Periodic Delta Function, and Expansion in Legendre Polynomials

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Abstract Let $f(x)$ be defined on $[-1,1]$, so that $f(1) = f(-1)$, and let $P_n(x)$ be the Legendre Polynomials on $[-1,1]$,

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \tfrac{3}{2}x^2 - \tfrac{1}{2},\ P_3(x) = \tfrac{5}{2}x^3 - \tfrac{3}{2}x, \ldots$$

The Legendre Series associated with $f(x)$ is

$$a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + \ldots.$$

where

$$a_n = \tfrac{2n+1}{2} \int_{\xi=-1}^{\xi=1} f(\xi) P_n(\xi) d\xi$$

are the Legendre coefficients.

The Legendre Series Theorem supplies the conditions under which the Legendre Series associated with $f(x)$ equals $f(x)$.

It is believed to hold in the Calculus of Limits for smooth enough function. In fact,

The Theorem cannot be proved in the Calculus of Limits under any conditions,

because the summation of the Legendre Series requires integration over the periodically singular Legendre Kernel.

Plots of partial sums of the Legendre Series speak volumes about the sensibility of the claims to have infinities bound by epsilons.

In Infinitesimal Calculus, the Legendre Kernel

$$\tfrac{1}{2}P_0(\cos\theta_\xi)P_0(\cos\theta_x) + \tfrac{3}{2}P_1(\cos\theta_\xi)P_1(\cos\theta_x) + \tfrac{5}{2}P_2(\cos\theta_\xi)P_2(\cos\theta_x) + \ldots$$

is the Periodic Delta Function,

$$\delta_{Periodic}(\theta_\xi - \theta_x) = \ldots + \delta(\theta_\xi - \theta_x + 2\pi) + \delta(\theta_\xi - \theta_x) + \delta(\theta_\xi - \theta_x - 2\pi) + \ldots.$$

$\delta_{Periodic}(\theta_\xi - \theta_x)$ equals its Legendre Series, and the Legendre Series associated with any periodic hyper-real integrable $f(x)$, equals $f(x)$.

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Contents

The Origin of the Legendre Series Theorem

The Legendre Polynomials on $[-1,1]$

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \tfrac{3}{2}x^2 - \tfrac{1}{2},\ P_3(x) = \tfrac{5}{2}x^3 - \tfrac{3}{2}x, \ldots,$$

are the orthogonal functions generated in the Gram-Schmidt orthogonalization so that.

$$\int_{x=-1}^{x=1} P_m(x)P_n(x)dx = \tfrac{2}{2n+1}\delta_{mn}.$$

0.1 Legendre

generated the polynomials by expanding

$$\frac{1}{\sqrt{1-2\alpha x+\alpha^2}} = [1-\alpha(2x-\alpha)]^{-\frac{1}{2}}$$

by the Binomial Theorem.

In the Binomial Expansion,

$$1 + \tfrac{1}{2}\alpha(2x-\alpha) + \tfrac{1\cdot 3}{2\cdot 4}\alpha^2(2x-\alpha)^2 + \tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\alpha^3(2x-\alpha)^3 + \ldots,$$

the coefficient of α^0 is

$$P_0(x) = 1,$$

the coefficient of α^1 is

$$P_1(x) = x,$$

the coefficient of α^2 is

$$P_2(x) = \tfrac{3}{2}x^2 - \tfrac{1}{2},$$

the coefficient of α^3 is

$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x),$$

....

0.2 Legendre Differential Equation

Laplace Equation in radial coordinates for a potential $V(r,\theta,\phi)$ is

$$\partial_r(r^2\partial_r V) + \tfrac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta V) + \tfrac{1}{\sin^2\theta}\partial_\phi^2 V = 0.$$

Assuming that

$$V(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi),$$

the Laplace equation becomes

$$\tfrac{1}{R}(r^2R')' + \tfrac{1}{\Theta\sin\theta}(\sin\theta\Theta')' + \tfrac{1}{\Phi\sin^2\theta}\Phi'' = 0.$$

Then,

$$\tfrac{1}{R}(r^2R')' = const \equiv C_1.$$

Substituting a linear combination of

$$r^n, \text{ and } \tfrac{1}{r^{n+1}}$$

the equation for R yields

$$n(n+1) = C_1$$

and the Laplace Equation becomes

$$n(n+1)\sin^2\theta + \tfrac{\sin\theta}{\Theta}(\sin\theta\Theta')' + \tfrac{1}{\Phi}\Phi'' = 0.$$

Then,

$$\tfrac{1}{\Phi}\Phi'' = const \equiv C_2.$$

To obtain solutions $e^{im\phi}$, and $e^{-im\phi}$, we set

$$C \equiv -m^2,$$

and the Laplace Equation becomes an equation for $\Theta(\theta)$,

$$n(n+1)\Theta + \tfrac{1}{\sin\theta}(\Theta'\sin\theta)' - \tfrac{m^2}{\sin^2\theta}\Theta = 0.$$

Denoting

$$x \equiv \cos\theta,$$

$$X(x) = \Theta(\theta),$$

$$\Theta' = \frac{d\Theta}{d\theta} = \frac{dX}{dx}\frac{dx}{d\theta} = -X'\sin\theta,$$

$$\Theta'' = \frac{d^2\Theta}{d\theta^2} = \frac{d}{d\theta}\{-X'\sin\theta\},$$

$$= -\frac{dX'}{dx}\frac{dx}{d\theta}\sin\theta - X'\cos\theta,$$

$$= X''\sin^2\theta - X'\cos\theta,$$

$$\tfrac{1}{\sin\theta}(\Theta'\sin\theta)' = \Theta'' + \Theta'\tfrac{\cos\theta}{\sin\theta},$$

$$= X''\sin^2\theta - X'\cos\theta - X'\sin\theta\tfrac{\cos\theta}{\sin\theta},$$

$$= (1-x^2)X'' - 2xX',$$

$$= [(1-x^2)X']',$$

the Laplace Equation becomes

$$[(1-x^2)X']' + [n(n+1) - \tfrac{m^2}{1-x^2}]X = 0.$$

For $m = 0$, it is the Legendre equation

$$(1-x^2)X'' - 2xX' + n(n+1)X = 0.$$

Substituting in it

$$X(x) = c_0 + c_1 x + c_2 x^2 + ... + c_l x^l + c_{l+1} x^{l+1} + c_{l+2} x^{l+2} + ...,$$

we have

$$\underbrace{D_x^2 \sum_{l=0}^{l=\infty} c_l x^l}_{\sum_{l=2}^{l=\infty} (l-1) l c_l x^{l-2}} - x^2 \underbrace{D_x^2 \sum_{l=0}^{l=\infty} c_l x^l}_{\sum_{l=2}^{l=\infty} (l-1) l c_l x^{l-2}} - 2x \underbrace{D_x \sum_{l=0}^{l=\infty} c_l x^l}_{\sum_{l=1}^{l=\infty} l c_l x^{l-1}} + n(n+1) \sum_{l=0}^{l=\infty} c_l x^l = 0,$$

$$\sum_{l=0}^{l=\infty} \{(l+1)(l+2) c_{l+2} - l(l-1) c_l - 2l c_l + n(n+1) c_l\} x^l = 0,$$

