

Linear Differential Equations Driven by Delta Functions

H. Vic Dannon
vic0@comcast.net
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Abstract We solve the linear differential equation $L(x)G(x, \xi) = \delta(x - \xi)$, for the Hyper-real Green function $G(x, \xi)$, where $L(x)$ is a linear differential operator, and $\delta(x - \xi)$ is the hyper-real Delta function.

Keywords: Sturm-Liouville Expansions, Infinitesimal, Infinite-Hyper-Real, Hyper-Real, infinite Hyper-real, Infinitesimal Calculus, Delta Function, Fourier Series, Laguerre Polynomials, Legendre Functions, Bessel Functions, Delta Function,

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21. Green Function for Legendre Sturm-Liouville

$$y''(\theta) + \left[\lambda + \left(\frac{1}{4} - m^2\right) \frac{1}{\cos^2 \theta}\right]y(\theta) = 0, \quad -\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$$

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$$u''(x) + \left(\lambda - \frac{\nu^2 - \frac{1}{4}}{x^2}\right)u(x) = 0 \ \& \ 0 < x < b$$

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30. Green's Function for Bessel's

$$y''(x) + (\lambda + e^{2x})y(x) = 0, \quad -\infty < x < \infty$$

31. Green's Function for Bessel's

$$y''(x) + (\lambda - e^{2x})y(x) = 0, \quad -\infty < x < \infty$$

References

0.

Linear Differential Equation driven by Delta Function

A linear Differential Equation for the Hyper-real function $y(x)$, forced by the Hyper-real function $f(x)$ is

$$L(x, D)y(x) = f(x),$$

where the Differential Operator $L(x, D)$ is linear.

If there is a Hyper-real Green function,

$$G(x, \xi)$$

that satisfies the equation

$$L(x, D)G(x, \xi) = \delta(x, \xi),$$

with the same Boundary condition that $y(x)$ satisfies

then, $G(x, \xi)$ is the Kernel of an Integral Operator that transforms a function $f(x)$ into the solution $y(x)$ of the linear differential equation $L(x, D)y(x) = f(x)$.

Indeed, if

$$y(x) = \int_{\xi=a}^{\xi=b} G(x, \xi)f(\xi)d\xi,$$

then

$$\begin{aligned}
L(x, D)y(x) &= L(x, D) \int_{\xi=a}^{\xi=b} G(x, \xi) f(\xi) d\xi, \\
&= L(x, D) \sum_{\xi=a}^{\xi=b} G(x, \xi) f(\xi) d\xi, \\
&= \sum_{\xi=a}^{\xi=b} \underbrace{L(x, D) G(x, \xi)}_{\delta(x-\xi)} f(\xi) d\xi \\
&= f(x).
\end{aligned}$$

We shall see that the expansion of $\delta(x - \xi)$ in series, or integrals guarantees the existence of a Green Function that can be expanded similarly.

The Delta Function can be defined only as a hyper-real function in infinitesimal Calculus.

We proceed with the definition of the Hyper-real line.

1.

Hyper-real Line

Each real number α can be represented by a Cauchy sequence of rational numbers, (r_1, r_2, r_3, \dots) so that $r_n \rightarrow \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, \dots)$ is a constant hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences (l_1, l_2, l_3, \dots) constitutes a family of infinitesimal hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{l_1}, \frac{1}{l_2}, \frac{1}{l_3}, \dots\right)$ are the infinite hyper-reals.
4. The infinite hyper-reals are greater than any real number, yet strictly smaller than infinity.
5. The infinite hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a non-constant hyper-real.
7. The Hyper-reals are the totality of constant hyper-reals, a family of infinitesimals, a family of infinitesimals with

- negative sign, a family of infinite hyper-reals, a family of infinite hyper-reals with negative sign, and non-constant hyper-reals.
8. The hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.
 9. That line includes the real numbers separated by the non-constant hyper-reals. Each real number is the center of an interval of hyper-reals, that includes no other real number.
 10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.
 11. Zero is not an infinitesimal, because zero is not strictly greater than zero.
 12. We do not add infinity to the hyper-real line.
 13. The infinitesimals, the infinitesimals with negative signs, the infinite hyper-reals, and the infinite hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.
 14. The hyper-real line is embedded in \mathbb{R}^∞ , and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the hyper-real onto the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal hyper-reals, or to the infinite hyper-reals, or to the non-constant hyper-reals.
16. No neighbourhood of a hyper-real is homeomorphic to an \mathbb{R}^n ball. Therefore, the hyper-real line is not a manifold.
17. The hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

2.

Hyper-real Function

2.1 Definition of a hyper-real function

$f(x)$ is a hyper-real function, iff it is from the hyper-reals into the hyper-reals.

This means that any number in the domain, or in the range of a hyper-real $f(x)$ is either one of the following

- real
- real + infinitesimal
- real – infinitesimal
- infinitesimal
- infinitesimal with negative sign
- infinite hyper-real
- infinite hyper-real with negative sign

Clearly,

2.2 *Every function from the reals into the reals is a hyper-real function.*

3.

Integral of a Hyper-real Function

In [Dan3], we defined the integral of a Hyper-real Function.

Let $f(x)$ be a hyper-real function on the interval $[a, b]$.

The interval may not be bounded.

$f(x)$ may take infinite hyper-real values, and need not be bounded.

At each

$$a \leq x \leq b,$$

there is a rectangle with base $[x - \frac{dx}{2}, x + \frac{dx}{2}]$, height $f(x)$, and area

$$f(x)dx.$$

We form the **Integration Sum** of all the areas for the x 's that start at $x = a$, and end at $x = b$,

$$\sum_{x \in [a, b]} f(x)dx.$$

If for any infinitesimal dx , the Integration Sum has the same hyper-real value, then $f(x)$ is integrable over the interval $[a, b]$.

Then, we call the Integration Sum the integral of $f(x)$ from $x = a$, to $x = b$, and denote it by

$$\int_{x=a}^{x=b} f(x)dx .$$

If the hyper-real is infinite, then it is the integral over $[a, b]$,

If the hyper-real is finite,

$$\int_{x=a}^{x=b} f(x)dx = \text{real part of the hyper-real.} \square$$

3.1 The countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers,

$Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and

we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = \dots = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = \dots \equiv \infty .$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[a, b]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(x)dx$.

