The Gamma Function Role in the Summation of the Catalan Series

H. Vic Dannon
vic0@comcast.net
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Abstract Catalan's Series

$$1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots$$

sums up to approximately

G = 0.9159 6559 4177 2190 1505 4603 5149 3238 4110 774....

Euler's well known

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8},$$

suggests that there is some -perhaps rational- number multiplying π^2 to which the alternating Catalan series sums up to.

But it remained open for centuries, and to date, no such number was found.

The Residue Calculus of Cauchy offered easier paths to the results obtained by Euler:

If $f(z)$ is an analytic function that has poles, Cauchy's Residue Theorem applies to the summation of series of terms of $f(z)$.

But the alternating series that aim to obtain the Catalan Series have terms at negative integers that cancel terms at positive

integers, and sum up to zero.

The Gamma dependent functions

$$\Gamma(z),\ \log\Gamma(z), \text{ and } \frac{\Gamma'(z)}{\Gamma(z)}.$$

are singular only at the non-positive integers

$$z = 0, -1, -2, -3, \dots$$

Could either of these Gamma functions serve as auxiliary function in the summation of the Catalan Series?

We show that neither one of these Gamma functions does.

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The Catalan's Series in Residue Calculus

Catalan's Series

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sums up to approximately

G = 0.9159 6559 4177 2190 1505 4603 5149 3238 4110 774....

Euler's well known

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But it remained open for centuries, and to date, no such number was found.

The Residue Calculus of Cauchy offered easier paths to the results obtained by Euler:

If $f(z)$ is an analytic function that has poles, Cauchy's Residue Theorem applies to the summation of series of terms of $f(z)$.

Four functions are utilized in series summation:

$$\pi\cot\pi z, \quad \frac{\pi}{\sin\pi z}, \quad \frac{\pi}{\cos\pi z}, \quad \text{and} \quad \pi\tan\pi z$$

0.1 $\underline{\pi\cot(\pi z)}\Big| = \pi\dfrac{\cos\pi z}{\sin\pi z}$

has first order poles at the zeros of $\sin\pi z$,

$$z = n = ..., -3, -2, -1, 0, 1, 2, 3...,$$

with residues

$$\operatorname{Res}\big[\pi\cot(\pi z)\big]_{z=n} = 1\,.$$

Then,

$$\sum \operatorname{Res}\big[\pi\cot(\pi z)f(z)\big]_{z=...,-3,-2,-1,0,1,2,3...} =$$

$$= ... + f(-3) + f(-2) + f(-1) + f(0) + f(1) + f(2) + f(3) + ...$$

By the Residue Theorem,

$$\sum \operatorname{Res}\big[\pi\cot(\pi z)f(z)\big]_{z=0,\pm1,\pm2,...} = -\sum \operatorname{Res}\big[\pi\cot(\pi z)f(z)\big]_{z=\text{poles of } f(z)}$$

Therefore,

$$... + f(-3) + f(-2) + f(-1) + f(0) + f(1) + f(2) + f(3) + ... =$$

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0.2 $\underline{\dfrac{\pi}{\sin\pi z}}\Big|$

has first order poles at the zeros of $\sin\pi z$,

$$z = n = ..., -3, -2, -1, 0, 1, 2, 3...$$

with residues

$$\operatorname{Res}\left[\frac{\pi}{\sin\pi z}\right]_{z=n} = (-1)^n\,.$$

Then,

$$\sum \operatorname{Res}\left[\frac{\pi}{\sin\pi z}f(z)\right]_{z=...,-2,-1,0,1,2,..} =$$

$$= ... - f(-3) + f(-2) - f(-1) + f(0) - f(1) + f(2) - f(3) + ...$$

By the Residue Theorem,

$$\sum \text{Res}\left[\frac{\pi}{\sin \pi z} f(z)\right]_{z=...,-2,-1,0,1,2,..} = -\sum \text{Res}\left[\frac{\pi}{\sin \pi z} f(z)\right]_{z=\text{poles of } f(z)}$$

Therefore,

$$... - f(-3) + f(-2) - f(-1) + f(0) - f(1) + f(2) - f(3) + ... =$$

$$= -\sum \text{Res}\left[\frac{\pi}{\sin \pi z} f(z)\right]_{z=\text{poles of } f(z)}$$

0.3 $\underline{\pi \tan \pi z}$

has first order poles at the zeros of $\cos \pi z$,

$$z = n - \tfrac{1}{2} = ..., -\tfrac{5}{2}, -\tfrac{3}{2} - \tfrac{1}{2}, \tfrac{1}{2}\tfrac{3}{2}, \tfrac{5}{2}, ...$$

with residues

$$\text{Res}\big[\pi \tan(\pi z)\big]_{z=...,-\frac{5}{2},-\frac{3}{2}-\frac{1}{2},\frac{1}{2}\frac{3}{2},\frac{5}{2},...} = -1.$$

Then,

$$-\sum \text{Res}\big[\pi \tan(\pi z) f(z)\big]_{z=...,-\frac{5}{2},-\frac{3}{2}-\frac{1}{2},\frac{1}{2}\frac{3}{2},\frac{5}{2},...} =$$

$$= ... + f(-\tfrac{5}{2}) + f(-\tfrac{3}{2}) + f(-\tfrac{1}{2}) + f(\tfrac{1}{2}) + f(\tfrac{3}{2}) + f(\tfrac{5}{2}) + f(\tfrac{7}{2}) + ...$$

By the Residue Theorem,

$$-\sum \text{Res}\big[\pi \tan(\pi z) f(z)\big]_{z=n-\frac{1}{2}} = \sum \text{Res}\big[\pi \tan(\pi z) f(z)\big]_{z=\text{poles of } f(z)}$$

Therefore,

$$... + f(-\tfrac{5}{2}) + f(-\tfrac{3}{2}) + f(-\tfrac{1}{2}) + f(\tfrac{1}{2}) + f(\tfrac{3}{2}) + f(\tfrac{5}{2}) + f(\tfrac{7}{2}) + ... =$$

$$= \sum \text{Res}\big[\pi \tan(\pi z) f(z)\big]_{z=\text{poles of } f(z)}$$

0.4 $\dfrac{\pi}{\cos \pi z}$

has first order poles at the zeros of $\cos \pi z$,

$$z = n - \tfrac{1}{2} = ..., -\tfrac{5}{2}, -\tfrac{3}{2} - \tfrac{1}{2}, \tfrac{1}{2} \tfrac{3}{2}, \tfrac{5}{2}, ...$$

with residues

$$\text{Res}\left[\frac{\pi}{\cos \pi z}\right]_{z=n-\frac{1}{2}} = (-1)^n .$$

Then,

$$-\sum \text{Res}\left[\frac{\pi}{\cos \pi z} f(z)\right]_{z=n-\frac{1}{2}} =$$

$$= ... - f(-\tfrac{5}{2}) + f(-\tfrac{3}{2}) - f(-\tfrac{1}{2}) + f(\tfrac{1}{2}) - f(\tfrac{3}{2}) + f(\tfrac{5}{2}) - f(\tfrac{7}{2}) + ...$$

By the Residue Theorem,

$$-\sum \text{Res}\left[\frac{\pi}{\cos \pi z} f(z)\right]_{z=n-\frac{1}{2}} = \sum \text{Res}\left[\frac{\pi}{\cos \pi z} f(z)\right]_{z=\text{poles of } f(z)}$$

Therefore,

$$... - f(-\tfrac{5}{2}) + f(-\tfrac{3}{2}) - f(-\tfrac{1}{2}) + f(\tfrac{1}{2}) - f(\tfrac{3}{2}) + f(\tfrac{5}{2}) - f(\tfrac{7}{2}) + ... =$$

$$= \sum \text{Res}\left[\frac{\pi}{\cos \pi z} f(z)\right]_{z=\text{poles of } f(z)}$$

0.5 The alternating series have terms at negative integers that cancel terms at positive integers, and sum up to zero.

If $\dfrac{\pi}{\sin \pi z}$ is used with $f(z) = \dfrac{1}{(z + \frac{1}{2})^2}$, then

$$... - f(-3) + f(-2) - f(-1) + f(0) - f(1) + f(2) - ... =$$

$$= ... \underbrace{-\frac{1}{(-3+\frac{1}{2})^2}}_{***} + \underbrace{\frac{1}{(-2+\frac{1}{2})^2}}_{**} \underbrace{-\frac{1}{(-1+\frac{1}{2})^2}}_{*} + \underbrace{\frac{1}{(0+\frac{1}{2})^2}}_{*} \underbrace{-\frac{1}{(1+\frac{1}{2})^2}}_{**} + \underbrace{\frac{1}{(2+\frac{1}{2})^2}}_{***} - ... = 0.$$

If $\dfrac{\pi}{\cos \pi z}$ is used with $f(z) = \dfrac{1}{z^2}$, then

$$... - f(-\tfrac{5}{2}) + f(-\tfrac{3}{2}) - f(-\tfrac{1}{2}) + f(\tfrac{1}{2}) - f(\tfrac{3}{2}) + f(\tfrac{5}{2}) - ... =$$

$$= ... \underbrace{-\frac{1}{(-\frac{5}{2})^2}}_{***} + \underbrace{\frac{1}{(-\frac{3}{2})^2}}_{**} \underbrace{-\frac{1}{(-\frac{1}{2})^2}}_{*} + \underbrace{\frac{1}{(\frac{1}{2})^2}}_{*} \underbrace{-\frac{1}{(\frac{3}{2})^2}}_{**} + \underbrace{\frac{1}{(\frac{5}{2})^2}}_{***} - ... = 0.$$

0.6 The Gamma dependent functions

$$\Gamma(z), \ \log\Gamma(z), \text{ and } \frac{\Gamma'(z)}{\Gamma(z)}.$$

Each is singular only at the non-positive integers

$$z = 0, -1, -2, -3, \ldots$$

Could either of these Gamma functions serve as auxiliary function in the summation of the Catalan Series?