$$(l+1)(l+2) c_{l+2} - l(l-1) c_l - 2l c_l + n(n+1) c_l = 0,$$

$$c_{l+2} = \frac{l(l-1) + 2l - n(n+1)}{(l+1)(l+2)} c_l$$

$$c_{l+2} = (-1) \frac{(n-l)(n+l-1)}{(l+1)(l+2)} c_l$$

The solution is

$$X(x) = c_0 + c_1 x - c_0 \tfrac{n(n+1)}{1\cdot 2} x^2 - c_1 \tfrac{(n-1)(n+2)}{1\cdot 2\cdot 3} x^3 +$$

$$+ c_0 \tfrac{n(n+1)(n-2)(n+3)}{1\cdot 2\cdot 3\cdot 4} x^4 + c_1 \tfrac{(n-1)(n+2)(n-3)(n+4)}{1\cdot 2\cdot 3\cdot 4\cdot 5} x^5 + ...$$

$$= c_0 \{1 - \tfrac{n(n+1)}{1\cdot 2} x^2 + \tfrac{n(n+1)(n-2)(n+3)}{1\cdot 2\cdot 3\cdot 4} x^4 - ...\} +$$

$$+ c_1 x \{1 - \tfrac{(n-1)(n+2)}{1\cdot 2\cdot 3} x^2 + \tfrac{(n-1)(n+2)(n-3)(n+4)}{1\cdot 2\cdot 3\cdot 4\cdot 5} x^4 + ...\}.$$

Therefore, for $n = 2k$, the c_0 series terms vanish for

$$2k + 2, 2k + 4, ...,$$

and we obtain the $P_{2k}(x)$ Legendre Polynomials.

for $n = 2k+1$, the c_1 series terms vanish for

$$2k+3, 2k+5, ...$$

and we obtain the $P_{2k+1}(x)$ Legendre Polynomials.

A solution for $\Theta(\theta)$ is the infinite linear combination

$$\alpha_0 P_0(x) + \alpha_1 P_1(x) + \alpha_2 P_2(x) + ...$$

0.3 The Legendre Series Associated with a periodic $f(x)$

Let $f(x)$ be integrable on $[-1,1]$, so that $f(1) = f(-1)$, and let $P_n(x)$ be the Legendre Polynomials on $[-1,1]$,

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \tfrac{3}{2}x^2 - \tfrac{1}{2},\ P_3(x) = \tfrac{5}{2}x^3 - \tfrac{3}{2}x, ...$$

The Polynomials are orthogonal on $[-1,1]$. That is,

$$\int_{x=-1}^{x=1} P_m(x) P_n(x) dx = \tfrac{2}{2n+1}\delta_{mn}$$

If $f(x)$ can be expanded in the Legendre Polynomials,

$$f(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + ...,$$

Then,

$$\int_{x=-1}^{x=1} f(x) P_n(x) dx = \int_{x=-1}^{x=1} \{a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + ...\} P_n(x) dx$$

$$= a_0 \underbrace{\int_{x=-1}^{x=1} P_0(x)P_n(x)}_{\frac{2}{2n+1}\delta_{0n}} + a_1 \underbrace{\int_{x=-1}^{x=1} P_1(x)P_n(x)}_{\frac{2}{2n+1}\delta_{1n}} + a_2 \underbrace{\int_{x=-1}^{x=1} P_2(x)P_n(x)}_{\frac{2}{2n+1}\delta_{2n}} + ...$$

$$= a_n \tfrac{2}{2n+1}.$$

Thus, the Legendre coefficients are

$$a_n = \tfrac{2n+1}{2} \int_{\xi=-1}^{\xi=1} f(\xi)P_n(\xi)d\xi .$$

The Legendre Series associated with $f(x)$ is

$$a_0P_0(x) + a_1P_1(x) + a_2P_2(x) + ...$$

Since

$$x = \cos\theta ,$$

the Legendre Polynomials,

$$P_n(x) = P_n(\cos\theta),$$

are periodic in θ, with period 2π.

Thus, the Legendre Polynomials are defined for any real

$$\theta = \arccos(x) + 2m\pi ,$$

where $-1 \leq x \leq 1$, and m is an integer.

Therefore, the Legendre Series is periodic in θ, with period 2π.

The Legendre Series Theorem supplies the conditions under which the Legendre Series associated with $f(x)$ equals $f(x)$.

Then, *the function must be periodic too*.

1.

Divergence of the Legendre Kernel in the Calculus of Limits

for the Legendre Series to equal its function reflect the belief that a smooth enough function equals its Legendre Series.

In fact, in the Calculus of Limits, no smoothness of the function guarantees even the convergence of the Legendre Series.

1.1 The Legendre Kernel is either singular or zero

In the Calculus of Limits, the Legendre Series is the limit of the sequence of Partial Sums

$$\mathcal{L}_{egendre}\mathcal{S}_n\left\{f(x)\right\} = a_0P_0(x) + a_1P_1(x) + ... + a_nP_n(x)$$

$$= \left(\tfrac{1}{2}\int_{\xi=-1}^{\xi=1} f(\xi)P_0(\xi)d\xi\right)P_0(x) + \left(\tfrac{3}{2}\int_{\xi=-1}^{\xi=1} f(\xi)P_1(\xi)d\xi\right)P_1(x) + ... + \left(\tfrac{2n+1}{2}\int_{\xi=-1}^{\xi=1} f(\xi)P_n(\xi)d\xi\right)P_n(x)$$

$$= \int_{\xi=-1}^{\xi=1} f(\xi)\left\{\tfrac{1}{2}P_0(\xi)P_0(x) + \tfrac{3}{2}P_1(\xi)P_1(x) + ... + \tfrac{2n+1}{2}P_n(\xi)P_n(x)\right\}d\xi\,.$$

As $n \to \infty$, the Legendre Sequence

$$\tfrac{1}{2}P_0(\xi)P_0(x) + \tfrac{3}{2}P_1(\xi)P_1(x) + ... + \tfrac{2n+1}{2}P_n(\xi)P_n(x)$$

becomes the Legendre Kernel,

$$\tfrac{1}{2}P_0(\xi)P_0(x)+\tfrac{3}{2}P_1(\xi)P_1(x)+...+\tfrac{2n+1}{2}P_n(\xi)P_n(x)+...,$$

To see that it diverges for $\xi = x$, we apply the Christoffel-Darboux Summation Formula, [Abramowitz, p.335, #8.9.1], or [Gradshteyn, p.986, #8.9.15]. And [Szego, p.43].

$$P_0(\xi)P_0(x)+...+(2n+1)P_n(\xi)P_n(x)=(n+1)\frac{P_{n+1}(\xi)P_n(x)-P_n(\xi)P_{n+1}(x)}{\xi-x}.$$

