Catalan Constant is Irrational and Sequentially Transcendental Number

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Abstract A number is algebraic if it is a zero of a polynomial in integer coefficients.

The algebraic numbers form a field with respect to addition , and multiplication. The sum, and product of two algebraic numbers is an algebraic number.

A number that is not algebraic, transcends algebraic numbers, and is called transcendental.

The following are believed to be transcendental

e (Hermit),

π (Lindemann),

e^{rational} (Hermit),

$e^{\text{algebraic}}$ (Lindemann),

$(\text{algebraic})^{\text{irrational algebraic}}$ (Gelfond-Schneider).

$$e^{\pi} = (e^{i\pi})^{-i} = (-1)^{-i}$$

$$e^{n\pi} = (e^{i\pi})^{-ni} = (-1)^{-ni}$$

$\log(\text{algebraic})$ for algebraic $\neq 0,1$ (Lindemann–Weierstrass).

Recently, we showed[1] that Euler's Constant γ is a limit of a sequence of transcendental numbers.

And we showed[2] that Riemann's $\zeta(3)$ is infinitesimally close to a transcendental number.

Catalan's Constant G is the infinite sum

$$1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+...$$

The series sum has a denominator that includes the product of all the prime numbers. But there is no number that is divided by all the primes. Hence, no number for the denominator of a rational number that equals the series. And we conclude that

G **is not a rational number**.

It is approximately

$$G = 0.915\,965\,594\,177\,219\,015\,054\,603\,514\,932\,384\,110\,774......$$

Euler sought an answer[3] to the question whether the G Series sums up to a rational number times π^2.

Recently, we answered Euler's question in the negative. We showed[4] that the series of G does Not sum up to a rational number times π^2.

We show here that

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[1] H. Vic Dannon, "Euler's Gamma is Irrational and Sequentially Transcendental Number", Gauge Institute Journal, Vol. 19, No 3. August 2023

[2] H. Vic Dannon, "Zeta(3) is Irrational and Sequentially Transcendental Number", Gauge Institute Journal, Vol. 19, No 4. November 2023

[3] Euler's letters to James Stirling in "James Stirling, This about Series and such things" Scottish Academic Press 1988, pp. 142-151.

[4] H. Vic Dannon, "All the π^2 series, and the Catalan series". Gauge Institute Journal, Vol. 18, No. 3, August 2022.

G **is Not a Liouville number**.

And we cannot say that $\zeta(3)$ is transcendental on account of its being a Liouville number which it is not.

Ramanujan gave a formula[5] for G by

$$G = \frac{\pi}{2} - \frac{\pi}{2}\log\left(\frac{\pi}{2}\right) + \left(\frac{\pi}{2}\right)^3 \frac{1}{2(3!)}B_2 - \left(\frac{\pi}{2}\right)^5 \frac{1}{4(5!)}B_4 + \left(\frac{\pi}{2}\right)^7 \frac{1}{6(7!)}B_6 + \ldots$$

Using this expansion, we show that

G **is the limit of a sequence** G_j **of transcendental numbers.**

This **Does Not** mean that G is transcendental But that

As far as we can ever compute, for any finite j**,**

the partial sum G_j **is a transcendental number**.

Indeed, a sequence of transcendental numbers need not converge to a transcendental number.

In 2022, we derived[6] an expansion for 1, which we named **The Archimedes Series for 1**

$$1 = \tfrac{\pi}{4} + \tfrac{1}{3}\left(\tfrac{\pi}{4}\right)^3 + \tfrac{2}{15}\left(\tfrac{\pi}{4}\right)^5 + \tfrac{17}{315}\left(\tfrac{\pi}{4}\right)^7 + \tfrac{62}{2835}\left(\tfrac{\pi}{4}\right)^9 +$$

$$+ \tfrac{(819)(691)}{3^6 5^2 7(11)(91)}\left(\tfrac{\pi}{4}\right)^{11} + \tfrac{5461}{3^5 5^2 7(11)(13)}\left(\tfrac{\pi}{4}\right)^{13} + \ldots. + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!}B_n\left(\tfrac{\pi}{4}\right)^{2n-1} + ..$$