The Lower Integral is the Integration Sum where $f(x)$ is replaced

by its lowest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

$$\mathbf{3.2} \quad \sum_{x \in [a, b]} \left(\inf_{x - \frac{dx}{2} \leq t \leq x + \frac{dx}{2}} f(t) \right) dx$$

The Upper Integral is the Integration Sum where $f(x)$ is replaced by its largest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

$$\mathbf{3.3} \quad \sum_{x \in [a, b]} \left(\sup_{x - \frac{dx}{2} \leq t \leq x + \frac{dx}{2}} f(t) \right) dx$$

If the integral is a finite hyper-real, we have

3.4 *A hyper-real function has a finite integral if and only if its upper integral and its lower integral are finite, and differ by an infinitesimal.*

4.

Delta Function

In [Dan5], we have defined the Delta Function, and established its properties

1. The Delta Function is a hyper-real function defined from the

hyper-real line into the set of two hyper-reals $\left\{0, \frac{1}{dx}\right\}$. The

hyper-real 0 is the sequence $\langle 0, 0, 0, \dots \rangle$. The infinite hyper-

real $\frac{1}{dx}$ depends on our choice of dx .

2. We will usually choose the family of infinitesimals that is

spanned by the sequences $\left\langle \frac{1}{n} \right\rangle, \left\langle \frac{1}{n^2} \right\rangle, \left\langle \frac{1}{n^3} \right\rangle, \dots$. It is a

semigroup with respect to vector addition, and includes all

the scalar multiples of the generating sequences that are

non-zero. That is, the family includes infinitesimals with

negative sign. Therefore, $\frac{1}{dx}$ will mean the sequence $\langle n \rangle$.

Alternatively, we may choose the family spanned by the

sequences $\left\langle \frac{1}{2^n} \right\rangle, \left\langle \frac{1}{3^n} \right\rangle, \left\langle \frac{1}{4^n} \right\rangle, \dots$. Then, $\frac{1}{dx}$ will mean the

sequence $\langle 2^n \rangle$. Once we determined the basic infinitesimal dx , we will use it in the Infinite Riemann Sum that defines an Integral in Infinitesimal Calculus.

3. The Delta Function is strictly smaller than ∞

4. We define, $\delta(x) \equiv \frac{1}{dx} \mathcal{X}_{[-\frac{dx}{2}, \frac{dx}{2}]}(x)$,

$$\text{where } \mathcal{X}_{[-\frac{dx}{2}, \frac{dx}{2}]}(x) = \begin{cases} 1, & x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right] \\ 0, & \text{otherwise} \end{cases}$$

5. Hence,

$$\diamond \text{ for } x < 0, \delta(x) = 0$$

$$\diamond \text{ at } x = -\frac{dx}{2}, \delta(x) \text{ jumps from } 0 \text{ to } \frac{1}{dx},$$

$$\diamond \text{ for } x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right], \delta(x) = \frac{1}{dx}.$$

$$\diamond \text{ at } x = 0, \delta(0) = \frac{1}{dx}$$

$$\diamond \text{ at } x = \frac{dx}{2}, \delta(x) \text{ drops from } \frac{1}{dx} \text{ to } 0.$$

$$\diamond \text{ for } x > 0, \delta(x) = 0.$$

$$\diamond x\delta(x) = 0$$

6. If $dx = \langle \frac{1}{n} \rangle$, $\delta(x) = \left\langle \mathcal{X}_{[-\frac{1}{2}, \frac{1}{2}]}(x), 2\mathcal{X}_{[-\frac{1}{4}, \frac{1}{4}]}(x), 3\mathcal{X}_{[-\frac{1}{6}, \frac{1}{6}]}(x) \dots \right\rangle$

7. If $dx = \langle \frac{2}{n} \rangle$, $\delta(x) = \left\langle \frac{1}{2 \cosh^2 x}, \frac{2}{2 \cosh^2 2x}, \frac{3}{2 \cosh^2 3x}, \dots \right\rangle$

8. If $dx = \langle \frac{1}{n} \rangle$, $\delta(x) = \langle e^{-x}\chi_{[0,\infty)}, 2e^{-2x}\chi_{[0,\infty)}, 3e^{-3x}\chi_{[0,\infty)}, \dots \rangle$

9.
$$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1.$$

10.
$$\delta(\xi - x) = \frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} e^{-ik(\xi-x)}dk$$

5.

Convergent Series

In [Dan8], we defined convergence of infinite series in Infinitesimal Calculus

5.1 Sequence Convergence to a finite hyper-real a

$$a_n \rightarrow a \text{ iff } \langle a_n \rangle - \langle a \rangle = \text{infinitesimal.}$$

5.2 Sequence Convergence to an infinite hyper-real A

$$a_n \rightarrow A \text{ iff } \langle a_n \rangle \text{ represents the infinite hyper-real } A.$$

5.3 Series Convergence to a finite hyper-real s

$$a_1 + a_2 + \dots \rightarrow s \text{ iff } \langle a_1 + \dots + a_n \rangle - \langle s \rangle = \text{infinitesimal.}$$

5.4 Series Convergence to an Infinite Hyper-real S

$$a_1 + a_2 + \dots \rightarrow S \text{ iff}$$

$$\langle a_1 + \dots + a_n \rangle \text{ represents the infinite hyper-real } S.$$

6.

Green's Function for

$$y'(x) + P(x)y(x) = f(x), \text{ with } y(0) = 0$$

6.1

$$G(x, \xi) = e^{\int_{\theta=0}^{\theta=\xi} P(\theta)d\theta} - \int_{\theta=0}^{\theta=x} P(\theta)d\theta = e^{\int_{\theta=0}^{\theta=\xi} P(\theta)d\theta}$$

Proof:

We solve the equation

$$\partial_x G(x, \xi) + P(x)G(x, \xi) = \delta(x - \xi), \text{ with } G(0, \xi) = 0$$

Multiplying both sides by the integrating factor

$$e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} \partial_x G(x, \xi) + e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} P(x)G(x, \xi) = e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} \delta(x - \xi)$$

$$\partial_x \left\{ e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} G(x, \xi) \right\} = e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} \delta(x - \xi)$$

Integrating with respect to x ,

$$e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} G(x, \xi) = \int_{x=-\infty}^{x=\infty} e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} \delta(x - \xi) dx$$

$$\begin{aligned}
&= \sum_{x=-\infty}^{x=\infty} e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} \delta(x - \xi) dx \\
&= e^{\int_{\theta=0}^{\theta=\xi} P(\theta)d\theta}
\end{aligned}$$

Therefore,

$$\begin{aligned}
G(x, \xi) &= e^{\int_{\theta=0}^{\theta=\xi} P(\theta)d\theta} - e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} \\
&= e^{\int_{\theta=x}^{\theta=\xi} P(\theta)d\theta} . \square
\end{aligned}$$

6.2

$$y(x) = \int_{\xi=0}^{\xi=x} e^{\int_{\theta=0}^{\theta=\xi} P(\theta)d\theta} f(x) dx$$

7.

