Polar Delta Function, the 2-D Fourier-Bessel Transform, and the 2-D Fourier-Bessel Integral

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Abstract In [Dan6], we have shown that $\delta(x,y) = \delta(x)\delta(y)$ is the 2-dimensional Fourier Transform of the function 1.

Here, we show that its Polar form is the 2-dimensional Fourier-Bessel Transform of the function 1:

$$\frac{\delta(\rho)}{\rho} = 2\pi\delta(x)\delta(y), \qquad \rho = \sqrt{x^2 + y^2}$$

$$= (2\pi)^2 \int_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\rho)\nu d\nu,$$

is a Polarly Symmetric Delta Function.

Similarly,

$$\frac{\delta(\rho-\sigma)}{\sigma} = 2\pi\delta(x-\xi)\delta(y-\eta), \qquad \rho = \sqrt{x^2+y^2}\,; \quad \sigma = \sqrt{\xi^2+\eta^2}$$

$$= (2\pi)^3 \int_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\rho)J_0(2\pi\nu\sigma)\nu d\nu,$$

is Polarly symmetric Delta Function.

The formulas above are exclusively Hyper-real. They cannot be derived by means of the Calculus of Limits, and are unknown in the Calculus of Limits.

The Polar Delta Function $\frac{\delta(\rho - \sigma)}{\sigma}$ is a Discontinuous, Hyper-Real function, that spikes at $\rho = \sigma$,and vanishes for $\rho \neq \sigma$.

The Fourier-Bessel Integral Theorem for a Polarly symmetric function $f(\rho)$ guarantees that the 2-dimensional Fourier-Bessel Transform and its Inverse are well defined operations, so that inversion yields the original function that generated the Transform.

It is believed to hold in the Calculus of Limits under given conditions. In fact, it does not hold in the Calculus of Limits because the integration of the Fourier-Bessel Integral requires the integration of $\int\limits_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\rho)J_0(2\pi\nu\sigma)\nu d\nu$ that diverges at $\sigma = \rho$.

Only in Infinitesimal Calculus, we can integrate over singularities, and the Fourier Integral Theorem holds.

$$f(\rho) = 2\pi \int\limits_{\nu=0}^{\nu=\infty} \left(2\pi \int\limits_{\sigma=0}^{\sigma=\infty} f(\sigma)J_0(2\pi\nu\sigma)\sigma d\sigma \right) J_0(2\pi\nu\rho)\nu d\nu$$

$$= \int_{\sigma=0}^{\sigma=\infty} f(\sigma)\left((2\pi)^2 \int_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\sigma) J_0(2\pi\nu\rho)\nu d\nu \right)\sigma d\sigma$$

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Contents

Introduction

By the Fourier-Bessel Integral Theorem

$$f(\rho) = 2\pi \int_{\nu=0}^{\nu=\infty} \left(2\pi \int_{\sigma=0}^{\sigma=\infty} f(\sigma) J_0(2\pi\nu\sigma)\sigma d\sigma \right) J_0(2\pi\nu\rho)\nu d\nu$$

$$= \int_{\sigma=0}^{\sigma=\infty} f(\sigma) \left((2\pi)^2 \int_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\sigma) J_0(2\pi\nu\rho)\nu d\nu \right) \sigma d\sigma \, .$$

However, in the Calculus of Limits, at $\sigma = \rho$,

$$\int_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\sigma) J_0(2\pi\nu\rho)\nu d\nu = \infty \, .$$

Avoiding the singularity at $\sigma = \rho$ does not recover the Fourier-Bessel Integral Theorem, because without the singularity the integral equals zero.

Thus, the Fourier Integral Theorem cannot be written in the Calculus of Limits.

In Infinitesimal Calculus [Dan4], the singularity can be integrated over, and defines the Polar Delta Function

$$\frac{\delta(\rho - \sigma)}{\sigma} = (2\pi)^3 \int_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\sigma) J_0(2\pi\nu\rho)\nu d\nu \, .$$

Then, the Fourier-Bessel Integral Theorem is the sifting property of the Polar Delta Function.

$$f(\rho) = \int_{\sigma=0}^{\sigma=\infty} f(\sigma)\frac{\delta(\rho-\sigma)}{\sigma}\sigma d\sigma\,,$$

and for any Hyper-real function $f(\rho)$, the Fourier-Bessel Integral Theorem holds.

In the Calculus of Limits, the Polar Delta Function cannot be defined, and its sifting property does not apply.

1.

Hyper-real Line

Each real number α can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant Hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of infinitesimal Hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the infinite Hyper-reals.
4. The infinite Hyper-reals are greater than any real number, yet strictly smaller than infinity.
5. The infinite Hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a non-constant Hyper-real.
7. The Hyper-reals are the totality of constant Hyper-reals, a family of infinitesimals, a family of infinitesimals with

negative sign, a family of infinite Hyper-reals, a family of infinite Hyper-reals with negative sign, and non-constant Hyper-reals.

8. The Hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.
9. That line includes the real numbers separated by the non-constant Hyper-reals. Each real number is the center of an interval of Hyper-reals, that includes no other real number.
10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.
11. Zero is not an infinitesimal, because zero is not strictly greater than zero.
12. We do not add infinity to the Hyper-real line.
13. The infinitesimals, the infinitesimals with negative signs, the infinite Hyper-reals, and the infinite Hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.
14. The Hyper-real line is embedded in $\mathbb{R}^{\infty}$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the Hyper-real onto the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal Hyper-reals, or to the infinite Hyper-reals, or to the non-constant Hyper-reals.

16. No neighbourhood of a Hyper-real is homeomorphic to an $\mathbb{R}^n$ ball. Therefore, the Hyper-real line is not a manifold.

17. The Hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

2.

Hyper-real Integral

In [Dan3], we defined the integral of a Hyper-real Function.

Let $f(x)$ be a Hyper-real function on the interval $[a,b]$.

The interval may not be bounded.

$f(x)$ may take infinite Hyper-real values, and need not be bounded.