We show that neither one of these Gamma functions does.

1.

Euler Infinite Product for Gamma

$$\Gamma(z) = \frac{1}{z} \cdot \frac{\left(1+\frac{1}{1}\right)^z}{1+\frac{1}{1}z} \cdot \frac{\left(1+\frac{1}{2}\right)^z}{1+\frac{1}{2}z} \cdot \frac{\left(1+\frac{1}{3}\right)^z}{1+\frac{1}{3}z} \cdot \frac{\left(1+\frac{1}{4}\right)^z}{1+\frac{1}{4}z} \cdot \ldots$$

Proof:

The Gamma Function extends the factorial function

$$\Gamma(n) = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \equiv (n-1)!$$

from the positive integers to the complex plane.

Then,

$$\Gamma(1) = (1-1)! = 0! \text{ is defined as } 1.$$

The recursive formula

$$n\Gamma(n) = \Gamma(n+1)$$

suggests that the extension to non-positive integers, $0, -1, -2, -3, \ldots$

has indefinitely large value:

$$0 \cdot \Gamma(0) = \Gamma(0+1) = 1 \quad \Rightarrow \Gamma(0) \text{ is indefinitely large}$$

Then

$$(-1) \cdot \Gamma(-1) = \Gamma(-1+1) = \Gamma(0) \quad \Rightarrow \Gamma(-1) \text{ is indefinitely large}$$

Thus, $\Gamma(z)$ has the factors

$$\frac{1}{z}, \ \frac{1}{1+\frac{1}{1}z}, \ \frac{1}{1+\frac{1}{2}z}, \ \frac{1}{1+\frac{1}{3}z}, \ \frac{1}{1+\frac{1}{4}z} \cdot \ldots$$

And depends on the infinite product

$$\frac{1}{z}\cdot\frac{1}{1+\frac{1}{1}z}\cdot\frac{1}{1+\frac{1}{2}z}\cdot\frac{1}{1+\frac{1}{3}z}\cdot\frac{1}{1+\frac{1}{4}z}\cdot ... =$$

$$=\frac{1}{z}\cdot\frac{1}{z+1}\cdot\frac{2}{z+2}\cdot\frac{3}{z+3}\cdot\frac{4}{z+4}\cdot ...$$

If z is a natural number, then for any n, large as it may be,

$$\frac{1}{(z-1)!}\frac{1}{z}\cdot\frac{1}{z+1}\cdot\frac{2}{z+2}\cdots\frac{n}{z+n}\cdot n^z\cdot\frac{n+1}{n}\cdot\frac{n+2}{n}\cdots\frac{n+z}{n}=1$$

Therefore,

$$(z-1)!=\frac{1}{z}\cdot\frac{1}{z+1}\cdot\frac{2}{z+2}\cdots\frac{n}{z+n}\cdot n^z\cdot\frac{n+1}{n}\cdot\frac{n+2}{n}\cdots\frac{n+z}{n}$$

$$=\frac{1}{z}\cdot\frac{1}{1+z}\cdot\frac{1}{1+\frac{1}{2}z}\cdots\frac{1}{1+\frac{1}{n}z}\cdot n^z\cdot\frac{n+1}{n}\cdot\frac{n+2}{n}\cdots\frac{n+z}{n}$$

For very large n, $\frac{n+1}{n}\cdot\frac{n+2}{n}\cdot\frac{n+3}{n}\cdots\frac{n+z}{n}\approx 1$, and

$$(z-1)!\approx\frac{1}{z}\cdot\frac{1}{1+z}\cdot\frac{1}{1+\frac{1}{2}z}\cdot\frac{1}{1+\frac{1}{3}z}\cdots\frac{1}{1+\frac{1}{n}z}\cdot n^z$$

$$\approx\frac{1}{z}\cdot\frac{2^z}{1+z}\cdot\frac{(\frac{3}{2})^z}{1+\frac{1}{2}z}\cdot\frac{(\frac{4}{3})^z}{1+\frac{1}{3}z}\cdots\frac{(\frac{n}{n-1})^z}{1+\frac{1}{n-1}z}\cdots$$

For n indefinitely large, these partial products converge to **Euler's** infinite product for Gamma

$$\boxed{\Gamma(z)=\frac{1}{z}\cdot\frac{\left(1+\frac{1}{1}\right)^z}{1+\frac{1}{1}z}\cdot\frac{\left(1+\frac{1}{2}\right)^z}{1+\frac{1}{2}z}\cdot\frac{\left(1+\frac{1}{3}\right)^z}{1+\frac{1}{3}z}\cdot\frac{\left(1+\frac{1}{4}\right)^z}{1+\frac{1}{4}z}\cdot ...}$$

$\Rightarrow$

$$\boxed{\Gamma(1)=1}$$

The recursion formula, $\Gamma(z+1) = z\Gamma(z)$ **holds:**

$$z\Gamma(z) = \frac{\left(1+\frac{1}{1}\right)^z}{1+\frac{1}{1}z} \cdot \frac{\left(1+\frac{1}{2}\right)^z}{1+\frac{1}{2}z} \cdot \frac{\left(1+\frac{1}{3}\right)^z}{1+\frac{1}{3}z} \cdot \frac{\left(1+\frac{1}{4}\right)^z}{1+\frac{1}{4}z} \cdot ....$$

$$= \frac{\frac{1}{2}(1+\frac{1}{1})^{z+1}}{1+z} \cdot \frac{\frac{2}{3}\left(1+\frac{1}{2}\right)^{z+1}}{\frac{1}{2}+\frac{1}{2}(z+1)} \cdot \frac{\frac{3}{4}\left(1+\frac{1}{3}\right)^{z+1}}{\frac{2}{3}+\frac{1}{3}(z+1)} \cdot \frac{\frac{4}{5}\left(1+\frac{1}{4}\right)^{z+1}}{\frac{3}{4}+\frac{1}{4}(z+1)} \cdot ..$$

$$= \frac{(1+\frac{1}{1})^{z+1}}{1+z} \cdot \frac{1}{2} \frac{\left(1+\frac{1}{2}\right)^{z+1}}{\frac{1}{2}+\frac{1}{2}(z+1)} \cdot \frac{2}{3} \frac{\left(1+\frac{1}{3}\right)^{z+1}}{\frac{2}{3}+\frac{1}{3}(z+1)} \cdot \frac{3}{4} \frac{\left(1+\frac{1}{4}\right)^{z+1}}{\frac{3}{4}+\frac{1}{4}(z+1)} \cdot \frac{4}{5} ..$$

$$= \frac{(1+\frac{1}{1})^{z+1}}{1+z} \cdot \frac{\left(1+\frac{1}{2}\right)^{z+1}}{1+(z+1)} \cdot \frac{\left(1+\frac{1}{3}\right)^{z+1}}{1+\frac{1}{2}(z+1)} \cdot \frac{\left(1+\frac{1}{4}\right)^{z+1}}{1+\frac{1}{3}(z+1)} \cdot \frac{4}{5} ..$$

$$= \frac{1}{1+z} \cdot \frac{\left(1+\frac{1}{1}\right)^{z+1}}{1+(z+1)} \cdot \frac{\left(1+\frac{1}{2}\right)^{z+1}}{1+\frac{1}{2}(z+1)} \cdot \frac{\left(1+\frac{1}{3}\right)^{z+1}}{1+\frac{1}{3}(z+1)} \cdot ... = \Gamma(z+1)$$

$$\boxed{\Gamma(z+1) = z\Gamma(z)}$$

$\Rightarrow$

$$\boxed{\Gamma(n+1) = n\Gamma(n) = n(n-1)(n-2)...\Gamma(1) = n!}$$

Also,

$$\boxed{\Gamma(-z) = \frac{1}{-z} \cdot \frac{\left(1+\frac{1}{1}\right)^{-z}}{1-\frac{1}{1}z} \cdot \frac{\left(1+\frac{1}{2}\right)^{-z}}{1-\frac{1}{2}z} \cdot \frac{\left(1+\frac{1}{3}\right)^{-z}}{1-\frac{1}{3}z} \cdot \frac{\left(1+\frac{1}{4}\right)^{-z}}{1-\frac{1}{4}z} \cdot ....}$$

$$\boxed{\Gamma(-z+1) = -z\Gamma(-z) = \frac{\left(1+\frac{1}{1}\right)^{-z}}{1-\frac{1}{1}z} \cdot \frac{\left(1+\frac{1}{2}\right)^{-z}}{1-\frac{1}{2}z} \cdot \frac{\left(1+\frac{1}{3}\right)^{-z}}{1-\frac{1}{3}z} \cdot \frac{\left(1+\frac{1}{4}\right)^{-z}}{1-\frac{1}{4}z} \cdot ..}$$

$\Rightarrow$

$$\boxed{\Gamma(-z+1)\Gamma(z) = -z\Gamma(-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}}$$

2.

