Delta Function, and Expansion in Hermite Functions

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June, 2012

Abstract Let $f(x)$ be defined on the real numbers, and let $H_n(x)$ be the Hermite Polynomials on the real numbers,

$$H_0(x) = 1,\ H_1(x) = 2x,\ H_2(x) = 4x^2 - 2,\ H_3(x) = 8x^3 - 12x, \ldots$$

The Hermite Series associated with $f(x)$ is

$$a_0 H_0(x) + a_1 H_1(x) + a_2 H_2(x) + \ldots.$$

where

$$a_n = \frac{1}{2^n n! \sqrt{\pi}} \int_{\xi=-\infty}^{\xi=\infty} e^{-\xi^2} f(\xi) H_n(\xi) d\xi$$

are the Hermite coefficients.

The Hermite Series Theorem supplies the conditions under which the Hermite Series associated with $f(x)$ equals $f(x)$.

It is believed to hold in the Calculus of Limits for smooth enough function. In fact,

The Theorem cannot be proved in the Calculus of Limits under any conditions,

because the summation of the Hermite Series requires integration of the singular Hermite Kernel.

Plots of partial sums of the Hermite Series speak volumes about the sensibility of the claims to have infinity bound by epsilon.

In Infinitesimal Calculus, the Hermite Kernel

$$\frac{1}{\sqrt{\pi}} e^{-\xi^2} \left\{ H_0(\xi) H_0(x) + \ldots + \frac{1}{2^n n!} H_n(\xi) H_n(x) + \ldots \right\}$$

is the Delta Function, $\delta(\xi - x)$.

$\delta(\xi - x)$ equals its Hermite Series, and the Hermite Series associated with any hyper-real integrable $f(x)$, equals $f(x)$

Keywords: Infinitesimal, Infinite-Hyper-Real, Hyper-Real, infinite Hyper-real, Infinitesimal Calculus, Delta Function, Hermite Polynomials, Hermite Coefficients, Delta Function, Hermite Series, Hermite Kernel, Expansion in Hermite Functions,

2000 Mathematics Subject Classification 26E35; 26E30; 26E15; 26E20; 26A06; 26A12; 03E10; 03E55; 03E17; 03H15; 46S20; 97I40; 97I30.

Contents

The Origin of the Hermite Series Theorem

The Hermite Polynomials on $(-\infty, \infty)$

$$H_0(x) = 1,\ H_1(x) = 2x,\ H_2(x) = 4x^2 - 2,\ H_3(x) = 8x^3 - 12x, \ldots,$$

are orthogonal so that

$$\int_{x=-\infty}^{x=\infty} e^{-x^2} H_m(x) H_n(x) dx = 2^n n! \sqrt{\pi} \delta_{mn}.$$

The Hermite Polynomials can be generated by expanding

$$e^{2x\alpha-\alpha^2} = 1 + [2x\alpha - \alpha^2] + \tfrac{1}{2!}[2x\alpha - \alpha^2]^2 + \tfrac{1}{3!}[2x\alpha - \alpha^2]^3 + \ldots$$

$$= 1 + 2x\alpha - \alpha^2 + \tfrac{1}{2!}[4x^2\alpha^2 - 4x\alpha^3 + \alpha^4] +$$

$$+ \tfrac{1}{3!}[8x^3\alpha^3 - 12x^2\alpha^4 + 6x\alpha^5 - \alpha^6] + \ldots$$

the coefficient of $\frac{1}{0!}\alpha^0$ is

$$H_0(x) = 1,$$

the coefficient of $\frac{1}{1!}\alpha^1$ is

$$H_1(x) = 2x,$$

the coefficient of $\frac{1}{2!}\alpha^2$ is

$$H_2(x) = 4x^2 - 2,$$

the coefficient of $\frac{1}{3!}\alpha^3$ is

$$H_3(x) = 8x^3 - 12x,$$

....

0.1 Schrodinger Equation for atomic size particle in linear harmonic motion

An atomic size particle with mass m, oscillates along a segment of wire $[-A, A]$, at frequency ν, under the force $-kx$.

The particle's position is

$$x(t) = A\cos\omega t\,, \quad \omega = 2\pi\nu\,.$$

Thus,

$$\dot{x} = -\omega A\sin\omega t$$

$$\ddot{x} = -\omega^2 A\cos\omega t = -\omega^2 x$$

The force equation is

$$-kx = m\ddot{x} = m(-\omega^2 x)\,.$$

Hence, the force constant is

$$k = m\omega^2\,,$$

and the potential energy of the particle is

$$V = \tfrac{1}{2}kx^2 = \tfrac{1}{2}m\omega^2 x^2\,.$$

De Broglie associated with the moving particle a wave of length

$$\lambda = \frac{h}{mv}\,,$$

where v is the velocity of the particle, and h is Planck's constant.

The wave's frequency is

$$\nu = \frac{v}{\lambda} = \frac{v}{\frac{h}{mv}} = \frac{mv^2}{h}\,.$$

The wave's angular frequency is

$$\omega = 2\pi\nu = 2\pi \frac{mv^2}{h}$$

In terms of the De Broglie wave, the particle's energy is a multiple of Planck's radiation energy,

$$E = \varepsilon h\nu = \varepsilon \hbar\omega, \quad \hbar = \tfrac{h}{2\pi}, \ \varepsilon \text{ is the multiplier.}$$

The kinetic energy of the particle is

$$\tfrac{1}{2}mv^2 = E - V.$$

Hence,

$$mv = \sqrt{2m(E - V)},$$

$$\lambda = \frac{h}{\sqrt{2m(E - V)}},$$

$$v = \lambda \underbrace{\nu}_{\frac{1}{2\pi}\omega} = \frac{\hbar\omega}{\sqrt{2m(E - V)}}.$$

$$\frac{1}{v^2} = \frac{2m(E - V)}{\hbar^2\omega^2}$$

Schrodinger postulated a complex valued potential

$$\Psi(x,t) = \psi(x)e^{i\omega t}$$

that satisfies the wave equation

$$\partial_x^2 \Psi(x,t) = \frac{1}{v^2}\partial_t^2 \Psi(x,t).$$

Then,

$$0 = \partial_x^2 \Psi(x,t) - \frac{1}{v^2}\partial_t^2 \Psi(x,t)$$

$$= \psi''(x)e^{i\omega t} - \frac{2m(E-V)}{\hbar^2\omega^2}\psi(x)(-\omega^2)e^{i\omega t}.$$

The Schrodinger equation for the linear harmonic oscillator is

$$\psi''(x) + \frac{2m}{\hbar^2}(E-V)\psi(x) = 0.$$

Substituting E, and V,

$$\psi'' + \frac{2m}{\hbar^2}(\varepsilon\hbar\omega - \tfrac{1}{2}m\omega^2x^2)\psi = 0$$