For $\xi \to x$,

$$P_0(\xi)P_0(x)+...+(2n+1)P_n(\xi)P_n(x)\to P_0^2(x)+...+(2n+1)P_n^2(x),$$

and

$$(n+1)\frac{P_{n+1}(\xi)P_n(x)-P_n(\xi)P_{n+1}(x)}{\xi-x}\to(n+1)\frac{0}{0}.$$

Applying Bernoulli's rule to the indeterminate limit,

$$\lim_{\xi\to x}\frac{P_{n+1}(\xi)P_n(x)-P_n(\xi)P_{n+1}(x)}{\xi-x}=\lim_{\xi\to x}\frac{D_\xi P_{n+1}(\xi)P_n(x)-D_\xi P_n(\xi)P_{n+1}(x)}{D_\xi(\xi-x)}$$

$$=\lim_{\xi\to x}[P_{n+1}{}'(\xi)P_n(x)-P_n{}'(\xi)P_{n+1}(x)]$$

$$=P_{n+1}{}'(x)P_n(x)-P_n{}'(x)P_{n+1}(x)$$

Therefore,

$$P_0^2(x)+...+(2n+1)P_n^2(x)=(n+1)[P_{n+1}{}'(x)P_n(x)-P_n{}'(x)P_{n+1}(x)].$$

Since both $P_n(x)$, and $P_{n+1}(x)$ solve the Legendre differential equation, [Spiegel, p.146],

$$\underbrace{(1-x^2)}_{a(x)}y''(x)\underbrace{-2x}_{b(x)}y'(x)+n(n+1)y(x)=0,$$

we have,

$$P_{n+1}'(x)P_n(x) - P_n'(x)P_{n+1}(x) = (const)e^{-\int \frac{b(x)}{a(x)}dx}$$

$$= (const)e^{\int \frac{2x}{1-x^2}dx}$$

$$= (const)e^{-\log(1-x^2)}$$

$$= \frac{const}{1-x^2} \geq const\,,$$

for any $-1 < x < 1$.

Hence,

$$\tfrac{1}{2}P_0^2(x) + ... + \tfrac{2n+1}{2}P_n^2(x) \geq \tfrac{1}{2}(n+1) \times const\,.$$

and the Legendre Kernel diverges to ∞ at any $\xi = x$.

Therefore, while the partial sums of the Legendre Series exist, their limit does not. That is, due to the singularity at $\xi = x$, the Legendre Series does not converge in the Calculus of Limits.

Avoiding the singularity at $\xi = x$, by using the Cauchy Principal Value of the integral does not recover the Theorem, because at any $\xi \neq x$, the Legendre Kernel vanishes, and the integral will be identically zero, for any function $f(x)$.

To see that the kernel vanishes for any $\xi \neq x$, we apply the Christoffel-Darboux Summation Formula, with $\xi \neq x$.

$$P_0(\xi)P_0(x) + ... + (2n+1)P_n(\xi)P_n(x) = (n+1)\frac{P_{n+1}(\xi)P_n(x) - P_n(\xi)P_{n+1}(x)}{\xi - x}.$$

By Rodrigue’s Formula [Spiegel, p. 146],

$$P_n(x) = \frac{1}{2^n n!} D_x^n (x^2 - 1)^n .$$

$$= \frac{n+1}{\xi - x} \frac{1}{2^{2n+1} n!(n+1)!} \{D_\xi^{n+1}(\xi^2-1)^{n+1} D_x^n (x^2-1)^n - D_\xi^n (\xi^2-1)^n D_x^{n+1}(x^2-1)^{n+1}\}$$

$$= \frac{1}{\xi - x} \underbrace{\frac{1}{2^{2n+1} n! n!}}_{\to 0,\ n\to\infty} \underbrace{\{D_\xi^{n+1}(\xi^2-1)^{n+1} D_x^n (x^2-1)^n - D_\xi^n (\xi^2-1)^n D_x^{n+1}(x^2-1)^{n+1}\}}_{\to 0,\ n\to\infty}$$

$$\to 0, \text{ as } n \to \infty .$$

That is, the Legendre Kernel vanishes for any $\xi \neq x$.

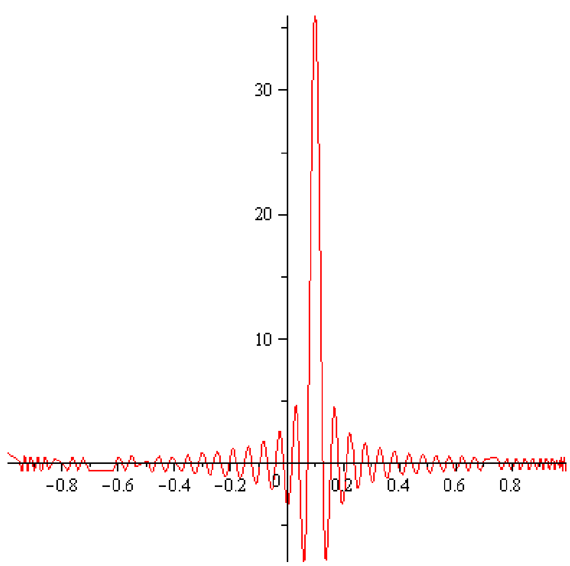
Plots of the Legendre sequence confirm that

In the Calculus of Limits,

the Legendre kernel is either singular or zero

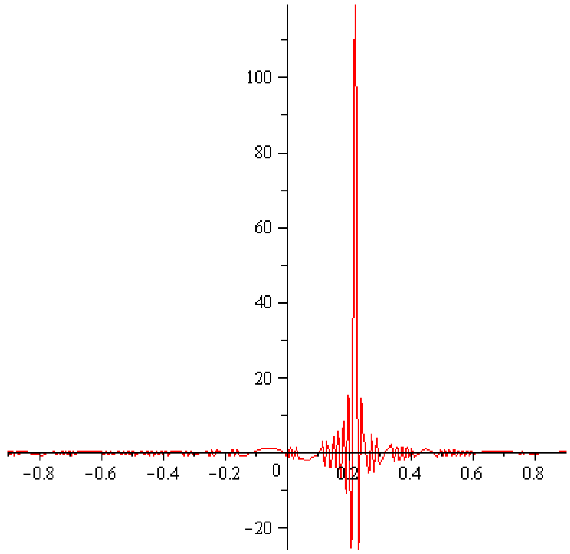
1.2 Plots of $\frac{1}{2} P_0(\xi) P_0(x) + ... + \frac{2\cdot n+1}{2} P_n(\xi) P_n(x)$

In Maple, plot($\sum_{i=0}^{111} \frac{2i+1}{2}$*LegendreP(i, 0.1)*LegendreP(i, x), x = -.99 .. .99)



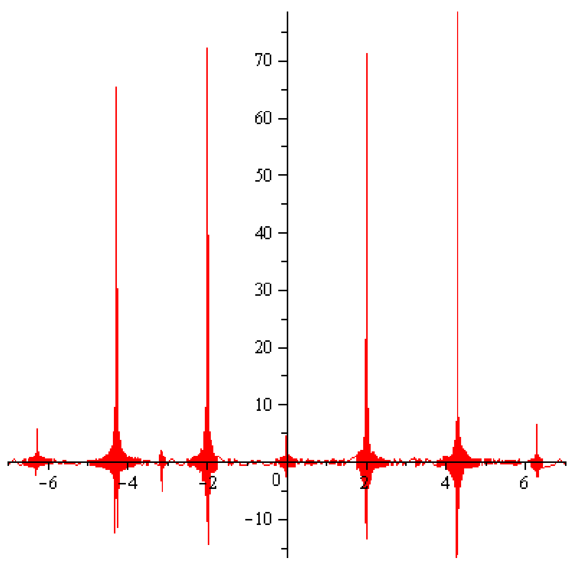
The pulse narrows with more terms:

