Fourier Series Convergence in the Mean, and Parseval's Theorem,

H. Vic Dannon
vic0@comcast.net
June, 2012

Abstract In the Calculus of Limits, the Fourier Series of a function is guaranteed to converge in the mean to that function, and the Parseval Equality for the Fourier Coefficients follows.

In fact,

Fourier Series Convergence in the mean, and the Parseval Theorem cannot be proved in the Calculus of Limits ,

because integration over the singular Dirichlet Kernel is not possible in the Calculus of Limits.

In Infinitesimal Calculus, the Dirichlet Kernel is the Periodic Delta Function,

$$\delta_{period}(x) = ... + \delta(x+4) + \delta(x+2) + \delta(x) + \delta(x-2) + \delta(x-4) + ...$$

It equals its Fourier Series,

$$\mathcal{FS}\left\{\delta_{period}(x)\right\} = ... + \tfrac{1}{2}e^{-in\pi x} + ... + \tfrac{1}{2}e^{-i\pi x} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi x} + ... + \tfrac{1}{2}e^{in\pi x} + ...$$

Then, the Fourier Series of a periodic hyper-real $f(x)$, equals $f(x)$

converges in the mean to $f(x)$, and the Parseval Equality follows.

Keywords: Infinitesimal, Infinite-Hyper-Real, Hyper-Real, infinite Hyper-real, Infinitesimal Calculus, Delta Function, Fourier Transform, Periodic Delta Function, Delta Comb, Fourier Series, Dirichlet Kernel, Convergence in the Mean, Parseval Theorem,

2000 Mathematics Subject Classification 26E35; 26E30; 26E15; 26E20; 26A06; 26A12; 03E10; 03E55; 03E17; 03H15; 46S20; 97I40; 97I30.

Contents

Introduction

Let $f(x)$ be a function defined on the interval $[-1,1]$, so that $f(1) = f(-1)$.

If for each $n = ..., -3, -2, -1, 0, 1, 2, 3, ...$,

$$c_n = \frac{1}{2}\int_{u=-1}^{u=1} f(u)e^{-in\pi u}du$$

Then, the Fourier Series associated with $f(x)$ is

$$... + c_{-n}e^{i(-n)\pi x} + ... + c_{-1}e^{i(-1)\pi x} + c_0 + c_1e^{i(1)\pi x} + ... + c_ne^{i(n)\pi x} + ...$$

The function is believed to equal the series under the Dirichlet Conditions. In [Dan8], we established that the equality cannot be proved in the Calculus of Limits under any conditions.

Here, we disprove the claim that in the Calculus of Limits, the Fourier Series associated with a square integrable $f(x)$ converges to $f(x)$ in the mean.

It follows that Parseval Equality for the Fourier Coefficients cannot be proved in the Calculus of Limits.

In Infinitesimal Calculus, a hyper-real $f(x)$ equals its Fourier Series, Convergence in the Mean follows, and the Parseval Theorem can be proved.

1.

Convergence in the Mean

In the Calculus of Limits, the Fourier Series partial sums $\mathcal{S}_n\left\{f(x)\right\}$ of a square integrable function $f(x)$, are claimed to converge in the mean to $f(x)$.

This Convergence in the Mean is supposed to get around the singularity of the Dirichlet Kernel that prevents the convergence of $\mathcal{S}_n\left\{f(x)\right\}$ to $f(x)$, in the Calculus of Limits.

We found proofs of the Convergence in the Mean in [Rogosinski, pp 56-58], [Carslaw, pp 285-287], [Zygmund, pp 12-37], and [Tolstov, pp. 54-56].

Rogosinski attempts to avoid the Dirichlet Kernel singularity.

Carslaw attempts to avoid the Fejer Kernel singularity.

Zygmund and Tolstov attempt to avoid the singularity with arguments of completeness.

But success of any of these arguments would amount to the definition of the Delta Function in the Calculus of Limits.

In [Dan5] we outlined the failed attempts of Cauchy, Poisson, Riemann, Dirac, Laurent Schwartz, and Mikusinski to define the Delta Function in the Calculus of Limits.