Multiplying by $\dfrac{\hbar}{m\omega}$,

$$\frac{\hbar}{m\omega}\psi''(x) + (2\varepsilon - \underbrace{\frac{m\omega}{\hbar}x^2}_{\xi^2})\psi(x) = 0.$$

The change of variable $\xi = \sqrt{\frac{m\omega}{\hbar}}x$, gives

$$\frac{d\psi}{dx} = \frac{d\psi}{d\xi}\frac{d\xi}{dx} = \psi'(\xi)\sqrt{\tfrac{m\omega}{\hbar}},$$

$$\frac{d^2\psi}{dx^2} = \frac{d}{d\xi}\left\{\psi'(\xi)\sqrt{\tfrac{m\omega}{\hbar}}\right\}\frac{d\xi}{dx} = \psi''(\xi)\tfrac{m\omega}{\hbar},$$

and the equation becomes

$$\psi''(\xi) + (2\varepsilon - \xi^2)\psi(x) = 0.$$

0.2 Hermite Differential Equation

The Schrodinger equation

$$\psi''(\xi) - \xi^2\psi(x) = -2\varepsilon\psi(x)$$

can be factored

$$(D_\xi - \xi)(D_\xi + \xi)\psi(x) = -2\varepsilon\psi(x).$$

To solve the homogeneous equation

$$(D_\xi - \xi)(D_\xi + \xi)\psi(x) = 0,$$

we solve

$$(D_\xi - \xi)\psi(x) = 0 \Rightarrow \frac{\psi'}{\psi} = \xi \Rightarrow \log\psi = \tfrac{1}{2}\xi^2 + c \Rightarrow \psi_1 = Ce^{\frac{1}{2}\xi^2}.$$

As $\xi \to \infty$, $\psi_1 \to \infty$, and is discarded.

$$(D_\xi + \xi)\psi(x) = 0 \Rightarrow \frac{\psi'}{\psi} = -\xi \Rightarrow \log\psi = -\tfrac{1}{2}\xi^2 + c \Rightarrow \psi_2 = Ce^{-\frac{1}{2}\xi^2}.$$

Now, substituting

$$\psi(\xi) = H(\xi)e^{-\frac{1}{2}\xi^2}$$

in $\psi''(\xi) + (2\varepsilon - \xi^2)\psi(\xi) = 0$, we have

$$0 = D_\xi^2\left(H(\xi)e^{-\frac{1}{2}\xi^2}\right) + (2\varepsilon - \xi^2)H(\xi)e^{-\frac{1}{2}\xi^2}$$

$$= D_\xi\left(H'(\xi)e^{-\frac{1}{2}\xi^2} - \xi H(\xi)e^{-\frac{1}{2}\xi^2}\right) + (2\varepsilon - \xi^2)H(\xi)e^{-\frac{1}{2}\xi^2}$$

$$= H''(\xi)e^{-\frac{1}{2}\xi^2} - 2\xi H'(\xi)e^{-\frac{1}{2}\xi^2} - H(\xi)e^{-\frac{1}{2}\xi^2} + \xi^2 H(\xi)e^{-\frac{1}{2}\xi^2} +$$

$$+(2\varepsilon - \xi^2)H(\xi)e^{-\frac{1}{2}\xi^2}$$

$$= \left[H''(\xi) - 2\xi H'(\xi) + (2\varepsilon - 1)H(\xi)\right]e^{-\frac{1}{2}\xi^2}.$$

The Schrodinger equation becomes Hermit Differential Equation

$$H''(\xi) - 2\xi H'(\xi) + (2\varepsilon - 1)H(\xi) = 0,$$

Substituting in it

$$H(\xi) = c_0 + c_1\xi + c_2\xi^2 + ... + c_l\xi^l + c_{l+1}\xi^{l+1} + c_{l+2}\xi^{l+2} + ...,$$

we have

$$\underbrace{D_\xi^2 \sum_{l=0}^{l=\infty} c_l \xi^l}_{\sum_{l=2}^{l=\infty}(l-1)lc_l\xi^{l-2}} - 2\xi \underbrace{D_\xi \sum_{l=0}^{l=\infty} c_l \xi^l}_{\sum_{l=1}^{l=\infty} lc_l\xi^{l-1}} + (2\varepsilon - 1)\sum_{l=0}^{l=\infty} c_l \xi^l = 0,$$

$$\sum_{l=0}^{l=\infty} \{(l+1)(l+2)c_{l+2} - 2lc_l + (2\varepsilon-1)c_l\}\xi^l = 0,$$

$$(l+1)(l+2)c_{l+2} - [2l + 1 - 2\varepsilon]c_l = 0$$

$$c_{l+2} = \frac{2l+1-2\varepsilon}{(l+1)(l+2)} c_l$$

The solution is

$$H(\xi) = c_0 + c_1\xi + c_0 \tfrac{1-2\varepsilon}{1\cdot 2}\xi^2 + c_1 \tfrac{3-2\varepsilon}{2\cdot 3}\xi^3 +$$

$$+c_0 \tfrac{(1-2\varepsilon)(5-2\varepsilon)}{1\cdot 2\cdot 3\cdot 4}\xi^4 + c_1 \tfrac{(3-2\varepsilon)(7-2\varepsilon)}{2\cdot 3\cdot 4\cdot 5}\xi^5 + ...$$

$$= c_0\{1 + \tfrac{1-2\varepsilon}{1\cdot 2}\xi^2 + \tfrac{(1-2\varepsilon)(5-2\varepsilon)}{1\cdot 2\cdot 3\cdot 4}\xi^4 + ...\} +$$

$$+c_1\xi\{1 + \tfrac{3-2\varepsilon}{2\cdot 3}\xi^2 + \tfrac{(3-2\varepsilon)(7-2\varepsilon)}{2\cdot 3\cdot 4\cdot 5}\xi^4 + ...\}.$$

To keep the solution from diverging at $\xi \to \infty$,

for $n = 2k$, the c_0 series terms vanish for

$$2\varepsilon = 1, 5, 9, 13, ...4k+1, ...,$$

and we obtain the $H_{2k}(\xi)$ Hermite Polynomials.

for $n = 2k+1$, the c_1 series terms vanish for

$$2\varepsilon = 3, 7, 11, ..., 4k+3, ...$$

and we obtain the $H_{2k+1}(\xi)$ Hermite Polynomials.