In Maple, $\text{plot}(\sum_{i=0}^{365} \frac{2i+1}{2}\text{*LegendreP(i, 0.22)*LegendreP(i, x)}, \ x = \text{-.9 .. .9})$

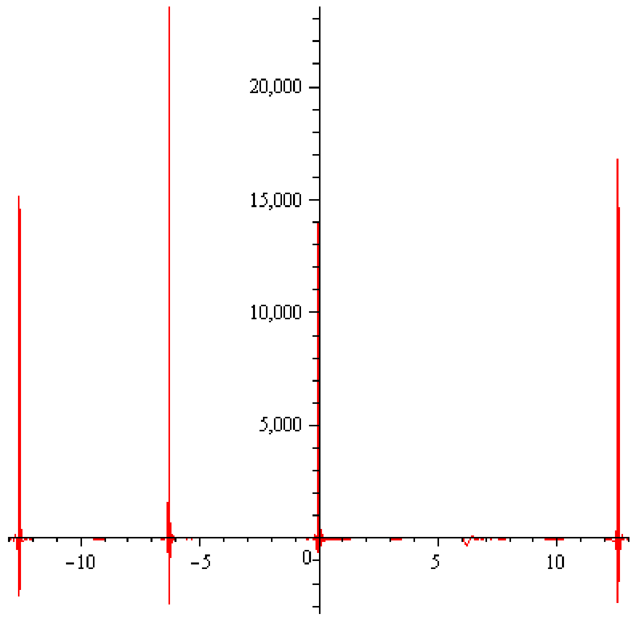


For $\theta = \arccos(x)$, the pulses are periodic

In Maple, $\text{plot}(\sum_{i=0}^{223} \frac{2i+1}{2}\text{*LegendreP(i, cos(2))*LegendreP(i, cos}(\theta))), \ \theta = \text{-7 .. 7})$



In Maple, $\text{plot}(\sum_{i=0}^{223} \frac{2i+1}{2}*\text{LegendreP}(i, \cos(0))*\text{LegendreP}(i, \cos(x)),\ x = -13 .. 13)$



Thus, the Legendre Series Theorem cannot be proved in the Calculus of Limits.

1.3 Infinitesimal Calculus Solution

By resolving the problem of the infinitesimals [Dan2], we obtained the Infinite Hyper-reals that are strictly smaller than ∞, and constitute the value of the Delta Function at the singularity.

The controversy surrounding the Leibnitz Infinitesimals derailed the development of the Infinitesimal Calculus, and the Delta Function could not be defined and investigated properly.

In Infinitesimal Calculus, [Dan3], we can differentiate over jump discontinuities, and integrate over singularities.

The Delta Function, the idealization of an impulse in Radar circuits, is a Discontinuous Hyper-Real function which definition requires Infinite Hyper-reals, and which analysis requires Infinitesimal Calculus.

In [Dan5], we show that in infinitesimal Calculus, the hyper-real

$$\delta(x) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x} d\omega$$

is zero for any $x \neq 0$,

it spikes at $x = 0$, so that its Infinitesimal Calculus integral is $\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1$,

and $\delta(0) = \dfrac{1}{dx} < \infty$.

Here, we show that in Infinitesimal calculus, the Legendre Kernel is a periodic hyper-real Delta Function: A periodic train of Delta Functions.

And the Legendre Series $\mathcal{L}_{egendre}\mathcal{S}\left\{f(x)\right\}$ associated with a Hyper-real periodic function $f(x)$, equals $f(x)$.

2.

Hyper-real Line

Each real number α can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of infinitesimal hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the infinite hyper-reals.
4. The infinite hyper-reals are greater than any real number, yet strictly smaller than infinity.
5. The infinite hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a non-constant hyper-real.
7. The Hyper-reals are the totality of constant hyper-reals, a family of infinitesimals, a family of infinitesimals with

negative sign, a family of infinite hyper-reals, a family of infinite hyper-reals with negative sign, and non-constant hyper-reals.

8. The hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.
9. That line includes the real numbers separated by the non-constant hyper-reals. Each real number is the center of an interval of hyper-reals, that includes no other real number.
10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.
11. Zero is not an infinitesimal, because zero is not strictly greater than zero.
12. We do not add infinity to the hyper-real line.
13. The infinitesimals, the infinitesimals with negative signs, the infinite hyper-reals, and the infinite hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.
14. The hyper-real line is embedded in $\mathbb{R}^\infty$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the hyper-real onto the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal hyper-reals, or to the infinite hyper-reals, or to the non-constant hyper-reals.

16. No neighbourhood of a hyper-real is homeomorphic to an $\mathbb{R}^n$ ball. Therefore, the hyper-real line is not a manifold.

17. The hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

3.

Integral of a Hyper-real Function

In [Dan3], we defined the integral of a Hyper-real Function.

Let $f(x)$ be a hyper-real function on the interval $[a,b]$.

The interval may not be bounded.

$f(x)$ may take infinite hyper-real values, and need not be bounded.

At each

$$a \le x \le b,$$

there is a rectangle with base $[x - \frac{dx}{2}, x + \frac{dx}{2}]$, height $f(x)$, and area

$$f(x)dx.$$

We form the **Integration Sum** of all the areas for the x's that start at $x = a$, and end at $x = b$,

$$\sum_{x \in [a,b]} f(x)dx.$$

If for any infinitesimal dx, the Integration Sum has the same hyper-real value, then $f(x)$ is integrable over the interval $[a,b]$.

Then, we call the Integration Sum the integral of $f(x)$ from $x = a$, to $x = b$, and denote it by

$$\int_{x=a}^{x=b} f(x)dx\,.$$

If the hyper-real is infinite, then it is the integral over $[a,b]$,

If the hyper-real is finite,

$$\int_{x=a}^{x=b} f(x)dx = \text{ real part of the hyper-real}\,.\square$$

3.1 The countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers, $Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty\,.$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[a,b]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(x)dx$.

The Lower Integral is the Integration Sum where $f(x)$ is replaced by its lowest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

3.2
$$\sum_{x \in [a,b]} \left(\inf_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

The Upper Integral is the Integration Sum where $f(x)$ is replaced by its largest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

3.3
$$\sum_{x \in [a,b]} \left(\sup_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

If the integral is a finite hyper-real, we have

3.4 *A hyper-real function has a finite integral if and only if its upper integral and its lower integral are finite, and differ by an infinitesimal*.

4.