At each

$$a \le x \le b,$$

there is a rectangle with base $[x - \frac{dx}{2}, x + \frac{dx}{2}]$, height $f(x)$, and area

$$f(x)dx.$$

We form the **Integration Sum** of all the areas for the x's that start at $x = a$, and end at $x = b$,

$$\sum_{x \in [a,b]} f(x)dx.$$

If for any infinitesimal dx, the Integration Sum has the same Hyper-real value, then $f(x)$ is integrable over the interval $[a,b]$.

Then, we call the Integration Sum the integral of $f(x)$ from $x = a$, to $x = b$, and denote it by

$$\int_{x=a}^{x=b} f(x)dx\,.$$

If the Hyper-real is infinite, then it is the integral over $[a,b]$,

If the Hyper-real is finite,

$$\int_{x=a}^{x=b} f(x)dx = \text{ real part of the hyper-real}\,.\square$$

2.1 The countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers, $Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty.$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[a,b]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(x)dx$.

The Lower Integral is the Integration Sum where $f(x)$ is replaced by its lowest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

2.2
$$\sum_{x \in [a,b]} \left(\inf_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

The Upper Integral is the Integration Sum where $f(x)$ is replaced by its largest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

2.3
$$\sum_{x \in [a,b]} \left(\sup_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

If the integral is a finite Hyper-real, we have

2.4 *A Hyper-real function has a finite integral if and only if its upper integral and its lower integral are finite, and differ by an infinitesimal.*

3.

Delta Function

In [Dan5], we defined the Delta Function, and established its properties

1. The Delta Function is a Hyper-real function defined from the Hyper-real line into the set of two Hyper-reals $\left\{0, \frac{1}{dx}\right\}$. The Hyper-real 0 is the sequence $\langle 0, 0, 0, ... \rangle$. The infinite Hyper-real $\frac{1}{dx}$ depends on our choice of dx.
2. We will usually choose the family of infinitesimals that is spanned by the sequences $\left\langle \frac{1}{n} \right\rangle, \left\langle \frac{1}{n^2} \right\rangle, \left\langle \frac{1}{n^3} \right\rangle, \ldots$ It is a semigroup with respect to vector addition, and includes all the scalar multiples of the generating sequences that are non-zero. That is, the family includes infinitesimals with negative sign. Therefore, $\frac{1}{dx}$ will mean the sequence $\langle n \rangle$. Alternatively, we may choose the family spanned by the sequences $\left\langle \frac{1}{2^n} \right\rangle, \left\langle \frac{1}{3^n} \right\rangle, \left\langle \frac{1}{4^n} \right\rangle, \ldots$ Then, $\frac{1}{dx}$ will mean the

sequence $\left\langle 2^n \right\rangle$. Once we determined the basic infinitesimal dx, we will use it in the Infinite Riemann Sum that defines an Integral in Infinitesimal Calculus.

3. The Delta Function is strictly smaller than ∞

4. We define, $\delta(x) \equiv \frac{1}{dx}\chi_{[-\frac{dx}{2},\frac{dx}{2}]}(x)$,

where $\chi_{[-\frac{dx}{2},\frac{dx}{2}]}(x) = \begin{cases} 1, x \in \left[-\frac{dx}{2},\frac{dx}{2}\right] \\ 0, \text{otherwise} \end{cases}$.

5. Hence,

   - for $x < 0$, $\delta(x) = 0$
   - at $x = -\frac{dx}{2}$, $\delta(x)$ jumps from 0 to $\frac{1}{dx}$,
   - for $x \in \left[-\frac{dx}{2},\frac{dx}{2}\right]$, $\delta(x) = \frac{1}{dx}$.
   - at $x = 0$, $\delta(0) = \frac{1}{dx}$
   - at $x = \frac{dx}{2}$, $\delta(x)$ drops from $\frac{1}{dx}$ to 0.
   - for $x > 0$, $\delta(x) = 0$.
   - $x\delta(x) = 0$

6. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle \chi_{[-\frac{1}{2},\frac{1}{2}]}(x), 2\chi_{[-\frac{1}{4},\frac{1}{4}]}(x), 3\chi_{[-\frac{1}{6},\frac{1}{6}]}(x) ... \right\rangle$

7. If $dx = \left\langle \frac{2}{n} \right\rangle$, $\delta(x) = \left\langle \frac{1}{2\cosh^2 x}, \frac{2}{2\cosh^2 2x}, \frac{3}{2\cosh^2 3x}, ... \right\rangle$

8. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle e^{-x}\chi_{[0,\infty)}, 2e^{-2x}\chi_{[0,\infty)}, 3e^{-3x}\chi_{[0,\infty)}, ... \right\rangle$

9. $$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1.$$

4.

The Fourier Transform

In [Dan6], we defined the Fourier Transform and established its properties

1. $\mathcal{F}\left\{\delta(x)\right\} = 1$

2. $\delta(x) =$ *the inverse Fourier Transform of the unit function* 1

$$= \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x} d\omega$$

$$= \int_{\nu=-\infty}^{\nu=\infty} e^{2\pi i x} d\nu\,, \quad \omega = 2\pi\nu$$

3. $$\frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x} d\omega \Bigg|_{x=0} = \frac{1}{dx} = \text{ an infinite Hyper-real}$$

$$\int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x} d\omega \Bigg|_{x\neq 0} = 0$$

4. Fourier Integral Theorem

$$f(x) = \frac{1}{2\pi}\int_{k=-\infty}^{k=\infty}\left(\int_{\xi=-\infty}^{\xi=\infty} f(\xi)e^{-ik\xi}d\xi\right)e^{ikx}dk$$

does not hold in the Calculus of Limits, under any conditions.