where B_n = Bernoulli Numbers

[5] Bernhard Candelpergher, "Ramanujan Summation of Divergent Series", p. 52, Springer, 2017.
[6] H. Vic Dannon, "Archimedes Series", Gauge Institute Journal, Vol. 18, No 3. August 2022, pp 1-11

if π is transcendental, the Partial Sums of the Archimedes Series for 1 are transcendental numbers that converge to the algebraic number 1

And the

Transcendental partial sums 1_n **transform to Algebraic** 1

We shall say that besides being an Algebraic Number,

1 is a Sequentially Transcendental Number

That is, there is a sequence of transcendentals that converges to 1.

In fact, for an infinite hyper-real N we cannot compute the algebraic partial sum with N transcendental terms.

As far as we can ever compute, for any finite n**,**

the partial sum 1_n **is Transcendental.**

Similarly, for G

As far as we can ever compute, for any finite j**,**

the partial sum G_j **is a transcendental number.**

Contents

1.

G is Irrational

$$\boxed{G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \text{...is irrational}}$$

<u>*Proof*</u>: For $n = 1,2,3,...$

$$G_n = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + ... + \frac{(-1)^{n+1}}{(2n-1)^2}$$

sums up to the quotient

$$\frac{p_n}{3^2 \cdot 5^2 \cdot 7^2 \cdot .... \cdot (2n-1)^2}.$$

For an infinite hyper-real N,

G_N has a common denominator $3^2 \cdot 5^2 \cdot 7^2 \cdot .... \cdot (2N-1)^2$

This denominator includes the product of all the prime numbers

$$p_1 \cdot p_2 \cdot p_3 \cdot ....$$

There is no finite natural number q that is divided by all the primes.

Thus, there is no rational number $\frac{p}{q}$ that equals G.

Therefore,

G is Not a rational number.

That is,

G is Irrational.$\square$

2.

G is Not a Liouville Number

Liouville showed that

If

$$\alpha = \text{the zero of a reduced polynomial}$$

$$P_n(x) \text{ of order } n,$$

and if α is the limit of a sequence of rational numbers,

$$\frac{p_m}{q_m},$$

so that

$$p_m, \text{ and } q_m \text{ are relatively prime,}$$

And if there is a constant $C_m > 0$ so that

$$q_m > C_m$$

Then,

$$\boxed{\left|\alpha - \frac{p_m}{q_m}\right| > \left(\frac{1}{q_m}\right)^{n+1}}$$

The negation of this statement is a criteria for transcendence.

If

$$\tau = \text{limit of a sequence of rational numbers,}$$

$$\frac{p_m}{q_m}, \text{ so that } p_m, \text{ and } q_m \text{ are relatively prime,}$$

and if for each $m = 1, 2, 3, \ldots,$

$$\boxed{\left|\tau - \frac{p_m}{q_m}\right| < \left(\frac{1}{q_m}\right)^{m+1}}$$

<u>Then</u>,

$$\tau = transcendental\,.$$

The partial sums of the expansion of G are such rationals

$$\frac{p_m}{q_m} = 1 - \frac{1}{3^2} + \frac{1}{5^2} + ... + \frac{(-1)^{m+1}}{(2m-1)^2} = \frac{p_m}{[(2m-1)!]^2}$$

$$\frac{1}{q_m} = \frac{1}{[(2m-1)!]^2}$$

$$\left|G - \frac{p_m}{q_m}\right| = \frac{1}{(2m+1)^2} - \frac{1}{(2m+3)^2} + ...$$

is not bounded by

$$\left(\frac{1}{q_m}\right)^{m+1} = \frac{1}{[(2m-1)!]^{2(m+1)}} \text{ for all } m = 1, 2, 3, ...$$

Therefore, G is not a Liouville Number.□

3.