Weierstrass Infinite Product for Gamma

$$\boxed{\Gamma(z) = \frac{1}{ze^{\gamma z}} \cdot \frac{e^{\frac{1}{1}z}}{1+\frac{1}{1}z} \cdot \frac{e^{\frac{1}{2}z}}{1+\frac{1}{2}z} \cdot \frac{e^{\frac{1}{3}z}}{1+\frac{1}{3}z} \cdot \frac{e^{\frac{1}{4}z}}{1+\frac{1}{4}z} \cdot ...}$$

<u>Proof</u>:

For any n

$$(z-1)! \approx \frac{1}{z} \cdot \frac{1}{1+z} \cdot \frac{1}{1+\frac{1}{2}z} \cdot \frac{1}{1+\frac{1}{3}z} \cdots \frac{1}{1+\frac{1}{n}z} \cdot n^z,$$

and $\Gamma(z)$ is the infinite product

$$\Gamma(z) = n^z \frac{1}{z} \cdot \frac{1}{1+z} \cdot \frac{1}{1+\frac{1}{2}z} \cdot \frac{1}{1+\frac{1}{3}z} \cdots \frac{1}{1+\frac{1}{n}z} \cdots$$

$$= e^{z \log n} \frac{1}{z} \cdot \frac{1}{1+z} \cdot \frac{1}{1+\frac{1}{2}z} \cdot \frac{1}{1+\frac{1}{3}z} \cdots \frac{1}{1+\frac{1}{n}z} \cdots$$

For N an infinite hyper-real number,

$$= \frac{1}{z} \cdot \frac{1}{e^{\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...\right)z - z\log N}} \frac{e^{\frac{1}{1}z}}{1+\frac{1}{1}z} \cdot \frac{e^{\frac{1}{2}z}}{1+\frac{1}{2}z} \cdot \frac{e^{\frac{1}{3}z}}{1+\frac{1}{3}z} \cdot \frac{e^{\frac{1}{4}z}}{1+\frac{1}{4}z} \cdot ...$$

Substituting Euler's constant $\gamma = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... - \log N$,

Weierstrass infinite product for Gamma is

$$\boxed{\Gamma(z) = \frac{1}{ze^{\gamma z}} \cdot \frac{e^{\frac{1}{1}z}}{1+\frac{1}{1}z} \cdot \frac{e^{\frac{1}{2}z}}{1+\frac{1}{2}z} \cdot \frac{e^{\frac{1}{3}z}}{1+\frac{1}{3}z} \cdot \frac{e^{\frac{1}{4}z}}{1+\frac{1}{4}z} \cdot ...}$$

The infinite product converges because its Log, the infinite series

$$\log\frac{e^{\frac{1}{1}z}}{1+\frac{1}{1}z}+\log\frac{e^{\frac{1}{2}z}}{1+\frac{1}{2}z}+\log\frac{e^{\frac{1}{3}z}}{1+\frac{1}{3}z}+\log\frac{e^{\frac{1}{4}z}}{1+\frac{1}{4}z}+\dots,$$

converges absolutely.

For $2\left|z\right|<n_0$, $\frac{|z|}{n_0}<\frac{1}{2}$, and

$$\left|\log\frac{e^{\frac{1}{n_0}z}}{1+\frac{1}{n_0}z}\right|=\left|\log\left(1+\frac{z}{n_0}\right)-\frac{z}{n_0}\right|$$

$$=\left|\left(\frac{z}{n_0}-\frac{1}{2}\left[\frac{z}{n_0}\right]^2+\frac{1}{3}\left[\frac{z}{n_0}\right]^3-\frac{1}{4}\left[\frac{z}{n_0}\right]^4+\dots\right)-\frac{z}{n_0}\right|$$

$$=\left|-\frac{1}{2}\left[\frac{z}{n_0}\right]^2+\frac{1}{3}\left[\frac{z}{n_0}\right]^3-\frac{1}{4}\left[\frac{z}{n_0}\right]^4+\dots.\right|$$

$$\leq\frac{1}{2}\left|\frac{z}{n_0}\right|^2+\frac{1}{3}\left|\frac{z}{n_0}\right|^3+\frac{1}{4}\left|\frac{z}{n_0}\right|^4+\dots$$

$$\leq\frac{1}{2}\left|\frac{z}{n_0}\right|^2\left\{1+\left|\frac{z}{n_0}\right|+\left|\frac{z}{n_0}\right|^2+\dots\right\}$$

$$\leq\frac{1}{2}\left|\frac{z}{n_0}\right|^2\underbrace{\left\{1+\frac{1}{2}+\left[\frac{1}{2}\right]^2+\dots\right\}}_{2}=\left|\frac{z}{n_0}\right|^2$$

$\Rightarrow$ The tail of the series is bounded:

$$\left|\log\frac{e^{\frac{1}{n_0}z}}{1+\frac{1}{n_0}z}\right|+\left|\log\frac{e^{\frac{1}{n_0+1}z}}{1+\frac{1}{n_0+1}z}\right|+\left|\log\frac{e^{\frac{1}{n_0+2}z}}{1+\frac{1}{n_0+2}z}\right|+\left|\log\frac{e^{\frac{1}{n_0+3}z}}{1+\frac{1}{n_0+3}z}\right|+\dots$$

$$\leq\left|\frac{z}{n_0}\right|^2+\left|\frac{z}{n_0+1}\right|^2+\left|\frac{z}{n_0+2}\right|^2+\left|\frac{z}{n_0+3}\right|^2\dots.$$

$$\leq \left|z^2\right|\left(\frac{1}{n_0^2} + \frac{1}{(n_0+1)^2} + \frac{1}{(n_0+2)^2} + \frac{1}{(n_0+3)^2} + ...\right)$$

$$\leq \left|z^2\right|\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n_0^2} + \frac{1}{(n_0+1)^2} + ...\right) = \left|z^2\right|\frac{\pi^2}{6}$$

$$\Gamma(z) = \frac{1}{ze^{\gamma z}} \cdot \frac{e^{\frac{1}{1}z}}{1+\frac{1}{1}z} \cdot \frac{e^{\frac{1}{2}z}}{1+\frac{1}{2}z} \cdot \frac{e^{\frac{1}{3}z}}{1+\frac{1}{3}z} \cdot \frac{e^{\frac{1}{4}z}}{1+\frac{1}{4}z} \cdot ..$$

$\Rightarrow$

$$\boxed{\Gamma(-z) = \frac{1}{-ze^{-\gamma z}} \cdot \frac{e^{-\frac{1}{1}z}}{1-\frac{1}{1}z} \cdot \frac{e^{-\frac{1}{2}z}}{1-\frac{1}{2}z} \cdot \frac{e^{-\frac{1}{3}z}}{1-\frac{1}{3}z} \cdot \frac{e^{-\frac{1}{4}z}}{1-\frac{1}{4}z} \cdot ...}$$

$$\Gamma(z)\Gamma(-z) = \frac{1}{-z^2} \cdot \frac{1}{1-\frac{1}{1}z^2} \cdot \frac{1}{1-\frac{1}{2^2}z^2} \cdot \frac{1}{1-\frac{1}{3^2}z^2} \cdot \frac{1}{1-\frac{1}{4^2}z^2} \cdot ...$$

$$= -\frac{1}{z^2}\frac{\pi z}{\sin \pi z}$$

$\Rightarrow$

$$\boxed{\Gamma(z)\Gamma(-z) = -\frac{\pi}{z}\frac{1}{\sin \pi z}}$$

3.

The Residues of Gamma

3.1 $$\boxed{\mathrm{Res}\{\Gamma(\mathrm{z})\}_{z=-n} = \frac{(-1)^n}{n!}}$$

<u>*Proof*</u>:

$$\mathrm{Res}\{\Gamma(\mathrm{z})\}_{z=-n} = [(z+n)\Gamma(z)]_{z=-n}$$

$$= \left[(z+n)\frac{\Gamma(z+1)}{z}\right]_{z=-n}$$

$$= \left[(z+n)\frac{\Gamma(z+2)}{(z+1)z}\right]_{z=-n}$$

$$= \left[(z+n)\frac{\Gamma(z+n+1)}{(z+n)(z+n-1)...(z+2)(z+1)z}\right]_{z=-n}$$

$$= \left[\frac{\Gamma(z+n+1)}{(z+n-1)...(z+2)(z+1)z}\right]_{z=-n}$$

$$= \frac{\Gamma(-n+n+1)}{(-n+n-1)...(-n+2)(-n+1)(-n)}$$

$$= \frac{\Gamma(1)}{(-1)^n 1...(n-2)(n-1)n}$$

$$= \frac{1}{(-1)^n n!}. \square$$

3.2 Gamma Function does not yield the Catalan Series

<u>*Proof*</u>:

If $\Gamma(z)$ is used with $f(z) = \dfrac{1}{(z+\frac{1}{2})^2}$,

Then,

$$\sum \mathrm{Res}\left[\Gamma(z)\frac{1}{(z+\frac{1}{2})^2}\right]_{z=0,-1,-2,..} =$$

$$\frac{(-1)^0}{0!}\frac{1}{(\frac{1}{2})^2} + \frac{(-1)^1}{1!}\frac{1}{(-1+\frac{1}{2})^2} + \frac{(-1)^2}{2!}\frac{1}{(-2+\frac{1}{2})^2} + \frac{(-1)^3}{3!}\frac{1}{(-3+\frac{1}{2})^2} + ... =$$

$$= \frac{1}{(\frac{1}{2})^2} - \frac{1}{(-\frac{1}{2})^2} + \frac{1}{2!}\frac{1}{(-\frac{3}{2})^2} - \frac{1}{3!}\frac{1}{(-\frac{5}{2})^2} + ...$$

$$= \frac{1}{2!}\frac{4}{3^2} - \frac{1}{3!}\frac{4}{5^2} + \frac{1}{4!}\frac{4}{7^2}...$$

By the Residue Theorem,

$$\sum \mathrm{Res}\left[\Gamma(z)\frac{1}{(z+\frac{1}{2})^2}\right]_{z=0,-1,-2,...} = -\sum \mathrm{Res}\left[\Gamma(z)\frac{1}{(z+\frac{1}{2})^2}\right]_{z=\text{poles of } \frac{1}{(z+\frac{1}{2})^2}}$$

Therefore,

$$\frac{1}{2!}\frac{4}{3^2} - \frac{1}{3!}\frac{4}{5^2} + \frac{1}{4!}\frac{4}{7^2}... = -\sum \mathrm{Res}\left[\Gamma(z)\frac{1}{(z+\frac{1}{2})^2}\right]_{z=\text{poles of } \frac{1}{(z+\frac{1}{2})^2}} .\Box$$

4.