The Dirichlet Kernel singularity prevents a proof of Convergence in the Mean in the Calculus of Limits:

1.1 *In the Calculus of Limits,* $f(x) \in L^2_{[-1,1]}$ *does not guarantee the convergence* $\mathcal{S}_n\left\{f(x)\right\} \to f(x)$ *in the mean*

Proof: In the Calculus of Limits, the Fourier Series is the limit of the sequence of Partial Sums

$$\mathcal{S}_n\left\{f(x)\right\} = c_n e^{in\pi x} + ... + c_1 e^{i\pi x} + c_0 + c_{-1}e^{-i\pi x} + ...c_{-n}e^{-in\pi x}$$

$$= \left(\tfrac{1}{2}\int_{\xi=-1}^{\xi=1} f(\xi)e^{-in\pi\xi}d\xi\right)e^{in\pi x} + ... + \left(\tfrac{1}{2}\int_{\xi=-1}^{\xi=1} f(\xi)d\xi\right) + ... + \left(\tfrac{1}{2}\int_{\xi=-1}^{\xi=1} f(\xi)e^{in\pi\xi}d\xi\right)e^{-in\pi x}$$

$$= \int_{\xi=-1}^{\xi=1} f(\xi)\left\{\tfrac{1}{2}e^{-in\pi(\xi-x)} + ... + \tfrac{1}{2}e^{-i\pi(\xi-x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi-x)} + ... + \tfrac{1}{2}e^{in\pi(\xi-x)}\right\}d\xi .$$

As $n \to \infty$, the Dirichlet Sequence

$$D_n(\xi - x) = \tfrac{1}{2}e^{-in\pi(\xi-x)} + ... + \tfrac{1}{2}e^{-i\pi(\xi-x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi-x)} + ... + \tfrac{1}{2}e^{in\pi(\xi-x)}$$

becomes the Dirichlet Kernel, the infinite series

$$... + \tfrac{1}{2}e^{-in\pi(\xi-x)} + ... + \tfrac{1}{2}e^{-i\pi(\xi-x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi-x)} + ... + \tfrac{1}{2}e^{in\pi(\xi-x)} + ...$$

Clearly, the Dirichlet Kernel is singular at any even $\xi - x$.

In particular,

$$x = \xi \Rightarrow e^{in\pi(x-\xi)} = 1,$$

and the Dirichlet Kernel diverges to

$$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+...=\infty.$$

Due to the singularity at $x=\xi$, the partial sums of the Fourier Series have no limit in the Calculus of Limits. That is,

$$\mathcal{S}_n\left\{f(x)\right\} \not\to f(x),$$

$$\left|\mathcal{S}_n\left\{f(x)\right\} - f(x)\right| \not\to 0,$$

$$\int_{x=-1}^{x=1}\left|\mathcal{S}_n\left\{f(x)\right\} - f(x)\right|^2 dx \not\to 0$$

Avoiding the singularity at $\xi = x$, by using the Cauchy Principal Value of the integral does not yield convergence in the mean.

At any uneven $\xi - x$, the Dirichlet Kernel is known to vanish. That is, at any uneven $\xi - x$,