Delta Function

In [Dan5], we have defined the Delta Function, and established its properties

1. The Delta Function is a hyper-real function defined from the hyper-real line into the set of two hyper-reals $\left\{0, \frac{1}{dx}\right\}$. The hyper-real 0 is the sequence $\langle 0, 0, 0, ... \rangle$. The infinite hyper-real $\frac{1}{dx}$ depends on our choice of dx.
2. We will usually choose the family of infinitesimals that is spanned by the sequences $\left\langle \frac{1}{n} \right\rangle, \left\langle \frac{1}{n^2} \right\rangle, \left\langle \frac{1}{n^3} \right\rangle, \ldots$ It is a semigroup with respect to vector addition, and includes all the scalar multiples of the generating sequences that are non-zero. That is, the family includes infinitesimals with negative sign. Therefore, $\frac{1}{dx}$ will mean the sequence $\langle n \rangle$. Alternatively, we may choose the family spanned by the sequences $\left\langle \frac{1}{2^n} \right\rangle, \left\langle \frac{1}{3^n} \right\rangle, \left\langle \frac{1}{4^n} \right\rangle, \ldots$ Then, $\frac{1}{dx}$ will mean the

sequence $\langle 2^n \rangle$. Once we determined the basic infinitesimal dx, we will use it in the Infinite Riemann Sum that defines an Integral in Infinitesimal Calculus.

3. The Delta Function is strictly smaller than ∞

4. We define, $\delta(x) \equiv \frac{1}{dx}\chi_{[-\frac{dx}{2},\frac{dx}{2}]}(x)$,

$$\text{where} \quad \chi_{[-\frac{dx}{2},\frac{dx}{2}]}(x) = \begin{cases} 1, x \in \left[-\frac{dx}{2},\frac{dx}{2}\right] \\ 0, \text{otherwise} \end{cases}.$$

5. Hence,

   ❖ for $x < 0$, $\delta(x) = 0$

   ❖ at $x = -\frac{dx}{2}$, $\delta(x)$ jumps from 0 to $\frac{1}{dx}$,

   ❖ for $x \in \left[-\frac{dx}{2},\frac{dx}{2}\right]$, $\delta(x) = \frac{1}{dx}$.

   ❖ at $x = 0$, $\delta(0) = \frac{1}{dx}$

   ❖ at $x = \frac{dx}{2}$, $\delta(x)$ drops from $\frac{1}{dx}$ to 0.

   ❖ for $x > 0$, $\delta(x) = 0$.

   ❖ $x\delta(x) = 0$

6. If $dx = \langle \frac{1}{n} \rangle$, $\delta(x) = \left\langle \chi_{[-\frac{1}{2},\frac{1}{2}]}(x), 2\chi_{[-\frac{1}{4},\frac{1}{4}]}(x), 3\chi_{[-\frac{1}{6},\frac{1}{6}]}(x)... \right\rangle$

7. If $dx = \langle \frac{2}{n} \rangle$, $\delta(x) = \left\langle \frac{1}{2\cosh^2 x}, \frac{2}{2\cosh^2 2x}, \frac{3}{2\cosh^2 3x}, ... \right\rangle$

8. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle e^{-x}\chi_{[0,\infty)}, 2e^{-2x}\chi_{[0,\infty)}, 3e^{-3x}\chi_{[0,\infty)}, ... \right\rangle$

9. $$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1.$$

10. $$\delta(\xi - x) = \frac{1}{2\pi}\int_{k=-\infty}^{k=\infty} e^{-ik(\xi - x)}dk$$

5.

Periodic Delta $\delta_{Periodic}(\theta_\xi - \theta_x)$

5.1 Periodic Delta Function Definition

$$\delta_{Periodic}(\theta_\xi - \theta_x) = ... + \delta(\theta_\xi - \theta_x + 2\pi) + \delta(\theta_\xi - \theta_x) + \delta(\theta_\xi - \theta_x - 2\pi) + ...$$

is a periodic hyper-real Delta function, with period $T = 2\pi$.

5.2 Fourier Transform of $\delta_{Periodic}(\theta)$

$$\mathcal{F}\left\{\delta_{Periodic}(\theta)\right\} = ... + e^{-i4\pi^2\nu} + 1 + e^{i4\pi^2\nu} + ...$$

Proof: $\mathcal{F}\left\{\delta_{Periodic}(\theta)\right\} = ... + \mathcal{F}\left\{\delta(\theta + 2\pi)\right\} + \mathcal{F}\left\{\delta(\theta)\right\} + \mathcal{F}\left\{\delta(\theta - 2\pi)\right\} + ...$

$$= ... + \underbrace{\int_{\theta=-\infty}^{\theta=\infty} \delta(\theta + 2\pi)e^{-i2\pi\nu\theta}d\theta}_{e^{i2\pi2\pi\nu}} + \underbrace{\int_{\theta=-\infty}^{\theta=\infty} \delta(\theta)e^{-i2\pi\nu\theta}d\theta}_{1} + \underbrace{\int_{\theta=-\infty}^{\theta=\infty} \delta(\theta - 2\pi)e^{-i2\pi\nu\theta}d\theta}_{e^{-i2\pi2\pi\nu}} + ...$$

5.3 Fourier Integral Theorem for $\delta_{Periodic}(\theta)$

$$\mathcal{F}^{-1}\mathcal{F}\left\{\delta_{Periodic}(\theta)\right\} = \delta_{Periodic}(\theta)$$

Proof: $\mathcal{F}^{-1}\mathcal{F}\left\{\delta_{Periodic}(\theta)\right\} = ... + \mathcal{F}^{-1}\left\{e^{2\pi i2\pi\nu}\right\} + \mathcal{F}^{-1}\left\{1\right\} + \mathcal{F}^{-1}\left\{e^{-2\pi i2\pi\nu}\right\} + ...$

$$= ... + \underbrace{\int_{\nu=-\infty}^{\nu=\infty} e^{i2\pi\nu2\pi}e^{i2\pi\nu\theta}d\nu}_{\delta(\theta+2\pi)} + \underbrace{\int_{\nu=-\infty}^{\nu=\infty} e^{i2\pi\nu\theta}d\nu}_{\delta(\theta)} + \underbrace{\int_{\nu=-\infty}^{\nu=\infty} e^{-i2\pi\nu2\pi}e^{i2\pi\nu\theta}d\nu}_{\delta(\theta-2\pi)} + ... \square$$

6.

Convergent Series

In [Dan8], we defined convergence of infinite series in Infinitesimal Calculus

6.1 Sequence Convergence to a finite hyper-real a

$$a_n \to a \;\; \textit{iff} \;\; \langle a_n \rangle - \langle a \rangle = \text{infinitesimal}.$$

6.2 Sequence Convergence to an infinite hyper-real A

$$a_n \to A \;\; \textit{iff} \;\; \langle a_n \rangle \text{ represents the infinite hyper-real } A.$$

6.3 Series Convergence to a finite hyper-real s

$$a_1 + a_2 + ... \to s \;\; \textit{iff} \;\; \langle a_1 + ... + a_n \rangle - \langle s \rangle = \text{infinitesimal}.$$

6.4 Series Convergence to an Infinite Hyper-real S

$$a_1 + a_2 + ... \to S \;\; \textit{iff}$$

$$\langle a_1 + ... + a_n \rangle \text{ represents the infinite hyper-real } S.$$

7.