Series Expansion of Gamma

$$\Gamma(\text{z}) = \frac{1}{z} - \frac{1}{z+1} + \frac{1}{2!}\frac{1}{z+2} - \frac{1}{3!}\frac{1}{z+3} + ... + \int_{t=1}^{t=\infty} t^{z-1}e^{-t}dt$$

Proof: By Euler,

$$\Gamma(z) = \int_{t=0}^{t=\infty} t^{z-1}e^{-t}dt$$

Thus,

$$\Gamma(z) = \int_{t=0}^{t=1} t^{z-1}e^{-t}dt + \int_{t=1}^{t=\infty} t^{z-1}e^{-t}dt$$

$$= \int_{t=0}^{t=1} t^{z-1}\{1 - t + \tfrac{1}{2}t^2 - \tfrac{1}{3!}t^3 + ...\}dt + \int_{t=1}^{t=\infty} t^{z-1}e^{-t}dt$$

$$= \int_{t=0}^{t=1} \{t^{z-1} - t^z + \tfrac{1}{2}t^{z+1} - \tfrac{1}{3!}t^{z+2} + ...\}dt + \int_{t=1}^{t=\infty} t^{z-1}e^{-t}dt$$

$$= \int_{t=0}^{t=1} t^{z-1}dt - \int_{t=0}^{t=1} t^z dt + \tfrac{1}{2}\int_{t=0}^{t=1} t^{z+1}dt - \tfrac{1}{3!}\int_{t=0}^{t=1} t^{z+2}dt + ... + \int_{t=1}^{t=\infty} t^{z-1}e^{-t}dt$$

$$= \frac{1}{z}\Big[t^z\Big]_{t=0}^{t=1} - \frac{1}{2!}\frac{1}{z+1}\Big[t^{z+1}\Big]_{t=0}^{t=1} + \frac{1}{3!}\frac{1}{z+2}\Big[t^{z+2}\Big]_{t=0}^{t=1} + ... + \int_{t=1}^{t=\infty} t^{z-1}e^{-t}dt$$

$$= \frac{1}{z} - \frac{1}{z+1} + \frac{1}{2!}\frac{1}{z+2} - \frac{1}{3!}\frac{1}{z+3} + ... + \int_{t=1}^{t=\infty} t^{z-1}e^{-t}dt\,.\square$$

5.

Series Expansion for Log Gamma

5.1 Weierstrass Series Expansion for Gamma

$$\log\Gamma(z) = -\log z - \gamma z + \left\{ z - \log(1+z) \right\} + \left\{ \tfrac{1}{2}z - \log(1+\tfrac{1}{2}z) \right\} + \left\{ \tfrac{1}{3}z - \log(1+\tfrac{1}{3}z) \right\} + \left\{ \tfrac{1}{4}z - \log(1+\tfrac{1}{4}z) \right\} + ...$$

Proof:

By Weierstrass,

$$\Gamma(z) = \frac{1}{ze^{\gamma z}} \cdot \frac{e^{\frac{1}{1}z}}{1+\frac{1}{1}z} \cdot \frac{e^{\frac{1}{2}z}}{1+\frac{1}{2}z} \cdot \frac{e^{\frac{1}{3}z}}{1+\frac{1}{3}z} \cdot \frac{e^{\frac{1}{4}z}}{1+\frac{1}{4}z} \cdot ...$$

$$\log\Gamma(z) = -\log z - \gamma z$$

$$+\left\{ z - \log(1+z) \right\}$$

$$+\left\{ \tfrac{1}{2}z - \log(1+\tfrac{1}{2}z) \right\}$$

$$+\left\{ \tfrac{1}{3}z - \log(1+\tfrac{1}{3}z) \right\}$$

$$+\left\{ \tfrac{1}{4}z - \log(1+\tfrac{1}{4}z) \right\} + ...$$

....

Thus, $\log\Gamma(z)$ becomes indefinitely large at the poles of $\Gamma(z)$

$$z = 0, -1, -2, -3, ...$$

5.2

$$\boxed{\begin{aligned}\log\Gamma(z+1) = -\gamma z &+ \left\{ z - \log(1+z) \right\} \\ &+ \left\{ \tfrac{1}{2}z - \log(1+\tfrac{1}{2}z) \right\} \\ &+ \left\{ \tfrac{1}{3}z - \log(1+\tfrac{1}{3}z) \right\} \\ &+ \left\{ \tfrac{1}{4}z - \log(1+\tfrac{1}{4}z) \right\} + ...\end{aligned}}$$

<u>*Proof*</u>: By 5.1,

$$\begin{aligned}-\gamma z &+ \left\{ z - \log(1+z) \right\} \\ &+ \left\{ \tfrac{1}{2}z - \log(1+\tfrac{1}{2}z) \right\} \\ &+ \left\{ \tfrac{1}{3}z - \log(1+\tfrac{1}{3}z) \right\} \\ &+ \left\{ \tfrac{1}{4}z - \log(1+\tfrac{1}{4}z) \right\} + ... = \log\Gamma(z) + \log z \\ &\qquad\qquad\qquad\qquad\qquad\quad = \log z\Gamma(z) \\ &\qquad\qquad\qquad\qquad\qquad\quad = \log\Gamma(z+1). \square\end{aligned}$$

5.3 Euler Series Expansion for Log Gamma

$$\boxed{\begin{aligned}\log\Gamma(z) = -\log z &+ \left\{ z\log\left(1+\tfrac{1}{1}\right) - \log(1+\tfrac{1}{1}z) \right\} \\ &+ \left\{ z\log\left(1+\tfrac{1}{2}\right) - \log(1+\tfrac{1}{2}z) \right\} \\ &+ \left\{ z\log\left(1+\tfrac{1}{3}\right) - \log(1+\tfrac{1}{3}z) \right\} \\ &+ \left\{ z\log\left(1+\tfrac{1}{4}\right) - \log(1+\tfrac{1}{4}z) \right\} + ...\end{aligned}}$$

<u>*Proof*</u>:

By Euler,

$$\Gamma(z) = \frac{1}{z} \cdot \frac{\left(1+\frac{1}{1}\right)^z}{1+\frac{1}{1}z} \cdot \frac{\left(1+\frac{1}{2}\right)^z}{1+\frac{1}{2}z} \cdot \frac{\left(1+\frac{1}{3}\right)^z}{1+\frac{1}{3}z} \cdot \frac{\left(1+\frac{1}{4}\right)^z}{1+\frac{1}{4}z} \cdot ...$$

$$\log\Gamma(z) = -\log z$$

$$+z\log\left(1+\tfrac{1}{1}\right) - \log(1+\tfrac{1}{1}z)$$

$$+z\log\left(1+\tfrac{1}{2}\right) - \log(1+\tfrac{1}{2}z)$$

$$+z\log\left(1+\tfrac{1}{3}\right)-\log(1+\tfrac{1}{3}z)$$

$$+z\log\left(1+\tfrac{1}{4}\right)-\log(1+\tfrac{1}{4}z)+...$$

....

5.4 $\log\Gamma(z)$ **has essential singularities at the poles of** $\Gamma(z)$**.**

The Residue Theorem applies only to poles.

5.5 In the Anus Disk $\boxed{0<\left|z\right|<1}$,

$\log\Gamma(1+z)$ **has the Taylor Series**

$$\boxed{\log\Gamma(1+z)=-\gamma z+\tfrac{1}{2}\zeta(2)z^2-\tfrac{1}{3}\zeta(3)z^3+\tfrac{1}{3}\zeta(4)z^4-...}$$

The Taylor Series is not available in $\left|z\right|>1$**.**

Proof: By 5.2,

$$\log\Gamma(z+1)=-\gamma z$$

$$+\left\{z-\log(1+z)\right\}$$

$$+\left\{\tfrac{1}{2}z-\log(1+\tfrac{1}{2}z)\right\}$$

$$+\left\{\tfrac{1}{3}z-\log(1+\tfrac{1}{3}z)\right\}$$

$$+\left\{\tfrac{1}{4}z-\log(1+\tfrac{1}{4}z)\right\}+...$$

For $\left|z\right|<1$,

$$\tfrac{1}{n}z-\log(1+\tfrac{1}{n}z)=\tfrac{1}{2}\tfrac{1}{n^2}z^2-\tfrac{1}{3}\tfrac{1}{n^3}z^3+\tfrac{1}{4}\tfrac{1}{n^4}z^4-\tfrac{1}{5}\tfrac{1}{n^5}z^5+...$$

Therefore,

$$\log\Gamma(z+1) = -\gamma z$$

$$+\frac{1}{2}\frac{1}{1^2}z^2 - \frac{1}{3}\frac{1}{1^3}z^3 + \frac{1}{4}\frac{1}{1^4}z^4 - \frac{1}{5}\frac{1}{1^5}z^5 + \dots$$

$$+\frac{1}{2}\frac{1}{2^2}z^2 - \frac{1}{3}\frac{1}{2^3}z^3 + \frac{1}{4}\frac{1}{2^4}z^4 - \frac{1}{5}\frac{1}{2^5}z^5 + \dots$$

$$+\frac{1}{2}\frac{1}{3^2}z^2 - \frac{1}{3}\frac{1}{3^3}z^3 + \frac{1}{4}\frac{1}{3^4}z^4 - \frac{1}{5}\frac{1}{3^5}z^5 + \dots$$

$$+\frac{1}{2}\frac{1}{4^2}z^2 - \frac{1}{3}\frac{1}{4^3}z^3 + \frac{1}{4}\frac{1}{4^4}z^4 - \frac{1}{5}\frac{1}{4^5}z^5 + \dots$$

and we sum along the columns.□

5.6 $\log\Gamma(1+z)$ is Inutile to the Catalan Series Summation in Residue Calculus

6.