5. Fourier Integral Theorem in Infinitesimal Calculus

If $f(x)$ *is a Hyper-real function,*

Then,

- *the Fourier Integral Theorem holds.*
- $\int_{x=-\infty}^{x=\infty} f(x)e^{-i\alpha x}dx$ ***converges to*** $F(\alpha)$
- $\frac{1}{2\pi}\int_{\alpha=-\infty}^{\alpha=\infty} F(\alpha)e^{-i\alpha x}d\alpha$ ***converges to*** $f(x)$

6. 2-Dimesional Fourier Transform

$$\mathcal{F}\left\{f(x,y)\right\} = \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} f(x,y)e^{-i\omega_x x - i\omega_y y}dxdy$$

$$= \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} f(x,y)e^{-2\pi i(\nu_x x+\nu_y y)}dxdy\,, \quad \begin{matrix}\omega_x = 2\pi\nu_x \\ \omega_y = 2\pi\nu_y\end{matrix}$$

8. 2-Dimesional Inverse Fourier Transform

$$\mathcal{F}^{-1}\left\{F(\omega_x,\omega_y)\right\} = \frac{1}{(2\pi)^2}\int_{\omega_y=-\infty}^{\omega_y=\infty}\int_{\omega_x=-\infty}^{\omega_x=\infty} F(\omega_x,\omega_y)e^{i(\omega_x x+\omega_y y)}d\omega_x d\omega_y$$

$$= \int_{\nu_y=-\infty}^{\nu_y=\infty}\int_{\nu_x=-\infty}^{\nu_x=\infty} F(2\pi\nu_x, 2\pi\nu_y)e^{2\pi i(\nu_x x+\nu_y y)}d\nu_x d\nu_y\,, \quad \begin{matrix}\omega_x = 2\pi\nu_x \\ \omega_y = 2\pi\nu_y\end{matrix}$$

9. 2-Dimesional Fourier Integral Theorem

$$f(x,y) = \frac{1}{(2\pi)^2}\int_{\omega_y=-\infty}^{\omega_y=\infty}\int_{\omega_x=-\infty}^{\omega_x=\infty}\left(\int_{\eta=-\infty}^{\eta=\infty}\int_{\xi=-\infty}^{\xi=\infty} f(\xi,\eta)e^{-i\omega_x\xi-i\omega_y\eta}d\xi d\eta\right)e^{i(\omega_x x+\omega_y y)}d\omega_x d\omega_y$$

$$= \int_{\eta=-\infty}^{\eta=\infty}\int_{\xi=-\infty}^{\xi=\infty} f(\xi,\eta)\left(\frac{1}{2\pi}\int_{\omega_x=-\infty}^{\omega_x=\infty} e^{i\omega_x(x-\xi)}d\omega_x\right)d\xi\left(\frac{1}{2\pi}\int_{\omega_y=-\infty}^{\omega_y=\infty} e^{i\omega_y(y-\eta)}d\omega_y\right)d\eta$$

$$= \int_{\eta=-\infty}^{\eta=\infty}\int_{\xi=-\infty}^{\xi=\infty} f(\xi,\eta)\left(\int_{\nu_x=-\infty}^{\nu_x=\infty} e^{2\pi i\nu_x(x-\xi)}d\nu_x\right)d\xi\left(\int_{\nu_y=-\infty}^{\nu_y=\infty} e^{2\pi i\nu_y(y-\eta)}d\nu_y\right)d\eta, \quad \begin{aligned}\omega_x &= 2\pi\nu_x\\ \omega_y &= 2\pi\nu_y\end{aligned}$$

10. 2-Dimesional Delta Function

$$\delta(x,y) = \left(\frac{1}{2\pi}\int_{\omega_x=-\infty}^{\omega_x=\infty} e^{i\omega_x x}d\omega_x\right)\left(\frac{1}{2\pi}\int_{\omega_y=-\infty}^{\omega_y=\infty} e^{i\omega_y y}d\omega_y\right)$$

$$= \left(\int_{\nu_x=-\infty}^{\nu_x=\infty} e^{2\pi i\nu_x x}d\nu_x\right)\left(\int_{\nu_y=-\infty}^{\nu_y=\infty} e^{2\pi i\nu_y y}d\nu_y\right), \quad \begin{aligned}\omega_x &= 2\pi\nu_x\\ \omega_y &= 2\pi\nu_y\end{aligned}$$

5.

Polar Delta Function $\frac{\delta(\rho)}{\rho}$

5.1 $$\delta(\rho) = \frac{1}{d\rho}\chi_{[-\frac{d\rho}{2},\frac{d\rho}{2}]}(\rho), \quad \rho \geq 0$$

5.2 $$\delta(\phi) = \frac{1}{d\phi}\chi_{[-\frac{d\phi}{2},\frac{d\phi}{2}]}(\phi), \quad 0 \leq \phi \leq 2\pi$$

5.3 $$\boxed{\delta(\rho)\delta(\phi) = \rho\delta(x)\delta(y)}$$

Proof: Transforming from Polar to Cartesian Coordinates,

$$\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \end{aligned},$$

$$\delta(\rho)\delta(\phi)d\rho d\phi = \delta(x)\delta(y)dxdy$$

$$\delta(\rho)\delta(\phi) = \delta(x)\delta(y)\left|\frac{\partial(x,y)}{\partial(\rho,\phi)}\right|$$

$$= \delta(x)\delta(y)\underbrace{\begin{vmatrix} \cos\phi & \sin\phi \\ -\rho\sin\phi & \rho\cos\phi \end{vmatrix}}_{\rho}.\square$$

5.4 $$\boxed{2\pi\delta(x)\delta(y) = \frac{\delta(\rho)}{\rho}}$$

Proof: Integrating 5.3 over ϕ,

$$\delta(\rho)\underbrace{\int_{\phi=0}^{\phi=2\pi}\delta(\phi)d\phi}_{1} = \rho\delta(x)\delta(y)\underbrace{\int_{\phi=0}^{\phi=2\pi}d\phi}_{2\pi}.\square$$

Thus,

5.5 The Polar Delta Function

$$\boxed{\frac{\delta(\rho)}{\rho} = 2\pi\delta(x)\delta(y)}$$

is a Polarly Symmetric Delta Function.