Legendre Sequence and $\delta_{Periodic}(\theta_\xi - \theta_x)$

7.1 Legendre Sequence Definition

The Legendre Series partial sums

$$\mathcal{L}_{egendre}\mathcal{S}_n\left\{f(x)\right\} = \int_{\xi=-1}^{\xi=1} f(\xi)\left\{\tfrac{1}{2}P_0(\xi)P_0(x) + ... + \tfrac{2n+1}{2}P_n(\xi)P_n(x)\right\}d\xi\,.$$

give rise to the Legendre Sequence

$$P_n(\xi,x) = \tfrac{1}{2}P_0(\xi)P_0(x) + \tfrac{3}{2}P_1(\xi)P_1(x) + ... + \tfrac{2n+1}{2}P_n(\xi)P_n(x)\,.$$

7.2 Legendre Sequence is a Periodic Delta Sequence

For each $n = 0,1,2,3,...,$

$$P_n(\xi,x) = \tfrac{1}{2}P_0(\xi)P_0(x) + \tfrac{3}{2}P_1(\xi)P_1(x) + ... + \tfrac{2n+1}{2}P_n(\xi)P_n(x),$$

1. *has the sifting property on each interval,*

$$..\int_{\xi=-3}^{\xi=-1} P_n(\xi,x)d\xi = 1; \quad \int_{\xi=-1}^{\xi=1} P_n(\xi,x)d\xi = 1; \quad \int_{\xi=1}^{\xi=3} P_n(\xi,x)d\xi = 1\,..$$

2. *is a continuous function*

3. *peaks on each of these intervals to* $\lim_{\xi\to x} P_n(\xi,x) \geq (n+1)const.$

Proof of (1)

$$\int_{\xi=-1}^{\xi=1} P_n(\xi,x)d\xi = \int_{\xi=-1}^{\xi=1} \{\tfrac{1}{2}P_0(\xi)P_0(x) + \tfrac{3}{2}P_1(\xi)P_1(x) + ... + \tfrac{2n+1}{2}P_n(\xi)P_n(x)\}d\xi$$

$$= \underbrace{\tfrac{1}{2}\underbrace{P_0(x)}_{1}\underbrace{\int_{\xi=-1}^{\xi=1}\underbrace{P_0(\xi)}_{1}d\xi}_{2}}_{1} + \tfrac{3}{2}P_1(x)\int_{\xi=-1}^{\xi=1} P_1(\xi)d\xi + ... + \tfrac{2n+1}{2}P_n(x)\int_{\xi=-1}^{\xi=1} P_n(\xi)d\xi$$

We show that the terms with $k = 1,2,...,n$, vanish.

By [Spiegel, p.145, #25.34], for $k = 1,2,...,n$,

$$\int_{\xi=-1}^{\xi=1} P_k(\xi)d\xi = \tfrac{1}{2k+1}\Big(P_{k+1}(\xi) - P_{k-1}(\xi)\Big)_{\xi=-1}^{\xi=1}.$$

By [Spiegel, p.145],

$$P_k(1) = 1; \quad P_k(-1) = (-1)^k$$

Therefore,

$$\int_{\xi=-1}^{\xi=1} P_k(\xi)d\xi = \tfrac{1}{2k+1}\underbrace{[\underbrace{P_{k+1}(1)}_{1} - \underbrace{P_{k-1}(1)}_{1}]}_{0} - \tfrac{1}{2k+1}\underbrace{[\underbrace{P_{k+1}(-1)}_{(-1)^{k+1}} - \underbrace{P_{k-1}(-1)}_{(-1)^{k-1}}]}_{0} = 0$$

Hence, for each $n = 0,1,2,3,...$

$$\int_{\xi=-1}^{\xi=1} P_n(\xi,x)d\xi = 1.\square$$

Proof of (3)

By the Christoffel-Darboux Summation Formula, [Abramowitz, p.335, #8.9.1], or [Gradshteyn, p.986, #8.9.15]. And [Szego, p.43].

$$P_0(\xi)P_0(x) + ... + (2n+1)P_n(\xi)P_n(x) = (n+1)\frac{P_{n+1}(\xi)P_n(x) - P_n(\xi)P_{n+1}(x)}{\xi - x}.$$

For $\xi \to x$,

$$P_0(\xi)P_0(x) + ... + (2n+1)P_n(\xi)P_n(x) \to P_0^2(x) + ... + (2n+1)P_n^2(x),$$

and

$$(n+1)\frac{P_{n+1}(\xi)P_n(x) - P_n(\xi)P_{n+1}(x)}{\xi - x} \to (n+1)\frac{0}{0}.$$

Applying Bernoulli's rule to the indeterminate limit,

$$\lim_{\xi \to x}\frac{P_{n+1}(\xi)P_n(x) - P_n(\xi)P_{n+1}(x)}{\xi - x} = \lim_{\xi \to x}\frac{D_\xi P_{n+1}(\xi)P_n(x) - D_\xi P_n(\xi)P_{n+1}(x)}{D_\xi(\xi - x)}$$

$$= \lim_{\xi \to x}[P_{n+1}'(\xi)P_n(x) - P_n'(\xi)P_{n+1}(x)]$$

$$= P_{n+1}'(x)P_n(x) - P_n'(x)P_{n+1}(x)$$

Therefore,

$$P_0^2(x) + ... + (2n+1)P_n^2(x) = (n+1)[P_{n+1}'(x)P_n(x) - P_n'(x)P_{n+1}(x)].$$

Since both $P_n(x)$, and $P_{n+1}(x)$ solve the Legendre differential equation, [Spiegel, p.146],

$$\underbrace{(1 - x^2)}_{a(x)}y''(x)\underbrace{-2x}_{b(x)}y'(x) + n(n+1)y(x) = 0,$$

we have,

$$P_{n+1}'(x)P_n(x) - P_n'(x)P_{n+1}(x) = (const)e^{-\int \frac{b(x)}{a(x)}dx}$$

$$= (const)e^{\int \frac{2x}{1-x^2}dx}$$

$$= (const)e^{-\log(1-x^2)}$$

$$= \frac{const}{1 - x^2} \geq const\,,$$

for any $-1 < x < 1$.

Hence,

$$\tfrac{1}{2}P_0^2(x) + ... + \tfrac{2n+1}{2}P_n^2(x) \geq (n+1) \times const\,.\square$$

8.

Legendre Kernel and $\delta_{Periodic}(\theta_\xi - \theta_x)$

8.1 Legendre Kernel in the Calculus of Limits

The Legendre Series partial sums

$$\mathcal{L}_{egendre}\mathcal{S}_n\left\{f(x)\right\} = \int_{\xi=-1}^{\xi=1} f(\xi)\underbrace{\left\{\tfrac{1}{2}P_0(\xi)P_0(x) + ... + \tfrac{2\cdot n+1}{2}P_n(\xi)P_n(x)\right\}}_{\text{Legendre Sequence}}d\xi\,.$$

give rise to the Legendre Sequence.