G is Sequentially Transcendental Number

By Ramanujan[7],

$$G = \frac{\pi}{2} - \frac{\pi}{2}\log\left(\frac{\pi}{2}\right)$$

$$+\left(\frac{\pi}{2}\right)^3 \frac{1}{2(3!)} B_2 - \left(\frac{\pi}{2}\right)^5 \frac{1}{4(5!)} B_4 + \left(\frac{\pi}{2}\right)^7 \frac{1}{6(7!)} B_6 + ...$$

$$B_2 = \tfrac{1}{6}$$

$$B_4 = \tfrac{-1}{30}$$

$$B_6 = \tfrac{1}{42}$$

$$B_8 = \tfrac{-1}{30}$$

$$B_{10} = \tfrac{5}{66}$$

$$B_{12} = \tfrac{-691}{2730}$$

$$B_{14} = \tfrac{7}{6}$$

$$B_{16} = \tfrac{-3617}{510}$$

$$B_{18} = \tfrac{43867}{798}$$

$$B_{20} = \tfrac{-174611}{330}$$

[7] Bernhard Candelpergher, "Ramanujan Summation of Divergent Series", p. 52, Springer, 2017.

....

We will use this expansion for G to show that G is sequentially transcendental

By Leibniz,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$$

$$\frac{\pi}{2} = 2\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...\right]$$

For $j = 1, 2, 3, \ldots$, define the sequence of rationals

$$p_j = 2\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... + \frac{(-1)^{j+1}}{2j-1}\right] \to \frac{\pi}{2}$$

$$(p_j)^3 = 2^3\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... + \frac{(-1)^{j+1}}{2j-1}\right]^3 \to \left(\frac{\pi}{2}\right)^3$$

$$(p_j)^5 = 2^5\left[\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... + \frac{(-1)^{j+1}}{2j-1}\right]^5 \to \left(\frac{\pi}{2}\right)^5$$

$$(p_j)^7 = 2^7\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... + \frac{(-1)^{j+1}}{2j-1}\right]^7 \to \left(\frac{\pi}{2}\right)^7$$

....

The partial sums

$$G_j = p_j - p_j \log p_j + \frac{(p_j)^3}{2(3!)}B_2 - \frac{(p_j)^5}{4(5!)}B_4 + \frac{(p_j)^7}{6(7!)}B_6 + ... \to G$$

are transcendental numbers.

Because if $G_j =$algebraic, then by the field property of algebraic numbers, $\log p_j =$algebraic.

But since $p_j = \text{algebraic} \neq 0, 1$,

$$\log p_j = \text{transcendental (Lindemann–Weierstrass)}$$

From that contradiction, it follows that $G_j = $ transcendental.

This holds for any $j = 1, 2, 3, 4, \ldots$ □

$$G_N \sim p_{1,N} - p_{1,N} \log p_{1,N} + \text{higher order reciprocals.}$$

$$G_{N+1} \sim p_{1,N+1} - p_{1,N+1} \log p_{1,N+1} + \text{ higher order reciprocals.}$$

$$\left|G_{N+1} - G_N\right| \leq \frac{1}{2N+1} + \underbrace{\log \underbrace{\frac{(p_{1,N+1})^{p_{1,N+1}}}{(p_{1,N})^{p_{1,N}}}}_{\approx 1}}_{\approx 0}$$

$$= \text{Order of } \frac{1}{2N+1}$$

That is, G_N is infinitesimally close to G.

That is,

$G =$ **is the limit of the transcendental partial sums** G_j

This **Does Not** mean that G is transcendental But that

As far as we can ever compute, for any finite j**,**

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Indeed, a sequence of transcendental numbers need not converge to a transcendental number.