$\frac{\delta(\rho)}{\rho}$ has a Bessel Function Integral representation because

5.6 $$\delta(x)\delta(y) = \underbrace{2\pi\int_{\nu=0}^{\nu=\infty}J_0(2\pi\nu\rho)\nu d\nu}_{\text{Inverse 2-Fourier-Bessel Transform of 1}}$$

Proof: By 4.2,

$$\delta(x)\delta(y) = \left(\int_{\nu_x=-\infty}^{\nu_x=\infty}e^{2\pi i\nu_x x}d\nu_x\right)\left(\int_{\nu_y=-\infty}^{\nu_y=\infty}e^{2\pi i\nu_y y}d\nu_y\right)$$

Substitute

$$x = \rho\cos\theta \qquad \nu_x = \nu\cos\gamma$$
$$y = \rho\sin\theta \qquad \nu_y = \nu\sin\gamma$$

Then,

$$\nu_x x + \nu_y y = \nu\rho(\cos\gamma\cos\theta + \sin\gamma\sin\theta) = \nu\rho\cos(\gamma-\theta).$$

Integrating with respect to ν, and γ,

$$= \int_{\nu=0}^{\nu=\infty}\left(\int_{\gamma=0}^{\gamma=2\pi} e^{2\pi i\nu\rho\cos(\gamma-\theta)}d\gamma\right)\nu d\nu$$

Denoting

$$\alpha = \gamma - \theta,$$

$$\int_{\gamma=0}^{\gamma=2\pi} e^{2\pi i\nu\rho\cos(\gamma-\theta)}d\gamma = \int_{\alpha=-\theta}^{\alpha=2\pi-\theta} e^{2\pi i\nu\rho\cos\alpha}d\alpha$$

Since $e^{2\pi i\nu\rho\cos\alpha}$ is periodic with period 2π,

$$= \int_{\alpha=0}^{\alpha=2\pi} e^{2\pi i\nu\rho\cos\alpha}d\alpha$$

$$= \int_{\alpha=0}^{\alpha=2\pi}\left(1 + \frac{2\pi i\nu\rho\cos\alpha}{1} + \frac{(2\pi i\nu\rho\cos\alpha)^2}{2!} + \frac{(2\pi i\nu\rho\cos\alpha)^3}{3!} + \dots\right)d\alpha.$$

The integrals of the odd powers vanish, and we have

$$= 2\pi - \frac{(2\pi\nu\rho)^2}{2^2}2\pi + \frac{(2\pi\nu\rho)^4}{2^2\cdot 4^2}2\pi - \frac{(2\pi\nu\rho)^6}{2^2\cdot 4^2\cdot 6^2}2\pi + \dots$$

$$= 2\pi J_0(2\pi\nu\rho).$$

Therefore,

$$\delta(x)\delta(y) = 2\pi \int_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\rho)\nu d\nu \,.\square$$

5.7 Bessel Integral Representation of $\dfrac{\delta(\rho)}{\rho}$

$$\boxed{\begin{aligned} \frac{\delta(\rho)}{\rho} &= 2\pi\delta(x)\delta(y) \\ &= (2\pi)^2 \int_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\rho)\nu d\nu \\ &= \int_{\omega=0}^{\omega=\infty} J_0(\omega\rho)\omega d\omega \end{aligned}}$$

6.

Polar Delta Function $\dfrac{\delta(\rho-\sigma)}{\sigma}$

6.1 $$\delta(\rho-\sigma) = \frac{1}{d\rho}\chi_{[\sigma-\frac{d\rho}{2},\sigma+\frac{d\rho}{2}]}(\rho), \ \ \rho \ge 0 \text{ and } \sigma \ge 0$$

6.2 $$\delta(\phi-\theta) = \frac{1}{d\phi}\chi_{[\theta-\frac{d\phi}{2},\theta+\frac{d\phi}{2}]}(\phi), \ \ \phi \text{ and } \theta \text{ in } [0,2\pi]$$

6.3 $$\boxed{\delta(x-\xi)\delta(y-\eta) = \frac{1}{\sigma}\delta(\rho-\sigma)\delta(\phi-\theta)},$$

Proof: Transforming from Polar to Cartesian Coordinates

$$\begin{matrix} x = \rho\cos\phi \\ y = \rho\sin\phi \end{matrix}, \qquad \begin{matrix} \xi = \sigma\cos\theta \\ \eta = \sigma\sin\theta \end{matrix}, \qquad \sigma > 0,$$

$$\delta(\rho-\sigma)\delta(\phi-\theta)d\rho d\phi = \delta(x-\xi)\delta(y-\eta)dxdy,$$

$$\delta(\rho-\sigma)\delta(\phi-\theta) = \delta(x-\xi)\delta(y-\eta)\underbrace{\left|\frac{\partial(x,y)}{\partial(\rho,\phi)}\right|}_{\underbrace{\begin{vmatrix} \cos\phi & \sin\phi \\ -\rho\sin\phi & \rho\cos\phi \end{vmatrix}}_{\rho}}.$$

Both sides vanish unless $\rho = \sigma + \text{infinitesimal} \approx \sigma$.

Therefore, we can replace ρ with σ. $\square$

Integrating 6.3 over ϕ

6.4 $$\boxed{2\pi\delta(x-\xi)\delta(y-\eta) = \frac{\delta(\rho-\sigma)}{\sigma} = \frac{\delta(\rho-\sigma)}{\rho}}$$

Proof: $$\delta(x-\xi)\delta(y-\eta)\underbrace{\int_{\phi=0}^{\phi=2\pi} d\phi}_{2\pi} = \frac{1}{\rho}\delta(\rho-\sigma)\underbrace{\int_{\phi=0}^{\phi=2\pi}\delta(\phi-\theta)d\phi}_{1}$$

Due to the symmetry between ρ, and σ,

$$= \frac{1}{\sigma}\underbrace{\delta(\sigma-\rho)}_{\delta(\rho-\sigma)}.\square$$

Thus,

6.5 The Polar Delta Function

$$\boxed{\frac{\delta(\rho-\sigma)}{\sigma} = 2\pi\delta(x-\xi)\delta(y-\eta)}$$

is the Polarly Symmetric Delta Function,

$\dfrac{\delta(\rho-\sigma)}{\sigma}$ is represented as a Bessel Function Integral because

6.6 $$\boxed{\delta(x-\xi)\delta(y-\eta) = (2\pi)^2\int_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\rho)J_0(2\pi\nu\sigma)\nu d\nu}$$

Proof: By 4.2,

$$\delta(x-\xi)\delta(y-\eta) = \left(\int_{\nu_x=-\infty}^{\nu_x=\infty} e^{2\pi i\nu_x(x-\xi)}d\nu_x\right)\left(\int_{\nu_y=-\infty}^{\nu_y=\infty} e^{2\pi i\nu_y(y-\eta)}d\nu_y\right)$$