Green's Function for

$y''(x) + P(x)y'(x) + Q(x)y(x) = f(x)$, with $y(0) = 0$,
and $y'(0) = 0$

7.1 $y''(x) + P(x)y'(x) + Q(x)y(x) = f(x)$, with $y(0) = 0$, $y'(0) = 0$

$u(x)$ solves $u''(x) + P(x)u'(x) + Q(x)u(x) = 0$, with $u(0) = 0$

$v(x)$ solves $v''(x) + P(x)v'(x) + Q(x)v(x) = 0$, with $v(0) = 0$

Then,

$$G(x, \xi) = \frac{\begin{vmatrix} u(\xi) & v(\xi) \\ u(x) & v(x) \end{vmatrix}}{\begin{vmatrix} u(\xi) & v(\xi) \\ u'(\xi) & v'(\xi) \end{vmatrix}}$$

Proof:

We solve the equation

$$\partial_x^2 G(x, \xi) + P(x)\partial_x G(x, \xi) + Q(x)G(x, \xi) = \delta(x - \xi),$$

$$\text{with } G(0, \xi) = 0, \text{ and } \partial_x G(x, \xi)|_{x=0} = 0$$

Let $u(x)$ solve the equation

$$u''(x) + P(x)u'(x) + Q(x)u(x) = 0, \text{ with } u(0) = 0$$

Then,

$$uG'' + PuG' + QuG = u\delta(x - \xi),$$

$$Gu'' + PGu' + QGu = 0,$$

Subtracting,

$$\underbrace{uG'' - Gu''}_{(uG' - Gu')'} + P(uG' - Gu') = u\delta(x - \xi)$$

Multiplying by the integrating factor $e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta}$,

$$\partial_x \left\{ e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} [u(x)G'(x, \xi) - G(x, \xi)u'(x)] \right\} = e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} u(x)\delta(x - \xi).$$

Integrating,

$$e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} [u(x)G'(x, \xi) - G(x, \xi)u'(x)] = \int_{\tau=0}^{\tau=x} e^{\int_{\theta=0}^{\theta=\tau} P(\theta)d\theta} u(\tau)\delta(\tau - \xi)d\tau. \quad (\text{I})$$

Similarly, for $v(x)$ that solves the equation

$$v''(x) + P(x)v'(x) + Q(x)v(x) = 0, \text{ with } v(0) = 0$$

$$e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} [v(x)G'(x, \xi) - G(x, \xi)v'(x)] = \int_{\tau=0}^{\tau=x} e^{\int_{\theta=0}^{\theta=\tau} P(\theta)d\theta} v(\tau)\delta(\tau - \xi)d\tau. \quad (\text{II})$$

To eliminate G' , subtract $(\text{I}) \times v(x) - (\text{II}) \times u(x)$,

$$e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} \begin{vmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{vmatrix} G(x, \xi) = \int_{\tau=0}^{\tau=x} e^{\int_{\theta=0}^{\theta=\tau} P(\theta)d\theta} \begin{vmatrix} u(\tau) & v(\tau) \\ u(x) & v(x) \end{vmatrix} \delta(\tau - \xi)d\tau$$

Substituting, by Abel equality,

$$e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} \begin{vmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{vmatrix} = \begin{vmatrix} u(0) & v(0) \\ u'(0) & v'(0) \end{vmatrix},$$

and

$$e^{\int_{\theta=0}^{\theta=\tau} P(\theta)d\theta} = \frac{\begin{vmatrix} u(0) & v(0) \\ u'(0) & v'(0) \end{vmatrix}}{\begin{vmatrix} u(\tau) & v(\tau) \\ u'(\tau) & v'(\tau) \end{vmatrix}}$$

$$\begin{vmatrix} u(0) & v(0) \\ u'(0) & v'(0) \end{vmatrix} G(x, \xi) = \int_{\tau=0}^{\tau=x} \frac{\begin{vmatrix} u(0) & v(0) \\ u'(\tau) & v'(\tau) \end{vmatrix}}{\begin{vmatrix} u(\tau) & v(\tau) \\ u'(\tau) & v'(\tau) \end{vmatrix}} \begin{vmatrix} u(\tau) & v(\tau) \\ u(x) & v(x) \end{vmatrix} \delta(\tau - \xi) d\tau$$

$$G(x, \xi) = \int_{\tau=0}^{\tau=x} \frac{1}{\begin{vmatrix} u(\tau) & v(\tau) \\ u'(\tau) & v'(\tau) \end{vmatrix}} \begin{vmatrix} u(\tau) & v(\tau) \\ u(x) & v(x) \end{vmatrix} \delta(\tau - \xi) d\tau$$

$$= \frac{\begin{vmatrix} u(\xi) & v(\xi) \\ u(x) & v(x) \end{vmatrix}}{\begin{vmatrix} u(\xi) & v(\xi) \\ u'(\xi) & v'(\xi) \end{vmatrix}} \cdot \square$$

8.

Green's Function for

$y''(x) + P(x)y'(x) + Q(x)y(x) = f(x)$, with $y(a) = 0$,
and $y(b) = 0$

8.1 $y''(x) + P(x)y'(x) + Q(x)y(x) = f(x)$, with $y(a) = 0$, $y(b) = 0$,
 $u''(x) + P(x)u'(x) + Q(x)u(x) = 0$, with $u(a) = 0$, $u(b) \neq 0$,
 $v''(x) + P(x)v'(x) + Q(x)v(x) = 0$, with $v(a) \neq 0$, $v(b) = 0$

Then,
$$G(x, \xi) = \frac{1}{\begin{vmatrix} u(\xi) & v(\xi) \\ u'(\xi) & v'(\xi) \end{vmatrix}} \begin{cases} u(\xi)v(x), & a \leq \xi \leq x \\ u(x)v(\xi), & b \geq \xi \geq x \end{cases}$$

Proof:

We solve the equation

$$y''(x) + P(x)y'(x) + Q(x)y(x) = f(x),$$

with $y(a) = 0$, and $y(b) = 0$.