Series Expansion for $\psi(z) = \dfrac{\Gamma'(z)}{\Gamma(z)}$

6.1 $$\boxed{\psi(z) = -\gamma - \frac{1}{z} + \left\{1 - \frac{1}{1+z}\right\} + \left\{\tfrac{1}{2} - \frac{1}{2+z}\right\} + \left\{\tfrac{1}{3} - \frac{1}{3+z}\right\} + \ldots}$$

Proof: By (5.1),

$$\frac{\Gamma'(z)}{\Gamma(z)} = \frac{d}{dz}\log\Gamma(z)$$

$$= -\frac{d}{dz}\gamma z - \frac{d}{dz}\log z + \frac{d}{dz}\left\{z - \log(1+z)\right\}$$

$$+\frac{d}{dz}\left\{\tfrac{1}{2}z - \log(1+\tfrac{1}{2}z)\right\}$$

$$+\frac{d}{dz}\left\{\tfrac{1}{3}z - \log(1+\tfrac{1}{3}z)\right\}$$

$$+\frac{d}{dz}\left\{\tfrac{1}{4}z - \log(1+\tfrac{1}{4}z)\right\} + \ldots.$$

$$= -\gamma - \frac{1}{z} + \left\{1 - \frac{1}{1+z}\right\} + \left\{\tfrac{1}{2} - \tfrac{1}{2}\frac{1}{1+\tfrac{1}{2}z}\right\} + \left\{\tfrac{1}{3} - \tfrac{1}{3}\frac{1}{1+\tfrac{1}{3}z}\right\} + \ldots$$

$$= -\gamma - \frac{1}{z} + \left\{1 - \frac{1}{1+z}\right\} + \left\{\tfrac{1}{2} - \frac{1}{2+z}\right\} + \left\{\tfrac{1}{3} - \frac{1}{3+z}\right\} + \ldots.\square$$

6.2 $$\boxed{\psi(1) = -\gamma}$$

Proof: $$\psi(1) = -\gamma - \frac{1}{1} + \left\{1 - \frac{1}{1+1}\right\} + \left\{\tfrac{1}{2} - \frac{1}{2+1}\right\} + \left\{\tfrac{1}{3} - \frac{1}{3+1}\right\} + \ldots$$

6.3 $$\boxed{\psi(z) = -\gamma - \frac{1}{z} + \frac{z}{1+z} + \frac{z}{2(2+z)} + \frac{z}{3(3+z)} + \frac{z}{4(4+z)}...}$$

Proof: $\frac{1}{n} - \frac{1}{n+z} = \frac{z}{n(n+z)}$

6.4 $$\boxed{\frac{\psi(z)}{z} \xrightarrow[-n\neq z\to\infty]{} 0}$$

Proof: $\frac{\psi(z)}{z} = -\frac{\gamma}{z} - \frac{1}{z^2} + \frac{1}{1+z} + \frac{1}{2(2+z)} + \frac{1}{3(3+z)} + \frac{1}{4(4+z)}..$

For $z = N =$ infinite hyper real, each term

$$\frac{1}{n(n+N)}$$

is an infinitesimal, and the sum of the terms is an infinitesimal.

By [Abramowitz, p.259],

6.5 In the Anus Disk $\boxed{0 < |z| < 1}$,

the DiGamma has the Taylor Series

$$\boxed{\psi(1+z) = -\gamma + \zeta(2)z - \zeta(3)z^2 + \zeta(4)z^3 - ....}$$

The Taylor Series is not available in $|z| > 1$**.**

In **12**, we will see that such Taylor series for the DiGamma would have enabled the expansion of the Catalan Series in zeta function values.

Here, we supply the proof to **6.5**

Proof:

$$\psi(z) = -\gamma - \left\{\frac{1}{z}\right\} + \left\{1 - \frac{1}{1+z}\right\} + \left\{\tfrac{1}{2} - \frac{1}{2+z}\right\} + \left\{\tfrac{1}{3} - \frac{1}{3+z}\right\} + ...$$

$$\psi(z+1) = -\gamma - \left\{\frac{1}{z+1}\right\} + \left\{1 - \frac{1}{2+z}\right\} + \left\{\tfrac{1}{2} - \frac{1}{3+z}\right\} + \left\{\tfrac{1}{3} - \frac{1}{4+z}\right\} +$$

$$= -\gamma + \left\{1 - \frac{1}{z+1}\right\} + \left\{\tfrac{1}{2} - \frac{1}{2+z}\right\} + \left\{\tfrac{1}{3} - \frac{1}{3+z}\right\} + \left\{\tfrac{1}{4} - \frac{1}{4+z}\right\} + ...$$

$$= -\gamma +$$

$$+\left\{1 - \frac{1}{1+z}\right\} + \tfrac{1}{2}\left\{1 - \frac{1}{1+\frac{1}{2}z}\right\} + \tfrac{1}{3}\left\{\tfrac{1}{3} - \frac{1}{1+\frac{1}{3}z}\right\} + \tfrac{1}{4}\left\{1 - \frac{1}{1+\frac{1}{4}z}\right\} + ...$$

For $|z| < 1$,

$$= -\gamma + \left\{z - z^2 + z^3 - z^4 + ...\right\}$$

$$+\tfrac{1}{2}\left\{\tfrac{1}{2}z - (\tfrac{1}{2}z)^2 + (\tfrac{1}{2}z)^3 - (\tfrac{1}{2}z)^4 + ...\right\}$$

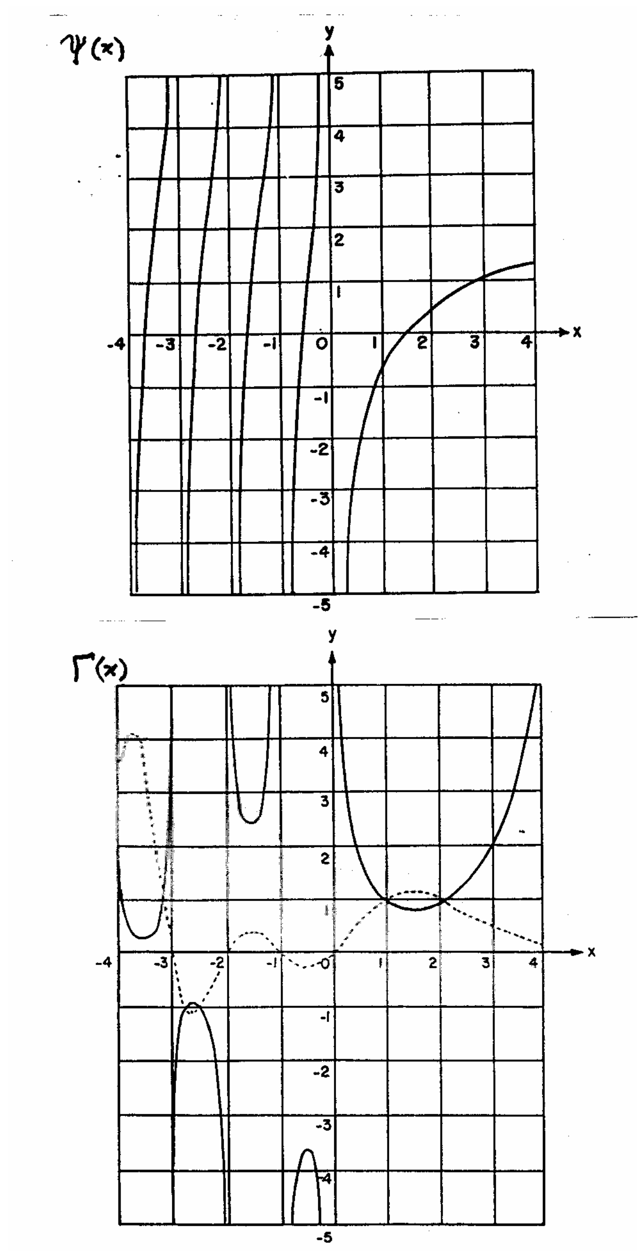
$$+\tfrac{1}{3}\left\{\tfrac{1}{3}z - (\tfrac{1}{3}z)^2 + (\tfrac{1}{3}z)^3 - (\tfrac{1}{3}z)^4 + ...\right\}$$

$$+\tfrac{1}{4}\left\{\tfrac{1}{4}z - (\tfrac{1}{4}z)^2 + (\tfrac{1}{4}z)^3 - (\tfrac{1}{4}z)^4 + ...\right\} + ...$$

and we sum along the columns.□

7.

The Graph of $\psi(z)$ vs. $\Gamma(z)$



$\psi(z)$ vanishes at the extreme points of $\Gamma(z)$

Both have the same first order poles.

8.

The Residues of $\psi(z)$

8.1 $\psi(z)$ **has poles at** $z = -n = 0, -1, -2, -3, \ldots$

$$\boxed{\text{Res}\{\psi(\text{z})\}_{z=-n} = -1}$$

<u>*Proof*</u>: By **6.1**,

$$\psi(z) = -\gamma - \frac{1}{z} + \left\{1 - \frac{1}{1+z}\right\} + \left\{\tfrac{1}{2} - \frac{1}{2+z}\right\} + \left\{\tfrac{1}{3} - \frac{1}{3+z}\right\} + \ldots$$

Since $\text{Res}\{\psi(\text{z})\}$, keeps its sign at $z = 0, -1, -2, -3, \ldots$, we use $\psi(z)$ to generate the series

$$\text{(I)} \qquad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{15^2} + ..$$

and

$$\text{(II)} \qquad \frac{1}{1^2} + \frac{1}{5^2} + \frac{1}{9^2} + \frac{1}{13^2} + \frac{1}{17^2} - \ldots$$

The Catalan Series is $1 + (\text{I}) - (\text{II})$

9.

