Radial Delta Function and the 3-D Fourier-Bessel Transform

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Abstract In [Dan6], we have shown that $\delta(x,y,z)=\delta(x)\delta(y)\delta(z)$ is the 3-dimensional Fourier Transform of the function 1. Here, we show that its Radial form is the 3-dimensional Fourier-Bessel Transform of the function 1,

$$\delta(x)\delta(y)\delta(z) = 4\int_{\nu=0}^{\nu=\infty} \frac{\sin\left(2\pi\nu r\right)}{r}\nu d\nu\,, \qquad r=\sqrt{x^2+y^2+z^2}\,.$$

Thus, $\delta(x)\delta(y)\delta(z)$ is a Radially Symmetric Delta Function that we will denote $\delta_{\text{Radial}}(r)$.

Similarly,

$$\delta(x-\xi)\delta(y-\eta)\delta(z-\zeta) = 4^2\frac{1}{r\sigma}\int_{\nu=0}^{\nu=\infty}\sin\left(2\pi\nu r\right)\sin\left(2\pi\nu\sigma\right)d\nu$$

is Radially symmetric, and defines the Radial Delta $\delta_{\text{Radial}}(r-\sigma)$.

The formulas above are exclusively Hyper-real. They cannot be derived by means of the Calculus of Limits, and are unknown in the Calculus of Limits.

The Radial Delta Function $\delta_{\text{Radial}}(r-\sigma)$ is a Discontinuous, Hyper-real function, that spikes at $r=\sigma$, and vanishes for $\sigma \neq r$.

The Fourier-Bessel Integral Theorem for Radially symmetric function $f(r)$ guarantees that the 3-dimesional Fourier-Bessel Transform and its Inverse are well defined operations, so that inversion yields the original function that generated the Transform.

It is believed to hold in the Calculus of Limits under given conditions. In fact, it does not hold in the Calculus of Limits because the integration of the Fourier Integral requires the integration of $\int\limits_{\nu=0}^{\nu=\infty} \sin(2\pi\nu\sigma)\sin(2\pi\nu r)\,d\nu$ that diverges at $\sigma = r$.

Only in Infinitesimal Calculus, we can integrate over singularities, and the Fourier Integral Theorem holds.

$$f(r) = 4\int\limits_{\nu=0}^{\nu=\infty}\left(4\int\limits_{\sigma=0}^{\sigma=\infty} f(\sigma)\frac{\sin(2\pi\nu\sigma)}{\nu\sigma}\sigma^2 d\sigma\right)\frac{\sin(2\pi\nu r)}{\nu r}\nu^2 d\nu$$

$$= \int\limits_{\sigma=0}^{\sigma=\infty} f(\sigma)\left(4^2\frac{1}{r\sigma}\int\limits_{\nu=0}^{\nu=\infty}\sin(2\pi\nu\sigma)\sin(2\pi\nu r)\,d\nu\right)\sigma^2 d\sigma$$

$$= \int\limits_{\sigma=0}^{\sigma=\infty} f(\sigma)\left(4^2\int\limits_{\nu=0}^{\nu=\infty}\sin(2\pi\nu\sigma)\sin(2\pi\nu r)\,d\nu\right)d\sigma .$$

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Contents

Introduction

It is believed that a radially symmetric $f(r)$, satisfies the 3-dimensional Fourier-Bessel Integral Theorem

$$f(r) = 4\int_{\nu=0}^{\nu=\infty}\left(4\int_{\sigma=0}^{\sigma=\infty} f(\sigma)\frac{\sin\left(2\pi\nu\sigma\right)}{\nu\sigma}\sigma^2 d\sigma\right)\frac{\sin\left(2\pi\nu r\right)}{\nu r}\nu^2 d\nu$$

$$= \int_{\sigma=0}^{\sigma=\infty} f(\sigma)\left(4^2\frac{1}{r\sigma}\int_{\nu=0}^{\nu=\infty}\sin\left(2\pi\nu\sigma\right)\sin\left(2\pi\nu r\right)d\nu\right)\sigma^2 d\sigma$$

However, in the Calculus of Limits, at $\sigma = r$,

$$\int_{\nu=0}^{\nu=\infty}\sin\left(2\pi\nu\sigma\right)\sin\left(2\pi\nu r\right)d\nu = \infty .$$

Avoiding the singularity at $\sigma = r$ does not recover the Fourier-Bessel Integral Theorem, because without the singularity the integral equals zero.

Thus, the Fourier Integral Theorem cannot be written in the Calculus of Limits.

In Infinitesimal Calculus [Dan4], the singularity can be integrated over, and defines the Spherical Delta Function $\delta_{\text{Radial}}(r-\sigma)$.

Then, for any Hyper-real function $f(r)$, the Fourier-Bessel Integral Theorem is the sifting property of $\delta_{\text{Radial}}(r-\sigma)$.

In the calculus of limits, $\delta_{\text{Radial}}(r-\sigma)$ cannot be defined.

1.

Hyper-real Line

Each real number α can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant Hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of infinitesimal Hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the infinite Hyper-reals.
4. The infinite Hyper-reals are greater than any real number, yet strictly smaller than infinity.
5. The infinite Hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a non-constant Hyper-real.
7. The Hyper-reals are the totality of constant Hyper-reals, a family of infinitesimals, a family of infinitesimals with

negative sign, a family of infinite Hyper-reals, a family of infinite Hyper-reals with negative sign, and non-constant Hyper-reals.

8. The Hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.
9. That line includes the real numbers separated by the non-constant Hyper-reals. Each real number is the center of an interval of Hyper-reals, that includes no other real number.
10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.
11. Zero is not an infinitesimal, because zero is not strictly greater than zero.
12. We do not add infinity to the Hyper-real line.
13. The infinitesimals, the infinitesimals with negative signs, the infinite Hyper-reals, and the infinite Hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.
14. The Hyper-real line is embedded in $\mathbb{R}^{\infty}$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the Hyper-real onto the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal Hyper-reals, or to the infinite Hyper-reals, or to the non-constant Hyper-reals.

16. No neighbourhood of a Hyper-real is homeomorphic to an $\mathbb{R}^n$ ball. Therefore, the Hyper-real line is not a manifold.

17. The Hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

2.

Hyper-real Integral

In [Dan3], we defined the Hyper-real integral of a Hyper-real Function.

Let $f(x)$ be a Hyper-real function on the interval $[a,b]$.

The interval may not be bounded.

$f(x)$ may take infinite Hyper-real values, and need not be bounded.

At each

$$a \le x \le b\,,$$

there is a rectangle with base $[x - \frac{dx}{2}, x + \frac{dx}{2}]$, height $f(x)$, and area

$$f(x)dx\,.$$

We form the **Integration Sum** of all the areas for the x's that start at $x = a$, and end at $x = b$,

$$\sum_{x \in [a,b]} f(x)dx\,.$$

If for any infinitesimal dx, the Integration Sum has the same Hyper-real value, then $f(x)$ is integrable over the interval $[a,b]$.

Then, we call the Integration Sum the integral of $f(x)$ from $x = a$, to $x = b$, and denote it by

$$\int_{x=a}^{x=b} f(x)dx\,.$$

If the Hyper-real is infinite, then it is the integral over $[a,b]$,

If the Hyper-real is finite,

$$\int_{x=a}^{x=b} f(x)dx = \text{ real part of the hyper-real}\,.\square$$

2.1 The countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers,

$Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty\,.$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[a,b]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(x)dx$.

The Lower Integral is the Integration Sum where $f(x)$ is replaced by its lowest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

2.2
$$\sum_{x \in [a,b]} \left(\inf_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

The Upper Integral is the Integration Sum where $f(x)$ is replaced by its largest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

2.3
$$\sum_{x \in [a,b]} \left(\sup_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

If the integral is a finite Hyper-real, we have

2.4 *A Hyper-real function has a finite integral if and only if its upper integral and its lower integral are finite, and differ by an infinitesimal.*

3.