$$\mathcal{S}_n\left\{f(x)\right\} \to 0,$$

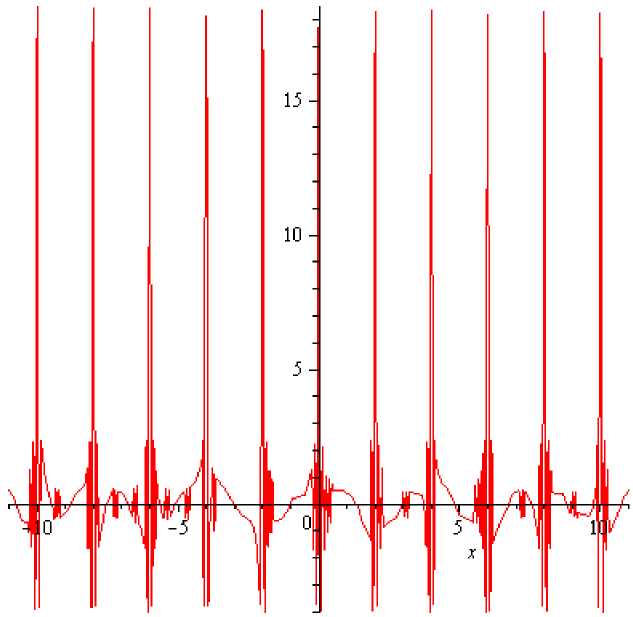
and

$$\int_{x=-1}^{x=1}\left|\mathcal{S}_n\left\{f(x)\right\} - f(x)\right|^2 dx \to \int_{x=-1}^{x=1}\left|f(x)\right|^2 dx > 0.$$

Consequently, Fourier Series does not converge in the mean in the Calculus of Limits.□

We demonstrate this with a Maple plot of the 37^{th} element of the Dirichlet sequence which limit is the Dirichlet Kernel.

$$plot\left(\frac{\sin\left(\pi\frac{37\,x}{2}\right)}{2\ \sin\left(\pi\frac{x}{2}\right)},\ x=-11\,..11\right)$$ gives 11 spikes



As $n \to \infty$, the terms of the Dirichlet Sequence evolve into the Dirichlet Kernel, spikes separated by intervals of zeros.

The Dirichlet Kernel values cement the failure of the proofs of Convergence in the Mean.

This failure of L^2 Convergence demonstrates another failure of the Lebesgue theory of Integration. We have pointed out more failures of Lebesgue's Integration theory in [Dan9].

2.

Parseval Theorem

In the Calculus of Limits, we can derive the Bessel Inequality. Parseval Theorem follows from the Bessel Inequality, provided that the Fourier Series of $f(x)$ converges in the mean to $f(x)$.

Since Convergence in the Mean can be proved only in Infinitesimal Calculus, Parseval theorem can be proved only in Infinitesimal Calculus.

To derive Bessel's Inequality, let $f(x)$ be periodic with period $T = 2$

For each $n = ..., -3, -2, -1, 0, 1, 2, 3, ...,$

$$c_n = \frac{1}{2} \int_{u=-1}^{u=1} f(u) e^{-in\pi u} du$$

And the Fourier Series partial sums are

$$\mathcal{S}_N\{f(x)\} = c_{-N} e^{i(-N)\pi x} + ... + c_{-1} e^{i(-1)\pi x} + c_0 + c_1 e^{i(1)\pi x} + ... + c_N e^{i(N)\pi x}.$$

We evaluate

$$\int_{x=-1}^{x=1} \left| f(x) - \mathcal{S}_N\{f(x)\} \right|^2 dx = \int_{x=-1}^{x=1} (f(x) - \mathcal{S}_N\{f(x)\})(\overline{f}(x) - \overline{\mathcal{S}}_N\{f(x)\}) dx$$

$$= \int_{x=-1}^{x=1} \left|f(x)\right|^2 dx - \int_{x=-1}^{x=1} \mathcal{S}_N\{f\}\overline{f}(x)dx - \int_{x=-1}^{x=1} \overline{\mathcal{S}}_N\{f\}f(x)dx + \int_{x=-1}^{x=1} \mathcal{S}_N\{f\}\overline{\mathcal{S}}_N\{f\}dx\,.$$

We have,

$$\int_{x=-1}^{x=1} \mathcal{S}_N\{f\}\overline{f}(x)dx = \int_{x=-1}^{x=1} \sum_{n=-N}^{n=N} c_n e^{-in\pi x}\overline{f}(x)dx$$

$$= \sum_{n=-N}^{n=N} c_n \underbrace{\int_{x=-1}^{x=1} \overline{f}(x)e^{-in\pi x}dx}_{2\overline{c}_n}$$

$$= 2\sum_{n=-N}^{n=N} \left|c_n\right|^2$$

Similarly we have,

$$\int_{x=-1}^{x=1} \overline{\mathcal{S}}_N\{f\}f(x)dx = 2\sum_{n=-N}^{n=N} \left|c_n\right|^2.$$

