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Infinitesimal Calculus

Infinitesimals **H. Vic Dannon**

Abstract: Leibnitz wrote the derivative as a quotient of two differentials,

$$f'(x) = \frac{df}{dx}.$$

The differential df is the difference between two extremely close values of f, attained on two extremely close values of x. The differential dx is the difference between those values of x.

Since $f'(x)$ may vanish, $df = f'(x)dx$ can vanish. But dx cannot vanish because division by zero is undefined.

The differential dx is a positive infinitesimal. it is smaller than any positive real number, yet it is greater than zero.

That characterization was met with understandable scepticism.

How can there be positive numbers that are smaller than any positive number and yet are greater than zero?

The concept of a limit seems to answer that question.

$f'(x_0) = \frac{df}{dx}(x_0)$ means that if for any sequence $x_n \to x_0$, so that $x_n \neq x_0$, the sequence of the quotient differences

$$\frac{f(x_n) - f(x_0)}{x_n - x_0}$$

converges to a number A, then $f'(x_0) = A$.

In other words, the positive differential dx may be interpreted as a shorthand for all the sequences

$$a_n = x_n - x_0, \text{ so that } a_n > 0, \text{ and } a_n \to 0.$$

However, the real numbers have the Archimedean property.

It means that if $a > 0$, and $b > 0$, are real numbers so that

$$a < b,$$

then there is some natural number N_0, so that

$$N_0 a > b.$$

This property does not hold for the two sequences

$$\left\langle \frac{1}{n} \right\rangle, \text{ and } \left\langle \frac{1}{n^2} \right\rangle$$

because there is no N_0 so that $N_0 \frac{1}{n^2}$ is strictly greater than $\frac{1}{n}$, for all $n = 1,2,3,\ldots$

Consequently, the Archimedean real number system does not contain infinitesimals, and the addition of infinitesimals to the real numbers, extends the real numbers into a Non-Archimedean number system.

Thus, to establish the existence of infinitesimals, we have to construct a system of the real numbers, that includes the Non-Archimedean infinitesimals. A Hyper-Reals Number system that will contain infinitesimal hyper-reals.

To obtain infinitesimals, Schmieden, and Laugwitz [Laugwitz], added in 1958, an infinity to the rational numbers.

"…adjoined to the field of rational numbers

an infinitely large natural number Ω…"

Clearly, this must be the first infinite ordinal

$$\omega,$$

that is equal to the first infinite Cardinal

$$Card\mathbb{N}.$$

According to [Laugwitz],

> *To adjoin Ω was meant in a more general sense than in algebra but defined by the following* <u>*Postulate:*</u>
>
> *Whatever is true for all finite large natural numbers will, by definition, remain true for Ω.*
>
> *More precisely, if $A(n)$ is a formula that is true for all sufficiently large finite natural n, then $A(\Omega)$ is true. (Leibnitz Principle)*
>
> *Since $n > 10^{100}$ for all large n it follows that $\Omega > 10^{100}$, etc.*

Thus Ω is indeed infinitely large.

Since n^n, and 2^{n^2} are natural numbers, and since $n^n < 2^{n^2}$ for all large n, the inequality

$$\Omega^\Omega < 2^{\Omega^2}$$

is true, and both sides are infinitely large natural numbers.

The Schmieden and Laugwitz Postulate is wrong.

What is true for natural numbers, need not hold for infinities. For instance, for any natural number n

$$n < n + n$$

But it is well known that

$$Card\mathbb{N} = Card\mathbb{N} + Card\mathbb{N}$$

Similarly, while $n^n < 2^{n^2}$, we have $\Omega^\Omega = 2^{\Omega^2}$.

Indeed, it is well known that

$$\Omega^\Omega = (Card\mathbb{N})^{Card\mathbb{N}} = 2^{Card\mathbb{N}} = 2^{(Card\mathbb{N})^2} = 2^{\Omega^2}.$$

Consequently, Schmieden and Laugwitz approach to the infinitesimals is founded on ignorance of infinities, and has to be rejected.

In 1961, Robinson produced, in the spirit of the Schmieden and Laugwitz, the

Transfer Axiom.

It says [Keisler, p.908],

Every real statement that holds for all real numbers holds for all hyper-real numbers.

How can a generalization be restricted the same as the special case?

The Transfer Axiom is a guess that guarantees that conjecture from the special to the general, must be true, contradictory to logical deduction.

To ensure the validity of his theory, Robinson stated his

Extension Axiom

that claims amongst other things [Keisler, p. 906]

There is a positive hyper-real number that is smaller than any positive real number.

In other words, Robinson's theory of Non Standard Analysis produces infinitesimals, by postulating them, and determines their properties by adding the right axioms.

This wrong answer to Leibnitz problem of infinitesimals demonstrates the divide between Robinson's Logic, and Mathematical Analysis. Logicians never constructed Leibnitz infinitesimals and proved their being on a line. They only postulated it.

Here, we observe that the solution to the infinitesimals problem was overlooked when the real numbers were constructed from Cauchy sequences.

That is, when we represent the Real Line as the infinite dimensional space of all the Cauchy sequences of rational numbers, we can construct families of infinitesimal hyper-reals.

In the following, we investigate the structure and properties of such families of infinitesimal hyper-reals, and construct the associated families of their reciprocals, the infinite hyper-reals.