = \frac{1}{\frac{d}{dz}e^{z}} = \frac{1}{e^{z}} = \frac{1}{w}$$

Applying the Binomial Theorem with the infinite hyper-real N, we expand the infinite product into an infinite series in powers of w,

$$\log(1+w) = N\left((1+w)^{\frac{1}{N}} - 1\right)$$

$$= \underbrace{N\frac{1}{N}}_{1}w + \frac{1}{2!}N\frac{1}{N}\underbrace{(\frac{1}{N}-1)}_{-1}w^2 + \frac{1}{3!}N\frac{1}{N}\underbrace{(\frac{1}{N}-1)(\frac{1}{N}-2)}_{2}w^3 +$$

$$+\frac{1}{4!}N\frac{1}{N}\underbrace{(\frac{1}{N}-1)(\frac{1}{N}-2)(\frac{1}{N}-3)}_{-2\cdot 3}w^4 + \ldots$$

$$\log(1+w) = w - \frac{1}{2}w^2 + \frac{1}{3}w^3 - \frac{1}{4}w^4 + \ldots$$

$$w = u + iv = \underbrace{\sqrt{u^2 + v^2}}_{|w|}\left(\underbrace{\frac{u}{\sqrt{u^2 + v^2}}}_{\cos\phi} + i\underbrace{\frac{v}{\sqrt{u^2 + v^2}}}_{\sin\phi}\right) = \left|w\right|e^{i\phi}$$

$$\underbrace{\log w}_{x+iy} = \log\left|w\right|e^{i\phi}$$

$$= \log\left|w\right| + \log e^{i\phi}$$

$$= \log\left|w\right| + i\phi$$

$$= \underbrace{\log\sqrt{u^2 + v^2}}_{x} + i\underbrace{\phi}_{y}$$

$$x = \log\sqrt{u^2 + v^2}$$

$$y = \phi$$

$$\log\omega = \log\sqrt{u^2 + v^2} + i\phi \leftrightarrow \begin{bmatrix} \log\sqrt{u^2 + v^2} & -\phi \\ \phi & \log\sqrt{u^2 + v^2} \end{bmatrix}$$

6.

The i Rotation

$$i = e^{\log i} = e^{\log\left|i\right| + i\frac{1}{2}\pi} = e^{i\frac{1}{2}\pi} = \text{rotation by } \tfrac{1}{2}\pi$$

$$i^2 = e^{2\log i} = e^{2\log\left|i\right| + 2i\frac{1}{2}\pi} = e^{i\pi} = \text{rotation by } \pi$$

$$i^3 = e^{3\log i} = e^{3\log\left|i\right| + 3i\frac{1}{2}\pi} = e^{i\frac{3}{2}\pi} = \text{rotation by } \tfrac{3}{2}\pi$$

$$i^4 = e^{4\log i} = e^{4\log\left|i\right| + 4i\frac{1}{2}\pi} = e^{i2\pi} = \text{rotation by } 2\pi$$

$$-i = \frac{1}{i} = \frac{1}{e^{i\frac{1}{2}\pi}} = e^{-i\frac{1}{2}\pi} = \text{rotation by } -\tfrac{1}{2}\pi$$

$$i\log i = ii\tfrac{1}{2}\pi = -\tfrac{1}{2}\pi$$

7.

α^z

$$\alpha^z = (\rho e^{i\phi})^z = e^{z\log\alpha} = e^{(x+iy)\log\rho e^{i\phi}} = e^{(x+iy)(\log\rho+i\phi)}$$

$$= e^{x\log\rho - y\phi + i(y\log\rho + x\phi)}$$

$$= e^{x\log\rho-y\phi}\cos(y\log\rho + x\phi) + ie^{x\log\rho-y\phi}\sin(y\log\rho + x\phi)$$

$$i^i = e^{i\log i} = e^{i(\log 1 + i\frac{1}{2}\pi)} = e^{-\frac{1}{2}\pi} = \text{rotation by } -\tfrac{1}{2}\pi$$