Substitute

$$x = \rho\cos\theta \quad \xi = \sigma\cos\phi \quad \nu_x = \nu\cos\gamma$$
$$y = \rho\sin\theta \quad \eta = \sigma\sin\phi \quad \nu_y = \nu\sin\gamma$$

Then,

$$\nu_x x + \nu_y y = \nu\rho(\cos\gamma\cos\theta + \sin\gamma\sin\theta) = \nu\rho\cos(\gamma-\theta)$$

$$\nu_x\xi + \nu_y\eta = \nu\sigma(\cos\gamma\cos\phi + \sin\gamma\sin\phi) = \nu\sigma\cos(\gamma-\phi)$$

Integrating with respect to ν, and γ,

$$= \int_{\nu=0}^{\nu=\infty}\left(\int_{\gamma=0}^{\gamma=2\pi} e^{2\pi i\nu\rho\cos(\gamma-\theta)}e^{-2\pi i\nu\sigma\cos(\gamma-\phi)}d\gamma\right)\nu d\nu$$

Denoting

$$\alpha = \gamma-\theta, \qquad \beta = \gamma-\phi,$$

$$\int_{\gamma=0}^{\gamma=2\pi} e^{2\pi i\nu\rho\cos(\gamma-\theta)}e^{-2\pi i\nu\sigma\cos(\gamma-\phi)}d\gamma = \int_{\alpha=-\theta}^{\alpha=2\pi-\theta} e^{2\pi i\nu\rho\cos\alpha}d\alpha \int_{\beta=-\phi}^{\beta=2\pi-\phi} e^{-2\pi i\nu\sigma\cos\beta}d\beta$$

Since $e^{2\pi i\nu\rho\cos\alpha}$, and $e^{-2\pi i\nu\sigma\cos\beta}$ are periodic with period 2π,

$$= \int_{\alpha=0}^{\alpha=2\pi} e^{2\pi i\nu\rho\cos\alpha}d\alpha \int_{\beta=0}^{\beta=2\pi} e^{-2\pi i\nu\sigma\cos\beta}d\beta$$

Integrating,

$$\int_{\alpha=0}^{\alpha=2\pi} e^{2\pi i\nu\rho\cos\alpha}d\alpha = \int_{\alpha=0}^{\alpha=2\pi}\left(1 + \frac{2\pi i\nu\rho\cos\alpha}{1} + \frac{(2\pi i\nu\rho\cos\alpha)^2}{2!} + \frac{(2\pi i\nu\rho\cos\alpha)^3}{3!} + \ldots\right)d\alpha\,.$$

The integrals of the odd powers vanish, and we have

$$= 2\pi - \frac{(2\pi\nu\rho)^2}{2^2}2\pi + \frac{(2\pi\nu\rho)^4}{2^2\cdot 4^2}2\pi - \frac{(2\pi\nu\rho)^6}{2^2\cdot 4^2\cdot 6^2}2\pi + ...$$

$$= 2\pi J_0(2\pi\nu\rho).$$

Similarly,

$$\int_{\beta=0}^{\beta=2\pi} e^{-2\pi i\nu\sigma\cos\beta}d\beta = \int_{\alpha=0}^{\alpha=2\pi}\left(1 - \frac{2\pi i\nu\sigma\cos\beta}{1} + \frac{(2\pi i\nu\sigma\cos\beta)^2}{2!} - \frac{(2\pi i\nu\sigma\cos\beta)^3}{3!} + ...\right)d\beta$$

$$= 2\pi - \frac{(2\pi\nu\sigma)^2}{2^2}2\pi + \frac{(2\pi\nu\sigma)^4}{2^2\cdot 4^2}2\pi - \frac{(2\pi\nu\sigma)^6}{2^2\cdot 4^2\cdot 6^2}2\pi + ...$$

$$= 2\pi J_0(2\pi\nu\sigma).$$

Therefore,

$$\delta(x-\xi)\delta(y-\eta) = (2\pi)^2\int_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\rho)J_0(2\pi\nu\sigma)\nu d\nu\ .\square$$

6.7 Bessel Integral Representation of $\frac{\delta(\rho-\sigma)}{\sigma}$

$$\boxed{\begin{aligned}\frac{\delta\left(\rho-\sigma\right)}{\sigma} &= (2\pi)^3\int_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\rho)J_0(2\pi\nu\sigma)\nu d\nu \\ &= 2\pi\int_{\omega=0}^{\omega=\infty} J_0(\omega\rho)J_0(\omega\sigma)\omega d\omega\end{aligned}}$$

6.8

$$\delta(\rho - \sigma) = 2\pi\sigma \int_{\omega=0}^{\omega=\infty} J_0(\omega\rho) J_0(\omega\sigma)\omega d\omega$$

$$= (2\pi)^3 \sigma \int_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\rho) J_0(2\pi\nu\sigma)\nu d\nu$$

6.9 Sifting by $\frac{\delta(\rho - \sigma)}{\sigma}$

$$\int_{\sigma=0}^{\sigma=\infty} \frac{\delta(\rho - \sigma)}{\sigma} \sigma d\sigma = 1$$

7.

Fourier-Bessel Transform of $\dfrac{\delta(\rho-\sigma)}{\sigma}$

The Fourier-Bessel Transform of the hyper-real function $f(\rho)$ of a hyper-real ρ is the Integration Sum

$$2\pi \sum_{\rho=0}^{\rho=\infty} f(\rho) J_0(\omega\rho)\rho d\rho .$$

As ρ varies, the infinitesimal projections of $2\pi f(\rho)d\rho$ on $J_0(\omega\rho)$, namely $2\pi f(\rho)J_0(\omega\rho)d\rho$, sum up to the Fourier Transform of $f(\rho)$.