Let $u(x)$ solve the equation

$$u''(x) + P(x)u'(x) + Q(x)u(x) = 0, \quad \text{with } u(a) = 0, \text{ and } u(b) \neq 0,$$

and let $v(x)$ solve the equation

$$v''(x) + P(x)v'(x) + Q(x)v(x) = 0, \quad \text{with } v(a) \neq 0, \text{ and } v(b) = 0.$$

By **7** a particular solution to the equation for $y(x)$ is

$$y_p(x) = \int_{\xi=a}^{\xi=x} \frac{1}{\begin{vmatrix} u(\xi) & v(\xi) \\ u'(\xi) & v'(\xi) \end{vmatrix}} \begin{vmatrix} u(\xi) & v(\xi) \\ u(x) & v(x) \end{vmatrix} f(\xi) d\xi,$$

and the general solution is

$$y(x) = c_1 u(x) + c_2 v(x) + y_p(x).$$

$$0 = y(a) = c_1 \underbrace{u(a)}_0 + c_2 \underbrace{v(a)}_{\neq 0} + \underbrace{y_p(a)}_0 \Rightarrow \underline{c_2 = 0}$$

$$0 = y(b) = c_1 \underbrace{u(b)}_{\neq 0} + c_2 \underbrace{v(b)}_0 + y_p(b) \Rightarrow$$

$$c_1 = -\frac{1}{u(b)} y_p(b)$$

$$= -\frac{1}{u(b)} \int_{\xi=a}^{\xi=b} \frac{1}{\begin{vmatrix} u(\xi) & v(\xi) \\ u'(\xi) & v'(\xi) \end{vmatrix}} \underbrace{\begin{vmatrix} u(\xi) & v(\xi) \\ u(b) & v(b) \end{vmatrix}}_{\underbrace{u(\xi)v(b) - u(b)v(\xi)}_0} f(\xi) d\xi$$

$$= \int_{\xi=a}^{\xi=b} \frac{v(\xi)}{\begin{vmatrix} u(\xi) & v(\xi) \\ u'(\xi) & v'(\xi) \end{vmatrix}} f(\xi) d\xi$$

$$y(x) = u(x) \int_{\xi=a}^{\xi=b} \frac{v(\xi)}{\begin{vmatrix} u(\xi) & v(\xi) \\ u'(\xi) & v'(\xi) \end{vmatrix}} f(\xi) d\xi + \int_{\xi=a}^{\xi=x} \frac{u(\xi)v(x) - u(x)v(\xi)}{\begin{vmatrix} u(\xi) & v(\xi) \\ u'(\xi) & v'(\xi) \end{vmatrix}} f(\xi) d\xi$$

$$= \int_{\xi=x}^{\xi=b} \frac{u(x)v(\xi)}{\begin{vmatrix} u(\xi) & v(\xi) \\ u'(\xi) & v'(\xi) \end{vmatrix}} f(\xi) d\xi + \int_{\xi=a}^{\xi=x} \frac{u(\xi)v(x)}{\begin{vmatrix} u(\xi) & v(\xi) \\ u'(\xi) & v'(\xi) \end{vmatrix}} f(\xi) d\xi$$

Therefore,

$$G(x, \xi) = \frac{1}{\begin{vmatrix} u(\xi) & v(\xi) \\ u'(\xi) & v'(\xi) \end{vmatrix}} \begin{cases} u(\xi)v(x), & a \leq \xi \leq x \\ u(x)v(\xi), & b \geq \xi \geq x \end{cases} . \square$$

9.

Hyper-real Sturm-Liouville Problem

The Hyper-real Sturm-Liouville equation is the second order Hyper-real linear differential equation for the Hyper-real function $y(x)$,

$$-y''(x) + q(x) = \lambda y(x),$$

on an interval that may be bounded, or may be the whole Hyper-real line.

The hyper-real function $q(x)$ is (assumed in the literature to be) continuous on the interval, and bounded at its endpoints. The choice of the number λ (which may be real or complex), allows the equation with boundary conditions, at the interval endpoints to become an eigen-value problem:

λ_n is an eigen-value, and $\psi_n(x)$ is the corresponding Hyper-real eigen-function iff

$$-\psi_n''(x) + q(x)\psi_n(x) = \lambda_n \psi_n(x).$$

The eigen-functions are orthogonal, over the interval.

$$\int_{x=a}^{x=b} \psi_n(x)\psi_m(x)dx = 0, \quad \text{for } n \neq m.$$

10.

Delta Expansion in Non-Normalized Eigen-functions

As described in [Titchmarsh, Chapter I], given numbers

$$\alpha, \text{ and } \beta,$$

the Sturm-Liouville Problem on the interval with endpoints a , and b has solutions

$$\phi_\alpha(x, \lambda), \text{ with } \phi_\alpha(a, \lambda) = \sin \alpha, \text{ and } \phi_\alpha'(a, \lambda) = -\cos \alpha,$$

$$\chi_\beta(x, \lambda), \text{ with } \chi_\beta(b, \lambda) = \sin \beta, \text{ and } \chi_\beta'(b, \lambda) = -\cos \beta,,$$

which are entire functions of λ .

Then,

$$\begin{aligned} \frac{d}{dx} \underbrace{\begin{vmatrix} \phi_\alpha(x, \lambda) & \chi_\beta(x, \lambda) \\ \phi_\alpha'(x, \lambda) & \chi_\beta'(x, \lambda) \end{vmatrix}}_{W[\phi_\alpha, \chi_\beta]} &= \underbrace{\begin{vmatrix} \phi_\alpha'(x, \lambda) & \chi_\beta'(x, \lambda) \\ \phi_\alpha'(x, \lambda) & \chi_\beta'(x, \lambda) \end{vmatrix}}_0 + \begin{vmatrix} \phi_\alpha(x, \lambda) & \chi_\beta(x, \lambda) \\ \phi_\alpha''(x, \lambda) & \chi_\beta''(x, \lambda) \end{vmatrix} \\ &= \phi_\alpha \chi_\beta'' - \chi_\beta \phi_\alpha'' \\ &= \phi_\alpha (q - \lambda) \chi_\beta - \chi_\beta (q - \lambda) \phi_\alpha \\ &= 0. \end{aligned}$$

Hence, the Wronskian $W[\phi_\alpha, \chi_\beta]$ is a function of λ alone:

$$W[\phi_\alpha, \chi_\beta] = \omega(\lambda).$$

Now, if the only zeros of $\omega(\lambda)$ are the simple zeros

$$\lambda_0, \lambda_1, \lambda_2, \lambda_3, \dots,$$

Then, for each $n = 0, 1, 2, 3, \dots$

$$0 = \omega(\lambda_n) = \begin{vmatrix} \phi_\alpha(x, \lambda_n) & \chi_\beta(x, \lambda_n) \\ \phi_\alpha'(x, \lambda_n) & \chi_\beta'(x, \lambda_n) \end{vmatrix}.$$

That is, for each $n = 0, 1, 2, 3, \dots$ there is a number k_n so that

$$\chi_\beta(x, \lambda_n) = k_n \phi_\alpha(x, \lambda_n).$$

Titchmarsh applied the Residue Theorem to obtain the coefficients in the Sturm-Liouville expansion of $f(x)$.