A solution for $\psi(\xi)$ is the infinite linear combination

$$\alpha_0 H_0(\xi) e^{-\frac{1}{2}\xi^2} + \alpha_1 H_1(\xi) e^{-\frac{1}{2}\xi^2} + \alpha_2 H_2(\xi) e^{-\frac{1}{2}\xi^2} + ...$$

0.3 The Hermite Series Associated with $f(x)$

Let $f(x)$ be defined on $(-\infty, \infty)$, and let $H_n(x)$ be the Hermite Polynomials

$$H_0(x) = 1,\ H_1(x) = 2x,\ H_2(x) = 4x^2 - 2,\ H_3(x) = 8x^3 - 12x, ...$$

The Polynomials are orthogonal on $(-\infty, \infty)$. That is,

$$\int_{x=-\infty}^{x=\infty} e^{-x^2} H_m(x) H_n(x) dx = 2^n n! \sqrt{\pi} \delta_{mn}$$

We define the *Orthonormalized Hermite Functions*

$$\varphi_n(x) = \frac{1}{(2^n n! \sqrt{\pi})^{\frac{1}{2}}} e^{-\frac{1}{2}x^2} H_n(x)$$

If $f(x)$ can be expanded in the $\varphi_n(x)$,

$$f(x) = \alpha_0 \varphi_0(x) + \alpha_1 \varphi_1(x) + \alpha_2 \varphi_2(x) + ...,$$

Then,

$$\int_{x=-\infty}^{x=\infty} f(x)\varphi_n(x)dx = \int_{x=-\infty}^{x=\infty} \{\alpha_0\varphi_0(x) + \alpha_1\varphi_1(x) + \alpha_2\varphi_2(x) + ...\}\varphi_n(x)dx$$

$$= \alpha_0 \underbrace{\int_{x=-\infty}^{x=\infty} \varphi_0(x)\varphi_n(x)dx}_{\delta_{0n}} + \alpha_1 \underbrace{\int_{x=-\infty}^{x=\infty} \varphi_1(x)\varphi_n(x)dx}_{\delta_{1n}} + \alpha_2 \underbrace{\int_{x=-\infty}^{x=\infty} \varphi_2(x)\varphi_n(x)dx}_{\delta_{2n}} + ..$$

$$= \alpha_n .$$

Thus, the Hermite coefficients with respect to the $\varphi_n(x)$ are

$$\alpha_n = \int_{\xi=-\infty}^{\xi=\infty} f(\xi)\varphi_n(\xi)d\xi .$$

The Orthonormal Hermite Series associated with $f(x)$ is

$$\alpha_0\varphi_0(x) + \alpha_1\varphi_1(x) + \alpha_2\varphi_2(x) + ...$$

1.

Divergence of the Hermit Kernel in the Calculus of Limits

Calculus of Limits Conditions for the Hermite Series to equal its function reflect the belief that a smooth enough function equals its Hermite Series.

In fact, in the Calculus of Limits, no smoothness of the function guarantees even the convergence of the Hermite Series.

1.1 The Hermite Kernel is either singular or zero

In the Calculus of Limits, the Hermite Series is the limit of the sequence of Partial Sums

$$\mathcal{H}_{ermite}\mathcal{S}_n\left\{f(x)\right\} = \alpha_0\varphi_0(x) + ... + \alpha_n\varphi_n(x)$$

$$= \left(\int_{\xi=-\infty}^{\xi=\infty} f(\xi)\varphi_0(\xi)d\xi\right)\varphi_0(x) + .. + \left(\int_{\xi=-\infty}^{\xi=\infty} f(\xi)\varphi_n(\xi)d\xi\right)\varphi_n(x)$$

$$= \int_{\xi=-\infty}^{\xi=\infty} f(\xi)\left\{\varphi_0(\xi)\varphi_0(x) + ... + \varphi_n(\xi)\varphi_n(x)\right\}d\xi\,.$$

As $n \to \infty$, the orthonormal Hermite Sequence

$$\varphi_0(\xi)\varphi_0(x) + ... + \varphi_n(\xi)\varphi_n(x)$$

becomes the orthonormal Hermite Kernel,

$$\varphi_0(\xi)\varphi_0(x) + ... + \varphi_n(\xi)\varphi_n(x) + ...,$$

To see that it is singular at $\xi = x$, we apply the Christoffel Summation Formula, [Sansone, p.371],

$$\varphi_0(\xi)\varphi_0(x) + ... + \varphi_n(\xi)\varphi_n(x) = \sqrt{\frac{n+1}{2}}\frac{\varphi_{n+1}(\xi)\varphi_n(x) - \varphi_n(\xi)\varphi_{n+1}(x)}{\xi - x}.$$

For $\xi \to x$,

$$\varphi_0(\xi)\varphi_0(x) + ... + \varphi_n(\xi)\varphi_n(x) \to \varphi_0^2(x) + ... + \varphi_n^2(x),$$

and

$$\sqrt{\frac{n+1}{2}}\frac{\varphi_{n+1}(\xi)\varphi_n(x) - \varphi_n(\xi)\varphi_{n+1}(x)}{\xi - x} \to \sqrt{\frac{n+1}{2}}\frac{0}{0}.$$

Applying Bernoulli's rule to the indeterminate limit,

$$\lim_{\xi\to x}\frac{\varphi_{n+1}(\xi)\varphi_n(x) - \varphi_n(\xi)\varphi_{n+1}(x)}{\xi - x} = \lim_{\xi\to x}\frac{D_\xi\varphi_{n+1}(\xi)\varphi_n(x) - D_\xi\varphi_n(\xi)\varphi_{n+1}(x)}{D_\xi(\xi - x)}$$

$$= \lim_{\xi\to x}[\varphi_{n+1}'(\xi)\varphi_n(x) - \varphi_n'(\xi)\varphi_{n+1}(x)]$$

$$= \varphi_{n+1}'(x)\varphi_n(x) - \varphi_n'(x)\varphi_{n+1}(x)$$

Therefore,

$$\varphi_0^2(x) + ... + \varphi_n^2(x) = \sqrt{\frac{n+1}{2}}[\varphi_{n+1}'(x)\varphi_n(x) - \varphi_n'(x)\varphi_{n+1}(x)].$$

Since $\varphi_n(x)$, and $\varphi_{n+1}(x)$ solve the differential equation, [Szego, p.105, #5.5.2],

$$\underbrace{1}_{a(x)} \cdot z''(x) + \underbrace{0}_{b(x)} \cdot z'(x) + (2n+1-x^2)z(x) = 0,$$

we have,

$$\varphi_{n+1}{}'(x)\varphi_n(x) - \varphi_n{}'(x)\varphi_{n+1}(x) = (const)e^{-\int \frac{b(x)}{a(x)}dx}$$

$$= (const)e^{\int 0 dx}$$

$$= const,$$

for any $-\infty < x < \infty$.

Hence,

$$\varphi_0^2(x) + ... + \varphi_n^2(x) = \sqrt{\frac{n+1}{2}} const$$

and the Hermite Kernel diverges to ∞ at any $\xi = x$.

Therefore, while the partial sums of the Hermite Series exist, their limit does not. That is, due to the singularity at $\xi = x$, the Hermite Series does not converge in the Calculus of Limits.

Avoiding the singularity at $\xi = x$, by using the Cauchy Principal Value of the integral does not recover the Theorem, because at any $\xi \neq x$, the Hermite Kernel vanishes, and the integral will be identically zero, for any function $f(x)$.

