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Delta Function

The Delta Function **H. Vic Dannon**

Abstract: The Dirac Delta Function, the idealization of an impulse in Radar circuits, is a Hyper-Real function which definition and analysis require Infinitesimal Calculus, and Infinite Hyper-reals.

The controversy surrounding the Leibnitz Infinitesimals derailed the development of the Infinitesimal Calculus, and the Delta Function could not be defined and investigated properly.

For instance, it is labeled a "Generalized Function" although it generalizes no function.

Dirac's intuitive definition by Delta's sampling property

$$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1,$$

that avoids specifying $\delta(0)$, remains the main definition of the delta function, although the Delta Function is not Riemann

integrable in the Calculus of Limits, and is not Lebesgue integrable in Measure Theory.

In fact, in the Calculus of Limits, only the Cauchy Principal Value Integral of the Delta Function exists, and it equals zero.

Only in Infinitesimal Calculus, can the Delta Function be defined, differentiated, and integrated.

Infinitesimal Calculus allows us to resolve open problems such as

What is $\delta(0)$?

How is $x\delta(x)$ defined at $x = 0$?

How is the Delta Function the derivative of a Step Function?

How do we integrate the Delta Function?

What is $\delta(x^2)$?

What is $\delta^2(x)$?

What is $\delta(x^3)$?

What is $\delta^3(x)$?

The Delta Function enables us to define the Fourier Transform with minimal requirements on the transformed function.

Delta Function, the Fourier Transform, and Fourier Integral Theorem **H. Vic Dannon**

Abstract: The Fourier Integral Theorem guarantees that the Fourier Transform and its Inverse are well defined operations, so that the inversed transform is the originally transformed function.

It is believed to hold in the Calculus of Limits under some highly restrictive sufficient conditions. In fact,

The Theorem does not hold in the Calculus of Limits under any conditions,

because evaluating the Fourier Integral requires the integration of

$$\int_{k=-\infty}^{k=\infty} e^{ik(x-\xi)}dk\,,$$

that diverges at $x = \xi$.

Only in Infinitesimal Calculus, the integral is the Hyper-real Delta Function

$$\delta(x-\xi) = \frac{1}{2\pi}\int_{k=-\infty}^{k=\infty} e^{ik(x-\xi)}dk\,,$$

and the Fourier Integral Theorem states the sifting property for the Delta Function

$$f(x) = \int_{\xi=-\infty}^{\xi=\infty} f(\xi)\delta(\xi - x)d\xi\,.$$

In infinitesimal Calculus we can integrate over singularities, and the Fourier Integral Theorem holds

$$f(x) = \frac{1}{2\pi}\int_{k=-\infty}^{k=\infty}\left(\int_{\xi=-\infty}^{\xi=\infty} f(\xi)e^{-ik\xi}d\xi\right)e^{ikx}dk\,,$$

where the Integrals are Hyper-real.

The highly restrictive conditions for the Fourier Integral Theorem, in the Calculus of Limits, are irrelevant to the simplest functions, such as constants, and useless for singular functions.

In particular, the singular $\delta(x)$ violates these conditions

- the Hyper-real Delta $\delta(x)$ is not defined in the Calculus of Limits, and is not Piecewise Continuous.
- $\delta'(x)$ is not defined, and is not Piecewise Continuous in any bounded interval.

But in Infinitesimal Calculus, $\delta(x)$ satisfies the Hyper-real Fourier integral Theorem

$$\delta(x) = \frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} \left(\int_{\xi=-\infty}^{\xi=\infty} \delta(\xi) e^{-ik\xi} d\xi \right) e^{ikx} dk.$$

Also, the constant function $f(x) \equiv 1$ violates the sufficient conditions' requirement of absolute integrability, $\int_{x=-\infty}^{x=\infty} |1| dx = \infty$.

But in Infinitesimal Calculus, $f(x) \equiv 1$ satisfies the Hyper-real Fourier integral Theorem

$$1 = \frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} \left(\int_{\xi=-\infty}^{\xi=\infty} e^{-ik\xi} d\xi \right) e^{ikx} dk.$$