In 2022, we derived[8] an expansion for 1, which we named **The Archimedes Series for 1**

$$1 = \frac{\pi}{4} + \frac{1}{3}\left(\frac{\pi}{4}\right)^3 + \frac{2}{15}\left(\frac{\pi}{4}\right)^5 + \frac{17}{315}\left(\frac{\pi}{4}\right)^7 + \frac{62}{2835}\left(\frac{\pi}{4}\right)^9 +$$

$$+ \frac{(819)(691)}{3^6 5^2 7(11)(91)}\left(\frac{\pi}{4}\right)^{11} + \frac{5461}{3^5 5^2 7(11)(13)}\left(\frac{\pi}{4}\right)^{13} + ... + \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_n \left(\frac{\pi}{4}\right)^{2n-1} + ..$$

where B_n = Bernoulli Numbers

Since π is transcendental, the Partial Sums of the Archimedes Series for 1 are transcendental numbers that converge to the algebraic number 1. And the

Transcendental partial sums 1_n transform to Algebraic 1

We shall say that besides being an Algebraic Number,

1 is a Sequentially Transcendental Number

That is, there is a sequence of transcendentals that converges to 1. In fact, for an infinite hyper-real N we cannot compute the algebraic partial sum with N transcendental terms.

As far as we can ever compute, for any finite n,

the partial sum 1_n is Transcendental.

Thus,

G is Sequentially Transcendental Number

Meaning that,

As far as we can ever compute, for any finite j,

the partial sum G_j is a transcendental number.

[8] H. Vic Dannon, "Archimedes Series", Gauge Institute Journal, Vol. 18, No 3. August 2022, pp 1-11

4.

Ge is Irrational and Sequentially Transcendental Number

Irrationality

$$Ge = \left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + ...\right)\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\right)$$

is the sum of infinitely many rational numbers with common denominator that includes the product of all the prime numbers

$$p_1 \cdot p_2 \cdot p_3 \cdot ....$$

There is no finite natural number

$$q$$

that is divided by all the primes.

Thus, there is no rational number $\frac{p}{q}$ that equals Ge

Therefore, Ge is Not a rational number.

That is, Ge is Irrational.□

Sequentially Transcendental

For $j = 1, 2, 3, ..., N$, $\boxed{G_j e_j = \text{transcendental}}$

Proof:

$$G_j e_j = (\text{transcendental})\underbrace{(1 + 1 + \tfrac{1}{2!} + \tfrac{1}{3!} + ... + \tfrac{1}{j!})}_{\frac{p}{j!} = \text{algebraic}}$$

Therefore, $\dfrac{G_j e_j}{\text{algebraic}} = $ transcendental.

If $G_j e_j = $ algebraic, then by the field property of algebraic numbers, $\dfrac{G_j e_j}{\text{algebraic}} = $ algebraic.

From that contradiction, it follows that $G_j e_j = $ transcendental.□

This holds for any $j = 1, 2, 3, 4, \ldots$,

Consequently,

$$\boxed{Ge = \text{Sequentially Transcendental Number}}$$

5.

$G\pi$ is Irrational and Sequentially Transcendental Number

Irrationality

$$G\pi = \left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + ...\right) 4\left(1 - \frac{1}{3} + \frac{1}{5} - ...\right)$$

is the sum of infinitely many rational numbers with common denominator that includes the product of all the prime numbers

$$p_1 \cdot p_2 \cdot p_3 \cdot ....$$

There is no finite natural number

$$q$$

that is divided by all the primes.

Thus, there is no rational number $\frac{p}{q}$ that equals $G\pi$

Therefore, $G\pi$ is Not a rational number.

That is, $G\pi$ is Irrational.□

Sequentially Transcendental

For $j = 1, 2, 3, ..., N$, $\boxed{G_j \pi_j = \text{transcendental}}$

Proof:

$$G_j\pi_j = (\text{transcendental})\underbrace{4(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+...+\tfrac{(-1)^{j+1}}{2j-1})}_{\text{algebraic}}$$

Therefore, $\dfrac{G_j\pi_j}{\text{algebraic}} =$ transcendental.

If $G_j\pi_j =$algebraic, then by the field property of algebraic numbers, $\dfrac{G_j\pi_j}{\text{algebraic}} =$algebraic.