To see that the kernel vanishes for any $\xi \neq x$, we apply the Christoffel Summation Formula, with $\xi \neq x$.

$$\varphi_0(\xi)\varphi_0(x) + ... + \varphi_n(\xi)\varphi_n(x) = \sqrt{\tfrac{n+1}{2}} \frac{\varphi_{n+1}(\xi)\varphi_n(x) - \varphi_n(\xi)\varphi_{n+1}(x)}{\xi - x}.$$

We have

$$\varphi_n(x) = \frac{1}{(2^n n!\sqrt{\pi})^{\frac{1}{2}}} e^{-\frac{1}{2}x^2} H_n(x)$$

$$= \frac{1}{(2^n n!\sqrt{\pi})^{\frac{1}{2}}} e^{\frac{1}{2}x^2} e^{-x^2} H_n(x)$$

By [Szego, p. 105, #5.5.3],

$$e^{-x^2} H_n(x) = (-1)^n D_x^n e^{-x^2}.$$

Thus,

$$\varphi_n(x) = \frac{(-1)^n}{(2^n n!\sqrt{\pi})^{\frac{1}{2}}} e^{\frac{1}{2}x^2} D_x^n\{e^{-x^2}\},$$

$$\varphi_{n+1}(x) = \frac{(-1)^{n+1}}{(2^{n+1}(n+1)!\sqrt{\pi})^{\frac{1}{2}}} e^{\frac{1}{2}x^2} D_x^{n+1}\{e^{-x^2}\}$$

and

$$\sqrt{\tfrac{n+1}{2}}\,\frac{\varphi_{n+1}(\xi)\varphi_n(x) - \varphi_n(\xi)\varphi_{n+1}(x)}{\xi - x} =$$

$$= \frac{1}{\xi - x}\underbrace{\sqrt{\tfrac{n+1}{2}}\,\frac{(-1)^{2n+1}}{2^n\sqrt{2\pi n!(n+1)!}}}_{\to 0,\ n\to\infty}\underbrace{\left\{D_\xi^{n+1}\{e^{-\xi^2}\}D_x^n\{e^{-x^2}\} - D_\xi^n\{e^{-\xi^2}\}D_x^{n+1}\{e^{-x^2}\}\right\}}_{\to 0,\ n\to\infty}$$

$$\to 0, \text{ as } n \to \infty.$$

That is, the Hermite Kernel vanishes for any $\xi \neq x$.

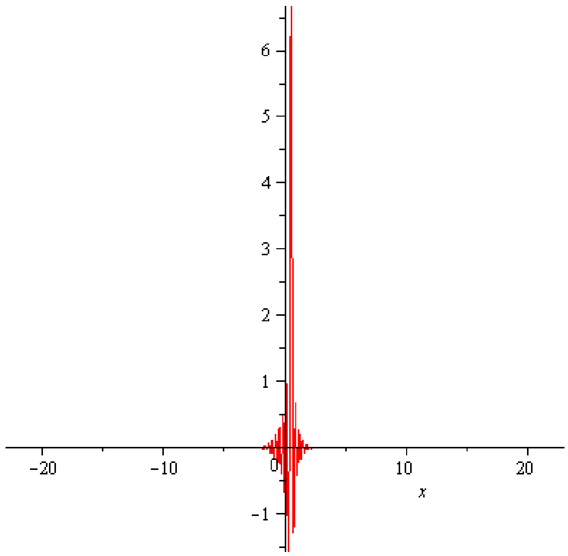
Plots of the Hermite Sequence confirm that

In the Calculus of Limits,

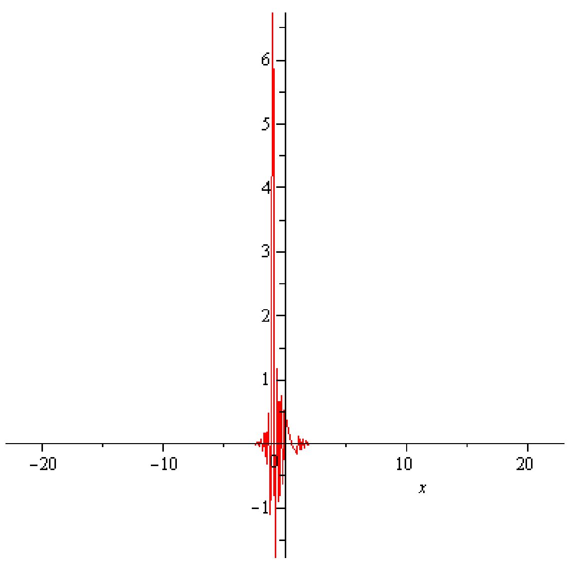
the Hermite Kernel is either singular or zero

1.2 Plots of $\frac{1}{\sqrt{\pi}} e^{-\xi^2} \{H_0(\xi)H_0(x) + ... + \frac{1}{2^n n!} H_n(\xi)H_n(x)\}$

In Maple, $plot(\sum_{i=0}^{223} \frac{1}{\sqrt{\pi}} e^{-x^2} \frac{1}{2^i i!} * HermiteH(i,.5) * HermiteH(i,x), x = -23..23)$

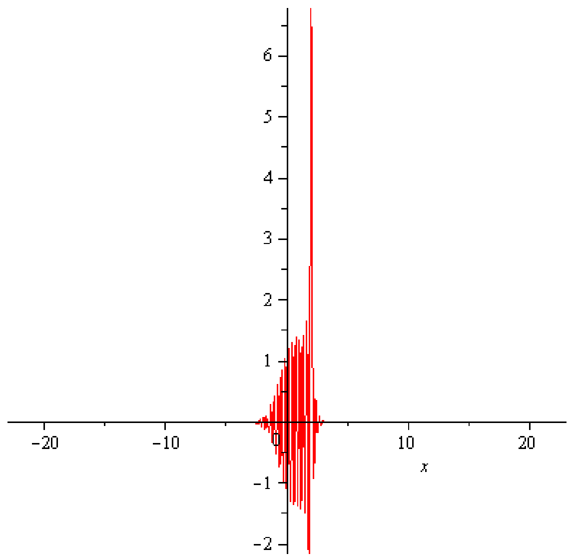


In Maple, $plot(\sum_{i=0}^{223} \frac{1}{\sqrt{\pi}} e^{-x^2} \frac{1}{2^i i!} * HermiteH(i,-1) * HermiteH(i,x), x = -23..23)$



e^{-x^2} that suppresses oscillations away from the origin, enhances them at the origin. Thus, a singularity away from the origin needs more terms

In Maple, $plot(\sum_{i=0}^{223} \frac{1}{\sqrt{\pi}} e^{-x^2} \frac{1}{2^i i!} * HermiteH(i,2) * HermiteH(i,x), x = -23..23)$



The plots confirm that the Hermite Series Theorem cannot be proved in the Calculus of Limits.

1.3 Infinitesimal Calculus Solution

By resolving the problem of the infinitesimals [Dan2], we obtained the Infinite Hyper-reals that are strictly smaller than ∞, and constitute the value of the Delta Function at the singularity.