Delta Function

In [Dan5], we defined the Delta Function, and established its properties

1. The Delta Function is a Hyper-real function defined from the Hyper-real line into the set of two Hyper-reals $\left\{0, \frac{1}{dx}\right\}$. The Hyper-real 0 is the sequence $\langle 0, 0, 0, ... \rangle$. The infinite Hyper-real $\frac{1}{dx}$ depends on our choice of dx.
2. We will usually choose the family of infinitesimals that is spanned by the sequences $\left\langle \frac{1}{n} \right\rangle, \left\langle \frac{1}{n^2} \right\rangle, \left\langle \frac{1}{n^3} \right\rangle, \ldots$ It is a semigroup with respect to vector addition, and includes all the scalar multiples of the generating sequences that are non-zero. That is, the family includes infinitesimals with negative sign. Therefore, $\frac{1}{dx}$ will mean the sequence $\langle n \rangle$. Alternatively, we may choose the family spanned by the sequences $\left\langle \frac{1}{2^n} \right\rangle, \left\langle \frac{1}{3^n} \right\rangle, \left\langle \frac{1}{4^n} \right\rangle, \ldots$ Then, $\frac{1}{dx}$ will mean the

sequence $\left\langle 2^n \right\rangle$. Once we determined the basic infinitesimal dx, we will use it in the Infinite Riemann Sum that defines an Integral in Infinitesimal Calculus.

3. The Delta Function is strictly smaller than ∞

4. We define, $\delta(x) \equiv \dfrac{1}{dx}\chi_{\left[-\frac{dx}{2},\frac{dx}{2}\right]}(x)$,

   where $\chi_{\left[-\frac{dx}{2},\frac{dx}{2}\right]}(x) = \begin{cases} 1, x \in \left[-\frac{dx}{2},\frac{dx}{2}\right] \\ 0, \text{otherwise} \end{cases}$.

5. Hence,

   - for $x < 0$, $\delta(x) = 0$
   - at $x = -\dfrac{dx}{2}$, $\delta(x)$ jumps from 0 to $\dfrac{1}{dx}$,
   - for $x \in \left[-\frac{dx}{2},\frac{dx}{2}\right]$, $\delta(x) = \dfrac{1}{dx}$.
   - at $x = 0$, $\delta(0) = \dfrac{1}{dx}$
   - at $x = \dfrac{dx}{2}$, $\delta(x)$ drops from $\dfrac{1}{dx}$ to 0.
   - for $x > 0$, $\delta(x) = 0$.
   - $x\delta(x) = 0$

6. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle \chi_{\left[-\frac{1}{2},\frac{1}{2}\right]}(x), 2\chi_{\left[-\frac{1}{4},\frac{1}{4}\right]}(x), 3\chi_{\left[-\frac{1}{6},\frac{1}{6}\right]}(x)... \right\rangle$

7. If $dx = \left\langle \frac{2}{n} \right\rangle$, $\delta(x) = \left\langle \dfrac{1}{2\cosh^2 x}, \dfrac{2}{2\cosh^2 2x}, \dfrac{3}{2\cosh^2 3x}, ... \right\rangle$

8. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle e^{-x}\chi_{[0,\infty)}, 2e^{-2x}\chi_{[0,\infty)}, 3e^{-3x}\chi_{[0,\infty)}, \ldots \right\rangle$

9. $$\int\limits_{x=-\infty}^{x=\infty} \delta(x)dx = 1.$$

4.

The Fourier Transform

In [Dan6], we defined the Fourier Transform and established its properties

1. $\mathcal{F}\left\{\delta(x)\right\} = 1$

2. $\delta(x)$ = *the inverse Fourier Transform of the unit function* 1

$$= \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x} d\omega$$

$$= \int_{\nu=-\infty}^{\nu=\infty} e^{2\pi i x} d\nu\,, \quad \omega = 2\pi\nu$$

3. $\left.\frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x} d\omega\right|_{x=0} = \frac{1}{dx} =$ an infinite Hyper-real

$$\left.\int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x} d\omega\right|_{x\neq 0} = 0$$

4. Fourier Integral Theorem

$$f(x) = \frac{1}{2\pi}\int_{k=-\infty}^{k=\infty}\left(\int_{\xi=-\infty}^{\xi=\infty} f(\xi)e^{-ik\xi}d\xi\right)e^{ikx}dk$$

does not hold in the Calculus of Limits, under any conditions.

5. Fourier Integral Theorem in Infinitesimal Calculus

If $f(x)$ *is Hyper-real function,*

Then,

❖ *the Hyper-real Fourier Integral Theorem holds.*

❖ $\int_{x=-\infty}^{x=\infty} f(x)e^{-i\alpha x}dx$ ***converges to*** $F(\alpha)$

❖ $\frac{1}{2\pi}\int_{\alpha=-\infty}^{\alpha=\infty} F(\alpha)e^{-i\alpha x}d\alpha$ ***converges to*** $f(x)$

6. 3-Dimesional Fourier Transform

$$\mathcal{F}\left\{f(x,y,z)\right\} = \int_{z=-\infty}^{z=\infty}\int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} f(x,y,z)e^{-i\omega_x x - i\omega_y y - i\omega_z z}dxdydz$$

$$= \int_{z=-\infty}^{z=\infty}\int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} f(x,y,z)e^{-2\pi i(\nu_x x + \nu_y y + \nu_z z)}dxdydz\,, \quad \begin{array}{l} \omega_x = 2\pi\nu_x \\ \omega_y = 2\pi\nu_y \\ \omega_z = 2\pi\nu_z \end{array}$$

8. 3-Dimesional Inverse Fourier Transform

$$\mathcal{F}^{-1}\left\{F(\omega_x,\omega_y,\omega_z)\right\} = \frac{1}{(2\pi)^3}\int_{\omega_z=-\infty}^{\omega_z=\infty}\int_{\omega_y=-\infty}^{\omega_y=\infty}\int_{\omega_x=-\infty}^{\omega_x=\infty} F(\omega_x,\omega_y,\omega_z)e^{i(\omega_x x + \omega_y y + \omega_z z)}d\omega_x d\omega_y d\omega_z$$

$$= \int_{\nu_z=-\infty}^{\nu_z=\infty}\int_{\nu_y=-\infty}^{\nu_y=\infty}\int_{\nu_x=-\infty}^{\nu_x=\infty} F(2\pi\nu_x,2\pi\nu_y,2\pi\nu_z)e^{2\pi i(\nu_x x + \nu_y y + \nu_z z)}d\nu_x d\nu_y d\nu_z\,, \quad \begin{array}{l} \omega_x = 2\pi\nu_x \\ \omega_y = 2\pi\nu_y \\ \omega_z = 2\pi\nu_z \end{array}$$

9. 3-Dimesional Fourier Integral Theorem

$$f(x,y,z) = \frac{1}{(2\pi)^3}\int_{\omega_z=-\infty}^{\omega_z=\infty}\int_{\omega_y=-\infty}^{\omega_y=\infty}\int_{\omega_x=-\infty}^{\omega_x=\infty}\left(\int_{\zeta=-\infty}^{\zeta=\infty}\int_{\eta=-\infty}^{\eta=\infty}\int_{\xi=-\infty}^{\xi=\infty} f(\xi,\eta,\zeta)e^{-i\omega_x\xi-i\omega_y\eta-i\omega_z\zeta}d\xi d\eta d\zeta\right)\times$$

$$\times e^{i(\omega_x x+i\omega_y y+i\omega_z z)}d\omega_x d\omega_y d\omega_z$$

$$= \int_{\zeta=-\infty}^{\zeta=\infty}\int_{\eta=-\infty}^{\eta=\infty}\int_{\xi=-\infty}^{\xi=\infty} f(\xi,\eta,\zeta)\left(\frac{1}{2\pi}\int_{\omega_x=-\infty}^{\omega_x=\infty} e^{i\omega_x(x-\xi)}d\omega_x\right)d\xi\left(\frac{1}{2\pi}\int_{\omega_y=-\infty}^{\omega_y=\infty} e^{i\omega_y(y-\eta)}d\omega_y\right)d\eta\times$$

$$\times\left(\frac{1}{2\pi}\int_{\omega_z=-\infty}^{\omega_z=\infty} e^{i\omega_z(z-\zeta)}d\omega_z\right)d\zeta$$

$$= \int_{\zeta=-\infty}^{\zeta=\infty}\int_{\eta=-\infty}^{\eta=\infty}\int_{\xi=-\infty}^{\xi=\infty} f(\xi,\eta,\zeta)\left(\int_{\nu_x=-\infty}^{\nu_x=\infty} e^{2\pi i\nu_x(x-\xi)}d\nu_x\right)d\xi\left(\int_{\nu_y=-\infty}^{\nu_y=\infty} e^{2\pi i\nu_y(y-\eta)}d\nu_y\right)d\eta\times$$

$$\times\left(\int_{\nu_z=-\infty}^{\nu_z=\infty} e^{2\pi i\nu_z(z-\zeta)}d\nu_z\right)d\zeta, \quad \begin{aligned}\omega_x &= 2\pi\nu_x\\ \omega_y &= 2\pi\nu_y\\ \omega_z &= 2\pi\nu_z\end{aligned}$$

10. 3-Dimesional Delta Function

$$\delta\left(x,y,z\right) = \left(\frac{1}{2\pi}\int_{\omega_x=-\infty}^{\omega_x=\infty} e^{i\omega_x x}d\omega_x\right)\left(\frac{1}{2\pi}\int_{\omega_y=-\infty}^{\omega_y=\infty} e^{i\omega_y y}d\omega_y\right)\left(\frac{1}{2\pi}\int_{\omega_z=-\infty}^{\omega_z=\infty} e^{i\omega_z z}d\omega_z\right)$$

$$= \left(\int_{\nu_x=-\infty}^{\nu_x=\infty} e^{2\pi i\nu_x x}d\nu_x\right)\left(\int_{\nu_y=-\infty}^{\nu_y=\infty} e^{2\pi i\nu_y y}d\nu_y\right)\left(\int_{\nu_z=-\infty}^{\nu_z=\infty} e^{2\pi i\nu_z z}d\nu_z\right), \quad \begin{aligned}\omega_x &= 2\pi\nu_x\\ \omega_y &= 2\pi\nu_y\\ \omega_z &= 2\pi\nu_z\end{aligned}$$

5.