The last term is

$$\int_{x=-1}^{x=1} \mathcal{S}_N\{f\}\overline{\mathcal{S}}_N\{f\}dx = \int_{x=-1}^{x=1} \sum_{n=-N}^{n=N} c_n e^{-in\pi x} \sum_{m=-N}^{m=N} \overline{c}_m e^{im\pi x}dx$$

$$= \sum_{n=-N}^{n=N} c_n \sum_{m=-N}^{m=N} \overline{c}_m \underbrace{\int_{x=-1}^{x=1} e^{-in\pi x}e^{im\pi x}dx}_{2\delta_{mn}}$$

$$= 2\sum_{n=-N}^{n=N} \left|c_n\right|^2$$

Therefore,

$$\underbrace{\int_{x=-1}^{x=1} \left| f(x) - \mathcal{S}_N\{f(x)\} \right|^2 dx}_{\geq 0} = \int_{x=-1}^{x=1} \left| f(x) \right|^2 dx - 2\sum_{n=-N}^{n=N} \left| c_n \right|^2 .$$

Thus, for any N,

$$\int_{x=-1}^{x=1} \left| f(x) \right|^2 dx \geq 2\sum_{n=-N}^{n=N} \left| c_n \right|^2 .$$

Hence, we obtain Bessel Inequality

$$\int_{x=-1}^{x=1} \left| f(x) \right|^2 dx \geq 2\sum_{n=-\infty}^{n=\infty} \left| c_n \right|^2 .$$

The Parseval Theorem is the Equality

$$\int_{x=-1}^{x=1} \left| f(x) \right|^2 dx = 2\sum_{n=-\infty}^{n=\infty} \left| c_n \right|^2 .$$

The equality requires that as $N \to \infty$,

$$\int_{x=-1}^{x=1} \left| f(x) - \mathcal{S}_N\{f(x)\} \right|^2 dx \to 0 .$$

We have seen above that in the Calculus of Limits, the singularity of the Dirichlet Kernel prevents this convergence. Thus, Parseval's Theorem cannot be proved in the Calculus of Limits.

3.

Hyper-real Line

Each real number α can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of infinitesimal hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the infinite hyper-reals.
4. The infinite hyper-reals are greater than any real number, yet strictly smaller than infinity.
5. The infinite hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a non-constant hyper-real.
7. The Hyper-reals are the totality of constant hyper-reals, a family of infinitesimals, a family of infinitesimals with

negative sign, a family of infinite hyper-reals, a family of infinite hyper-reals with negative sign, and non-constant hyper-reals.

8. The hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.

9. That line includes the real numbers separated by the non-constant hyper-reals. Each real number is the center of an interval of hyper-reals, that includes no other real number.

10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.

11. Zero is not an infinitesimal, because zero is not strictly greater than zero.

12. We do not add infinity to the hyper-real line.

13. The infinitesimals, the infinitesimals with negative signs, the infinite hyper-reals, and the infinite hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.

14. The hyper-real line is embedded in $\mathbb{R}^{\infty}$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the hyper-real onto the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal hyper-reals, or to the infinite hyper-reals, or to the non-constant hyper-reals.

16. No neighbourhood of a hyper-real is homeomorphic to an $\mathbb{R}^n$ ball. Therefore, the hyper-real line is not a manifold.

17. The hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

4.

Integral of a Hyper-real Function

In [Dan3], we defined the integral of a Hyper-real Function.

Let $f(x)$ be a hyper-real function on the interval $[a,b]$.

The interval may not be bounded.

$f(x)$ may take infinite hyper-real values, and need not be bounded.

At each

$$a \le x \le b,$$

there is a rectangle with base $[x - \frac{dx}{2}, x + \frac{dx}{2}]$, height $f(x)$, and area

$$f(x)dx.$$

We form the **Integration Sum** of all the areas for the x's that start at $x = a$, and end at $x = b$,

$$\sum_{x \in [a,b]} f(x)dx.$$

If for any infinitesimal dx, the Integration Sum has the same hyper-real value, then $f(x)$ is integrable over the interval $[a,b]$.