From that contradiction, it follows that $G_j\pi_j =$transcendental.□

This holds for any $n = 1,2,3,4,...,$

Consequently,

$$\boxed{G\pi = \text{Sequentially Transcendental Number}}$$

6.

$G+e$ is Irrational and Sequentially Transcendental Number

Irrationality

$$G+e=\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+...\right)+\left(\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+...\right)$$

is the sum of infinitely many rational numbers with common denominator that includes the product of all the prime numbers

$$p_1\cdot p_2\cdot p_3\cdot ....$$

There is no finite natural number

$$q$$

that is divided by all the primes.

Thus, there is no rational number $\frac{p}{q}$ that equals $G+e$

Therefore, $G+e$ is Not a rational number.

That is, $G+e$ is Irrational.□

Sequentially Transcendental

For $j=1,2,3,...,N$, $\boxed{G_j+e_j=\text{transcendental}}$

Proof:

$$G_j + e_j = (\text{transcendental}) + (\underbrace{1 + 1 + \tfrac{1}{2!} + \tfrac{1}{3!} + \ldots + \tfrac{1}{j!}}_{\frac{p}{j!} = \text{algebraic}})$$

Therefore, $G_j + e_j - \text{algebraic} =$ **transcendental.**

If $G_j + e_j =$ **algebraic, then by the field property of algebraic numbers,** $G_j + e_j - \text{algebraic} =$ **algebraic.**

From the contradiction, $G_j + e_j =$ **transcendental.**□

This holds for any $j = 1, 2, 3, 4, \ldots$,

Consequently,

$$\boxed{G + e = \text{Sequentially Transcendental Number}}$$

7.

$G+\pi$ is Irrational and Sequentially Transcendental Number

Irrationality

$$G+\pi=\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+...\right)+4\left(1-\frac{1}{3}+\frac{1}{5}-...\right)$$

is the sum of infinitely many rational numbers with common denominator that includes the product of all the prime numbers

$$p_1\cdot p_2\cdot p_3\cdot....$$

There is no finite natural number

$$q$$

that is divided by all the primes.

Thus, there is no rational number $\frac{p}{q}$ that equals $G+\pi$

Therefore, $G+\pi$ is Not a rational number.

That is, $G+\pi$ is Irrational.□

Sequentially Transcendental

For $j=1,2,3,...,N$, $\boxed{G_j+\pi_j=\text{transcendental}}$

Proof:

$$G_j + \pi_j = (\text{transcendental}) + \underbrace{4(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + ... + \tfrac{(-1)^{j+1}}{2j-1})}_{\text{algebraic}}$$

Therefore, $G_j + \pi_j - \text{algebraic} =$ transcendental.

If $G_j + \pi_j =$ algebraic, then by the field property of algebraic numbers, $G_j + \pi_j - \text{algebraic} =$ algebraic.

From the contradiction, it follows that

$G_j + \pi_j =$ transcendental.□

This holds for any $j = 1, 2, 3, 4, \ldots$,

Consequently,

$$\boxed{G + \pi = \text{Sequentially Transcendental Number}}$$

8.

Sequential Transcendence versus Transcendence

A number ξ is Transcendental if it is not the root of any n degree polynomial equation with rational coefficients, for any finite natural number n.

This definition excludes any infinite hyper-real number N. Indeed,

8.1

The Transcendental number π is the root of a polynomial equation with rational coefficients of degree N.