The limit of the Legendre Sequence is an infinite series called the Legendre Kernel

$$\tfrac{1}{2}P_0(\xi)P_0(x) + \tfrac{3}{2}P_1(\xi)P_1(x) + ... + \tfrac{2\cdot n+1}{2}P_n(\xi)P_n(x) + ...$$

8.2 *In the Calculus of Limits, the Legendre Kernel does not have the sifting property*

<u>*Proof*</u>: for $\xi \to x$,

$$\tfrac{1}{2}P_0(\xi)P_0(x) + \tfrac{3}{2}P_1(\xi)P_1(x) + ... + \tfrac{2\cdot n+1}{2}P_n(\xi)P_n(x) \ge (n+1)\times const$$

$$\underset{n\to\infty}{\longrightarrow} \infty$$

That is, for $\xi \to x$,

$\tfrac{1}{2}P_0(\xi)P_0(x) + \tfrac{3}{2}P_1(\xi)P_1(x) + .. + \tfrac{2\cdot n+1}{2}P_n(\xi)P_n(x) + ...$is singular.□

8.3 Hyper-real Legendre Kernel in Infinitesimal Calculus

Let $x = \cos\theta_x$, $\xi = \cos\theta_\xi$, $\langle n \rangle$ *an infinite Hyper-real.*

Then $\mathcal{L}_{egendre}(\theta_\xi - \theta_x) =$

$$= \tfrac{1}{2}P_0(\cos\theta_\xi)P_0(\cos\theta_x) + \tfrac{3}{2}P_1(\cos\theta_\xi)P_1(\cos\theta_x) + \tfrac{5}{2}P_2(\cos\theta_\xi)P_2(\cos\theta_x) + \ldots$$

$$= \begin{cases} \langle n \rangle & ,\theta_\xi - \theta_x = 2\pi m \\ 0 & ,\theta_\xi - \theta_x \neq 2\pi m \end{cases}$$

$$= \ldots + \delta(\theta_\xi - \theta_x + 2\pi) + \delta(\theta_\xi - \theta_x) + \delta(\theta_\xi - \theta_x - 2\pi) + \ldots$$

$$= \delta_{Periodic}(\theta_\xi - \theta_x).$$

Proof: $L_{egendre}(\theta_\xi - \theta_x) = \tfrac{1}{2}P_0(\cos\theta_\xi)P_0(\cos\theta_x) + \tfrac{3}{2}P_1(\cos\theta_\xi)P_1(\cos\theta_x) + \ldots$

$$= \begin{cases} \langle n \rangle & ,\theta_\xi - \theta_x = 2\pi m \\ 0 & ,\theta_\xi - \theta_x \neq 2\pi m \end{cases}$$

$$= \ldots + \begin{cases} 0, \theta_\xi - \theta_x \neq -2\pi \\ \frac{1}{d\theta}, \theta_\xi - \theta_x = -2\pi \end{cases} + \begin{cases} 0, \theta_\xi - \theta_x \neq 0 \\ \frac{1}{d\theta}, \theta_\xi - \theta_x = 0 \end{cases} + \begin{cases} 0, \theta_\xi - \theta_x \neq 2\pi \\ \frac{1}{d\theta}, \theta_\xi - \theta_x = 2\pi \end{cases} + \ldots$$

$$= \ldots + \delta(\theta_\xi - \theta_x + 2\pi) + \delta(\theta_\xi - \theta_x) + \delta(\theta_\xi - \theta_x - 2\pi) + \ldots$$

$$= \delta_{Periodic}(\theta_\xi - \theta_x). \square$$

9.

Legendre Series and $\delta_{Periodic}(\theta_\xi - \theta_x)$

9.1 Legendre Series of a Hyper-real Function

Let $f(x)$ be a hyper-real function integrable on $[-1,1]$.

For $x = \cos\theta$, $f(\cos\theta)$ is defined for each real θ.

The powers $x^k = \cos^k\theta$ in the Legendre Polynomial $P_n(\cos\theta)$ can be written in terms of $\cos\theta$, $\cos 2\theta, \dots, \cos n\theta$.

$$P_n(\cos\theta) = \alpha_{n,0} + \alpha_{n,1}\cos\theta + \dots\alpha_{n,n}\cos n\theta.$$

Then, for each $n = 0,1,2,3,\dots$, the integrals

$$a_n = \tfrac{2n+1}{2}\int_{x=-1}^{x=1} f(x)P_n(x)dx$$

exist, with finite, or infinite hyper-real values. The a_n are the Legendre Coefficients of $f(x)$.

The Legendre Series associated with $f(x)$ is

$$\mathcal{L}_{egendre}\mathcal{S}\{f(x)\} = a_0 + a_1P_1(x) + a_2P_2(x) + a_3P_3(x) + \dots$$

$$= a_0 + a_1P_1(\cos\theta) + a_2P_2(\cos\theta) + a_3P_3(\cos\theta) + \dots$$

For each x, it may assume finite or infinite hyper-real values.

9.2 $$\mathcal{L}_{egendre}\mathcal{S}\left\{\delta_{Periodic}(\theta_\xi - \theta_x)\right\} = \delta_{Periodic}(\theta_\xi - \theta_x)$$

Proof:

$$\mathcal{L}_{egendre}\mathcal{S}\left\{\delta_{Periodic}(\theta_\xi - \theta_x)\right\} = a_0 + a_1P_1(x) + a_2P_2(x) + a_3P_3(x)...$$

where

$$a_n = \tfrac{2n+1}{2}\int_{x=-1}^{x=1} \delta_{Periodic}(\theta_\xi - \theta_x)P_n(x)dx\,,$$

Substituting from 8.3,

$$\delta_{Periodic}(\theta_\xi - \theta_x) = \tfrac{1}{2}P_0(\xi)P_0(x) + \tfrac{3}{2}P_1(\xi)P_1(x) + \tfrac{5}{2}P_2(\xi)P_2(x) + ...,$$

$$a_n = \tfrac{2n+1}{2}\int_{x=-1}^{x=1}\left\{\tfrac{1}{2}P_0(\xi)P_0(x) + \tfrac{3}{2}P_1(\xi)P_1(x) + \tfrac{5}{2}P_2(\xi)P_2(x) + ...\right\}P_n(x)dx$$

$$= \tfrac{2n+1}{2}P_0(\xi)\underbrace{\int_{x=-1}^{x=1}\tfrac{1}{2}P_0(x)P_n(x)dx}_{0} + ...$$

$$... + \tfrac{2n+1}{2}P_m(\xi)\underbrace{\int_{x=-1}^{x=1}\tfrac{2m+1}{2}P_m(x)P_n(x)dx}_{\delta_{mn}} + ...$$

$$= \tfrac{2n+1}{2}P_n(\xi).$$

Therefore,

$$\mathcal{L}_{egendre}\mathcal{S}\left\{\delta_{Periodic}(\theta_\xi - \theta_x)\right\} = \tfrac{1}{2}P_0(\xi)P_0(x) + \tfrac{3}{2}P_1(\xi)P_1(x) + \tfrac{5}{2}P_2(\xi)P_2(x) + ...$$

$$= \delta_{Periodic}(\theta_\xi - \theta_x).\square$$

10.

Legendre Series Theorem

The Legendre Series Theorem for a hyper-real function, $f(x)$, is the Fundamental Theorem of Legendre Series.

It supplies the conditions under which the Legendre Series associated with $f(x)$ equals $f(x)$.