Then, we call the Integration Sum the integral of $f(x)$ from $x = a$, to $x = b$, and denote it by

$$\int_{x=a}^{x=b} f(x)dx.$$

If the hyper-real is infinite, then it is the integral over $[a,b]$,

If the hyper-real is finite,

$$\int_{x=a}^{x=b} f(x)dx = \text{ real part of the hyper-real}. \square$$

4.1 The countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers, $Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty.$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[a,b]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(x)dx$.

The Lower Integral is the Integration Sum where $f(x)$ is replaced by its lowest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

4.2
$$\sum_{x \in [a,b]} \left(\inf_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

The Upper Integral is the Integration Sum where $f(x)$ is replaced by its largest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

4.3
$$\sum_{x \in [a,b]} \left(\sup_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

If the integral is a finite hyper-real, we have

4.4 *A hyper-real function has a finite integral if and only if its upper integral and its lower integral are finite, and differ by an infinitesimal.*

5.

Delta Function

In [Dan5], we defined the Delta Function, and established its properties

1. The Delta Function is a hyper-real function defined from the hyper-real line into the set of two hyper-reals $\left\{0, \frac{1}{dx}\right\}$. The hyper-real 0 is the sequence $\langle 0, 0, 0, ... \rangle$. The infinite hyper-real $\frac{1}{dx}$ depends on our choice of dx.
2. We will usually choose the family of infinitesimals that is spanned by the sequences $\left\langle \frac{1}{n} \right\rangle, \left\langle \frac{1}{n^2} \right\rangle, \left\langle \frac{1}{n^3} \right\rangle, \ldots$ It is a semigroup with respect to vector addition, and includes all the scalar multiples of the generating sequences that are non-zero. That is, the family includes infinitesimals with negative sign. Therefore, $\frac{1}{dx}$ will mean the sequence $\langle n \rangle$. Alternatively, we may choose the family spanned by the sequences $\left\langle \frac{1}{2^n} \right\rangle, \left\langle \frac{1}{3^n} \right\rangle, \left\langle \frac{1}{4^n} \right\rangle, \ldots$ Then, $\frac{1}{dx}$ will mean the

sequence $\left\langle 2^n \right\rangle$. Once we determined the basic infinitesimal dx, we will use it in the Infinite Riemann Sum that defines an Integral in Infinitesimal Calculus.

3. The Delta Function is strictly smaller than ∞

4. We define, $\delta(x) \equiv \frac{1}{dx}\chi_{[-\frac{dx}{2},\frac{dx}{2}]}(x)$,

where $\chi_{[-\frac{dx}{2},\frac{dx}{2}]}(x) = \begin{cases} 1, x \in \left[-\frac{dx}{2},\frac{dx}{2}\right] \\ 0, \text{otherwise} \end{cases}$.

5. Hence,

❖ for $x < 0$, $\delta(x) = 0$

❖ at $x = -\frac{dx}{2}$, $\delta(x)$ jumps from 0 to $\frac{1}{dx}$,

❖ for $x \in \left[-\frac{dx}{2},\frac{dx}{2}\right]$, $\delta(x) = \frac{1}{dx}$.

❖ at $x = 0$, $\delta(0) = \frac{1}{dx}$

❖ at $x = \frac{dx}{2}$, $\delta(x)$ drops from $\frac{1}{dx}$ to 0.

❖ for $x > 0$, $\delta(x) = 0$.

❖ $x\delta(x) = 0$

6. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle \chi_{[-\frac{1}{2},\frac{1}{2}]}(x), 2\chi_{[-\frac{1}{4},\frac{1}{4}]}(x), 3\chi_{[-\frac{1}{6},\frac{1}{6}]}(x)... \right\rangle$

7. If $dx = \left\langle \frac{2}{n} \right\rangle$, $\delta(x) = \left\langle \frac{1}{2\cosh^2 x}, \frac{2}{2\cosh^2 2x}, \frac{3}{2\cosh^2 3x}, ... \right\rangle$

8. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle e^{-x}\chi_{[0,\infty)}, 2e^{-2x}\chi_{[0,\infty)}, 3e^{-3x}\chi_{[0,\infty)}, ... \right\rangle$

9. $$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1.$$

10. $$\delta(\xi - x) = \frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} e^{-ik(\xi - x)}dk$$

6.