The controversy surrounding the Leibnitz Infinitesimals derailed the development of the Infinitesimal Calculus, and the Delta

Function could not be defined and investigated properly.

In Infinitesimal Calculus, [Dan3], we can differentiate over jump discontinuities, and integrate over singularities.

The Delta Function, the idealization of an impulse in Radar circuits, is a Discontinuous Hyper-Real function which definition requires Infinite Hyper-reals, and which analysis requires Infinitesimal Calculus.

In [Dan5], we show that in infinitesimal Calculus, the hyper-real

$$\delta(x) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x} d\omega$$

is zero for any $x \neq 0$,

it spikes at $x = 0$, so that its Infinitesimal Calculus integral is $\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1$,

and $\delta(0) = \frac{1}{dx} < \infty$.

Here, we show that in Infinitesimal calculus, the Hermite Kernel is a hyper-real Delta Function.

And the Hermite Series $\mathcal{L}_{egendre}\mathcal{S}\left\{f(x)\right\}$ associated with a Hyper-real function $f(x)$, equals $f(x)$.

2.

Hyper-real Line

Each real number α can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of infinitesimal hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the infinite hyper-reals.
4. The infinite hyper-reals are greater than any real number, yet strictly smaller than infinity.
5. The infinite hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a non-constant hyper-real.
7. The Hyper-reals are the totality of constant hyper-reals, a family of infinitesimals, a family of infinitesimals with

negative sign, a family of infinite hyper-reals, a family of infinite hyper-reals with negative sign, and non-constant hyper-reals.

8. The hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.
9. That line includes the real numbers separated by the non-constant hyper-reals. Each real number is the center of an interval of hyper-reals, that includes no other real number.
10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.
11. Zero is not an infinitesimal, because zero is not strictly greater than zero.
12. We do not add infinity to the hyper-real line.
13. The infinitesimals, the infinitesimals with negative signs, the infinite hyper-reals, and the infinite hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.
14. The hyper-real line is embedded in $\mathbb{R}^{\infty}$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the hyper-real onto the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal hyper-reals, or to the infinite hyper-reals, or to the non-constant hyper-reals.

16. No neighbourhood of a hyper-real is homeomorphic to an $\mathbb{R}^n$ ball. Therefore, the hyper-real line is not a manifold.

17. The hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

3.

Integral of a Hyper-real Function

In [Dan3], we defined the integral of a Hyper-real Function.

Let $f(x)$ be a hyper-real function on the interval $[a,b]$.

The interval may not be bounded.

$f(x)$ may take infinite hyper-real values, and need not be bounded.

At each

$$a \le x \le b,$$

there is a rectangle with base $[x - \frac{dx}{2}, x + \frac{dx}{2}]$, height $f(x)$, and area

$$f(x)dx.$$

We form the **Integration Sum** of all the areas for the x's that start at $x = a$, and end at $x = b$,

$$\sum_{x \in [a,b]} f(x)dx.$$

If for any infinitesimal dx, the Integration Sum has the same hyper-real value, then $f(x)$ is integrable over the interval $[a,b]$.

Then, we call the Integration Sum the integral of $f(x)$ from $x = a$, to $x = b$, and denote it by

$$\int_{x=a}^{x=b} f(x)dx\,.$$

If the hyper-real is infinite, then it is the integral over $[a,b]$,

If the hyper-real is finite,

$$\int_{x=a}^{x=b} f(x)dx = \text{ real part of the hyper-real}\,.\square$$

3.1 The countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers,

$Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty\,.$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[a,b]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(x)dx$.

The Lower Integral is the Integration Sum where $f(x)$ is replaced by its lowest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

3.2 $$\sum_{x \in [a,b]} \left(\inf_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

The Upper Integral is the Integration Sum where $f(x)$ is replaced by its largest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

3.3 $$\sum_{x \in [a,b]} \left(\sup_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

If the integral is a finite hyper-real, we have

3.4 *A hyper-real function has a finite integral if and only if its upper integral and its lower integral are finite, and differ by an infinitesimal.*

4.

Delta Function

In [Dan5], we have defined the Delta Function, and established its properties

1. The Delta Function is a hyper-real function defined from the hyper-real line into the set of two hyper-reals $\left\{0, \frac{1}{dx}\right\}$. The hyper-real 0 is the sequence $\left\langle 0,0,0,...\right\rangle$. The infinite hyper-real $\frac{1}{dx}$ depends on our choice of dx.
2. We will usually choose the family of infinitesimals that is spanned by the sequences $\left\langle \frac{1}{n}\right\rangle, \left\langle \frac{1}{n^2}\right\rangle, \left\langle \frac{1}{n^3}\right\rangle,\ldots$ It is a semigroup with respect to vector addition, and includes all the scalar multiples of the generating sequences that are non-zero. That is, the family includes infinitesimals with negative sign. Therefore, $\frac{1}{dx}$ will mean the sequence $\left\langle n\right\rangle$. Alternatively, we may choose the family spanned by the sequences $\left\langle \frac{1}{2^n}\right\rangle, \left\langle \frac{1}{3^n}\right\rangle, \left\langle \frac{1}{4^n}\right\rangle,\ldots$ Then, $\frac{1}{dx}$ will mean the

sequence $\langle 2^n \rangle$. Once we determined the basic infinitesimal dx, we will use it in the Infinite Riemann Sum that defines an Integral in Infinitesimal Calculus.

3. The Delta Function is strictly smaller than ∞

4. We define, $\delta(x) \equiv \frac{1}{dx}\chi_{[-\frac{dx}{2},\frac{dx}{2}]}(x)$,

$$\text{where} \quad \chi_{[-\frac{dx}{2},\frac{dx}{2}]}(x) = \begin{cases} 1, x \in \left[-\frac{dx}{2},\frac{dx}{2}\right] \\ 0, \text{otherwise} \end{cases}.$$

5. Hence,

- for $x < 0$, $\delta(x) = 0$
- at $x = -\frac{dx}{2}$, $\delta(x)$ jumps from 0 to $\frac{1}{dx}$,
- for $x \in \left[-\frac{dx}{2},\frac{dx}{2}\right]$, $\delta(x) = \frac{1}{dx}$.
- at $x = 0$, $\delta(0) = \frac{1}{dx}$
- at $x = \frac{dx}{2}$, $\delta(x)$ drops from $\frac{1}{dx}$ to 0.
- for $x > 0$, $\delta(x) = 0$.
- $x\delta(x) = 0$

6. If $dx = \langle \frac{1}{n} \rangle$, $\delta(x) = \left\langle \chi_{[-\frac{1}{2},\frac{1}{2}]}(x), 2\chi_{[-\frac{1}{4},\frac{1}{4}]}(x), 3\chi_{[-\frac{1}{6},\frac{1}{6}]}(x)... \right\rangle$

7. If $dx = \langle \frac{2}{n} \rangle$, $\delta(x) = \left\langle \frac{1}{2\cosh^2 x}, \frac{2}{2\cosh^2 2x}, \frac{3}{2\cosh^2 3x}, ... \right\rangle$

8. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle e^{-x}\chi_{[0,\infty)}, 2e^{-2x}\chi_{[0,\infty)}, 3e^{-3x}\chi_{[0,\infty)}, \ldots \right\rangle$

9. $$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1.$$

10. $$\delta(\xi - x) = \frac{1}{2\pi}\int_{k=-\infty}^{k=\infty} e^{-ik(\xi - x)}dk$$

5.