Following Titchmarsh, we conclude that the Hyper-real Sturm-Liouville expansion of a Hyper-real function $f(x)$ in the Hyper-real eigen-functions $\phi_\alpha(x, \lambda_n)$ is

$$\begin{aligned} f(x) &= \sum_{n=0}^{n=\infty} \left(\frac{k_n}{\omega'(\lambda_n)} \int_{\xi=a}^{\xi=b} f(\xi) \phi_\alpha(\xi, \lambda_n) d\xi \right) \phi_\alpha(x, \lambda_n) \\ &= \sum_{n=0}^{n=\infty} \left(\frac{k_n}{\omega'(\lambda_n)} \sum_{\xi=a}^{\xi=b} f(\xi) \phi_\alpha(\xi, \lambda_n) d\xi \right) \phi_\alpha(x, \lambda_n) \end{aligned}$$

Exchanging summation order

$$\begin{aligned} &= \sum_{\xi=a}^{\xi=b} \sum_{n=0}^{n=\infty} \frac{k_n}{\omega'(\lambda_n)} \phi_\alpha(x, \lambda_n) \phi_\alpha(\xi, \lambda_n) f(\xi) d\xi \\ &= \int_{\xi=a}^{\xi=b} \underbrace{\sum_{n=0}^{n=\infty} \frac{k_n}{\omega'(\lambda_n)} \phi_\alpha(x, \lambda_n) \phi_\alpha(\xi, \lambda_n)}_{\delta(x-\xi)} f(\xi) d\xi \end{aligned}$$

Therefore,

10.1 *The Hyper-real Delta Function Expansion in non-normalized eigen-functions is*

$$\delta(x - \xi) = \sum_{n=0}^{n=\infty} \frac{k_n}{\omega'(\lambda_n)} \phi_\alpha(x, \lambda_n) \phi_\alpha(\xi, \lambda_n)$$

That is,

10.2 *The Hyper-real Delta Function is the infinite sequence*

$$\left\langle \frac{k_0}{\omega'(\lambda_0)} \phi_\alpha(x, \lambda_0) \phi_\alpha(\xi, \lambda_0) + \dots + \frac{k_n}{\omega'(\lambda_n)} \phi_\alpha(x, \lambda_n) \phi_\alpha(\xi, \lambda_n) \right\rangle.$$

11.

Green's Function Expansion in Non-Normalized Eigen-functions

Green's Function is defined by

$$\boxed{D_x^2 G(x, \xi, \lambda) + [\lambda - q(x)]G(x, \xi, \lambda) = \delta(x - \xi)}$$

with the Boundary conditions at the intervals endpoints.

Substituting Delta from **10.1**,

$$D_x^2 G(x, \xi, \lambda) + [\lambda - q(x)]G(x, \xi, \lambda) = \sum_{n=0}^{n=\infty} \frac{k_n}{\omega'(\lambda_n)} \phi_\alpha(x, \lambda_n) \phi_\alpha(\xi, \lambda_n)$$

Thus, Green's Function must have an expansion in $\phi_\alpha(x, \lambda_n) \phi_\alpha(\xi, \lambda_n)$,

$$G(x, \xi, \lambda) = \sum_{n=0}^{n=\infty} a_n(\lambda) \phi_\alpha(x, \lambda_n) \phi_\alpha(\xi, \lambda_n).$$

To determine the $a_n(\lambda)$, we substitute the expansion into the

Sturm-Liouville equation for $G(x, \xi, \lambda)$,

$$\sum_{n=0}^{n=\infty} a_n(\lambda) \underbrace{\left\{ \frac{\phi_\alpha''(x, \lambda_n)}{-[\lambda_n - q(x)]\phi_\alpha(x, \lambda_n)} + [\lambda - q(x)]\phi_\alpha(x, \lambda_n) \right\}}_{[\lambda - \lambda_n]\phi_\alpha(x, \lambda_n)} \phi_\alpha(\xi, \lambda_n) = \sum_{n=0}^{n=\infty} \frac{k_n}{\omega'(\lambda_n)} \phi_\alpha(x, \lambda_n) \phi_\alpha(\xi, \lambda_n)$$

$$\sum_{n=0}^{n=\infty} a_n(\lambda) [\lambda - \lambda_n] \phi_\alpha(x, \lambda_n) \phi_\alpha(\xi, \lambda_n) = \sum_{n=0}^{n=\infty} \frac{k_n}{\omega'(\lambda_n)} \phi_\alpha(x, \lambda_n) \phi_\alpha(\xi, \lambda_n),$$

Equating the coefficients, for each $n = 0, 1, 2, 3, \dots$

$$a_n(\lambda)[\lambda - \lambda_n] = \frac{k_n}{\omega'(\lambda_n)},$$

$$a_n(\lambda) = \frac{1}{\lambda - \lambda_n} \frac{k_n}{\omega'(\lambda_n)}.$$

11.1 The Hyper-real Green Function is

$$G(x, \xi, \lambda) = \sum_{n=0}^{n=\infty} \frac{1}{\lambda - \lambda_n} \frac{k_n}{\omega'(\lambda_n)} \phi_\alpha(x, \lambda_n) \phi_\alpha(\xi, \lambda_n).$$

That is,

11.2 The Hyper-real Green Function is the infinite sequence

$$\left\langle \frac{1}{\lambda - \lambda_0} \frac{k_0}{\omega'(\lambda_0)} \phi_\alpha(x, \lambda_0) \phi_\alpha(\xi, \lambda_0) + \dots + \frac{1}{\lambda - \lambda_n} \frac{k_n}{\omega'(\lambda_n)} \phi_\alpha(x, \lambda_n) \phi_\alpha(\xi, \lambda_n) \right\rangle_{n=0}^{n=\infty}.$$

11.3 The Hyper-real $y(x, \lambda)$ for Non-Normalized Eigen-functions

$$y(x, \lambda) = \int_{\xi=a}^{\xi=b} G(x, \xi, \lambda) f(\xi) d\xi$$

Proof: We substitute it into the Sturm-Liouville equation

$$y''(x, \lambda) = \int_{\xi=a}^{\xi=b} D_x^2 G(x, \xi, \lambda) f(\xi) d\xi,$$

$$\begin{aligned} y''(x, \lambda) + [\lambda - q(x)]y(x, \lambda) &= \int_{\xi=a}^{\xi=b} \underbrace{\{D_x^2 G(x, \xi, \lambda) + [\lambda - q(x)]G(x, \xi, \lambda)\}}_{\delta(x-\xi)} f(\xi) d\xi \\ &= f(x). \square \end{aligned}$$

12.