$\delta(r - r_0)\delta(\theta - \theta_0)\delta(\phi - \phi_0)$

5.1 $\delta(r - r_0) = \frac{1}{dr}\chi_{[r_0 - \frac{dr}{2}, r_0 + \frac{dr}{2}]}(r), \ \ r \geq 0$

5.2 $\delta(\theta - \theta_0) = \frac{1}{d\theta}\chi_{[\theta_0 - \frac{d\theta}{2}, \theta_0 + \frac{d\theta}{2}]}(\theta), \ \ 0 \leq \theta \leq \pi$

5.3 $\delta(\phi - \phi_0) = \frac{1}{d\phi}\chi_{[\phi_0 - \frac{d\phi}{2}, \phi_0 + \frac{d\phi}{2}]}(\phi), \ \ 0 \leq \phi \leq 2\pi$

The product $\delta(r - r_0)\delta(\theta - \theta_0)\delta(\phi - \phi_0)$ defines a Delta Function that sifts along a line through its singularity at (r_0, θ_0, ϕ_0).

5.4 $\delta(r - r_0, \theta - \theta_0, \phi - \phi_0) \equiv \delta(r - r_0)\delta(\theta - \theta_0)\delta(\phi - \phi_0)$

$$= \frac{1}{dr}\chi_{[r_0 - \frac{dr}{2}, r_0 + \frac{dr}{2}]}(r)\frac{1}{d\theta}\chi_{[\theta_0 - \frac{d\theta}{2}, \theta_0 + \frac{d\theta}{2}]}(\theta)\frac{1}{d\phi}\chi_{[\phi_0 - \frac{d\phi}{2}, \phi_0 + \frac{d\phi}{2}]}(\phi)$$

Transforming between Spherical and Cartesian Coordinates

$$\begin{aligned} x &= r\sin\theta\cos\phi & x_0 &= r_0\sin\theta_0\cos\phi_0 \\ y &= r\sin\theta\sin\phi\,, & y_0 &= r_0\sin\theta_0\sin\phi_0\,, \quad r_0 > 0 \\ z &= r\cos\theta & z_0 &= r_0\cos\theta_0 \end{aligned}$$

5.5 $\delta(x-x_0)\delta(y-y_0)\delta(z-z_0) = \dfrac{1}{r_0^2\sin\theta_0}\delta(r-r_0)\delta(\theta-\theta_0)\delta(\phi-\phi_0)$

Proof:

$$\delta(r-r_0)\delta(\theta-\theta_0)\delta(\phi-\phi_0)drd\theta d\phi = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)dxdydz$$

$$\delta(r-r_0)\delta(\theta-\theta_0)\delta(\phi-\phi_0) = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\underbrace{\left|\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\right|}_{r^2\sin\theta}$$

Both sides vanish unless

$$r = r_0 + \text{infinitesimal} \approx r_0,$$

and

$$\theta = \theta_0 + \text{infinitesimal} \approx \theta_0.$$

Therefore, we can replace r with r_0, and θ with θ_0.

$$\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)r_0^2\sin\theta_0 = \delta(r-r_0)\delta(\theta-\theta_0)\delta(\phi-\phi_0).\square$$

5.6 $\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi\,, \\ z &= r\cos\theta \end{aligned}$ $\quad r_0 = 0 \Rightarrow \quad \delta(x)\delta(y)\delta(z) = \dfrac{1}{4\pi r^2}\delta(r)$

$$= \frac{1}{4\pi r^2 dr}\chi_{[-\frac{dr}{2},\frac{dr}{2}]}(r)$$

Proof:

$$\delta(r)\delta(\theta)\delta(\phi)drd\theta d\phi = \delta(x)\delta(y)\delta(z)dxdydz$$

$$\delta(r)\delta(\theta)\delta(\phi) = \delta(x)\delta(y)\delta(z)\underbrace{\left|\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\right|}_{r^2\sin\theta}$$

Since $r_0 = 0$, ϕ may take any value in $[0, 2\pi]$, θ may take any value in $[0, \pi]$, and we integrate over them. Then,

$$r^2 \sin\theta d\theta \underbrace{\int_{\phi=0}^{\phi=2\pi} d\phi}_{2\pi}\, \delta(x)\delta(y)\delta(z) = \delta(r)\delta(\theta)d\theta \underbrace{\int_{\phi=0}^{\phi=2\pi} \delta(\phi)d\phi}_{1}$$

$$2\pi r^2 \delta(x)\delta(y)\delta(z) \underbrace{\int_{\theta=0}^{\theta=\pi} \sin\theta d\theta}_{-\cos\theta\big|_{\theta=0}^{\theta=\pi}=2} = \delta(r) \underbrace{\int_{\theta=0}^{\theta=\pi} \delta(\theta)d\theta}_{1}.$$

$$4\pi r^2 \delta(x)\delta(y)\delta(z) = \delta(r).\square$$

6.

Radial Delta $\delta_{\text{Radial}}(r)$

6.1 $$\delta(x)\delta(y)\delta(z) = 4\int_{\nu=0}^{\nu=\infty} \frac{\sin(2\pi\nu r)}{r}\nu d\nu$$

Proof: By 4.10,

$$\delta(x)\delta(y)\delta(z) = \left(\int_{\nu_x=-\infty}^{\nu_x=\infty} e^{2\pi i\nu_x x}d\nu_x\right)\left(\int_{\nu_y=-\infty}^{\nu_y=\infty} e^{2\pi i\nu_y y}d\nu_y\right)\left(\int_{\nu_z=-\infty}^{\nu_z=\infty} e^{2\pi i\nu_z z}d\nu_z\right)$$

Substitute

$$\begin{aligned} x &= r\sin\theta\cos\phi & \nu_x &= \nu\sin\gamma\cos\beta \\ y &= r\sin\theta\sin\phi & \nu_y &= \nu\sin\gamma\sin\beta \\ z &= r\cos\theta & \nu_z &= \nu\cos\gamma \end{aligned}$$

Then,

$$\nu_x x + \nu_y y + \nu_z z =$$

$$= \nu r\left[\sin\theta\sin\gamma\left\{\cos\phi\cos\beta + \sin\phi\sin\beta\right\} + \cos\theta\cos\gamma\right]$$

$$= \nu r\left[\sin\theta\sin\gamma\cos(\beta-\phi) + \cos\theta\cos\gamma\right]$$

Integrating with respect to ν, γ, and β,

$$= \int_{\nu=0}^{\nu=\infty}\left(\int_{\gamma=0}^{\gamma=\pi}\left[\int_{\beta=0}^{\beta=2\pi} e^{2\pi i\nu r\sin\theta\sin\gamma\cos(\beta-\phi)}d\beta\right]e^{2\pi i\nu r\cos\theta\cos\gamma}\sin\gamma d\gamma\right)\nu^2 d\nu$$

Denoting

$$\alpha = \beta - \phi,$$

$$\int_{\beta=0}^{\beta=2\pi} e^{2\pi i\nu r\sin\theta\sin\gamma\cos(\beta-\phi)}d\beta = \int_{\alpha=-\phi}^{\alpha=2\pi-\phi} e^{2\pi i\nu r\sin\theta\sin\gamma\cos\alpha}d\alpha$$