Convergent Series

In [Dan8], we defined convergence of infinite series in Infinitesimal Calculus

5.1 Sequence Convergence to a finite hyper-real a

$$a_n \to a \;\; \textit{iff} \;\; \langle a_n \rangle - \langle a \rangle = \text{infinitesimal}.$$

5.2 Sequence Convergence to an infinite hyper-real A

$$a_n \to A \;\; \textit{iff} \;\; \langle a_n \rangle \text{ represents the infinite hyper-real } A.$$

5.3 Series Convergence to a finite hyper-real s

$$a_1 + a_2 + ... \to s \;\; \textit{iff} \;\; \langle a_1 + ... + a_n \rangle - \langle s \rangle = \text{infinitesimal}.$$

5.4 Series Convergence to an Infinite Hyper-real S

$$a_1 + a_2 + ... \to S \;\; \textit{iff}$$

$$\langle a_1 + ... + a_n \rangle \text{ represents the infinite hyper-real } S.$$

6.

Hermite Sequence and $\delta(\xi - x)$

6.1 Hermite Sequence Definition

If $f(x)$ can be expanded in the $H_n(x)$,

$$f(x) = a_0 H_0(x) + a_1 H_1(x) + a_2 H_2(x) + \ldots,$$

Then,

$$\int_{x=-\infty}^{x=\infty} e^{-x^2} f(x) H_n(x) dx =$$

$$= \int_{x=\infty}^{x=\infty} f(x) e^{-x^2} \{a_0 H_0(x) + a_1 H_1(x) + a_2 H_2(x) + \ldots\} H_n(x) dx$$

$$= a_0 \underbrace{\int_{x=-\infty}^{x=\infty} e^{-x^2} H_0(x) H_n(x) dx}_{2^0 0! \sqrt{\pi} \delta_{0n}} + a_1 \underbrace{\int_{x=-\infty}^{x=\infty} e^{-x^2} H_1(x) H_n(x) dx}_{2^1 1! \sqrt{\pi} \delta_{1n}} + ..$$

$$= 2^n n! \sqrt{\pi} a_n .$$

The Hermite Series partial sums

$$\mathcal{H}_{ermite} \mathcal{S}_n \left\{ f(x) \right\} = a_0 H_0(x) + \ldots + a_n H_n(x)$$

$$= \int_{\xi=-\infty}^{\xi=\infty} f(\xi) \tfrac{1}{\sqrt{\pi}} e^{-\xi^2} \left\{ H_0(\xi) H_0(x) + \ldots + \tfrac{1}{2^n n!} H_n(\xi) H_n(x) \right\} d\xi .$$

give rise to the Hermite Sequence

$$H_n(\xi,x) = \tfrac{1}{\sqrt{\pi}} e^{-\xi^2} \left\{ H_0(\xi)H_0(x) + ... + \tfrac{1}{2^n n!} H_n(\xi)H_n(x) \right\}.$$

6.2 Hermite Sequence is a Delta Sequence

For each $n = 0,1,2,3,...$

$$H_n(\xi,x) = \tfrac{1}{\sqrt{\pi}} e^{-\xi^2} \left\{ H_0(\xi)H_0(x) + ... + \tfrac{1}{2^n n!} H_n(\xi)H_n(x) \right\},$$

1. *has the sifting property*

$$\int_{\xi=-\infty}^{\xi=\infty} \tfrac{1}{\sqrt{\pi}} e^{-\xi^2} \left\{ H_0(\xi)H_0(x) + ... + \tfrac{1}{2^n n!} H_n(\xi)H_n(x) \right\} d\xi = 1$$

2. *is a continuous function*

3. *peaks for each* $\xi \to x$ *to* $const \cdot \sqrt{n+1}$

Proof of (1)

$$\int_{\xi=-\infty}^{\xi=\infty} \tfrac{1}{\sqrt{\pi}} e^{-\xi^2} \left\{ H_0(\xi)H_0(x) + ... + \tfrac{1}{2^n n!} H_n(\xi)H_n(x) \right\} d\xi =$$

$$= \underbrace{H_0(x)}_{1} \underbrace{\tfrac{1}{\sqrt{\pi}} \int_{\xi=-\infty}^{\xi=\infty} e^{-\xi^2} \underbrace{H_0(\xi)}_{1} d\xi}_{1} + ... + H_n(x) \tfrac{1}{\sqrt{\pi}} \tfrac{1}{2^n n!} \int_{\xi=-\infty}^{\xi=\infty} e^{-\xi^2} H_n(\xi) d\xi$$

By [Spanier, p.222, #24:10:5], for $k = 1,2,...,n$,

$$\int_{\xi=-\infty}^{\xi=\infty} e^{-\xi^2} H_k(b\xi) d\xi = \begin{cases} 0, & k = 1,3,5,... \\ \sqrt{\pi} n!(b^2-1)^{\frac{1}{2}n}, & k = 2,4,6,... \end{cases}$$

Therefore, for $k = 1,2,...,n$,

$$\int_{\xi=-\infty}^{\xi=\infty} e^{-\xi^2} H_k(\xi) d\xi = 0.$$

Hence,

$$\int_{\xi=-\infty}^{\xi=\infty} \tfrac{1}{\sqrt{\pi}} e^{-\xi^2} \left\{ H_0(\xi)H_0(x) + ... + \tfrac{1}{2^n n!} H_n(\xi)H_n(x) \right\} d\xi = 1. \square$$

Proof of (3)

By the Christoffel Summation Formula, [Sansone, p.371],

$$\varphi_0(\xi)\varphi_0(x) + ... + \varphi_n(\xi)\varphi_n(x) = \sqrt{\frac{n+1}{2}} \frac{\varphi_{n+1}(\xi)\varphi_n(x) - \varphi_n(\xi)\varphi_{n+1}(x)}{\xi - x},$$

where

$$\varphi_n(x) = \frac{1}{(2^n n! \sqrt{\pi})^{\frac{1}{2}}} e^{-\frac{1}{2}x^2} H_n(x).$$

For $\xi \to x$,

$$\sqrt{\frac{n+1}{2}} \frac{\varphi_{n+1}(\xi)\varphi_n(x) - \varphi_n(\xi)\varphi_{n+1}(x)}{\xi - x} \to \sqrt{\frac{n+1}{2}} \frac{0}{0}.$$