It is believed to hold in the Calculus of Limits under the Picone Conditions, or under the Hobson Conditions [Sansone]. In fact,

The Theorem cannot be proved in the Calculus of Limits under any conditions,

because the summation of the Legendre Series requires integration of the singular Legendre Kernel.

10.1 Legendre Series Theorem cannot be proved in the Calculus of Limits

Proof: Let $f(x)$ be integrable on $[-1,1]$, so that $f(1) = f(-1)$

In the Calculus of Limits, the Legendre Series is the limit of

$$\mathcal{L}_{egendre}\mathcal{S}_n\left\{f(x)\right\} = a_0P_0(x) + a_1P_1(x) + ... + a_nP_n(x)$$

$$= \left(\tfrac{1}{2}\int\limits_{\xi=-1}^{\xi=1} f(\xi)P_0(\xi)d\xi\right)P_0(x) + \left(\tfrac{3}{2}\int\limits_{\xi=-1}^{\xi=1} f(\xi)P_1(\xi)d\xi\right)P_1(x) + ... + \left(\tfrac{2n+1}{2}\int\limits_{\xi=-1}^{\xi=1} f(\xi)P_n(\xi)d\xi\right)P_n(x)$$

$$= \int_{\xi=-1}^{\xi=1} f(\xi)\left\{\tfrac{1}{2}P_0(\xi)P_0(x) + \tfrac{3}{2}P_1(\xi)P_1(x) + \ldots + \tfrac{2n+1}{2}P_n(\xi)P_n(x)\right\}d\xi\,.$$

As $n \to \infty$, the Legendre Sequence

$$P_n(\xi, x) = \tfrac{1}{2}P_0(\xi)P_0(x) + \tfrac{3}{2}P_1(\xi)P_1(x) + \ldots + \tfrac{2n+1}{2}P_n(\xi)P_n(x)$$

becomes the Legendre Kernel, the infinite series

$$\tfrac{1}{2}P_0(\xi)P_0(x) + \tfrac{3}{2}P_1(\xi)P_1(x) + \ldots + \tfrac{2n+1}{2}P_n(\xi)P_n(x) + \ldots$$

By 8.2, The Legendre Kernel diverges to infinity at any $\theta_\xi - \theta_x = 2\pi m$. In particular,

$$x = \xi \Rightarrow \theta_\xi = \theta_x\,,$$

and the Legendre Kernel diverges to infinity.

Therefore, while the partial sums of the Legendre Series exist, their limit does not. Picone and Hobson failed to comprehend the sifting through the values of $f(\xi)$ by the Legendre Kernel, and the picking of $f(\xi)$ at $\xi = x$.

Avoiding the singularity at $\xi = x$, by using the Cauchy Principal Value of the integral does not recover the Theorem, because for any $\xi \neq x$, the Legendre Kernel vanishes, and the integral is identically zero, for any function $f(x)$.

Thus, the Legendre Series Theorem cannot be proved in the Calculus of Limits.□

10.2 Calculus of Limits Conditions are irrelevant to Legendre Series Theorem

Proof: The Picone Conditions [Sansone, p.203] are

1. $f(x)$ integrable in $[-1,1]$
2. $(1-x^2)f(x)$ of bounded variation in $[-1,1]$

It is clear from 10.1 that these conditions on $f(x)$ do not resolve the singularity of the Legendre kernel, and are not sufficient for the Legendre Series Theorem.□

The Hobson proof avoids the singularity at $\xi = x$, and interprets the resulting 0, as $f(x)$.□

In Infinitesimal Calculus, by 8.3, the Legendre Kernel is the Periodic Delta Function, and by 9.2, it equals its Legendre Series. Then, the Legendre Series Theorem holds for any periodic Hyper-Real Function:

10.3 Legendre Series Theorem for periodic Hyper-real $f(x)$

If $f(x)$ is hyper-real function integrable on $[-1,1]$, and $f(-1) = f(1)$

Then,
$$f(x) = \mathcal{L}_{egendre}\mathcal{S}\left\{f(x)\right\}$$

Proof:

$$f(x) = \int_{\xi=-1}^{\xi=1} f(\xi)\underbrace{\left\{... + \delta(\xi - x + 2) + \delta(\xi - x) + \delta(\xi - x - 2) + ...\right\}}_{\delta_{Periodic}(\xi - x),\text{ where the period of Delta is } 2} d\xi$$

$$= \int_{\theta_\xi=-\infty}^{\theta_\xi=\infty} f(\cos\theta_\xi)\underbrace{\left\{... + \delta(\theta_\xi - \theta_x + 2\pi) + \delta(\theta_\xi - \theta_x) + \delta(\theta_\xi - \theta_x - 2\pi) + ...\right\}}_{\delta_{Periodic}(\theta_\xi - \theta_x),\text{ where the period of Delta is } 2\pi} d\theta_\xi$$

By 8.2, $\delta_{Periodic}(\theta_\xi - \theta_x) = \mathcal{L}_{egendre}(\theta_\xi - \theta_x)$

$$= \int_{\theta_\xi=-\infty}^{\theta_\xi=\infty} f(\cos\theta_\xi)\left\{\tfrac{1}{2}P_0(\cos\theta_\xi)P_0(\cos\theta_x) + \tfrac{3}{2}P_1(\cos\theta_\xi)P_1(\cos\theta_x) + ...\right\} d\theta_\xi$$

$$= \int_{\xi=-1}^{\xi=1} f(\xi)\{\tfrac{1}{2}P_0(\xi)P_0(x) + \tfrac{3}{2}P_1(\xi)P_1(x) + \tfrac{5}{2}P_2(\xi)P_2(x) + ...\} d\xi$$

This Hyper-real Integral is the summation,

$$\sum_{\xi=-1}^{\xi=1} f(\xi)\{\tfrac{1}{2}P_0(\xi)P_0(x) + \tfrac{3}{2}P_1(\xi)P_1(x) + \tfrac{5}{2}P_2(\xi)P_2(x) + ...\} d\xi$$

which amounts to the hyper-real function $f(x)$,and is well-defined.

Hence, the summation of each term in the integrand exists, and we may write the integral as the sum

$$= \underbrace{\left(\tfrac{1}{2}\int_{\xi=-1}^{\xi=1} f(\xi)P_0(\xi)d\xi\right)}_{a_0} P_0(x) + \underbrace{\left(\tfrac{3}{2}\int_{\xi=-1}^{\xi=1} f(\xi)P_1(\xi)d\xi\right)}_{a_1} P_1(x) + \underbrace{\left(\tfrac{5}{2}\int_{\xi=-1}^{\xi=1} f(\xi)P_2(\xi)d\xi\right)}_{a_2} P_2(x) + ...$$

$$= a_0P_0(x) + a_1P_1(x) + a_2P_2(x) + ...$$

$$= \mathcal{L}_{egendre}\mathcal{S}\left\{f(x)\right\}.\square$$

In particular, the periodic Delta Function violates the Picone Conditions

- *The Hyper-real* $\delta(x)$*,is not defined in the Calculus of Limits, and is not integrable in* $[-1,1]$.
- $(1-x^2)\delta(x)$ *is not of bounded variation in* $[-1,1]$.

But by 9.2, $\delta_{Periodic}(x)$ satisfies the Legendre Series Theorem.

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