$$i^{-i} = e^{-i\log i} = e^{-i(\log 1 + i\frac{1}{2}\pi)} = e^{\frac{1}{2}\pi} = \text{rotation by } \tfrac{1}{2}\pi$$

$$(-1)^i = e^{i\log(-1)} = e^{i(\log 1 + i\pi)} = e^{-\pi} = \text{rotation by } -\pi$$

$$(-1)^{-i} = e^{-i\log(-1)} = e^{-i(\log 1 + i\pi)} = e^{\pi} = \text{rotation by } \pi$$

$$\log i^i = i\log i = i(\log 1 + i\tfrac{1}{2}\pi) = -\tfrac{1}{2}\pi$$

$$\log i^{-i} = -i\log i = -i(\log 1 + i\tfrac{1}{2}\pi) = \tfrac{1}{2}\pi$$

$$\log(-1)^i = i\log(-1) = i(\log 1 + i\pi) = -\pi$$

$$\log(-1)^{-i} = -i\log(-1) = -i(\log 1 + i\pi) = \pi$$

$$e^i = \cos 1 + i\sin 1 = \text{rotation by 1 radian}$$

$$i^e = e^{e\log i} = e^{e(\log 1 + i\frac{1}{2}\pi)} = e^{i\frac{1}{2}\pi e} = \text{rotation by } \tfrac{1}{2}\pi e$$

$$e^{i\frac{1}{2}\pi} = \cos(\tfrac{1}{2}\pi) + i\sin(\tfrac{1}{2}\pi) = i$$

$$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$$

$$e^{i\frac{3}{2}\pi} = \cos(\tfrac{3}{2}\pi) + i\sin(\tfrac{3}{2}\pi) = -i$$

$$e^{i2\pi} = \cos(2\pi) + i\sin(2\pi) = 1$$

8.

Expansion in Powers of $\log y$

To expand in powers of logarithms, we use the infinite series expansion for the exponential function

$$y = e^{x} = e^{\log y} = 1 + \log y + \frac{1}{2!}(\log y)^2 + \frac{1}{3!}(\log y)^3 + ...$$

$$y = 1 + \log y + \frac{1}{2!}(\log y)^2 + \frac{1}{3!}(\log y)^3 + ...$$

For an integer n,

$$n = 1 + \log n + \frac{1}{2!}(\log n)^2 + \frac{1}{3!}(\log n)^3 + ...$$

That is,

Every integer n has Power Series Expansion in $\log n$

Since the terms of the series are transcendental numbers, the partial sums of the series are transcendental numbers. Therefore,

Every Integer has Power Series Expansion in Transcendental Numbers

The same holds for any rational number $\frac{p}{q}$,

$$\boxed{\frac{p}{q} = 1 + \log\frac{p}{q} + \frac{1}{2!}\left(\log\frac{p}{q}\right)^2 + \frac{1}{3!}\left(\log\frac{p}{q}\right)^3 + ...}$$

Every Rational $\frac{p}{q}$ has Power Series Expansion in $\log\frac{p}{q}$

And

Every Rational has Power Series Expansion in Transcendental Numbers

9.

Euler's Constant Gamma Expanded in Logarithmic Powers

Euler's constant Gamma is

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+...+\frac{1}{n}-\log n\right)$$

This looks like $\infty - \infty$, which is undefined.