Proof One such polynomial equation of degree N with rational coefficients follows from our 2022 derivation[9] of an expansion for 1, which we named **The Archimedes Series for 1**

$$1 = \tfrac{\pi}{4} + \tfrac{1}{3}(\tfrac{\pi}{4})^3 + \tfrac{2}{15}(\tfrac{\pi}{4})^5 + \tfrac{17}{315}(\tfrac{\pi}{4})^7 + \tfrac{62}{2835}(\tfrac{\pi}{4})^9 +$$

$$+ \tfrac{(819)(691)}{3^6 5^2 7(11)(91)}(\tfrac{\pi}{4})^{11} + \tfrac{5461}{3^5 5^2 7(11)(13)}(\tfrac{\pi}{4})^{13} + \dots + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!} B_n (\tfrac{\pi}{4})^{2n-1} + ..$$

where B_n =Bernoulli Numbers.□

Similarly, our definition of sequential Transcendence breaks down for $n = N$

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[9] H. Vic Dannon, "Archimedes Series", Gauge Institute Journal, Vol. 18, No 3. August 2022, pp 1-11

We defined a number ξ to be sequentially transcendental if for any finite natural number n there a transcendental number ξ_n as close as we wish to ξ.

This definition excludes any infinite hyper-real number N. Indeed, if we allow $n = N$, then

8.2

For an Algebraic, and Sequentially Transcendental α,

ξ_N must be algebraic

For instance, for the Algebraic number 1, the partial sum

$$1_N = \frac{\pi}{4} + \frac{1}{3}\left(\frac{\pi}{4}\right)^3 + \frac{2}{15}\left(\frac{\pi}{4}\right)^5 + \frac{17}{315}\left(\frac{\pi}{4}\right)^7 + \frac{62}{2835}\left(\frac{\pi}{4}\right)^9 +$$

$$+ \frac{(819)(691)}{3^6 5^2 7(11)(91)}\left(\frac{\pi}{4}\right)^{11} + \frac{5461}{3^5 5^2 7(11)(13)}\left(\frac{\pi}{4}\right)^{13} + ... + \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_N \left(\frac{\pi}{4}\right)^{2n-1}$$

where $B_n =$ Bernoulli Numbers for $n = 1, 2, ... N$

is infinitesimally close to 1,

$$1 = 1_N + \text{infinitesimal}$$

Therefore,

$$\underbrace{1}_{\text{algebraic}} = \{\text{the standard part of } 1\} = 1_N$$

That is, for any finite n,

$$1_n = \text{transcendental}$$

But for an infinite hyper-real N

$$1_N = \text{algebraic}$$

It follows that

Our definitions of Transcendental, and Sequentially Transcendental apply only to finite n

In any event, we cannot compute with any infinite n.

But if we are limited to finite n, then the transcendental π is actually the Leibniz rational partial sum

$$\pi_n = 4\left\{1 - \frac{1}{3} + \frac{1}{5} - .... + \frac{(-1)^{n+1}}{2n-1}\right\}$$

that can be made as close as we can compute to π

And γ is actually the transcendental

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + ...\frac{1}{n} - \log(n+1)$$

that can be made as close as we can compute to γ

In other words, since transcendence breaks down at the forever incomprehensible infinity

Sequential Transcendence is way more informative then Transcendence

In 2022, we derived[10] an expansion for π, which we named **The Archimedes Series for π**

$$\pi = \tfrac{1}{2}\{\alpha(2\pi) + \tfrac{1}{3}\alpha^3(2\pi) + \tfrac{2}{15}\alpha^5(2\pi) + \tfrac{17}{315}\alpha^7(2\pi) + \tfrac{62}{2835}\alpha^9(2\pi) +$$

$$+\tfrac{(819)(691)}{3^6 5^2 7(11)(91)}\alpha^{11}(2\pi) + \tfrac{5461}{3^5 5^2 7(11)(13)}\alpha^{13}(2\pi) + ... + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!}B_n\alpha^{2n-1}(2\pi) + ...\}$$

where $\alpha(2\pi) = \arctan(2\pi) \approx 1.412965137..$

[10] H. Vic Dannon, "Archimedes Series", Gauge Institute Journal, Vol. 18, No 3. August 2022, pp 1-11

and B_n = Bernoulli Numbers.