Applying Bernoulli's rule to the indeterminate limit,

$$\lim_{\xi \to x} \frac{\varphi_{n+1}(\xi)\varphi_n(x) - \varphi_n(\xi)\varphi_{n+1}(x)}{\xi - x} = \lim_{\xi \to x} \frac{D_\xi \varphi_{n+1}(\xi)\varphi_n(x) - D_\xi \varphi_n(\xi)\varphi_{n+1}(x)}{D_\xi(\xi - x)}$$

$$= \lim_{\xi \to x} [\varphi_{n+1}'(\xi)\varphi_n(x) - \varphi_n'(\xi)\varphi_{n+1}(x)]$$

$$= \varphi_{n+1}'(x)\varphi_n(x) - \varphi_n'(x)\varphi_{n+1}(x)$$

Therefore,

$$\varphi_0^2(x) + ... + \varphi_n^2(x) = \sqrt{\frac{n+1}{2}}[\varphi_{n+1}'(x)\varphi_n(x) - \varphi_n'(x)\varphi_{n+1}(x)].$$

Since $\varphi_n(x)$, and $\varphi_{n+1}(x)$ solve the differential equation, [Szego, p.105, #5.5.2],

$$\underbrace{1}_{a(x)} \cdot z''(x) + \underbrace{0}_{b(x)} \cdot z'(x) + (2n+1-x^2)z(x) = 0,$$

we have,

$$\varphi_{n+1}'(x)\varphi_n(x) - \varphi_n'(x)\varphi_{n+1}(x) = (const)e^{-\int \frac{b(x)}{a(x)}dx}$$

$$= (const)e^{\int 0 dx}$$

$$= const,$$

for any $-\infty < x < \infty$.

Hence,

$$\varphi_0^2(x) + ... + \varphi_n^2(x) = \sqrt{n+1}const$$

Therefore, substituting

$$H_n(x) = (2^n n!\sqrt{\pi})^{\frac{1}{2}} e^{\frac{1}{2}x^2} \varphi_n(x),$$

$$H_n(\xi) = (2^n n!\sqrt{\pi})^{\frac{1}{2}} e^{\frac{1}{2}\xi^2} \varphi_n(\xi)$$

$$\tfrac{1}{\sqrt{\pi}} e^{-\xi^2}\{H_0(\xi)H_0(x) + ... + \tfrac{1}{2^n n!} H_n(\xi)H_n(x)\} =$$

$$= e^{-\frac{1}{2}(\xi^2 - x^2)}\left\{\varphi_0(\xi)\varphi_0(x) + ... + \varphi_n(\xi)\varphi_n(x)\right\}$$

$$\underset{\xi \to x}{\to} \left\{\varphi_0^2(x) + ... + \varphi_n^2(x)\right\} = \sqrt{n+1}const. \square$$

7.

Hermite Kernel and $\delta(\xi - x)$

7.1 Hermite Kernel in the Calculus of Limits

The Hermite Series partial sums

$$\mathcal{H}_{ermite}\mathcal{S}_n\left\{f(x)\right\} = \int_{\xi=-\infty}^{\xi=\infty} f(\xi)\underbrace{\tfrac{1}{\sqrt{\pi}}e^{-\xi^2}\left\{H_0(\xi)H_0(x) + ... + \tfrac{1}{2^n n!}H_n(\xi)H_n(x)\right\}}_{\text{Hermite Sequence}}d\xi\,.$$

give rise to the Hermite Sequence.

The limit of the Hermite Sequence is an infinite series called the Hermite Kernel

$$\mathcal{H}_{ermite}(\xi - x) = \tfrac{1}{\sqrt{\pi}}e^{-\xi^2}\left\{H_0(\xi)H_0(x) + ... + \tfrac{1}{2^n n!}H_n(\xi)H_n(x) + ...\right\}$$

7.2 *In the Calculus of Limits, the Hermite Kernel does not have the sifting property*

<u>*Proof*</u>: for $\xi \to x$,

$$\tfrac{1}{\sqrt{\pi}}e^{-\xi^2}\left\{H_0(\xi)H_0(x) + ... + \tfrac{1}{2^n n!}H_n(\xi)H_n(x) + ..\right\} = \lim_{n\to\infty}\sqrt{n+1}const$$

$$\underset{n\to\infty}{\longrightarrow} \infty$$

That is, for $\xi \to x$,

$\tfrac{1}{\sqrt{\pi}}e^{-\xi^2}\left\{H_0(\xi)H_0(x) + ... + \tfrac{1}{2^n n!}H_n(\xi)H_n(x) + ...\right\}$ is singular.□

7.3 Hyper-real Hermite Kernel in Infinitesimal Calculus

$$\mathcal{H}_{ermite}(\xi - x) = \tfrac{1}{\sqrt{\pi}} e^{-\xi^2} \left\{ H_0(\xi)H_0(x) + ... + \tfrac{1}{2^n n!} H_n(\xi)H_n(x) + ... \right\}$$

$$= \begin{cases} \left\langle \sqrt{n} \right\rangle & ,\xi = x \\ 0 & ,\ \xi \neq x \end{cases}$$

$$= \delta(\xi - x).$$

<u>Proof</u>:

$$\mathcal{H}_{ermite}(\xi - x) = \tfrac{1}{\sqrt{\pi}} e^{-\xi^2} \left\{ H_0(\xi)H_0(x) + ... + \tfrac{1}{2^n n!} H_n(\xi)H_n(x) + ... \right\}$$

$$= \begin{cases} \left\langle \sqrt{n} \right\rangle & ,\xi = x \\ 0 & ,\ \xi \neq x \end{cases}.$$

Denoting by $\frac{1}{dx}$ the infinite hyper-real $\left\langle \sqrt{n} \right\rangle$,

$$= \begin{cases} 0, \xi \neq x \\ \frac{1}{dx}, \xi = x \end{cases}$$

$$= \delta(\xi - x). \square$$

8.

Hermite Series and $\delta(\xi - x)$

8.1 Hermite Series of a Hyper-real Function

Let $f(x)$ be a hyper-real function integrable on $(-\infty, \infty)$.

Then, for each $n = 0, 1, 2, 3, ...$, the integrals

$$a_n = \frac{1}{2^n n! \sqrt{\pi}} \int_{x=-\infty}^{x=\infty} e^{-x^2} f(x) H_n(x) dx .$$

exist, with finite, or infinite hyper-real values. The a_n are the Hermite Coefficients of $f(x)$.

The Hermite Series associated with $f(x)$ is

$$\mathcal{H}_{ermite}\mathcal{S}\left\{f(x)\right\} = a_0 H_0(x) + a_1 H_1(x) + a_2 H_2(x) + ...$$

For each x, it may assume finite or infinite hyper-real values.