7.1 Fourier-Bessel Transform of $\dfrac{\delta(\rho-\sigma)}{\sigma}$

$$\mathcal{F}_{Bessel}\{\frac{\delta(\rho-\sigma)}{\sigma}\} = 2\pi \sum_{\sigma=0}^{\sigma=\infty} \delta(\rho-\sigma) J_0(\omega\sigma) d\sigma .$$

Therefore,

7.2

$$\boxed{\mathcal{F}_{Bessel}\{\frac{\delta(\rho-\sigma)}{\sigma}\} = 2\pi J_0(\omega\rho)}$$

Proof: $$\mathcal{F}_{Bessel}\{\frac{\delta(\rho-\sigma)}{\sigma}\} = 2\pi \int_{\sigma=0}^{\sigma=\infty} \delta(\rho-\sigma) J_0(\omega\sigma) d\sigma$$

$$= 2\pi J_0(\omega\sigma)\Big|_{\sigma=\rho}$$

$$= 2\pi J_0(\omega\rho).\square$$

8.

Fourier-Bessel Integral of $\frac{\delta(\rho-\sigma)}{\sigma}$

Since the polar Delta Function is the inverse Transform of $\hat{f}_{Bessel}(\omega) = 2\pi J_0(\omega\rho)$, we have

8.1 Fourier-Bessel Integral of $\frac{\delta(\rho-\sigma)}{\sigma}$

$$\boxed{\frac{\delta(\rho-\sigma)}{\sigma} = 2\pi \int_{\omega=0}^{\omega=\infty} J_0(\omega\rho)J_0(\omega\sigma)\omega d\omega}$$

Hence,

8.2 Fourier-Bessel representation of $\delta(\rho-\sigma)$

$$\boxed{\delta(\rho-\sigma) = 2\pi\sigma \int_{\omega=0}^{\omega=\infty} J_0(\omega\rho)J_0(\omega\sigma)\omega d\omega}$$

9.

2-D Fourier-Bessel Transform

The 2-dimensional Fourier-Bessel Transform is the 2-dimensional Fourier Transform applied to a Polarly symmetric function $f(\rho)$.

Then, integration of the Exponential Function with respect to the azimuth angle, yields a Bessel Function.

9.1 The 2-D Fourier-Bessel Transform Definition

$$\mathcal{F}_{Bess}\left\{f(\rho)\right\} = 2\pi \int_{\rho=0}^{\rho=\infty} f(\rho)J_0(\rho\omega)\rho d\rho$$

$$= 2\pi \int_{\rho=0}^{\rho=\infty} f(\rho)J_0(2\pi\nu\rho)\rho d\rho\,, \qquad \omega = 2\pi\nu\,.$$

<u>Proof</u>: The 2-dimensional Fourier Transform of a polarly symmetric function $f(x,y) = f(\rho)$ is

$$\mathcal{F}_{Bess}\left\{f(\rho)\right\} = \int_{y=-\infty}^{y=\infty} \int_{x=-\infty}^{x=\infty} f(\rho)e^{-i\omega_x x - i\omega_y y} dxdy$$

Substitute

$$x = \rho\cos\theta \qquad \omega_x = \omega\cos\phi$$
$$y = \rho\sin\theta \qquad \omega_y = \omega\sin\phi$$

$$= \int_{\rho=0}^{\rho=\infty} f(\rho)\left(\int_{\theta=0}^{\theta=2\pi} e^{-i\omega\rho(\cos\theta\cos\phi+\sin\theta\sin\phi)}d\theta\right)\rho d\rho$$

$$= \int_{\rho=0}^{\rho=\infty} f(\rho)\left(\int_{\theta=0}^{\theta=2\pi} e^{-i\omega\rho\cos(\theta-\phi)}d\theta\right)\rho d\rho$$

Denoting $\alpha = \theta - \phi$

$$= \int_{\rho=0}^{\rho=\infty} f(\rho)\left(\int_{\alpha=-\phi}^{\alpha=2\pi-\phi} e^{-i\omega\rho\cos\alpha}d\alpha\right)\rho d\rho$$

Since $e^{-i\omega\rho\cos\alpha}$ is periodic function of α with period 2π,

$$= \int_{\rho=0}^{\rho=\infty} f(\rho)\left(\int_{\alpha=0}^{\alpha=2\pi} e^{-i\omega\rho\cos\alpha}d\alpha\right)\rho d\rho$$

Integrating,

$$\int_{\alpha=0}^{\alpha=2\pi} e^{-i\omega\rho\cos\alpha}d\alpha = \int_{\alpha=0}^{\alpha=2\pi}\left(1 - \frac{i\omega\rho\cos\alpha}{1} + \frac{(i\omega\rho\cos\alpha)^2}{2!} - \frac{(i\omega\rho\cos\alpha)^3}{3!} + ...\right)d\alpha$$

$$= 2\pi - \frac{(\omega\rho)^2}{2^2}2\pi + \frac{(\omega\rho)^4}{2^2\cdot 4^2}2\pi - \frac{(\omega\rho)^6}{2^2\cdot 4^2\cdot 6^2}2\pi + ...$$

$$= 2\pi J_0(\rho\omega).$$

Therefore,

$$\mathcal{F}_{Bess}\left\{f(\rho)\right\} = 2\pi\int_{\rho=0}^{\rho=\infty} f(\rho)J_0(\rho\omega)\rho d\rho$$

$$= 2\pi \int_{\rho=0}^{\rho=\infty} f(\rho) J_0(2\pi\nu\rho)\rho d\rho\,, \qquad \omega = 2\pi\nu\,.\square$$

9.2 The 2-D Inverse Fourier-Bessel Transform

$$\mathcal{F}_{Bess}^{-1}\left\{F(\omega)\right\} = \frac{1}{2\pi}\int_{\omega=0}^{\omega=\infty} F(\omega) J_0(\rho\omega)\omega d\omega\,, \qquad \omega = \sqrt{\omega_x^2 + \omega_y^2}$$

$$= 2\pi \int_{\nu=0}^{\nu=\infty} F(2\pi\nu) J_0(2\pi\nu\rho)\nu d\nu\,, \quad \omega = 2\pi\nu$$

Proof: The 2-dimensional Inverse Fourier Transform of a polarly symmetric function $F(\omega_x, \omega_y) = F(\omega)$ is

$$\mathcal{F}_{Bess}^{-1}\left\{F(\omega)\right\} = \int_{\omega_y=-\infty}^{\omega_y=\infty}\int_{\omega_x=-\infty}^{\omega_x=\infty} F(\omega) e^{-i\omega_x x - i\omega_y y} d\omega_x d\omega_y$$