$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{7^2}+\frac{1}{11^2}+...$ and $\frac{1}{1^2}+\frac{1}{5^2}+\frac{1}{9^2}+\frac{1}{13^2}+...$

in Residue Calculus

9.1

$$\boxed{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{15^2}+..=-\frac{1}{16}\sum_{z=-n}\text{Res}\left[\psi(z)\frac{1}{(z+\frac{1}{4})^2}\right]_{z=0,-1,-2,..}}$$

Proof:

$$-\sum_{z=-n}\text{Res}\left[\psi(z)\frac{1}{(z+\frac{1}{4})^2}\right]_{z=0,-1,-2,..}=-\sum_{n}(-1)\frac{1}{(-n+\frac{1}{4})^2}$$

$$=\frac{1}{(\frac{1}{4})^2}+\frac{1}{(-1+\frac{1}{4})^2}+\frac{1}{(-2+\frac{1}{4})^2}+\frac{1}{(-3+\frac{1}{4})^2}+\frac{1}{(-4+\frac{1}{4})^2}-...$$

$$=\frac{4^2}{1^2}+\frac{4^2}{3^2}+\frac{4^2}{7^2}+\frac{4^2}{11^2}+\frac{4^2}{15^2}+....\square$$

9.2

$$\boxed{\frac{1}{1^2}+\frac{1}{5^2}+\frac{1}{9^2}+\frac{1}{13^2}+\frac{1}{17^2}-...=-\frac{1}{16}\sum_{z=-n}\text{Res}\left[\psi(z)\frac{1}{(z-\frac{1}{4})^2}\right]_{z=0,-1,-2,..}}$$

Proof:

$$-\sum_{z=-n}\text{Res}\left[\psi(z)\frac{1}{(z-\frac{1}{4})^2}\right]_{z=-n}=-\sum_{z=-n}(-1)\frac{1}{(n-\frac{1}{4})^2}$$

$$= \frac{1}{(\frac{1}{4})^2} + \frac{1}{(-1-\frac{1}{4})^2} + \frac{1}{(-2-\frac{1}{4})^2} + \frac{1}{(-3-\frac{1}{4})^2} + \frac{1}{(-4-\frac{1}{4})^2} - \dots$$

$$= \frac{4^2}{1^2} + \frac{4^2}{5^2} + \frac{4^2}{9^2} + \frac{4^2}{13^2} + \frac{4^2}{17^2} - \dots . \square$$

9.3

$$1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} + \frac{1}{13^2} - \frac{1}{15^2} + \frac{1}{17^2} - \dots =$$

$$= 1 + \frac{1}{16}\sum_{n=0}^{n=\infty} \operatorname{Res}\left[\psi(z)\frac{1}{(z+\frac{1}{4})^2}\right]_{z=-n} - \frac{1}{16}\sum_{n=0}^{n=\infty} \operatorname{Res}\left[\psi(z)\frac{1}{(z-\frac{1}{4})^2}\right]_{z=-n}$$

Proof:

By **9.1**,

$$-\frac{1}{1^2} - \frac{1}{3^2} - \frac{1}{7^2} - \frac{1}{11^2} - \frac{1}{15^2} + \dots = \frac{1}{16}\sum_n \operatorname{Res}\left[\psi(z)\frac{1}{(z+\frac{1}{4})^2}\right]_{z=0,-1,-2,..}$$

By **9.2**,

$$\frac{1}{1^2} + \frac{1}{5^2} + \frac{1}{9^2} + \frac{1}{13^2} + \frac{1}{17^2} - \dots = -\frac{1}{16}\sum_n \operatorname{Res}\left[\psi(z)\frac{1}{(z-\frac{1}{4})^2}\right]_{z=0,-1,-2,..}$$

10.

The Residue Theorem Applied to

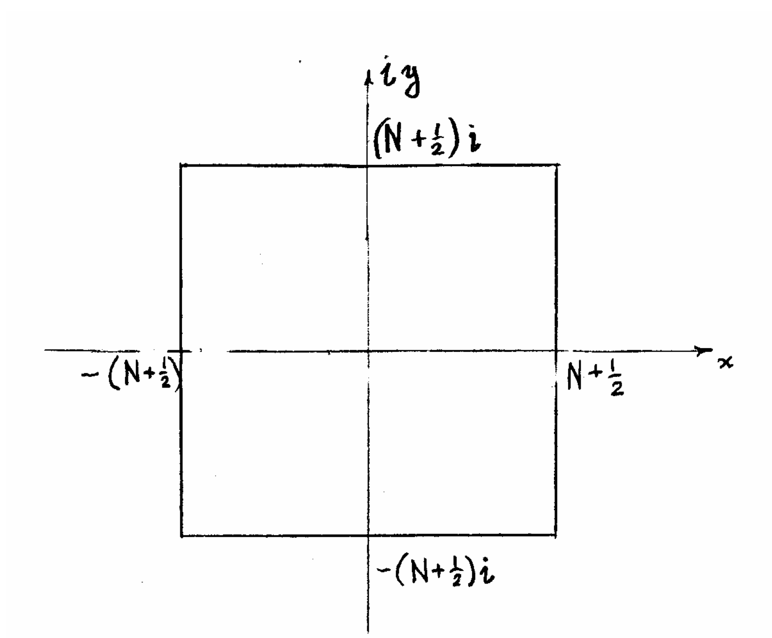
$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{7^2}+\frac{1}{11^2}+...$ and to $\frac{1}{1^2}+\frac{1}{5^2}+\frac{1}{9^2}+\frac{1}{13^2}+...$

10.1

$$\boxed{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{15^2}+...=\frac{1}{16}\operatorname{Res}\left[\psi(z)\frac{1}{(z+\frac{1}{4})^2}\right]_{z=-\frac{1}{4}}}$$

Proof:

Integrate $\psi(z)\frac{1}{(z+\frac{1}{4})^2}$ along the closed square $\square_{N+\frac{1}{2}}$



By the Residue Theorem,

$$\oint_{\square_{N+\frac{1}{2}}} \psi(\zeta)\frac{1}{(\zeta+\frac{1}{4})^2}d\zeta = \sum_n \text{Res}\left\{\psi(z)\frac{1}{(z+\frac{1}{4})^2}\right\}_{z=-n} +$$

$$+\text{Res}\left\{\psi(z)\frac{1}{(z+\frac{1}{4})^2}\right\}_{z=-\frac{1}{4}}$$

$$\left|\oint_{\square_{N+\frac{1}{2}}} \psi(\zeta)\frac{1}{(\zeta+\frac{1}{4})^2}d\zeta\right| \le \left|\frac{\psi(z)}{z}\right|\frac{1}{N}\underbrace{(\text{length}\square_{N+\frac{1}{2}})}_{8\left(N+\frac{1}{2}\right)} \xrightarrow[z\to\infty]{} 0$$

$$0 = \sum_n \text{Res}\left\{\psi(z)\frac{1}{(z+\frac{1}{4})^2}\right\}_{z=-n} + \text{Res}\left\{\psi(z)\frac{1}{(z+\frac{1}{4})^2}\right\}_{z=-\frac{1}{4}}$$

$$\text{Res}\left\{\psi(z)\frac{1}{(z+\frac{1}{4})^2}\right\}_{z=-\frac{1}{4}} = -\sum_n \text{Res}\left\{\psi(z)\frac{1}{(z+\frac{1}{4})^2}\right\}_{z=-n}$$

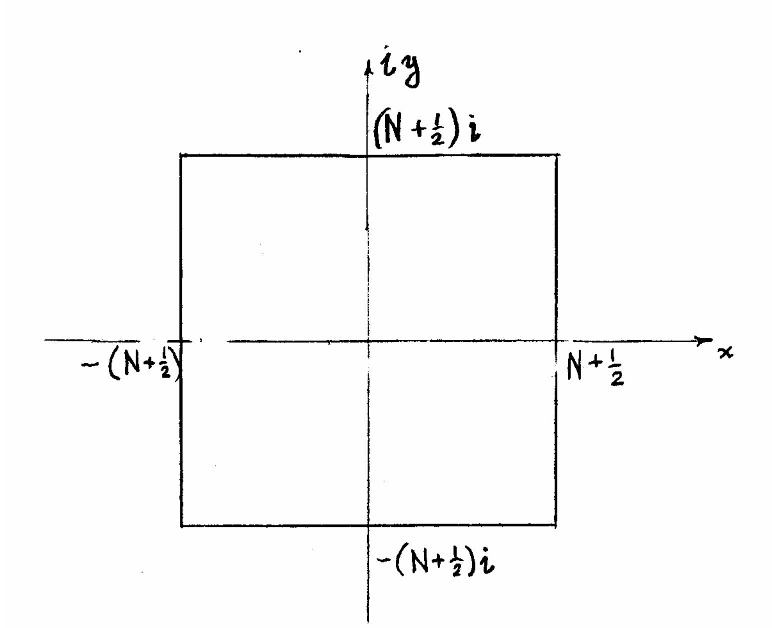
$$= \frac{4^2}{1^2}+\frac{4^2}{3^2}+\frac{4^2}{7^2}+\frac{4^2}{11^2}+\frac{4^2}{15^2}+....\square$$

10.2

$$\boxed{\frac{1}{1^2}+\frac{1}{5^2}+\frac{1}{9^2}+\frac{1}{13^2}+\frac{1}{17^2}+... = \frac{1}{16}\text{Res}\left[\psi(z)\frac{1}{(z-\frac{1}{4})^2}\right]_{z=\frac{1}{4}}}$$

Proof:

Integrate $\psi(z)\frac{1}{(z-\frac{1}{4})^2}$ along the closed square $\square_{N+\frac{1}{2}}$