Since $e^{2\pi i\nu r\sin\theta\sin\gamma\cos\alpha}$ is periodic in α with period 2π,

$$= \int_{\alpha=0}^{\alpha=2\pi} e^{2\pi i\nu r\sin\theta\sin\gamma\cos\alpha}d\alpha$$

$$= \int_{\alpha=0}^{\alpha=2\pi}\left(1 + \frac{2\pi i\nu r\sin\theta\sin\gamma\cos\alpha}{1} + \frac{(2\pi i\nu r\sin\theta\sin\gamma\cos\alpha)^2}{2!} + ...\right)d\alpha.$$

The integrals of the odd powers vanish, and we have

$$= 2\pi - \frac{(2\pi\nu r\sin\theta\sin\gamma)^2}{2^2}2\pi + \frac{(2\pi\nu r\sin\theta\sin\gamma)^4}{2^2\cdot 4^2}2\pi - \frac{(2\pi\nu r\sin\theta\sin\gamma)^6}{2^2\cdot 4^2\cdot 6^2}2\pi + ...$$

$$= 2\pi J_0(2\pi\nu r\sin\theta\sin\gamma).$$

Therefore,

$$\delta(x)\delta(y)\delta(z) = \int_{\nu=0}^{\nu=\infty}\left(\int_{\gamma=0}^{\gamma=\pi} 2\pi J_0(2\pi\nu r\sin\theta\sin\gamma)e^{2\pi i\nu r\cos\theta\cos\gamma}\sin\gamma d\gamma\right)\nu^2 d\nu$$

Denote

$$u = \cos\gamma,$$

Then,

$$du = -\sin\gamma d\gamma$$

$$\gamma = 0 \Rightarrow u = 1 \ ,$$

$$\gamma = \pi \Rightarrow u = -1$$

and the γ summation becomes

$$2\pi \int_{u=-1}^{u=1} J_0([2\pi\nu r \sin\theta]\sqrt{1-u^2}) e^{[2\pi i \nu r \cos\theta]u} du =$$

$$= 8\pi \int_{u=0}^{u=1} J_0([2\pi\nu r \sin\theta]\sqrt{1-u^2}) \cos([2\pi\nu r \cos\theta]u) du \,.$$

Denote

$$u = \sqrt{1-v^2} \,.$$

Then,

$$\begin{aligned} du &= -\frac{v}{\sqrt{1-v^2}} dv \\ u = 0 &\Rightarrow v = 1 \quad , \\ u = 1 &\Rightarrow v = 0 \end{aligned}$$

and the u summation becomes

$$= 8\pi \int_{v=0}^{v=1} J_0([2\pi\nu r \sin\theta]v) \cos([2\pi\nu r \cos\theta]\sqrt{1-v^2}) \frac{v}{\sqrt{1-v^2}} dv \,.$$

Denoting,

$$\begin{aligned} b &= 2\pi\nu r \cos\theta \\ c &= 2\pi\nu r \sin\theta \text{'} \end{aligned}$$

$$= 8\pi \int_{v=0}^{v=1} J_0(cv) \cos(b\sqrt{1-v^2}) \frac{v}{\sqrt{1-v^2}} dv \,.$$

By [Prudnikov, Vol.2, p.201, 2.12.21 #5], the γ summation equals

$$= 8\pi\sqrt{\frac{\pi}{2}} \left(b^2 + c^2\right)^{-\frac{1}{4}} J_{\frac{1}{2}}\left(\sqrt{b^2 + c^2}\right)$$

$$= 8\pi\sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{2\pi\nu r}}J_{\frac{1}{2}}\left(2\pi\nu r\right)$$

$$= 4\pi\frac{1}{\sqrt{\nu r}}J_{\frac{1}{2}}\left(2\pi\nu r\right)$$

Since $J_{\frac{1}{2}}\left(x\right) = \sqrt{\frac{2}{\pi x}}\sin x$, the γ summation equals

$$= 4\pi\frac{1}{\sqrt{\nu r}}\sqrt{\frac{2}{\pi 2\pi\nu r}}\sin\left(2\pi\nu r\right)$$

$$= 4\frac{1}{\nu r}\sin\left(2\pi\nu r\right)$$

Therefore,

$$\delta(x)\delta(y)\delta(z) = 4\int_{\nu=0}^{\nu=\infty}\frac{\sin\left(2\pi\nu r\right)}{r}\nu d\nu\,.\square$$

Thus, $\delta(x)\delta(y)\delta(z)$ is a Radially Symmetric Delta Function that we will denote $\delta_{\text{Radial}}(r)$.

6.2 The Radial Delta $\delta_{\text{Radial}}(r)$ Definition

$$\delta_{\text{Radial}}(r) \equiv 4\int_{\nu=0}^{\nu=\infty}\frac{\sin\left(2\pi\nu r\right)}{r}\nu d\nu$$

7.

Radial Delta $\delta_{\text{Radial}}(r-\sigma)$

By 6.2, we have,

7.1 $$\delta_{\text{Radial}}(r-\sigma) \equiv 4\int_{\nu=0}^{\nu=\infty} \frac{\sin\left(2\pi\nu[r-\sigma]\right)}{r-\sigma}\nu d\nu$$

Alternatively, we may derive similarly to 6.1,

7.2 $$\delta(x-\xi)\delta(y-\eta)\delta(z-\zeta) = 4^2\frac{1}{r\sigma}\int_{\nu=0}^{\nu=\infty}\sin(2\pi\nu r)\sin(2\pi\nu\sigma)d\nu$$

$$= 4^2\frac{1}{r^2}\int_{\nu=0}^{\nu=\infty}\sin(2\pi\nu r)\sin(2\pi\nu\sigma)d\nu$$

Proof: By 4.2,

$\delta(x-\xi)\delta(y-\eta)\delta(z-\zeta) =$

$$= \left(\int_{\nu_x=-\infty}^{\nu_x=\infty} e^{2\pi i\nu_x(x-\xi)}d\nu_x\right)\left(\int_{\nu_y=-\infty}^{\nu_y=\infty} e^{2\pi i\nu_y(y-\eta)}d\nu_y\right)\left(\int_{\nu_z=-\infty}^{\nu_z=\infty} e^{2\pi i\nu_z(z-\zeta)}d\nu_z\right)$$

Substitute

$$\begin{aligned} x &= r\sin\theta\cos\phi & \xi &= \sigma\sin\lambda\cos\mu & \nu_x &= \nu\sin\gamma\cos\beta \\ y &= r\sin\theta\sin\phi & \eta &= \sigma\sin\lambda\sin\mu & \nu_y &= \nu\sin\gamma\sin\beta \\ z &= r\cos\theta & \zeta &= \sigma\cos\lambda & \nu_z &= \nu\cos\gamma \end{aligned}$$

Then,

$$\nu_x x + \nu_y y + \nu_z z =$$

$$= \nu r\left[\sin\theta\sin\gamma\left\{\cos\phi\cos\beta + \sin\phi\sin\beta\right\} + \cos\theta\cos\gamma\right]$$

$$= \nu r\left[\sin\theta\sin\gamma\cos(\beta-\phi) + \cos\theta\cos\gamma\right]$$

$$\nu_x \xi + \nu_y \eta + \nu_z \zeta =$$

$$= \nu\sigma\left[\sin\lambda\sin\gamma\left\{\cos\mu\cos\beta + \sin\mu\sin\beta\right\} + \cos\lambda\cos\gamma\right]$$

$$= \nu\sigma\left[\sin\lambda\sin\gamma\cos(\beta-\mu) + \cos\lambda\cos\gamma\right]$$

Integrating with respect to ν, γ, and β,

$$= \int_{\nu=0}^{\nu=\infty}\left(\int_{\gamma=0}^{\gamma=\pi}\left(\int_{\beta=0}^{\beta=2\pi} e^{2\pi i\nu r\sin\theta\sin\gamma\cos(\beta-\phi)}e^{-2\pi i\nu\sigma\sin\lambda\sin\gamma\cos(\beta-\mu)}d\beta\right)\times\right.$$

$$\left.\times e^{2\pi i\nu r\cos\theta\cos\gamma}e^{-2\pi i\nu\sigma\cos\lambda\cos\gamma}\sin\gamma d\gamma\right)\nu^2 d\nu\,.$$

Denoting

$$A = \beta - \phi, \qquad B = \beta - \mu,$$

$$\int_{\beta=0}^{\beta=2\pi} e^{2\pi i\nu r\sin\theta\sin\gamma\cos(\beta-\phi)}e^{-2\pi i\nu\sigma\sin\lambda\sin\gamma\cos(\beta-\mu)}d\beta =$$

$$= \int_{A=-\phi}^{A=2\pi-\phi} e^{2\pi i\nu r\sin\theta\sin\gamma\cos A}dA \int_{B=-\mu}^{B=2\pi-\mu} e^{-2\pi i\nu\sigma\sin\lambda\sin\gamma\cos B}dB$$