In fact, we substitute

$$\log n = \log\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot ...\cdot\frac{n-1}{n-2}\cdot\frac{n}{n-1}$$

$$= \log 2 + \log\frac{3}{2} + \log\frac{4}{3} + ... + \log\frac{n}{n-1}$$

This gives the infinite series

$$\gamma = (1-\log 2) + (\frac{1}{2}-\log\frac{3}{2}) + (\frac{1}{3}-\log\frac{4}{3}) + (\frac{1}{4}-\log\frac{5}{4}) + ...$$

We shall obtain γ in logarithmic powers

Expanding in logarithmic series,

$$1 = 1 + \log 1 + \frac{1}{2!}(\log 1)^2 + \frac{1}{3!}(\log 1)^3 + \frac{1}{4!}(\log 1)^4 + ...$$

$$\frac{1}{2} = 1 + \log\frac{1}{2} + \frac{1}{2!}(\log\frac{1}{2})^2 + \frac{1}{3!}(\log\frac{1}{2})^3 + \frac{1}{4!}(\log\frac{1}{2})^4 + ...$$

$$\frac{2}{3} = 1 + \log\frac{2}{3} + \frac{1}{2!}(\log\frac{2}{3})^2 + \frac{1}{3!}(\log\frac{2}{3})^3 + \frac{1}{4!}(\log\frac{2}{3})^4 + ...$$

$$\frac{3}{4} = 1 + \log\frac{3}{4} + \frac{1}{2!}(\log\frac{3}{4})^2 + \frac{1}{3!}(\log\frac{3}{4})^3 + \frac{1}{4!}(\log\frac{3}{4})^4 + ...$$

....

$$1 = 1$$

$$-\frac{1}{2} = -\log 2 + \frac{1}{2!}(-\log 2)^2 + \frac{1}{3!}(-\log 2)^3 + \frac{1}{4!}(-\log 2)^4 + ...$$

$$-\frac{1}{3} = -\log\frac{3}{2} + \frac{1}{2!}(-\log\frac{3}{2})^2 + \frac{1}{3!}(-\log\frac{3}{2})^3 + \frac{1}{4!}(-\log\frac{3}{2})^4 + ...$$

$$-\frac{1}{4} = -\log\frac{4}{3} + \frac{1}{2!}(-\log\frac{4}{3})^2 + \frac{1}{3!}(-\log\frac{4}{3})^3 + \frac{1}{4!}(-\log\frac{4}{3})^4 + ...$$

....

$$1 = 1$$

$$\frac{1}{2} - \log 2 = -\frac{1}{2!}(\log 2)^2 + \frac{1}{3!}(\log 2)^3 - \frac{1}{4!}(\log 2)^4 + ...$$

$$\frac{1}{3} - \log\frac{3}{2} = -\frac{1}{2!}(\log\frac{3}{2})^2 + \frac{1}{3!}(\log\frac{3}{2})^3 - \frac{1}{4!}(\log\frac{3}{2})^4 + ...$$

$$\frac{1}{4} - \log\frac{4}{3} = -\frac{1}{2!}(\log\frac{4}{3})^2 + \frac{1}{3!}(\log\frac{4}{3})^3 - \frac{1}{4!}(-\log\frac{4}{3})^4 + ...$$

....

Summing both sides,

$$\underbrace{(1 - \log 2) + (\frac{1}{2} - \log\frac{3}{2}) + (\frac{1}{3} - \log\frac{4}{3}) + (\frac{1}{4} - \log\frac{5}{4}) + ...}_{\gamma} =$$

$$= 1 - \frac{1}{2!}\left\{\left(\log 2\right)^2 + \left(\log\frac{3}{2}\right)^2 + \left(\log\frac{4}{3}\right)^2 + \left(\log\frac{5}{4}\right)^2 + ...\right\}$$

$$+\frac{1}{3!}\left\{\left(\log 2\right)^3 + \left(\log\frac{3}{2}\right)^3 + \left(\log\frac{4}{3}\right)^3 + \left(\log\frac{5}{4}\right)^3 + ...\right\}$$

$$-\frac{1}{4!}\left\{\left(\log 2\right)^4 + \left(\log\frac{3}{2}\right)^4 + \left(\log\frac{4}{3}\right)^4 + \left(\log\frac{5}{4}\right)^4 + ...\right\}$$

....

References

1. [Maor] Eli Maor, "e The Story of a Number", Princeton U. Press, 1994.