Fourier Series of a Hyper-real Function

6.1 Let $f(x)$ be a hyper-real function defined on the bounded interval $[c-L, c+L]$.

Then, for each $n = ..., -3, -2, -1, 0, 1, 2, 3, ...$, the integrals

$$\frac{1}{2L}\int_{u=c-L}^{u=c+L} f(u)e^{-in\frac{\pi}{L}u}du \equiv c_n$$

exist, with finite, or infinite hyper-real values. The c_n are the Fourier Coefficients of $f(x)$.

The Fourier Series associated with $f(x)$ is

$$\mathcal{FS}\{f(x)\} = ... + c_{-n}e^{i(-n)\frac{\pi}{L}x} + ... + c_{-1}e^{i(-1)\frac{\pi}{L}x} + c_0 + c_1e^{i(1)\frac{\pi}{L}x} + ... + c_ne^{i(n)\frac{\pi}{L}x} + ...$$

For each x, it may assume finite or infinite hyper-real values.

In [Dan8], we established

6.2 Fourier Series Theorem for a periodic Hyper-real $f(x)$

If $f(x)$ *is a periodic hyper-real function defined on the bounded interval* $[c-L, c+L]$.

Then,
$$\mathcal{FS}\{f(x)\} = f(x)$$

7.

Periodic Delta Function, $\delta_{period}(x)$

In [Dan8], we defined the Periodic Delta Function, and established its properties

1. The Periodic Delta is a periodic train of delta functions.
2. $\delta_{period}(x) = ... + \delta(x+4) + \delta(x+2) + \delta(x) + \delta(x-2) + \delta(x-4) + ...$

   is a periodic hyper-real Delta function, with period $T = 2$.

   represented by the Dirichlet Sequence

   $$\left\langle \frac{\sin(\frac{1}{2})\pi x}{2\sin\frac{1}{2}\pi x}, \frac{\sin(\frac{3}{2})\pi x}{2\sin\frac{1}{2}\pi x}, \frac{\sin(\frac{5}{2})\pi x}{2\sin\frac{1}{2}\pi x}, ... \right\rangle =$$

   $$= \left\langle \frac{1}{2}, \frac{1}{2} + \cos\pi x, \frac{1}{2} + \cos\pi x + \cos 2\pi x, ... \right\rangle$$

   $$= \left\langle \tfrac{1}{2}e^{-in\pi x} + ... + \tfrac{1}{2}e^{-i\pi x} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi x} + ... + \tfrac{1}{2}e^{in\pi x} \right\rangle$$

3. *The Hyper-real Dirichlet Kernel*

   $$... + \tfrac{1}{2}e^{-in\pi x} + ... + \tfrac{1}{2}e^{-i\pi x} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi x} + ... + \tfrac{1}{2}e^{in\pi x} + ...$$

   is a Periodic Delta Function

4. *The Fourier Transform of* $\delta_{period}(x)$

   $$\mathcal{F}\left\{\delta_{period}(x)\right\} = ... + e^{-i4\pi\nu} + 1 + e^{i4\pi\nu} + ...$$

5. *Fourier Integral Theorem for* $\delta_{period}(x)$

$$\mathcal{F}^{-1}\mathcal{F}\left\{\delta_{period}(x)\right\} = \delta_{period}(x)$$

6. *Fourier Series Theorem for* $\delta_{period}(x)$

$$\mathcal{FS}\left\{\delta_{period}(x)\right\} = \delta_{period}(x)$$

8.

Convergence in the Mean, and Parseval Theorem for a Hyper-real Function

In [Dan8], we established that in Infinitesimal Calculus, the Dirichlet Kernel is the periodic hyper-real Delta Function: A periodic train of Delta Functions.

And the Fourier Series $\mathcal{FS}\left\{f(x)\right\}$ associated with a Hyper-real periodic function $f(x)$, equals $f(x)$.