Green Function for Sine Sturm-Liouville $y''(x) + \lambda y(x) = 0$ & $\alpha = \beta = 0$

Two independent solutions are

$$\cos \sqrt{\lambda}x, \text{ and } \sin \sqrt{\lambda}x.$$

For $\alpha = 0$,

$$\phi_0(x, \lambda) = -\frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda}(x - a)$$

satisfies the Boundary conditions

$$\phi_0(a, \lambda) = \sin 0 = 0,$$

and

$$\phi_0'(a, \lambda) = -\cos 0 = -1.$$

For $\beta = 0$,

$$\chi_0(x, \lambda) = \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda}(b - x)$$

satisfies the Boundary conditions

$$\chi_0(b, \lambda) = \sin 0 = 0,$$

and

$$\chi_0'(b, \lambda) = -\cos 0 = -1,$$

Therefore,

$$\begin{aligned}\omega(\lambda) &= \left| \begin{array}{cc} \frac{\sin \sqrt{\lambda}(x-a)}{\sqrt{\lambda}} & \frac{\sin \sqrt{\lambda}(b-x)}{\sqrt{\lambda}} \\ -\cos \sqrt{\lambda}(x-a) & -\cos \sqrt{\lambda}(b-x) \end{array} \right| \\ &= \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda}(b-a).\end{aligned}$$

Hence, the zeros of $\omega(\lambda)$ are

$$\lambda_n = \left(\frac{n\pi}{b-a} \right)^2, \quad n = 1, 2, 3, \dots$$

$$\begin{aligned}\omega'(\lambda) &= \frac{d}{d\lambda} \left\{ \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda}(b-a) \right\} \\ &= \frac{1}{\lambda} \left\{ \left[\cos \sqrt{\lambda}(b-a) \right] \frac{b-a}{2\sqrt{\lambda}} \sqrt{\lambda} - \frac{1}{2\sqrt{\lambda}} \sin \sqrt{\lambda}(b-a) \right\}.\end{aligned}$$

$$\begin{aligned}\omega'(\lambda_n) &= \frac{1}{2\lambda_n} (b-a) \cos \underbrace{\sqrt{\lambda_n}(b-a)}_{n\pi} \\ &= \frac{1}{2\lambda_n} (b-a) (-1)^n.\end{aligned}$$

$$\begin{aligned}k_n &= \frac{\chi_0(x, \lambda_n)}{\phi_0(x, \lambda_n)} \\ &= -\frac{\sin \sqrt{\lambda_n}(b-x)}{\sin \sqrt{\lambda_n}(x-a)} \\ &= -\frac{\sin \frac{b-x}{b-a} n\pi}{\sin \frac{x-a}{b-a} n\pi} \\ &= -\frac{\sin \left[n\pi - \frac{x-a}{b-a} n\pi \right]}{\sin \frac{x-a}{b-a} n\pi}\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sin \frac{x-a}{b-a} n\pi} \left\{ \underbrace{-\sin(n\pi) \cos \frac{x-a}{b-a} n\pi}_0 + \underbrace{\cos(n\pi) \sin \frac{x-a}{b-a} n\pi}_{(-1)^n} \right\} \\
&= (-1)^n.
\end{aligned}$$

Therefore,

12.1 The Hyper-real Green Function in $[a, b]$

$$\boxed{G(x, \xi, \lambda) = \frac{2}{b-a} \sum_{n=1}^{n=\infty} \frac{1}{\lambda - \left(\frac{n\pi}{b-a}\right)^2} \sin\left(n\pi \frac{x-a}{b-a}\right) \sin\left(n\pi \frac{\xi-a}{b-a}\right)}$$

Proof: $G(x, \xi, \lambda) = \sum_{n=1}^{n=\infty} \frac{1}{\lambda - \lambda_n} \frac{k_n}{\omega'(\lambda_n)} \phi_\alpha(\xi, \lambda_n) \phi_\alpha(x, \lambda_n)$

$$\begin{aligned}
&= \sum_{n=1}^{n=\infty} \frac{1}{\lambda - \left(\frac{n\pi}{b-a}\right)^2} \frac{(-1)^n}{\frac{b-a}{2\lambda_n} (-1)^n} \frac{\sin \sqrt{\lambda_n} (x-a)}{\sqrt{\lambda_n}} \frac{\sin \sqrt{\lambda_n} (\xi-a)}{\sqrt{\lambda_n}} \\
&= \sum_{n=1}^{n=\infty} \frac{1}{\lambda - \left(\frac{n\pi}{b-a}\right)^2} \frac{2}{b-a} \sin\left(n\pi \frac{x-a}{b-a}\right) \sin\left(n\pi \frac{\xi-a}{b-a}\right). \square
\end{aligned}$$

Namely,

12.2 The Hyper-real Green Function in $[a, b]$ is the infinite sequence

$$\begin{aligned}
&\frac{2}{b-a} \left\langle \frac{1}{\lambda - \left(\frac{\pi}{b-a}\right)^2} \sin\left(\pi \frac{x-a}{b-a}\right) \sin\left(\pi \frac{\xi-a}{b-a}\right) + \dots \right. \\
&\quad \left. + \frac{1}{\lambda - \left(\frac{n\pi}{b-a}\right)^2} \sin\left(n\pi \frac{x-a}{b-a}\right) \sin\left(n\pi \frac{\xi-a}{b-a}\right) \right\rangle_{n=1}^{n=\infty}
\end{aligned}$$

13.