Since $e^{2\pi i\nu r\sin\theta\sin\gamma\cos A}$, and $e^{-2\pi i\nu\sigma\sin\lambda\sin\gamma\cos B}$ are periodic with period 2π,

$$= \int_{A=0}^{A=2\pi} e^{2\pi i\nu r \sin\theta \sin\gamma \cos A} dA \int_{B=0}^{B=2\pi} e^{-2\pi i\nu\sigma \sin\lambda \sin\gamma \cos B} dB$$

Integrating,

$$\int_{A=0}^{A=2\pi} e^{2\pi i\nu r \sin\theta \sin\gamma \cos A} dA =$$

$$= \int_{A=0}^{A=2\pi} \left(1 + \frac{2\pi i\nu r \sin\theta \sin\gamma \cos A}{1} + \frac{(2\pi i\nu r \sin\theta \sin\gamma \cos A)^2}{2!} + \ldots \right) dA$$

The integrals of the odd powers vanish, and we have

$$= 2\pi - \frac{(2\pi\nu r \sin\theta \sin\gamma)^2}{2^2} 2\pi + \frac{(2\pi\nu r \sin\theta \sin\gamma)^4}{2^2 \cdot 4^2} 2\pi - \frac{(2\pi\nu r \sin\theta \sin\gamma)^6}{2^2 \cdot 4^2 \cdot 6^2} 2\pi + \ldots$$

$$= 2\pi J_0(2\pi\nu r \sin\theta \sin\gamma).$$

Similarly,

$$\int_{B=0}^{B=2\pi} e^{-2\pi i\nu\sigma \sin\lambda \sin\gamma \cos B} dB =$$

$$= \int_{B=0}^{B=2\pi} \left(1 + \frac{2\pi i\nu\sigma \sin\lambda \sin\gamma \cos B}{1} + \frac{(2\pi i\nu r \sin\lambda \sin\gamma \cos B)^2}{2!} + \ldots \right) dB$$

The integrals of the odd powers vanish, and we have

$$= 2\pi - \frac{(2\pi\nu\sigma \sin\lambda \sin\gamma)^2}{2^2} 2\pi + \frac{(2\pi\nu\sigma \sin\lambda \sin\gamma)^4}{2^2 \cdot 4^2} 2\pi - \frac{(2\pi\nu\sigma \sin\lambda \sin\gamma)^6}{2^2 \cdot 4^2 \cdot 6^2} 2\pi + \ldots$$

$$= 2\pi J_0(2\pi\nu\sigma \sin\lambda \sin\gamma).$$

Therefore, the γ summation is

$$\int_{\gamma=0}^{\gamma=\pi} 2\pi J_0(2\pi\nu r\sin\theta\sin\gamma)e^{2\pi i\nu r\cos\theta\cos\gamma}\times$$

$$\times 2\pi J_0(2\pi\nu\sigma\sin\lambda\sin\gamma)e^{-2\pi i\nu\sigma\cos\lambda\cos\gamma}\sin\gamma d\gamma =$$

Denoting,

$$\begin{aligned} b_1 &= 2\pi\nu r\cos\theta \qquad & b_2 &= 2\pi\nu\sigma\cos\lambda \\ c_1 &= 2\pi\nu r\sin\theta\text{'} \qquad & c_2 &= 2\pi\nu\sigma\sin\lambda\text{'} \end{aligned}$$

$$= \int_{\gamma=0}^{\gamma=\pi} 2\pi J_0(c_1\sin\gamma)e^{b_1\cos\gamma}2\pi J_0(c_2\sin\gamma)e^{-b_2\cos\gamma}\sin\gamma d\gamma$$

Denote

$$u = \cos\gamma,$$

Then,

$$du = -\sin\gamma d\gamma$$

$$\gamma = 0 \Rightarrow u = 1 \ ,$$

$$\gamma = \pi \Rightarrow u = -1$$

and the γ summation becomes

$$= \int_{u=-1}^{u=1} 2\pi J_0(c_1\sqrt{1-u^2})e^{b_1u}2\pi J_0(c_2\sqrt{1-u^2})e^{-b_2u}du$$

$$= \int_{u=-1}^{u=1} 2\pi J_0(c_1\sqrt{1-u^2})e^{b_1u}du\int_{u=-1}^{u=1} 2\pi J_0(c_2\sqrt{1-u^2})e^{-b_2u}du$$

The first integration is

$$\int_{u=-1}^{u=1} 2\pi J_0(c_1\sqrt{1-u^2})e^{b_1 u}du = 8\pi\int_{u=0}^{u=1} J_0(c_1\sqrt{1-u^2})\cos(b_1 u)du$$

Denote

$$u = \sqrt{1-v^2}\,.$$

Then,

$$du = -\frac{v}{\sqrt{1-v^2}}dv$$
$$u = 0 \Rightarrow v = 1 \quad ,$$
$$u = 1 \Rightarrow v = 0$$

and the u summation becomes

$$= 8\pi\int_{v=0}^{v=1} J_0(c_1 v)\cos(b_1\sqrt{1-v^2})\frac{v}{\sqrt{1-v^2}}dv\,.$$

By [Prudnikov, Vol.2, p.201, 2.12.21 #5], the v summation equals

$$= 8\pi\sqrt{\frac{\pi}{2}}\left(b_1{}^2 + c_1{}^2\right)^{-\frac{1}{4}} J_{\frac{1}{2}}\left(\sqrt{b_1{}^2 + c_1{}^2}\right)$$

Recalling $\begin{aligned} b_1 &= 2\pi\nu r\cos\theta \\ c_1 &= 2\pi\nu r\sin\theta \end{aligned}$,

$$= 8\pi\sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{2\pi\nu r}} J_{\frac{1}{2}}\left(2\pi\nu r\right)$$

$$= 4\pi\frac{1}{\sqrt{\nu r}} J_{\frac{1}{2}}\left(2\pi\nu r\right)$$

Since $J_{\frac{1}{2}}\left(x\right) = \sqrt{\frac{2}{\pi x}}\sin x$, the v summation equals

$$= 4\pi \frac{1}{\sqrt{\nu r}} \sqrt{\frac{2}{\pi 2\pi \nu r}} \sin\left(2\pi\nu r\right)$$

$$= 4 \frac{1}{\nu r} \sin\left(2\pi\nu r\right).$$

That is,

$$\int_{u=-1}^{u=1} 2\pi J_0(c_1\sqrt{1-u^2})e^{b_1 u}du = 4\frac{1}{\nu r}\sin\left(2\pi\nu r\right)$$

Similarly,

$$\int_{u=-1}^{u=1} 2\pi J_0(c_2\sqrt{1-u^2})e^{b_2 u}du = 4\frac{1}{\nu\sigma}\sin\left(2\pi\nu\sigma\right)$$

Therefore, the γ summation is

$$4\frac{1}{\nu r}\sin\left(2\pi\nu r\right)4\frac{1}{\nu\sigma}\sin\left(2\pi\nu\sigma\right)$$

and the ν summation is

$$\int_{\nu=0}^{\nu=\infty} 4\frac{1}{\nu r}\sin\left(2\pi\nu r\right)4\frac{1}{\nu\sigma}\sin\left(2\pi\nu\sigma\right)\nu^2 d\nu = 4^2\frac{1}{r\sigma}\int_{\nu=0}^{\nu=\infty}\sin\left(2\pi\nu\sigma\right)\sin\left(2\pi\nu r\right)d\nu$$

Therefore,

$$\delta(x-\xi)\delta(y-\eta)\delta(z-\zeta) = 4^2\frac{1}{r\sigma}\int_{\nu=0}^{\nu=\infty}\sin(2\pi\nu r)\sin(2\pi\nu\sigma)d\nu\,.\square$$

Thus, $\delta(x-\xi)\delta(y-\eta)\delta(z-\zeta)$ is a Radially Symmetric Delta function that we will denote $\delta_{\text{Radial}}(r-\sigma)$

7.3 The Radial Delta $\delta_{\text{Radial}}(r-\sigma)$ Definition

$$\delta_{\text{Radial}}(r-\sigma) \equiv 4^2 \frac{1}{r\sigma} \int_{\nu=0}^{\nu=\infty} \sin\left(2\pi\nu\sigma\right) \sin\left(2\pi\nu r\right) d\nu$$

spikes at $r=\sigma$*, and vanishes for* $r \neq \sigma$*.*

From 7.1, and 7.3,

7.4 $$\int_{\nu=0}^{\nu=\infty} \frac{\sin\left(2\pi\nu[r-\sigma]\right)}{r-\sigma} \nu d\nu = 4 \int_{\nu=0}^{\nu=\infty} \frac{\sin\left(2\pi\nu\sigma\right)\sin\left(2\pi\nu r\right)}{r\sigma} d\nu$$

<u>Common Tables do not have 7.4.</u>

8.