8.2 $$\mathcal{H}_{ermite}\mathcal{S}\left\{\delta(\xi - x)\right\} = \delta(\xi - x)$$

Proof:

$$\mathcal{H}_{ermite}\mathcal{S}\left\{\delta(\xi - x)\right\} = a_0 H_0(x) + a_1 H_1(x) + a_2 H_2(x) + ...$$

where

$$a_n = \frac{1}{2^n n! \sqrt{\pi}} \int_{x=-\infty}^{x=\infty} e^{-x^2} \delta(\xi - x) H_n(x) dx$$

$$= \frac{1}{2^n n! \sqrt{\pi}} e^{-\xi^2} H_n(\xi).$$

Therefore,

$$\mathcal{H}_{ermite}\mathcal{S}\left\{\delta(\xi - x)\right\} = \frac{1}{\sqrt{\pi}} e^{-\xi^2} \left\{H_0(\xi)H_0(x) + ... + \frac{1}{2^n n!} H_n(\xi)H_n(x) + ...\right\}$$

$$= \mathcal{H}_{ermite}(\xi - x), \text{ by 7.3,}$$

$$= \delta(\xi - x), \quad \text{by 7.3.}\square$$

9.

Hermite Series Theorem

The Hermite Series Theorem for a hyper-real function, $f(x)$, is the Fundamental Theorem of Hermite Series.

It supplies the conditions under which the Hermite Series associated with $f(x)$ equals $f(x)$.

It is believed to hold in the Calculus of Limits under the Picone Conditions, or under the Hobson Conditions [Sansone]. In fact,

The Theorem cannot be proved in the Calculus of Limits under any conditions,

because the summation of the Hermite Series requires integration of the singular Hermite Kernel.

9.1 Hermite Series Theorem cannot be proved in the Calculus of Limits

Proof: Let $f(x)$ be integrable on $(-\infty, \infty)$.

In the Calculus of Limits, the Hermite Series is the limit of the sequence of Partial Sums

$$\mathcal{H}_{ermite}\mathcal{S}_n\left\{f(x)\right\} = a_0 H_0(x) + ... + a_n H_n(x)$$

$$= \left(\frac{1}{\sqrt{\pi}} \int_{\xi=-\infty}^{\xi=\infty} f(\xi) e^{-\xi^2} H_0(\xi) d\xi \right) H_0(x) + \ldots$$

$$\ldots + \left(\frac{1}{2^n n!} \frac{1}{\sqrt{\pi}} \int_{\xi=-\infty}^{\xi=\infty} f(\xi) e^{-\xi^2} H_n(\xi) d\xi \right) H_n(x)$$

$$= \int_{\xi=-\infty}^{\xi=\infty} f(\xi) \frac{1}{\sqrt{\pi}} e^{-\xi^2} \left\{ H_0(\xi) H_0(x) + \ldots + \frac{1}{2^n n!} H_n(\xi) H_n(x) \right\} d\xi \,.$$

As $n \to \infty$, the Hermite Sequence

$$\frac{1}{\sqrt{\pi}} e^{-\xi^2} \left\{ H_0(\xi) H_0(x) + \ldots + \frac{1}{2^n n!} H_n(\xi) H_n(x) \right\}$$

becomes the Hermite Kernel,

$$\frac{1}{\sqrt{\pi}} e^{-\xi^2} \left\{ H_0(\xi) H_0(x) + \ldots + \frac{1}{2^n n!} H_n(\xi) H_n(x) + \ldots \right\},$$

By 7.2, the Hermite Kernel diverges to infinity at any $\xi = x$.

Therefore, while the partial sums of the Hermite Series exist, their limit does not. Conditions by Uspensky [Sansone] failed to comprehend the sifting through the values of $f(\xi)$ by the Hermite Kernel, and the picking of $f(\xi)$ at $\xi = x$.

Avoiding the singularity at $\xi = x$, by using the Cauchy Principal Value of the integral does not recover the Theorem, because for any $\xi \neq x$, the Hermite Kernel vanishes, and the integral is identically zero, for any function $f(x)$.

Thus, the Hermite Series Theorem cannot be proved in the Calculus of Limits.□

9.2 Calculus of Limits Conditions are irrelevant to Hermite Series Theorem

Proof: The Uspensky Conditions [Sansone, p.371] are

1. $f(x)$ integrable in any bounded interval
2. $f(x)$ integrable in $(-\infty, \infty)$

It is clear from 9.1 that these conditions on $f(x)$ do not resolve the singularity of the Hermite kernel, and are not sufficient for the Hermite Series Theorem.□

In Infinitesimal Calculus, by 7.3, the Hermite Kernel is the Delta Function, and by 8.2, it equals its Hermite Series.

Then, the Hermite Series Theorem holds for any Hyper-Real Function:

8.3 Hermite Series Theorem for Hyper-real $f(x)$

If $f(x)$ is hyper-real function integrable on $(-\infty, \infty)$

Then, $$f(x) = \mathcal{H}_{ermite}\mathcal{S}\left\{f(x)\right\}$$

Proof:

$$f(x) = \int_{\xi=-\infty}^{\xi=\infty} f(\xi)\delta(\xi - x)d\xi$$

Substituting from 7.3,

$$\delta(\xi - x) = \tfrac{1}{\sqrt{\pi}} e^{-\xi^2} \left\{ H_0(\xi)H_0(x) + ... + \tfrac{1}{2^n n!} H_n(\xi)H_n(x) + ... \right\},$$

$$f(x) = \int_{\xi=-\infty}^{\xi=\infty} f(\xi) \tfrac{1}{\sqrt{\pi}} e^{-\xi^2} \left\{ H_0(\xi)H_0(x) + ... + \tfrac{1}{2^n n!} H_n(\xi)H_n(x) + ... \right\} d\xi$$

This Hyper-real Integral is the summation,

$$\sum_{\xi=-\infty}^{\xi=\infty} f(\xi) \tfrac{1}{\sqrt{\pi}} e^{-\xi^2} \left\{ H_0(\xi)H_0(x) + ... + \tfrac{1}{2^n n!} H_n(\xi)H_n(x) + ... \right\} d\xi$$

which amounts to the hyper-real function $f(x)$,and is well-defined. Hence, the summation of each term in the integrand exists, and we may write the integral as the sum

$$= \underbrace{\left(\frac{1}{\sqrt{\pi}} \int_{\xi=-\infty}^{\xi=\infty} f(\xi) e^{-\xi^2} H_0(\xi) d\xi \right)}_{a_0} H_0(x) + ...$$

$$... + \underbrace{\left(\frac{1}{2^n n! \sqrt{\pi}} \int_{\xi=-\infty}^{\xi=\infty} f(\xi) e^{-\xi^2} H_n(\xi) d\xi \right)}_{a_n} H_n(x) + ...$$

$$= a_0 H_0(x) + a_1 H_1(x) + a_2 H_2(x) + ...$$

$$= \mathcal{H}_{ermite}\mathcal{S}\left\{ f(x) \right\}. \square$$

In particular, the Delta Function violates Uspensky's Conditions

- *The Hyper-real* $\delta(x)$, *is not defined in the Calculus of Limits, and is not integrable in any interval.*

But by 8.2, $\delta(\xi - x)$ satisfies the Hermite Series Theorem.

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