Substitute

$$x = \rho\cos\theta \qquad \omega_x = \omega\cos\phi$$
$$y = \rho\sin\theta \qquad \omega_y = \omega\sin\phi$$

$$= \frac{1}{2\pi}\int_{\omega=0}^{\omega=\infty} F(\omega)\left(\frac{1}{2\pi}\int_{\phi=0}^{\phi=2\pi} e^{i\omega\rho(\cos\theta\cos\phi + \sin\theta\sin\phi)} d\phi\right)\omega d\omega$$

$$= \frac{1}{2\pi}\int_{\omega=0}^{\omega=\infty} F(\omega)\left(\frac{1}{2\pi}\int_{\phi=0}^{\phi=2\pi} e^{i\omega\rho\cos(\phi-\theta)} d\phi\right)\omega d\omega$$

Denoting $\alpha = \phi - \theta$

$$= \frac{1}{2\pi}\int_{\omega=0}^{\omega=\infty} F(\omega)\left(\frac{1}{2\pi}\int_{\alpha=-\theta}^{\alpha=2\pi-\theta} e^{i\omega\rho\cos\alpha}d\alpha\right)\omega d\omega$$

Since $e^{i\omega\rho\cos\alpha}$ is a periodic function of α with period 2π,

$$= \frac{1}{2\pi}\int_{\omega=0}^{\omega=\infty} F(\omega)\left(\frac{1}{2\pi}\int_{\alpha=0}^{\alpha=2\pi} e^{i\omega\rho\cos\alpha}d\alpha\right)\omega d\omega$$

Integrating,

$$\int_{\alpha=0}^{\alpha=2\pi} e^{i\omega\rho\cos\alpha}d\alpha = \int_{\alpha=0}^{\alpha=2\pi}\left(1+\frac{i\omega\rho\cos\alpha}{1}+\frac{(i\omega\rho\cos\alpha)^2}{2!}+\frac{(i\omega\rho\cos\alpha)^3}{3!}+...\right)d\alpha$$

$$= 2\pi - \frac{(\omega\rho)^2}{2^2}2\pi + \frac{(\omega\rho)^4}{2^2\cdot 4^2}2\pi - \frac{(\omega\rho)^6}{2^2\cdot 4^2\cdot 6^2}2\pi + ...$$

$$= 2\pi J_0(\rho\omega).$$

Therefore,

$$\mathcal{F}_{Bess}^{-1}\left\{F(\omega)\right\} = \frac{1}{2\pi}\int_{\omega=0}^{\omega=\infty} F(\omega)J_0(\rho\omega)\omega d\omega$$

$$= 2\pi\int_{\nu=0}^{\nu=\infty} F(2\pi\nu)J_0(2\pi\nu\rho)\nu d\nu, \qquad \omega = 2\pi\nu. \square$$

10.

Fourier-Bessel Integral Theorem

The 2-dimensional Fourier-Bessel Integral Theorem guarantees that the 2-dimensional Fourier-Bessel Transform and its Inverse are well defined operations, so that inversion yields the original function that generated the Transform.

That is,

$$f(\rho) = 2\pi \int_{\nu=0}^{\nu=\infty} \left(2\pi \int_{\sigma=0}^{\sigma=\infty} f(\sigma) J_0(2\pi\nu\sigma)\sigma d\sigma \right) J_0(2\pi\nu\rho)\nu d\nu \,.$$

But in the Calculus of Limits, this integral is singular, and the Fourier-Bessel Integral Theorem does not hold in the Calculus of Limits under any conditions.

10.1 The 2-D Fourier-Bessel Integral Theorem Fails in the Calculus of Limits

Proof: By the 2-dimensional Fourier-Bessel Integral Theorem

$$f(\rho) = 2\pi \int_{\nu=0}^{\nu=\infty} \left(2\pi \int_{\sigma=0}^{\sigma=\infty} f(\sigma) J_0(2\pi\nu\sigma)\sigma d\sigma \right) J_0(2\pi\nu\rho)\nu d\nu$$

$$= \int_{\sigma=0}^{\sigma=\infty} f(\sigma) \left((2\pi)^2 \int_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\sigma) J_0(2\pi\nu\rho)\nu d\nu \right) \sigma d\sigma \,.$$

However, we can show that at $\sigma = \rho$,

$$(2\pi)^2 \int_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\sigma)J_0(2\pi\nu\rho)\nu d\nu = \int_{\omega=0}^{\omega=\infty} J_0{}^2(\omega\sigma)\omega d\omega$$

is infinite.

Indeed, $u(\omega) = J_0(\omega\sigma)$ satisfies the Bessel differential Equation,

$$(\omega u')' + \sigma^2\omega u = 0.$$

Multiplying the equation by $2\omega u'$,

$$2\omega u'(\omega u')' + 2\sigma^2\omega^2 uu' = 0,$$

$$\frac{d}{d\omega}(\omega u')^2 + 2\sigma^2\omega^2 uu' = 0,$$

$$\frac{d}{d\omega}\left\{\omega^2(u')^2 + \sigma^2\omega^2u^2\right\} - 2\sigma^2\omega u^2 = 0.$$

Integrating from $\omega = 0$, to $\omega = \infty$,

$$2\sigma^2 \int_{\omega=0}^{\omega=\infty} u^2\omega d\omega = \left\{\omega^2(u')^2 + \sigma^2\omega^2u^2\right\}_{\omega=0}^{\omega=\infty}.$$

Since $\omega^2(u')^2 = \omega^2\sigma^2\left[J_0'(\omega\sigma)\right]^2$, then dividing by $2\sigma^2$,

$$\int_{\omega=0}^{\omega=\infty} \left[J_0(\omega\sigma)\right]^2 \omega d\omega = \frac{1}{2}\omega^2\left\{\left[J_0'(\omega\sigma)\right]^2 + \left[J_0(\omega\sigma)\right]^2\right\}_{\omega=0}^{\omega=\infty}.$$