$\delta_{\text{Radial}}(r - r_0)$ and $\delta(r - r_0)$

8.1 $$\delta_{\text{Radial}}(r - r_0) = \frac{1}{r_0^2}\delta(r - r_0)$$

Proof: Denote
$$\begin{aligned} x &= r\sin\theta\cos\phi & x_0 &= r_0\sin\theta_0\cos\phi_0 \\ y &= r\sin\theta\sin\phi\,, & y_0 &= r_0\sin\theta_0\sin\phi_0\,, \\ z &= r\cos\theta & z_0 &= r_0\cos\theta_0 \end{aligned}$$

Then,

$$\delta(x - x_0)\delta(y - y_0)\delta(z - z_0)dxdydz = \delta(r - r_0)(dr)(\sin\theta d\theta)d\phi$$

$$\delta(x - x_0)\delta(y - y_0)\delta(z - z_0)\left|\frac{\partial(x, y, z)}{\partial(r, -\cos\theta, \phi)}\right| = \delta(r - r_0)$$

Since $\left|\dfrac{\partial(x, y, z)}{\partial(r, -\cos\theta, \phi)}\right| = r^2$,

$$\underbrace{\delta(x - x_0)\delta(y - y_0)\delta(z - z_0)}_{\delta_{\text{Radial}}(r - r_0)} = \frac{1}{r^2}\delta(r - r_0)$$

$$\delta_{\text{Radial}}(r - r_0) = \frac{1}{r_0^2}\delta(r - r_0).\square$$

8.2 Sifting by $\delta_{\text{Radial}}(r - \sigma)$

$$\int_{\sigma=0}^{\sigma=\infty} \delta_{\text{Radial}}(r-\sigma)\sigma^2 d\sigma = 1$$

Proof: By 8.1, $\int_{\sigma=0}^{\sigma=\infty} \delta_{\text{Radial}}(r-\sigma)\sigma^2 d\sigma = \int_{\sigma=0}^{\sigma=\infty} \delta(r-\sigma)d\sigma = 1.\square$

8.3 $$\delta(r-\sigma) = 4^2 \int_{\nu=0}^{\nu=\infty} \sin(2\pi\nu r)\sin(2\pi\nu\sigma)d\nu$$

$$= 4 \int_{\nu=0}^{\nu=\infty} \frac{\sin\left(2\pi\nu[r-\sigma]\right)}{r-\sigma}\nu d\nu$$

Proof: By 8.1,

$$\delta(r-\sigma) = \sigma^2 \delta_{\text{Radial}}(r-\sigma)$$

$$= \sigma^2 4^2 \int_{\nu=0}^{\nu=\infty} \frac{\sin(2\pi\nu r)\sin(2\pi\nu\sigma)}{r\sigma} d\nu$$

For $r \neq \sigma$, the integral vanishes. For $r = \sigma$, we have

$$= \sigma^2 4^2 \int_{\nu=0}^{\nu=\infty} \frac{\sin(2\pi\nu r)\sin(2\pi\nu\sigma)}{\sigma^2} d\nu$$

$$= 4^2 \int_{\nu=0}^{\nu=\infty} \sin(2\pi\nu r)\sin(2\pi\nu\sigma)d\nu$$

By 7.3,

$$= 4 \int_{\nu=0}^{\nu=\infty} \frac{\sin\left(2\pi\nu[r-\sigma]\right)}{r-\sigma}\nu d\nu .\square$$

<u>Common Tables do not have 8.3</u>.

9.

3-D Fourier-Bessel Transform

The 3-dimensional Fourier-Bessel Transform is the 3-dimensional Fourier Transform applied to a Radially symmetric function $f(r)$. Then, integration of the Exponential Function with respect to the azimuth angle, yields a Bessel Function.

9.1 The 3-dimensional Fourier-Bessel Transform

$$\mathcal{F}_{Bess}\left\{f(r)\right\} = 4\int_{r=0}^{r=\infty} f(r)\frac{\sin\left(2\pi\nu r\right)}{\nu} rdr$$

Proof: The 3-dimensional Fourier Transform of a radially symmetric function $f(x,y,z) = f(r)$ is

$$\mathcal{F}_{Bess}\left\{f(r)\right\} = \int_{z=-\infty}^{z=\infty}\int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} f(r)e^{-2\pi i\nu_x x-2\pi i\nu_y y-2\pi i\nu_z z}dxdydz$$

Substitute

$$\begin{aligned} x &= r\sin\theta\cos\phi & \nu_x &= \nu\sin\gamma\cos\beta \\ y &= r\sin\theta\sin\phi & \nu_y &= \nu\sin\gamma\sin\beta \\ z &= r\cos\theta & \nu_z &= \nu\cos\gamma \end{aligned}$$

Then,

$$\nu_x x + \nu_y y + \nu_z z =$$

$$= \nu r\left[\sin\theta\sin\gamma\left\{\cos\phi\cos\beta + \sin\phi\sin\beta\right\} + \cos\theta\cos\gamma\right]$$

$$= \nu r\left[\sin\theta\sin\gamma\cos(\beta-\phi) + \cos\theta\cos\gamma\right]$$

Integrating with respect to r, θ, and ϕ,

$$\mathcal{F}_{Bess}\left\{f(r)\right\} = \int_{r=0}^{r=\infty} f(r)\left(\int_{\theta=0}^{\theta=\pi}\left[\int_{\phi=0}^{\phi=2\pi} e^{-2\pi i\nu r\sin\theta\sin\gamma\cos(\phi-\beta)}d\phi\right]e^{-2\pi i\nu r\cos\theta\cos\gamma}\sin\theta d\theta\right)r^2 dr$$

Denoting

$$\alpha = \phi - \beta,$$

$$\int_{\phi=0}^{\phi=2\pi} e^{-2\pi i\nu r\sin\theta\sin\gamma\cos(\phi-\beta)}d\phi = \int_{\alpha=-\beta}^{\alpha=2\pi-\beta} e^{-2\pi i\nu r\sin\theta\sin\gamma\cos\alpha}d\alpha$$

Since $e^{-2\pi i\nu r\sin\theta\sin\gamma\cos\alpha}$ is periodic in α with period 2π,

$$= \int_{\alpha=0}^{\alpha=2\pi} e^{-2\pi i\nu r\sin\theta\sin\gamma\cos\alpha}d\alpha$$

$$= \int_{\alpha=0}^{\alpha=2\pi}\left(1 - \frac{2\pi i\nu r\,\mathrm{in}\,\theta\sin\gamma\cos\alpha}{1} + \frac{(2\pi i\nu r\sin\theta\sin\gamma\cos\alpha)^2}{2!} - ...\right)d\alpha.$$

The integrals of the odd powers vanish, and we have

$$= 2\pi - \frac{(2\pi\nu r\sin\theta\sin\gamma)^2}{2^2}2\pi + \frac{(2\pi\nu r\sin\theta\sin\gamma)^4}{2^2\cdot 4^2}2\pi - \frac{(2\pi\nu r\sin\theta\sin\gamma)^6}{2^2\cdot 4^2\cdot 6^2}2\pi + ...$$

$$= 2\pi J_0(2\pi\nu r\sin\theta\sin\gamma).$$

Therefore,

$$\mathcal{F}_{Bess}\left\{f(r)\right\} = \int_{r=0}^{r=\infty} f(r)\left(2\pi\int_{\theta=0}^{\theta=\pi} J_0(2\pi\nu r\sin\theta\sin\gamma)e^{-2\pi i\nu r\cos\theta\cos\gamma}\sin\theta d\theta\right)r^2 dr$$

Denoting,

$$\begin{aligned} b &= 2\pi\nu r\cos\gamma \\ c &= 2\pi\nu r\sin\gamma \end{aligned},$$

the θ summation becomes

$$2\pi\int_{\theta=0}^{\theta=\pi} J_0(c\sin\theta)e^{-b\cos\theta}\sin\theta d\theta\,.$$

Denote

$$u = \cos\theta\,.$$

Then,

$$\begin{aligned} du &= -\sin\theta d\theta \\ \theta = 0 &\Rightarrow u = 1 \\ \theta = \pi &\Rightarrow u = -1 \end{aligned},$$

and the θ summation becomes

$$2\pi\int_{u=-1}^{u=1} J_0(c\sqrt{1-u^2})e^{bu}du = 8\pi\int_{u=0}^{u=1} J_0(c\sqrt{1-u^2})\cos(bu)du$$