The Transcendental partial sums

$$\pi_n = \tfrac{1}{2}\{\alpha(2\pi) + \tfrac{1}{3}\alpha^3(2\pi) + \tfrac{2}{15}\alpha^5(2\pi) + \tfrac{17}{315}\alpha^7(2\pi) + \tfrac{62}{2835}\alpha^9(2\pi) +$$

$$+\tfrac{(819)(691)}{3^6 5^2 7(11)(91)}\alpha^{11}(2\pi)+\tfrac{5461}{3^5 5^2 7(11)(13)}\alpha^{13}(2\pi)+\ldots. + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!}B_n\alpha^{2n-1}(2\pi)\}$$

represent π better than any inapplicable statement about its not being a root of a polynomial equation.

We conclude that

Sequential Transcendence is

a superior characterization of a number.

Appendix
Transcendental Numbers

Liouville $\boxed{\frac{1}{10}+\frac{1}{10^{2!}}+\frac{1}{10^{3!}}+\frac{1}{10^{4!}}+....=transcendental}$

Hermit $\boxed{e^{\text{rational}}=transcendental}$

Lindemann $\boxed{e^{\text{algebraic}}=transcendental}$

$$\boxed{e^{\tau}=\text{algebraic}\Rightarrow \tau\text{=tanscendental}}$$

$$e^{i\pi}=-1\Rightarrow i\pi=\text{trans}\ \Rightarrow \pi=-i\cdot\text{trans}=\text{trans}$$

α_1,α_2 =**Algebraically Dependent over** $\mathbb{Q}$

<u>iff</u> $P(\alpha_1,\alpha_2)=0$ where $P(x,y)$ has coefficients from $\mathbb{Q}$

$\sqrt{\pi},\pi=$ algebraically dependent over $\mathbb{Q}$ with $P(x,y)=x^2-y$.

Lindemann-Weierstrass

$\alpha_1,\alpha_2,\alpha_3=$ algebraic

$\beta_1,\beta_2,\beta_3=$ algebraic, Linearly independent over $\mathbb{Q}$

$$\boxed{\Rightarrow \beta_1 e^{\alpha_1}+\beta_2 e^{\alpha_2}+\beta_3 e^{\alpha_3}=\text{transcendental}}$$

α algebraic $\neq 0 \ \Rightarrow\ \cos\alpha,\ \sin\alpha,\ \tan\alpha=$ transcendental,

α algebraic $\neq 0,1\Rightarrow\ \log\alpha=$transcendental

$\alpha_1,\alpha_2,\alpha_3=$algebraic, linearly independent over $\mathbb{Q}$

$$\Rightarrow e^{\alpha_1},e^{\alpha_2},e^{\alpha_2}=\text{ algebraically independent over } \mathbb{Q}$$

$\Rightarrow r_1 e^{\alpha_1} + r_2 e^{\alpha_2} + r_3 e^{\alpha_3} = transcendental$ for any $r_1, r_2, r_3 \in \mathbb{Q}$

Baker $\alpha_1 \neq \alpha_2 \neq \alpha_3$ algebraic

$\Rightarrow e^{\alpha_1}, e^{\alpha_2}, e^{\alpha_3} =$ linearly independent over $\mathbb{A}$

$\Rightarrow \beta_1 e^{\alpha_1} + \beta_2 e^{\alpha_2} + \beta_3 e^{\alpha_3} = transcendental$ for any $\beta_1, \beta_2 \in \mathbb{A}$

Gelfond

$\alpha \neq 0{,}1$ algebraic , $\beta =$ irrational algebraic $\boxed{\Rightarrow \alpha^{\beta} = \text{transcendental}}$