Green Function for Cosine Sturm-

Liouville $y''(x) + \lambda y(x) = 0$ & $\alpha = \beta = \frac{1}{2}\pi$

Similarly to the former section, for $\alpha = \beta = \frac{1}{2}\pi$, we obtain

13.1 *The Hyper-real Green Function in $[a, b]$*

$$G(x, \xi, \lambda) = \frac{1}{b-a} \left\{ \frac{1}{\lambda} + 2 \sum_{n=1}^{n=\infty} \frac{1}{\lambda - \left(\frac{n\pi}{b-a}\right)^2} \cos\left(n\pi \frac{x-a}{b-a}\right) \cos\left(n\pi \frac{\xi-a}{b-a}\right) \right\}$$

Namely,

13.2 *The Hyper-real Green Function in $[a, b]$ is the infinite sequence*

$$\frac{1}{b-a} \left\langle \frac{1}{\lambda} + \dots + 2 \frac{1}{\lambda - \left(\frac{n\pi}{b-a}\right)^2} \cos\left(n\pi \frac{x-a}{b-a}\right) \cos\left(n\pi \frac{\xi-a}{b-a}\right) \right\rangle_{n=1}^{n=\infty}$$

14.

Green's Function for Bessel

Sturm-Liouville $y''(x) + \lambda y(x) = 0$ & $\alpha = 0$

For $\alpha = 0$,

$$\phi_0(x, \lambda) = -\frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}}$$

satisfies the Boundary conditions

$$\phi_0(a, \lambda) = \sin 0 = 0,$$

and

$$\phi_0'(a, \lambda) = -\cos 0 = -1.$$

The solution

$$\chi_{\beta}(x, \lambda) = \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda}(b-x) \cos \beta + \cos[\sqrt{\lambda}(b-x)] \sin \beta$$

satisfies the Boundary conditions

$$\chi_{\beta}(b, \lambda) = \sin \beta,$$

and

$$\chi_{\beta}'(b, \lambda) = -\cos \beta,$$

Therefore, at $x = b$,

$$\omega(\lambda) = \begin{vmatrix} -\frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda}b & \sin \beta \\ -\cos \sqrt{\lambda}b & -\cos \beta \end{vmatrix}$$

Hence, the zeros of $\omega(\lambda)$ are the roots of

$$\frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda} b \cos \beta = -\cos \sqrt{\lambda} b \sin \beta.$$

That is,

$$\tan \sqrt{\lambda_n} b = -\sqrt{\lambda_n} \tan \beta.$$

[Titchmarsh, p. 17] obtains

$$\omega'(\lambda_n) = \frac{1}{2\lambda_n} b \cos \beta \cos(b\sqrt{\lambda_n}) \{1 + \lambda_n \tan^2 \beta + \frac{1}{b} \tan \beta\}$$

$$k_n = \cos \beta \cos(b\sqrt{\lambda_n}) \{1 + \lambda_n \tan^2 \beta\}$$

Therefore,

14.1 *The Green Function expansion in $[a, b]$ is*

$$G(x, \xi, \lambda) = \frac{2}{b} \sum_{n=0}^{n=\infty} \frac{1}{\lambda - \lambda_n} \frac{1 + \lambda_n \tan^2 \beta}{1 + \lambda_n \tan^2 \beta + \frac{1}{b} \tan \beta} \sin(x\sqrt{\lambda_n}) \sin(\xi\sqrt{\lambda_n})$$

Proof: $G(x, \xi, \lambda) = \sum_{n=0}^{n=\infty} \frac{1}{\lambda - \lambda_n} \frac{k_n}{\omega'(\lambda_n)} \phi_\alpha(\xi, \lambda_n) \phi_\alpha(x, \lambda_n). \square$

Namely,

14.2 *The Hyper-real Green Function in $[a, b]$ is the infinite sequence*

$$\frac{2}{b} \left\langle \sum_{j=0}^{j=n} \frac{1}{\lambda - \lambda_n} \frac{1 + \lambda_j \tan^2 \beta}{1 + \lambda_j \tan^2 \beta + \frac{1}{b} \tan \beta} \sin(x\sqrt{\lambda_j}) \sin(\xi\sqrt{\lambda_j}) \right\rangle_{n=0}^{n=\infty}$$

15.

Green's Function for Bessel Sturm

Liouville $u''(x) + \left(\lambda - \frac{\nu^2 - \frac{1}{4}}{x^2}\right)u(x) = 0$ & $\alpha = \beta = 0$

Put

$$u(x) = \sqrt{x}y(x),$$

to obtain Bessel's

$$y''(x) + \frac{1}{x}y'(x) + \left(\lambda - \frac{\nu^2}{x^2}\right)y(x) = 0.$$

Two independent solutions to Bessel's equation are

$$J_\nu(x\sqrt{\lambda}), \text{ and } Y_\nu(x\sqrt{\lambda}).$$

Two independent solutions to $u''(x) + \left(\lambda - \frac{\nu^2 - \frac{1}{4}}{x^2}\right)u(x) = 0$ are

$$\sqrt{x}J_\nu(x\sqrt{\lambda}), \text{ and } \sqrt{x}Y_\nu(x\sqrt{\lambda}).$$

For $\alpha = 0$,

$$\phi_0(x, \lambda) = \frac{1}{2}\pi\sqrt{ax}\{J_\nu(x\sqrt{\lambda})Y_\nu(a\sqrt{\lambda}) - Y_\nu(x\sqrt{\lambda})J_\nu(a\sqrt{\lambda})\}$$

satisfies the Boundary conditions

$$\phi_0(a, \lambda) = \sin 0 = 0,$$

and

$$\phi_0'(a, \lambda) = -\cos 0 = -1.$$

For $\beta = 0$,

$$\chi_0(x, \lambda) = \frac{1}{2}\pi\sqrt{bx}\{J_\nu(x\sqrt{\lambda})Y_\nu(b\sqrt{\lambda}) - Y_\nu(x\sqrt{\lambda})J_\nu(b\sqrt{\lambda})\}$$

satisfies the Boundary conditions

$$\chi_0(b, \lambda) = \sin 0 = 0,$$

and

$$\chi_0'(b, \lambda) = -\cos 0 = -1,$$

[Titchmarsh, p.18] obtains

$$\begin{aligned} \omega(\lambda) &= \begin{vmatrix} \phi_0(x, \lambda) & \chi_0(x, \lambda) \\ \partial_x \phi_0(x, \lambda) & \partial_x \chi_0(x, \lambda) \end{vmatrix} \\ &= \frac{1}{2} \pi \sqrt{ab} \{J_\nu(a\sqrt{\lambda})Y_\nu(b\sqrt{\lambda}) - Y_\nu(a\sqrt{\lambda})J_\nu(b\sqrt{\lambda})\}. \\ \omega'(\lambda) &= - \left\{ \frac{a}{2\sqrt{\lambda}} \frac{J_\nu'(a\sqrt{\lambda})}{J_\nu(b\sqrt{\lambda})} + \frac{b}{2\sqrt{\lambda}} \frac{J_\nu'(b\sqrt{\lambda})}{J_\nu(b\sqrt{\lambda})} \right\} \omega(\lambda) - \frac{\sqrt{ab}}{2\lambda} \left[\frac{J_\nu(b\sqrt{\lambda})}{J_\nu(a\sqrt{\lambda})} - \frac{J_\nu(a\sqrt{\lambda})}{J_\nu(b\sqrt{\lambda})} \right] \\ \omega'(\lambda_n) &= - \frac{\sqrt{ab}}{2\lambda_n} \left[\frac{J_\nu^2(b\sqrt{\lambda_n}) - J_\nu^2(a\sqrt{\lambda_n})}{J_\nu(a\sqrt{\lambda_n})J_\nu(b\sqrt{\lambda_n})} \right] \\ k_n &= \sqrt{\frac{b}{a} \frac{J_\nu(b\sqrt{\lambda_n})}{J_\nu(a\sqrt{\lambda_n})}}, \end{aligned}$$

λ_n are the zeros of $\omega(\lambda)$.