Denote

$$u = \sqrt{1-v^2}\,.$$

Then,

$$du = -\frac{v}{\sqrt{1-v^2}}dv$$
$$u = 0 \Rightarrow v = 1 \quad ,$$
$$u = 1 \Rightarrow v = 0$$

and the u summation becomes

$$= 8\pi \int_{v=0}^{v=1} J_0(cv)\cos(b\sqrt{1-v^2})\frac{v}{\sqrt{1-v^2}}dv$$

By [Prudnikov, Vol.2, p.201, 2.12.21 #5], this equals

$$= 8\pi\sqrt{\frac{\pi}{2}}\left(b^2+c^2\right)^{-\frac{1}{4}} J_{\frac{1}{2}}\left(\sqrt{b^2+c^2}\right)$$

Recalling $\begin{matrix} b = 2\pi\nu r\cos\gamma \\ c = 2\pi\nu r\sin\gamma \end{matrix}$,

$$= 8\pi\sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{2\pi\nu r}}J_{\frac{1}{2}}\left(2\pi\nu r\right)$$

$$= 4\pi\frac{1}{\sqrt{\nu r}}J_{\frac{1}{2}}\left(2\pi\nu r\right)$$

Since $J_{\frac{1}{2}}\left(x\right) = \sqrt{\frac{2}{\pi x}}\sin x$, the v summation equals

$$= 4\pi\frac{1}{\sqrt{\nu r}}\sqrt{\frac{2}{\pi 2\pi\nu r}}\sin\left(2\pi\nu r\right)$$

$$= 4\frac{1}{\nu r}\sin\left(2\pi\nu r\right)$$

Therefore,

$$\mathcal{F}_{Bess}\left\{f(r)\right\} = \int_{r=0}^{r=\infty} f(r)4\frac{1}{\nu r}\sin\left(2\pi\nu r\right)r^2dr$$

$$= 4\int_{r=0}^{r=\infty} f(r)\frac{\sin\left(2\pi\nu r\right)}{\nu}rdr\,.\square$$

9.2 3-D Inverse Fourier-Bessel Transform

$$\mathcal{F}_{Bess}^{-1}F(2\pi\nu) = 4\int_{\nu=0}^{\nu=\infty} F(2\pi\nu)\frac{\sin\left(2\pi\nu r\right)}{r}\nu d\nu$$

Proof: The 3-dimensional Inverse Fourier-Bessel Transform of a radially symmetric function $F(\nu_x,\nu_y,\nu_z) = F(2\pi\nu)$ is

$$\mathcal{F}_{Bess}^{-1}\left\{F(2\pi\nu)\right\} = \int_{\nu_z=-\infty}^{\nu_z=\infty}\int_{\nu_y=-\infty}^{\nu_y=\infty}\int_{\nu_x=-\infty}^{\nu_x=\infty} F(2\pi\nu)e^{2\pi i[\nu_x x+\nu_y y+\nu_z z]}d\nu_x d\nu_y d\nu_z$$

Substitute

$$\begin{aligned} x &= r\sin\theta\cos\phi & \nu_x &= \nu\sin\gamma\cos\beta \\ y &= r\sin\theta\sin\phi & \nu_y &= \nu\sin\gamma\sin\beta \\ z &= r\cos\theta & \nu_z &= \nu\cos\gamma \end{aligned}$$

Then,

$$\nu_x x + \nu_y y + \nu_z z =$$

$$= \nu r\left[\sin\theta\sin\gamma\left\{\cos\phi\cos\beta + \sin\phi\sin\beta\right\} + \cos\theta\cos\gamma\right]$$

$$= \nu r\left[\sin\theta\sin\gamma\cos(\beta-\phi) + \cos\theta\cos\gamma\right]$$

Integrating with respect to ν, γ, and β,

$$\mathcal{F}_{Bess}^{-1}\left\{F(2\pi\nu)\right\} = \int_{\nu=0}^{\nu=\infty} F(2\pi\nu)\left(\int_{\gamma=0}^{\gamma=\pi}\left[\int_{\beta=0}^{\beta=2\pi} e^{2\pi i\nu r\sin\theta\sin\gamma\cos(\beta-\phi)}d\beta\right]e^{2\pi i\nu r\cos\theta\cos\gamma}\sin\gamma d\gamma\right)\nu^2 d\nu$$

Denoting

$$\alpha = \beta - \phi,$$

$$\int_{\beta=0}^{\beta=2\pi} e^{2\pi i\nu r\sin\theta\sin\gamma\cos(\beta-\phi)}d\beta = \int_{\alpha=-\phi}^{\alpha=2\pi-\phi} e^{2\pi i\nu r\sin\theta\sin\gamma\cos\alpha}d\alpha$$

Since $e^{2\pi i\nu r\sin\theta\sin\gamma\cos\alpha}$ is periodic in α with period 2π,

$$= \int_{\alpha=0}^{\alpha=2\pi} e^{2\pi i\nu r\sin\theta\sin\gamma\cos\alpha}d\alpha$$

$$= \int_{\alpha=0}^{\alpha=2\pi}\left(1 + \frac{2\pi i\nu r\sin\theta\sin\gamma\cos\alpha}{1} + \frac{(2\pi i\nu r\sin\theta\sin\gamma\cos\alpha)^2}{2!} + ...\right)d\alpha.$$

The integrals of the odd powers vanish, and we have

$$= 2\pi - \frac{(2\pi\nu r\sin\theta\sin\gamma)^2}{2^2}2\pi + \frac{(2\pi\nu r\sin\theta\sin\gamma)^4}{2^2\cdot 4^2}2\pi - \frac{(2\pi\nu r\sin\theta\sin\gamma)^6}{2^2\cdot 4^2\cdot 6^2}2\pi + ...$$

$$= 2\pi J_0(2\pi\nu r\sin\theta\sin\gamma).$$

Therefore,

$$\mathcal{F}_{Bess}^{-1}F(2\pi\nu) = \int_{\nu=0}^{\nu=\infty} F(2\pi\nu)\left(\int_{\gamma=0}^{\gamma=\pi} 2\pi J_0(2\pi\nu r\sin\theta\sin\gamma)e^{2\pi i\nu r\cos\theta\cos\gamma}\sin\gamma d\gamma\right)\nu^2 d\nu$$

Denoting,

$$b = 2\pi\nu r\cos\theta$$
$$c = 2\pi\nu r\sin\theta\text{'}$$

the γ summation is

$$\int_{\gamma=0}^{\gamma=\pi} 2\pi J_0(c\sin\gamma)e^{b\cos\gamma}\sin\gamma d\gamma\,.$$

Denote

$$u = \cos\gamma$$

Then,

$$du = -\sin\gamma d\gamma$$

$$\gamma = 0 \Rightarrow u = 1\ ,$$

$$\gamma = \pi \Rightarrow u = -1$$

and the γ summation becomes

$$2\pi\int_{u=-1}^{u=1} J_0(c\sqrt{1-u^2})e^{bu}du = 8\pi\int_{u=0}^{u=1} J_0(c\sqrt{1-u^2})\cos(bu)du$$

Denote

$$u = \sqrt{1-v^2}\,.$$

Then,

$$du = -\frac{v}{\sqrt{1-v^2}}dv$$

$$u = 0 \Rightarrow v = 1\ ,$$

$$u = 1 \Rightarrow v = 0$$

and the u summation becomes

$$= 8\pi\int_{v=0}^{v=1} J_0(cv)\cos(b\sqrt{1-v^2})\frac{v}{\sqrt{1-v^2}}dv$$

By [Prudnikov, Vol.2, p.201, 2.12.21 #5], this equals

$$= 8\pi\sqrt{\frac{\pi}{2}}\left(b^2+c^2\right)^{-\frac{1}{4}}J_{\frac{1}{2}}\left(\sqrt{b^2+c^2}\right)$$

$$= 8\pi\sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{2\pi\nu r}}J_{\frac{1}{2}}\left(2\pi\nu r\right)$$

$$= 4\pi\frac{1}{\sqrt{\nu r}}J_{\frac{1}{2}}\left(2\pi\nu r\right)$$

Since $J_{\frac{1}{2}}\left(x\right) = \sqrt{\frac{2}{\pi x}}\sin x$,

$$= 4\pi\frac{1}{\sqrt{\nu r}}\sqrt{\frac{2}{\pi 2\pi\nu r}}\sin\left(2\pi\nu r\right)$$

$$= 4\frac{1}{\nu r}\sin\left(2\pi\nu r\right)$$

Therefore,

$$\mathcal{F}_{Bess}^{-1}F(2\pi\nu) = 4\int_{\nu=0}^{\nu=\infty} F(2\pi\nu)\frac{\sin\left(2\pi\nu r\right)}{r}\nu d\nu\,.\square$$

10.