Since $J_0(0)$, and $J_0'(0)$ are bounded,

$$\frac{1}{2}\omega^2\left\{\left[J_0'(\omega\sigma)\right]^2 + \left[J_0(\omega\sigma)\right]^2\right\}_{\omega=0} = 0.$$

By [Bowman, p.83], for very large ω,

$$\left[J_0{}'(\omega\sigma)\right]^2 \sim \frac{2}{\pi\omega\sigma}\sin^2(\omega\sigma - \tfrac{\pi}{4}),$$

$$\left[J_0(\omega\sigma)\right]^2 \sim \frac{2}{\pi\omega\sigma}\cos^2(\omega\sigma - \tfrac{\pi}{4})$$

Therefore, for very large ω,

$$\frac{1}{2}\omega^2\left\{\left[J_0{}'(\omega\sigma)\right]^2 + \left[J_0(\omega\sigma)\right]^2\right\} \sim \frac{1}{\pi\sigma}\omega$$

Thus, for $\sigma = \rho$,

$$\int_{\omega=0}^{\omega=\infty} J_0(\omega\sigma)J_0(\omega\rho)\omega d\omega = \lim_{\omega\to\infty}\left(\frac{1}{\pi\sigma}\omega\right) = \infty.$$

Hence, the 2-dimensinal Fourier-Bessel Integral diverges, and the 2-dimensional Fourier-Bessel Integral Theorem does not hold in the Calculus of Limits.□

Avoiding the singularity at $\sigma = \rho$ does not recover the Theorem, because without the singularity the integral equals zero.

Furthermore,

10.2 Calculus of Limits Conditions are insufficient for the 2-D Fourier-Bessel Integral Theorem

Proof: The following Calculus of Limits Conditions are stated in [Watson, p.458]

1. convergence of $\int_{\rho=0}^{\rho=\infty} \left|f(\rho)\right|\sqrt{\rho}d\rho$

2. Existence of $\lim_{\lambda\to\infty} \int_{\rho=0}^{\rho=\infty} f(\rho)\left(\int_{\omega=0}^{\omega=\lambda} J_0(\omega\sigma)J_0(\omega\rho)\omega d\omega\right)\rho d\rho$

It is clear from 10.1 that Condition 2. never holds. No conditions on $f(x)$ can resolve the singularity of the 2-dimensional Bessel Functions integral

Therefore, the Calculus of Limits Conditions are not sufficient for the 2-dimensional Fourier Integral Theorem.□

The 2-Dimensional Fourier-Bessel Integral Theorem holds only in Infinitesimal Calculus. Then , it holds for any Hyper-real $f(\rho)$:

10.3 2-D Fourier-Bessel Integral Theorem in Infinitesimal Calculus

If $f(\rho)$ is Hyper-real function,

Then, the Fourier-Bessel Integral Theorem holds.

$$f(\rho) = 2\pi \int_{\nu=0}^{\nu=\infty} \left(2\pi \int_{\sigma=0}^{\sigma=\infty} f(\sigma)J_0(2\pi\nu\sigma)\sigma d\sigma \right) J_0(2\pi\nu\rho)\nu d\nu$$

$$= \int_{\sigma=0}^{\sigma=\infty} f(\sigma)\left((2\pi)^2 \int_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\sigma)J_0(2\pi\nu\rho)\nu d\nu \right)\sigma d\sigma .$$

Proof:

In Infinitesimal Calculus,

$$f(\rho) = f(x,y)$$

$$= \int_{\eta=-\infty}^{\eta=\infty} \int_{\xi=-\infty}^{\xi=\infty} f(\xi,\eta)\delta(x-\xi)\delta(y-\eta)d\xi d\eta$$

By 9.2,

$$\delta(x-\xi)\delta(y-\eta) = (2\pi)^2 \int_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\rho)J_0(2\pi\nu\sigma)\nu d\nu\,,$$

and

$$f(\rho) = \int_{\eta=-\infty}^{\eta=\infty} \int_{\xi=-\infty}^{\xi=\infty} f(\xi,\eta)\left((2\pi)^2 \int_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\sigma)J_0(2\pi\nu\rho)\nu d\nu \right) d\xi d\eta$$

Integrating with respect to σ,

$$d\xi d\eta = \sigma d\sigma\,,$$

and

$$f(\rho) = \int_{\sigma=0}^{\sigma=\infty} f(\sigma)\left((2\pi)^2 \int_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\sigma)J_0(2\pi\nu\rho)\nu d\nu \right)\sigma d\sigma\,.$$

By changing the Summation order,

$$f(\rho) = 2\pi \int_{\nu=0}^{\nu=\infty} \left(2\pi \int_{\sigma=0}^{\sigma=\infty} f(\sigma)J_0(2\pi\nu\sigma)\sigma d\sigma \right) J_0(2\pi\nu\rho)\nu d\nu\,.\square$$

Thus,

10.4 ***The 2-D Fourier-Bessel Integral Theorem holds only in Infinitesimal Calculus***

Then, the 2-dimensional Fourier-Bessel Transform of $f(\rho)$,

$$2\pi \int_{\sigma=0}^{\sigma=\infty} f(\sigma) J_0(2\pi\nu\sigma)\sigma d\sigma ,$$

converges to a Hyper-real function $F(2\pi\nu)$, some of its values may be infinite Hyper-reals, like the Delta Function.

And the Inverse Fourier-Bessel Transform of $F(2\pi\nu)$

$$2\pi \int_{\nu=0}^{\nu=\infty} F(2\pi\nu) J_0(2\pi\nu\rho)\nu d\nu$$

converges to the Hyper-real function $f(\rho)$.

This re-establishes the existence of the Fourier-Bessel Transform in Infinitesimal Calculus that we proved in **7.** above:

10.5 The existence of the Fourier-Bessel Transform

If $f(\rho)$ is Hyper-real function, Then,

- ❖ $2\pi \int_{\sigma=0}^{\sigma=\infty} f(\sigma) J_0(2\pi\nu\sigma)\sigma d\sigma$ *converges to* $F(2\pi\nu)$
- ❖ $2\pi \int_{\nu=0}^{\nu=\infty} F(2\pi\nu) J_0(2\pi\nu\rho)\nu d\nu$ *converges to* $f(\rho)$

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