By the Residue Theorem,

$$\oint_{\square_{N+\frac{1}{2}}} \psi(\zeta)\frac{1}{(\zeta-\frac{1}{4})^2}d\zeta = \sum_n \text{Res}\left\{\psi(z)\frac{1}{(z-\frac{1}{4})^2}\right\}_{z=-n} + $$

$$+\text{Res}\left\{\psi(z)\frac{1}{(z-\frac{1}{4})^2}\right\}_{z=\frac{1}{4}}$$

$$\left|\oint_{\square_{N+\frac{1}{2}}} \psi(\zeta)\frac{1}{(\zeta-\frac{1}{4})^2}d\zeta\right| \le \left|\frac{\psi(z)}{z}\right|\frac{1}{N}\underbrace{(\text{length}\square_{N+\frac{1}{2}})}_{8\left(N+\frac{1}{2}\right)} \xrightarrow[z\to\infty]{} 0$$

$$0 = \sum_n \text{Res}\left\{\psi(z)\frac{1}{(z-\frac{1}{4})^2}\right\}_{z=-n} + \text{Res}\left\{\psi(z)\frac{1}{(z-\frac{1}{4})^2}\right\}_{z=\frac{1}{4}}$$

$$\text{Res}\left\{\psi(z)\frac{1}{(z-\frac{1}{4})^2}\right\}_{z=\frac{1}{4}} = -\sum_n \text{Res}\left\{\psi(z)\frac{1}{(z-\frac{1}{4})^2}\right\}_{z=0,-1,-2,-3,\dots}$$

$$= \frac{4^2}{1^2}+\frac{4^2}{5^2}+\frac{4^2}{9^2}+\frac{4^2}{13^2}+\frac{4^2}{17^2}+\dots$$

10.3

$$\underbrace{1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}-\frac{1}{11^2}+\frac{1}{13^2}-\frac{1}{15^2}+\frac{1}{17^2}-...}_{\text{Catalan Constant}}=$$

$$=1-\frac{1}{16}\text{Res}\left[\psi(z)\frac{1}{(z+\frac{1}{4})^2}\right]_{z=-\frac{1}{4}}+\frac{1}{16}\text{Res}\left[\psi(z)\frac{1}{(z-\frac{1}{4})^2}\right]_{z=\frac{1}{4}}$$

Proof:

By **10.1**

$$-\frac{1}{1^2}-\frac{1}{3^2}-\frac{1}{7^2}-\frac{1}{11^2}-\frac{1}{15^2}-...=-\frac{1}{16}\text{Res}\left[\psi(z)\frac{1}{(z+\frac{1}{4})^2}\right]_{z=-\frac{1}{4}}$$

By **10.2**

$$\frac{1}{1^2}+\frac{1}{5^2}+\frac{1}{9^2}+\frac{1}{13^2}+\frac{1}{17^2}-...=\frac{1}{16}\text{Res}\left[\psi(z)\frac{1}{(z-\frac{1}{4})^2}\right]_{z=\frac{1}{4}}$$

However, we show that evaluating

$$\text{Res}\left[\psi(z)\frac{1}{(z-\frac{1}{4})^2}\right]_{z=\frac{1}{4}}$$

with

$$\psi(z)=-\gamma-\frac{1}{z}+\frac{z}{1+z}+\frac{z}{2(2+z)}+\frac{z}{3(3+z)}+\frac{z}{4(4+z)}...$$

leads to a null result:

11.

The Residue Theorem Applied to the DiGamma Series yields a Trivial Result

11.1 If we apply the Residue theorem to

$$\psi(z) = -\gamma - \frac{1}{z} + \frac{z}{1+z} + \frac{z}{2(2+z)} + \frac{z}{3(3+z)} + \frac{z}{4(4+z)}...,$$

then, we obtain the trivial result

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{7^2} + \frac{1}{11^2} + ... = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{7^2} + \frac{1}{11^2} + ..$$

Proof:

$$\psi(z) = -\gamma - \frac{1}{z} + \frac{z}{1+z} + \frac{z}{2(2+z)} + \frac{z}{3(3+z)} + \frac{z}{4(4+z)}...$$

$$\psi(z)\frac{1}{(z+\frac{1}{4})^2} = -\gamma\frac{1}{(z+\frac{1}{4})^2}$$

$$-\frac{1}{z}\frac{1}{(z+\frac{1}{4})^2}$$

$$+\frac{z}{1+z}\frac{1}{(z+\frac{1}{4})^2}$$

$$+\frac{z}{2(2+z)}\frac{1}{(z+\frac{1}{4})^2}$$

$$+\frac{z}{3(3+z)}\frac{1}{(z+\frac{1}{4})^2}$$

$$+\frac{z}{4(4+z)}\frac{1}{(z+\frac{1}{4})^2}$$

$$+\frac{z}{5(5+z)}\frac{1}{(z+\frac{1}{4})^2}+...$$

$$\text{Res}\left[-\frac{1}{z}\frac{1}{(z+\frac{1}{4})^2}\right]_{z=-\frac{1}{4}} = \frac{d}{dz}\left\{-\frac{1}{z}(z+\tfrac{1}{4})^2\frac{1}{(z+\frac{1}{4})^2}\right\}_{z=-\frac{1}{4}}$$

$$= \frac{d}{dz}\left\{-\frac{1}{z}\right\}_{z=-\frac{1}{4}}$$

$$= \left\{\frac{1}{z^2}\right\}_{z=-\frac{1}{4}} = \frac{4^2}{1^2}$$

$$\text{Res}\left[\frac{z}{1+z}\frac{1}{(z+\frac{1}{4})^2}\right]_{z=-\frac{1}{4}} = \frac{d}{dz}\left\{(z+\tfrac{1}{4})^2\frac{1}{1+z}\frac{z}{(z+\frac{1}{4})^2}\right\}_{z=-\frac{1}{4}}$$

$$= \frac{d}{dz}\left\{\frac{z}{1+z}\right\}_{z=-\frac{1}{4}}$$

$$= \left[\frac{1}{(1+z)^2}\right]_{z=-\frac{1}{4}} = \frac{4^2}{3^2}.\square$$

Similarly,

11.2 If we apply the Residue theorem to

$$\psi(z) = -\gamma - \frac{1}{z} + \frac{z}{1+z} + \frac{z}{2(2+z)} + \frac{z}{3(3+z)} + \frac{z}{4(4+z)} \dots,$$

then, we obtain the trivial result

$$\frac{1}{1^2} + \frac{1}{5^2} + \frac{1}{9^2} + \frac{1}{13^2} + \dots = \frac{1}{1^2} + \frac{1}{5^2} + \frac{1}{9^2} + \frac{1}{13^2} + ..$$

The Taylor Series for $\psi(z)$ is known in $\left|z\right| < 1$. But not for $\left|z\right| > 1$. We proceed to show how such series for $\left|z\right| > 1$ would have expanded the Catalan Series in zeta function values.

12.

Had the DiGamma Taylor Series held in $\left|z\right| > 1$, the Catalan Series would have had a non-trivial sum

12.1 In $\boxed{\left|z\right| < 1}$,

$$\boxed{\psi(z) = -\gamma + \zeta(2)(z-1) - \zeta(3)(z-1)^2 + \zeta(4)(z-1)^3 - ...}$$

Proof: By **6.5**, for $\left|z\right| < 1$

$$\psi(1+z) = -\gamma + \zeta(2)z - \zeta(3)z^2 + \zeta(4)z^3 - .... \square$$

12.2 Had 12.1 held for $\left|z\right| > 1$**, then**

$$\operatorname{Res}\left[\psi(z)\frac{1}{(z+\frac{1}{4})^2}\right]_{z=-\frac{1}{4}} = \zeta(2) + 2(\tfrac{5}{4})\zeta(3) + 3(\tfrac{5}{4})^2\zeta(4) + 4(\tfrac{5}{4})^3\zeta(5) + ..$$

Proof:

$$\psi(z)\frac{1}{(z+\frac{1}{4})^2} = -\gamma\frac{1}{(z+\frac{1}{4})^2}$$

$$+\zeta(2)\frac{1}{(z+\frac{1}{4})^2}(z-1)$$

$$-\zeta(3)\frac{1}{(z+\frac{1}{4})^2}(z-1)^2$$

$$+\zeta(4)\frac{1}{(z+\frac{1}{4})^2}(z-1)^3$$

$$-\zeta(5)\frac{1}{(z+\frac{1}{4})^2}(z-1)^4 - ....$$

$$\zeta(2)\frac{z-1}{(z+\frac{1}{4})^2} = \zeta(2)\frac{-\frac{5}{4}+(z+\frac{1}{4})}{(z+\frac{1}{4})^2}$$

$$= \zeta(2)\frac{-\frac{5}{4}}{(z+\frac{1}{4})^2} + \boxed{\zeta(2)}\frac{1}{z+\frac{1}{4}}$$

$$-\zeta(3)\frac{[z-1]^2}{(z+\frac{1}{4})^2} = -\zeta(3)\frac{[-\frac{5}{4}+(z+\frac{1}{4})]^2}{(z+\frac{1}{4})^2}$$

$$= -\zeta(3)\frac{(\frac{5}{4})^2 - 2(\frac{5}{4})(z+\frac{1}{4}) + (z+\frac{1}{4})^2}{(z+\frac{1}{4})^2}$$

$$= -\zeta(3)\frac{(\frac{5}{4})^2}{(z+\frac{1}{4})^2} + \boxed{2(\tfrac{5}{4})\zeta(3)}\frac{1}{z+\frac{1}{4}} - \zeta(3)$$

$$\zeta(4)\frac{(z-1)^3}{(z+\frac{1}{4})^2} = \zeta(4)\frac{[-\frac{5}{4}+(z+\frac{1}{4})^3]}{(z+\frac{1}{4})^2}$$

$$= \zeta(4)\frac{(-\frac{5}{4})^3 + 3(-\frac{5}{4})^2(z+\frac{1}{4}) + 3(-\frac{5}{4})(z+\frac{1}{4})^2 + (z+\frac{1}{4})^3}{(z+\frac{1}{4})^2}$$

$$= -\frac{\zeta(4)(\frac{5}{4})^3}{(z+\frac{1}{4})^2} + \boxed{\zeta(4)3(\tfrac{5}{4})^2}\frac{1}{z+\frac{1}{4}} - \zeta(4)3(\tfrac{5}{4}) + \zeta(4)(z+\tfrac{1}{4})$$

$$-\zeta(5)\frac{(z-1)^4}{(z+\frac{1}{4})^2} = -\zeta(5)\frac{[-\frac{5}{4}+(z+\frac{1}{4})]^4}{(z+\frac{1}{4})^2}$$

$$= -\zeta(5)\frac{(-\frac{5}{4})^4 + 4(-\frac{5}{4})^3(z+\frac{1}{4}) + 6(-\frac{5}{4})^2(z+\frac{1}{4})^2 + 4(-\frac{5}{4})(z+\frac{1}{4})^3 + (z+\frac{1}{4})^4}{(z+\frac{1}{4})^2}$$

$$= -\frac{\zeta(5)(\frac{5}{4})^4}{(z+\frac{1}{4})^2} + \boxed{\zeta(5)4(\tfrac{5}{4})^3}\frac{1}{z+\frac{1}{4}} - \zeta(5)6(\tfrac{5}{4})^2 + \zeta(5)4(\tfrac{5}{4})(z+\tfrac{1}{4}) - \zeta(5)(z+\tfrac{1}{4})^2 . \square$$