That is, for a periodic hyper-real $f(x)$,

$$\mathcal{S}_n\left\{f(x)\right\} \to f(x).$$

Therefore, $\mathcal{S}_n\left\{f(x)\right\}$ converges to $f(x)$ in the Mean,

$$\int_{x=-1}^{x=1} \left|f(x) - \mathcal{S}_n\{f(x)\}\right|^2 dx \to 0.$$

Then, the Parseval Equality follows.

References

[Carslaw] Carslaw, H. S., "*Introduction to the Theory of Fourier Series and integrals*" Third Edition, Macmillan, 1930.

[Dan1] Dannon, H. Vic, "*Well-Ordering of the Reals, Equality of all Infinities, and the Continuum Hypothesis*" in Gauge Institute Journal Vol. 6 No. 2, May 2010;

[Dan2] Dannon, H. Vic, "*Infinitesimals*" in Gauge Institute Journal Vol.6 No. 4, November 2010;

[Dan3] Dannon, H. Vic, "*Infinitesimal Calculus*" in Gauge Institute Journal Vol. 7 No. 4, November 2011;

[Dan4] Dannon, H. Vic, "*Riemann's Zeta Function: the Riemann Hypothesis Origin, the Factorization Error, and the Count of the Primes*", in Gauge Institute Journal of Math and Physics, Vol. 5, No. 4, November 2009.

[Dan5] Dannon, H. Vic, "*The Delta Function*" in Gauge Institute Journal Vol. 8, No. 1, February, 2012;

[Dan6] Dannon, H. Vic, "*Delta Function, the Fourier Transform, and the Fourier Integral Theorem*" in Gauge Institute Journal Vol. 8, No. 2, May, 2012;

[Dan7] Dannon, H. Vic, "*Riemannian Trigonometric Series*", Gauge Institute Journal, Volume 7, No. 3, August 2011.

[Dan8] Dannon, H. Vic, "*Periodic Delta Function and Dirichlet Summation of Fourier Series*", October 2010 posted to www.gauge-institute.org

[Dan9] Dannon, H. Vic, "*Lebesgue Integration*" Gauge Institute Journal Vol.7 No. 1, February 2011; Also Posted to www.gauge-institute.org

[Euler1] Euler, Leonhard, "*Disquisitio Ulterior Super Seriebus Secundum Multipla Cuiusdam Anguli Progredientibus*", Leonhardi Euleri Opera Omnia, Series Prima, Volume XVI, pp.33-355. May 29, 1777.

[Euler2] Euler, Leonhard, "*Methodus Facilis Inveniendi Series per Sinus Cosinusve Angulorum Multiplorum Procedentes Quarum Usus In Universa Theoria Asronomiae Est Amplissimus*". in Leonhardi Euleri Opera Omnia, Series Prima, Volume XVI, pp.311-332, May 26, 1777.

[Euler3] Euler, Leonhard, "*De Serierum Determinantione Sev Nova Methodus Inveniendi Terminos Generales Serierum*", Leonhardi Euleri Opera Omnia, Series Prima, Volume XIV, pp.463-515. 1753.

[Hardy] Hardy, G. H., *Divergent Series*, Chelsea 1991.

[Riemann] Riemann, Bernhard, "*On the Representation of a Function by a Trigonometric Series*".

(1) In "*Collected Papers, Bernhard Riemann*", translated from the 1892 edition by Roger Baker, Charles Christenson, and Henry Orde, Paper XII, Part 5, Conditions for the existence of a definite integral, pages 231-232, Part 6, Special Cases, pages 232-234. Kendrick press, 2004

(2) In "*God Created the Integers*" Edited by Stephen Hawking, Part 5, and Part 6, pages 836-840, Running Press, 2005.

[Rogosinski] Rogosinski, Werner, "*Fourier Series*" Chelsea, 1950.

[Tolstov] Tolstov, Georgi, "*Fourier Series*" Prentice-Hall,1962

[Zygmund] Zygmund, A., "*Trigonometric Series*", Second Edition, Cambridge University Press, 1968.