$$2^{\sqrt{2}},$$

$$\sqrt{2}^{\sqrt{2}},$$

$$e^{\pi} = (e^{i\pi})^{-i} = (-1)^{-i},$$

$$e^{-\frac{1}{2}\pi} = (e^{i\frac{1}{2}\pi})^{i} = (i)^{i}.$$

Gelfond-Schneider

$\alpha_1, \alpha_2 =$ algebraic $\neq 0{,}1$

$\beta_1, \beta_2 =$ algebraic,

$1, \beta_1, \beta_2 =$ Linearly independent over $\mathbb{Q}$

$$\boxed{\Rightarrow \alpha_1^{\beta_1} \alpha_2^{\beta_2} = transcendental}$$

Baker

$\alpha_1 \neq \alpha_2$ algebraic $\neq 0{,}1$

$\beta_1, \beta_2 =$ irrational algebraic,

$1, \beta_1, \beta_2 =$ linearly independent over $\mathbb{Q}$

$$\boxed{\Rightarrow \alpha_1^{\beta_1}\alpha_2^{\beta_2} = transcendental}$$

Gelfond-Schneider

$\alpha_1, \alpha_2 =$ algebraic, $\neq 0,1$

$\beta_1, \beta_2 =$ algebraic,

$\log\alpha_1, \log\alpha_2 =$ Linearly independent over $\mathbb{Q}$

$$\boxed{\Rightarrow \beta_1 \log\alpha_1 + \beta_2 \log\alpha_2 = transcendental}$$

Gelfond-Schneider-Baker

$\alpha_1, \alpha_2, \alpha_3 =$ algebraic, $\neq 0,1$

$\beta_0, \beta_1, \beta_2, \beta_3 =$ algebraic,

$\beta_0, \beta_1, \beta_2, \beta_3 =$ Linearly independent over $\mathbb{Q}$

$$\boxed{\Rightarrow e^{\beta_0}\alpha_1^{\beta_1}\alpha_2^{\beta_2}\alpha_3^{\beta_3} = \text{transcendental}}$$

Gelfond-Schneider-Baker

$\alpha_1, \alpha_2, \alpha_3 =$ algebraic, $\neq 0,1$

$\beta_0 \neq 0, \beta_1, \beta_2, \beta_3 =$ algebraic,

$\log\alpha_1, \log\alpha_2, \log\alpha_3 =$ Linearly independent over $\mathbb{Q}$

$$\boxed{\Rightarrow \beta_0 + \beta_1 \log\alpha_1 + \beta_2 \log\alpha_2 + \beta_3 \log\alpha_3 = \text{transcendental}}$$

References

https://en.wikipedia.org/wiki/Particular_values_of_the_Riemann_zeta_function

https://en.wikipedia.org/wiki/Baker%27s_theorem

https://en.wikipedia.org/wiki/Gelfond%E2%80%93Schneider_theorem

https://en.wikipedia.org/wiki/Transcendental_number

https://en.wikipedia.org/wiki/Transcendental_number_theory

https://en.wikipedia.org/wiki/Lindemann%E2%80%93Weierstrass_theorem

https://en.wikipedia.org/wiki/List_of_representations_of_e

https://en.wikipedia.org/wiki/List_of_formulae_involving_%CF%80

https://en.wikipedia.org/wiki/Irrational_number

https://en.wikipedia.org/wiki/Algebraic_number

https://en.wikipedia.org/wiki/Algebraic_number_field

https://en.wikipedia.org/wiki/Algebraic_number_theory

https://en.wikipedia.org/wiki/Proof_that_pi_is_irrational

https://en.wikipedia.org/wiki/Proof_that_e_is_irrational

https://en.wikipedia.org/wiki/Riemann_zeta_function

https://en.wikipedia.org/wiki/Diophantine_approximation

[Dan1] H. Vic Dannon, "Archimedes Series", Gauge Institute Journal, Vol. 18, No 3. August 2022, pp 1-11

[Dan2] H. Vic Dannon, "Euler's Gamma is Irrational and Sequentially Transcendental Number", Gauge Institute Journal,Vol.19, No 3. August 2023

[Dan3] H. Vic Dannon, "Zeta(3) is Irrational and Sequentially Transcendental Number", Gauge Institute Journal, Vol. 19, No 4. November 2023

[Dan4] H. Vic Dannon, "All the π^2 series, and the Catalan series". Gauge Institute Journal, Vol. 18, No. 3, August 2022.