12.3 Had 12.1 held for $|z| > 1$, then

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{7^2}+\frac{1}{11^2}+... = \frac{1}{16}\left\{\zeta(2)+2(\tfrac{5}{4})\zeta(3)+3(\tfrac{5}{4})^2\zeta(4)+...\right\}$$

Proof:

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{7^2}+... = -\frac{1}{16}\sum_n \text{Res}\left[\psi(z)\frac{1}{(z+\frac{1}{4})^2}\right]_{z=-n}$$

$$= \frac{1}{16}\text{Res}\left[\psi(z)\frac{1}{(z+\frac{1}{4})^2}\right]_{z=-\frac{1}{4}}$$

$$= \frac{1}{16}\left\{\zeta(2)+2(\tfrac{5}{4})\zeta(3)+3(\tfrac{5}{4})^2\zeta(4)+...\right\}$$

12.4 Had 12.1 held for $|z| > 1$, then

$$\text{Res}\left[\psi(z)\frac{1}{(z-\frac{1}{4})^2}\right]_{z=\frac{1}{4}} = \zeta(2)+2(\tfrac{3}{4})\zeta(3)+3(\tfrac{3}{4})^2\zeta(4)+4(\tfrac{3}{4})^3\zeta(5)+...$$

Proof:

$$\psi(z)\frac{1}{(z-\frac{1}{4})^2} = -\gamma\frac{1}{(z-\frac{1}{4})^2}$$

$$+\zeta(2)\frac{1}{(z-\frac{1}{4})^2}(z-1)$$

$$-\zeta(3)\frac{1}{(z-\frac{1}{4})^2}(z-1)^2$$

$$+\zeta(4)\frac{1}{(z-\frac{1}{4})^2}(z-1)^3$$

$$-\zeta(5)\frac{1}{(z-\frac{1}{4})^2}(z-1)^4 - ....$$

$$\zeta(2)\frac{z-1}{(z-\frac{1}{4})^2} = \zeta(2)\frac{-\frac{3}{4}+(z-\frac{1}{4})}{(z-\frac{1}{4})^2}$$

$$= \zeta(2)\frac{-\frac{3}{4}}{(z-\frac{1}{4})^2} + \boxed{\zeta(2)}\frac{1}{z-\frac{1}{4}}$$

$$-\zeta(3)\frac{[z-1]^2}{(z-\frac{1}{4})^2} = -\zeta(3)\frac{[-\frac{3}{4}+(z-\frac{1}{4})]^2}{(z-\frac{1}{4})^2}$$

$$= -\zeta(3)\frac{(\frac{3}{4})^2 - 2(\frac{3}{4})(z-\frac{1}{4}) + (z-\frac{1}{4})^2}{(z-\frac{1}{4})^2}$$

$$= -\zeta(3)\frac{(\frac{3}{4})^2}{(z-\frac{1}{4})^2} + \boxed{2(\tfrac{3}{4})\zeta(3)}\frac{1}{z-\frac{1}{4}} - \zeta(3)$$

$$\zeta(4)\frac{(z-1)^3}{(z-\frac{1}{4})^2} = \zeta(4)\frac{[-\frac{3}{4}+(z-\frac{1}{4})^3]}{(z-\frac{1}{4})^2}$$

$$= \zeta(4)\frac{(-\frac{3}{4})^3 + 3(-\frac{3}{4})^2(z-\frac{1}{4}) + 3(-\frac{3}{4})(z-\frac{1}{4})^2 + (z-\frac{1}{4})^3}{(z-\frac{1}{4})^2}$$

$$= -\frac{\zeta(4)(\frac{3}{4})^3}{(z-\frac{1}{4})^2} + \boxed{\zeta(4)3(\frac{3}{4})^2}\frac{1}{z-\frac{1}{4}} - \zeta(4)3(\frac{3}{4}) + \zeta(4)(z-\frac{1}{4})$$

$$-\zeta(5)\frac{(z-1)^4}{(z-\frac{1}{4})^2} = -\zeta(5)\frac{[-\frac{3}{4} + (z-\frac{1}{4})]^4}{(z-\frac{1}{4})^2}$$

$$= -\zeta(5)\frac{(-\frac{3}{4})^4 + 4(-\frac{3}{4})^3(z-\frac{1}{4}) + 6(-\frac{3}{4})^2(z-\frac{1}{4})^2 + 4(-\frac{3}{4})(z-\frac{1}{4})^3 + (z-\frac{1}{4})^4}{(z-\frac{1}{4})^2}$$

$$= -\frac{\zeta(5)(\frac{3}{4})^4}{(z-\frac{1}{4})^2} + \boxed{\zeta(5)4(\frac{3}{4})^3}\frac{1}{z-\frac{1}{4}} - \zeta(5)6(\frac{3}{4})^2 + \zeta(5)4(\frac{3}{4})(z-\frac{1}{4}) - \zeta(5)(z-\frac{1}{4})^2.$$

$$\text{Res}\left[\psi(z)\frac{1}{(z-\frac{1}{4})^2}\right]_{z=\frac{1}{4}} = \zeta(2) + 2(\tfrac{3}{4})\zeta(3) + 3(\tfrac{3}{4})^2\zeta(4) + 4(\tfrac{3}{4})^3\zeta(5) + ... \square$$

12.5 Had 12.1 held for $|z| > 1$, then

$$\boxed{\frac{1}{1^2} + \frac{1}{5^2} + \frac{1}{9^2} + ... = \frac{1}{16}\left\{\zeta(2) + 2(\tfrac{3}{4})\zeta(3) + 3(\tfrac{3}{4})^2\zeta(4) + 4(\tfrac{3}{4})^3\zeta(5) + ...\right\}}$$

Proof:

$$\frac{1}{1^2} + \frac{1}{5^2} + \frac{1}{9^2} + ... = -\frac{1}{16}\sum_n \text{Res}\left[\psi(z)\frac{1}{(z-\frac{1}{4})^2}\right]_{z=-n}$$

$$= \frac{1}{16}\text{Res}\left[\psi(z)\frac{1}{(z-\frac{1}{4})^2}\right]_{z=\frac{1}{4}}$$

$$= \frac{1}{16}\left\{\zeta(2) + 2(\tfrac{3}{4})\zeta(3) + 3(\tfrac{3}{4})^2\zeta(4) + 4(\tfrac{3}{4})^3\zeta(5) + \ldots\right\}$$

12.6 Had 12.1 held for $\left|z\right| > 1$, then

$$\operatorname{Res}\left[\psi(z)\frac{1}{(z-\frac{1}{4})^2}\right]_{z=\frac{1}{4}} - \operatorname{Res}\left[\psi(z)\frac{1}{(z+\frac{1}{4})^2}\right]_{z=-\frac{1}{4}} =$$

$$= -\left\{\zeta(3) + 3\zeta(4) + 4\tfrac{1}{4^3}\left[5^3 - 3^3\right]\zeta(5) + \ldots + (n-1)\tfrac{1}{4^{n-2}}\left[5^{n-2} - 3^{n-2}\right]\zeta(n) + \ldots\right\}$$

Proof:

$$\operatorname{Res}\left[\psi(z)\frac{1}{(z-\frac{1}{4})^2}\right]_{z=\frac{1}{4}} - \operatorname{Res}\left[\psi(z)\frac{1}{(z+\frac{1}{4})^2}\right]_{z=-\frac{1}{4}} =$$

$$= \zeta(2) + 2(\tfrac{5}{4})\zeta(3) + 3(\tfrac{5}{4})^2\zeta(4) + 4(\tfrac{5}{4})^3\zeta(5) + \ldots$$

$$-\left[\zeta(2) + 2(\tfrac{3}{4})\zeta(3) + 3(\tfrac{3}{4})^2\zeta(4) + 4(\tfrac{3}{4})^3\zeta(5) + \ldots\right]$$

$$= \zeta(3) + 3\zeta(4) + 4\left[(\tfrac{5}{4})^3 - (\tfrac{3}{4})^3\right]\zeta(5) + .. + (n-1)\left[(\tfrac{5}{4})^{n-2} - (\tfrac{3}{4})^{n-2}\right]\zeta(n) + \ldots. \square$$

12.7 Had 12.1 held for $\left|z\right| > 1$, then

$$1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} + \frac{1}{13^2} - \frac{1}{15^2} + \frac{1}{17^2} - .. =$$

$$= 1 - \tfrac{1}{16}\left[\zeta(3) + 3\zeta(4) + \ldots + (n-1)\tfrac{1}{4^{n-2}}\left[5^{n-2} - 3^{n-2}\right]\zeta(n) + \ldots\right]$$

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