Fourier-Bessel Integral Theorem Holds only in Infinitesimal Calculus

The 3-dimensional Fourier-Bessel Integral Theorem guarantees that the 3-dimensional Fourier-Bessel Transform and its Inverse are well defined operations, so that inversion yields the original function that generated the Transform.

That is,

$$f(r) = 4\int_{\nu=0}^{\nu=\infty}\left(4\int_{\sigma=0}^{\sigma=\infty} f(\sigma)\frac{\sin\left(2\pi\nu\sigma\right)}{\nu}\sigma d\sigma\right)\frac{\sin\left(2\pi\nu r\right)}{r}\nu d\nu$$

But in the Calculus of Limits, this integral is singular, and the Fourier-Bessel Integral Theorem does not hold in the Calculus of Limits under any conditions.

10.1 The 3-D Fourier-Bessel Integral Theorem Fails in the Calculus of Limits

Proof: By the 3-dimensional Fourier-Bessel Integral Theorem

$$f(r) = 4\int_{\nu=0}^{\nu=\infty}\left(4\int_{\sigma=0}^{\sigma=\infty} f(\sigma)\frac{\sin\left(2\pi\nu\sigma\right)}{\nu\sigma}\sigma^2 d\sigma\right)\frac{\sin\left(2\pi\nu r\right)}{\nu r}\nu^2 d\nu$$

$$= \int_{\sigma=0}^{\sigma=\infty} f(\sigma)\left(4^2\frac{1}{r\sigma}\int_{\nu=0}^{\nu=\infty}\sin\left(2\pi\nu\sigma\right)\sin\left(2\pi\nu r\right)d\nu\right)\sigma^2 d\sigma$$

However, at $\sigma = r$,

$$\int_{\nu=0}^{\nu=\infty}\sin\left(2\pi\nu\sigma\right)\sin\left(2\pi\nu r\right)d\nu = \int_{\nu=0}^{\nu=\infty}\sin^2\left(2\pi\nu\sigma\right)d\nu$$

$$= \frac{1}{2}\int_{\nu=0}^{\nu=\infty}\left\{1-\cos\left(4\pi\nu\sigma\right)\right\}d\nu$$

$$= \left[\nu - \frac{\sin(4\pi\nu\sigma)}{4\pi\sigma}\right]_{\nu=0}^{\nu=\infty} = \infty .$$

Hence, the 3-dimensinal Fourier-Bessel Integral diverges, and the 3-dimensional Fourier-Bessel Integral Theorem does not hold in the Calculus of Limits.□

Avoiding the singularity at $\sigma = \rho$ does not recover the Theorem, because without the singularity the integral equals zero.

Furthermore,

10.2 Calculus of Limits Conditions are insufficient for the 3-D Fourier-Bessel Integral Theorem

Proof: According to [Watson, p.458], The following conditions guarantee the 3-dimensional Fourier-Bessel Integral Theorem in the Calculus of Limits

1. convergence of $\int\limits_{r=-\infty}^{r=\infty} \left|f(r)\right|\sqrt{r}dr$

2. Existence of $\lim\limits_{\lambda\to\infty} \int\limits_{r=0}^{r=\infty} f(r)\left(\int\limits_{\omega=0}^{\omega=\lambda} J_{\frac{1}{2}}(\omega\sigma)J_{\frac{1}{2}}(\omega r)\omega d\omega \right) rdr$

It is clear from 7.1 that Condition 2. never holds. No conditions on $f(x)$ can resolve the singularity, at $\sigma = r$, of

$$r\int\limits_{\omega=0}^{\omega=\infty} J_{\frac{1}{2}}(\omega\sigma)J_{\frac{1}{2}}(\omega r)\omega d\omega = 4^2 \int\limits_{\nu=0}^{\nu=\infty} \sin(2\pi\nu\sigma)\sin(2\pi\nu r)d\nu .$$

Therefore, the Calculus of Limits Conditions are insufficient for the 3-dimensional Fourier Integral Theorem.□

In Infinitesimal Calculus, the 3-dimensional Fourier-Bessel Integral Theorem holds for any radial Hyper-real function

10.3 3-dimensional Fourier Integral Theorem

If $f(x,y,z) = f(r)$ is Hyper-real function,

Then, the Fourier-Bessel Integral Theorem holds.

$$f(r) = 4\int\limits_{\nu=0}^{\nu=\infty}\left(4\int\limits_{\sigma=0}^{\sigma=\infty} f(\sigma)\frac{\sin\left(2\pi\nu\sigma\right)}{\nu\sigma}\sigma^2 d\sigma \right)\frac{\sin\left(2\pi\nu r\right)}{\nu r}\nu^2 d\nu$$

$$= \int_{\sigma=0}^{\sigma=\infty} f(\sigma)\left(4^2 \frac{1}{r\sigma} \int_{\nu=0}^{\nu=\infty} \sin\left(2\pi\nu\sigma\right)\sin\left(2\pi\nu r\right)d\nu\right)\sigma^2 d\sigma .$$

Proof:

$f(r) = f(x,y,z)$

In Infinitesimal Calculus,

$$= \int_{\zeta=-\infty}^{\zeta=\infty}\int_{\eta=-\infty}^{\eta=\infty}\int_{\xi=-\infty}^{\xi=\infty} f(\xi,\eta,\zeta)\delta(x-\xi)\delta(y-\eta)\delta(z-\zeta)d\xi d\eta d\zeta$$

By 7.2,

$$\delta(x-\xi, y-\eta, z-\zeta) = 4^2 \frac{1}{r\sigma}\int_{\nu=0}^{\nu=\infty} \sin\left(2\pi\nu\sigma\right)\sin\left(2\pi\nu r\right)d\nu ,$$

and

$$f(r) = \int_{\zeta=-\infty}^{\zeta=\infty}\int_{\eta=-\infty}^{\eta=\infty}\int_{\xi=-\infty}^{\xi=\infty} f(\xi,\eta,\zeta)\left(4^2 \frac{1}{r\sigma}\int_{\nu=0}^{\nu=\infty} \sin\left(2\pi\nu\sigma\right)\sin\left(2\pi\nu r\right)d\nu\right)d\xi d\eta d\zeta$$

Put

$$f(\xi,\eta,\zeta) = f(\sigma).$$

Integrating with respect to σ,

$$d\xi d\eta d\zeta = \sigma^2 d\sigma ,$$

and

$$f(r) = \int_{\sigma=0}^{\sigma=\infty} f(\sigma)\left(4^2 \frac{1}{r\sigma}\int_{\nu=0}^{\nu=\infty} \sin\left(2\pi\nu\sigma\right)\sin\left(2\pi\nu r\right)d\nu\right)\sigma^2 d\sigma$$

By changing the Summation order,

$$f(r) = 4\int_{\nu=0}^{\nu=\infty}\left(4\int_{\sigma=0}^{\sigma=\infty} f(\sigma)\frac{\sin\left(2\pi\nu\sigma\right)}{\nu\sigma}\sigma^2 d\sigma\right)\frac{\sin\left(2\pi\nu r\right)}{\nu r}\nu^2 d\nu\,.\square$$

Then, the 3-dimensional Fourier-Bessel Transform of $f(r)$,

$$4\int_{\sigma=0}^{\sigma=\infty} f(\sigma)\frac{\sin(2\pi\nu\sigma)}{\nu}\sigma d\sigma\,,$$

converges to a Hyper-real function $F(2\pi\nu)$, some of its values may be infinite Hyper-reals, like the Delta Function.

And the Inverse Fourier-Bessel Transform of $F(2\pi\nu)$

$$4\int_{\nu=0}^{\nu=\infty} F(2\pi\nu)\frac{\sin(2\pi\nu\sigma)}{\sigma}\nu d\nu$$

converges to the Hyper-real function $f(\rho)$.

10.4 *If $f(r)$ is Hyper-real function,*

Then,

- *the Hyper-real integral* $4\int_{\sigma=0}^{\sigma=\infty} f(\sigma)\frac{\sin(2\pi\nu\sigma)}{\nu}\sigma d\sigma$ *converges to* $F(2\pi\nu)$

- ***the Hyper-real integral*** $4\int_{\nu=0}^{\nu=\infty} F(2\pi\nu)\frac{\sin(2\pi\nu\sigma)}{\sigma}\nu d\nu$ ***converges to*** $f(r)$

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