15.1 *The Hyper-real Green Function in $[a, b]$ is*

$$\begin{aligned} G(x, \xi, \lambda) &= \frac{\pi^2}{2} \sqrt{x\xi} \sum_{n=0}^{n=\infty} \frac{1}{\lambda - \lambda_n} \frac{\lambda_n J_\nu^2(b\sqrt{\lambda_n})}{J_\nu^2(a\sqrt{\lambda_n}) - J_\nu^2(b\sqrt{\lambda_n})} \times \\ &\quad \times \left\{ J_\nu(x\sqrt{\lambda_n})Y_\nu(a\sqrt{\lambda_n}) - Y_\nu(x\sqrt{\lambda_n})J_\nu(a\sqrt{\lambda_n}) \right\} \times \\ &\quad \times \left\{ J_\nu(\xi\sqrt{\lambda_n})Y_\nu(a\sqrt{\lambda_n}) - Y_\nu(\xi\sqrt{\lambda_n})J_\nu(a\sqrt{\lambda_n}) \right\} \end{aligned}$$

Proof: $G(x, \xi, \lambda) = \sum_{n=0}^{n=\infty} \frac{1}{\lambda - \lambda_n} \frac{k_n}{\omega'(\lambda_n)} \phi_\alpha(\xi, \lambda_n) \phi_\alpha(x, \lambda_n). \square$

Namely,

15.2 *The Hyper-real Green Function in $[a, b]$ is the infinite sequence*

$$\begin{aligned} & \frac{\pi^2}{2} \sqrt{x\xi} \left\langle \sum_{j=0}^{j=n} \frac{1}{\lambda - \lambda_n} \frac{\lambda_j J_\nu^2(b\sqrt{\lambda_j})}{J_\nu^2(a\sqrt{\lambda_j}) - J_\nu^2(b\sqrt{\lambda_j})} \times \right. \\ & \quad \times \left\{ J_\nu(x\sqrt{\lambda_j}) Y_\nu(a\sqrt{\lambda_j}) - Y_\nu(x\sqrt{\lambda_j}) J_\nu(a\sqrt{\lambda_j}) \right\} \times \\ & \quad \times \left. \left\{ J_\nu(\xi\sqrt{\lambda_j}) Y_\nu(a\sqrt{\lambda_j}) - Y_\nu(\xi\sqrt{\lambda_j}) J_\nu(a\sqrt{\lambda_j}) \right\} \right\rangle_{n=0}^{n=\infty} \end{aligned}$$

16.

Delta Expansion in Orthonormal Eigen-functions

The Hyper-real eigen-functions of a Hyper-real Sturm-Liouville problem over the interval with endpoints a , and b ,

$$\psi_0(x), \psi_1(x), \psi_2(x), \dots$$

can be normalized so that

$$\int_{x=a}^{x=b} \psi_n(x)\psi_m(x)dx = \delta_{nm}.$$

Then, a hyper-real function $f(x)$ may be expanded in them by

$$\begin{aligned} f(x) &= \sum_{n=0}^{n=\infty} \left(\int_{\xi=a}^{\xi=b} f(\xi)\psi_n(\xi)d\xi \right) \psi_n(x). \\ &= \sum_{n=0}^{n=\infty} \sum_{\xi=a}^{\xi=b} f(\xi)\psi_n(\xi)d\xi \psi_n(x) \end{aligned}$$

Exchanging summation order,

$$\begin{aligned} &= \sum_{\xi=a}^{\xi=b} \left\{ \sum_{n=0}^{n=\infty} \psi_n(x)\psi_n(\xi) \right\} f(\xi)d\xi \\ &= \int_{\xi=a}^{\xi=b} \underbrace{\sum_{n=0}^{n=\infty} \psi_n(x)\psi_n(\xi)}_{\delta(x-\xi)} f(\xi)d\xi. \end{aligned}$$

Therefore,

16.1 *The Hyper-real Delta Function expansion in orthonormal Sturm-Liouville eigen-functions is*

$$\delta(x - \xi) = \sum_{n=0}^{n=\infty} \psi_n(x)\psi_n(\xi).$$

That is,

16.2 *The Hyper-real Delta Function is the infinite sequence*

$$\langle \psi_0(x)\psi_0(\xi) + \psi_1(x)\psi_1(\xi) + \dots + \psi_n(x)\psi_n(\xi) \rangle.$$

17.

Green's Function Expansion in Orthonormal Eigen-functions

Green's Function is defined as the solution to Sturm-Liouville problem. Namely, the Sturm-Liouville equation

$$\boxed{D_x^2 G(x, \xi, \lambda) + [\lambda - q(x)]G(x, \xi, \lambda) = \delta(x - \xi)}$$

with the Boundary conditions at the intervals endpoints.

That is,

$$D_x^2 G(x, \xi, \lambda) + [\lambda - q(x)]G(x, \xi, \lambda) = \sum_{n=0}^{n=\infty} \psi_n(x)\psi_n(\xi)$$

Hence, Green's Function must have an expansion in $\psi_n(x)\psi_n(\xi)$,

$$G(x, \xi, \lambda) = \sum_{n=0}^{n=\infty} a_n(\lambda)\psi_n(x)\psi_n(\xi).$$

To determine the $a_n(\lambda)$, we substitute into the Sturm-Liouville equation for $G(x, \xi, \lambda)$,

$$\sum_{n=0}^{n=\infty} a_n(\lambda) \left\{ \underbrace{\psi_n''(x) + [\lambda - q(x)]\psi_n(x)}_{-[\lambda_n - q(x)]\psi_n(x)} \right\} \psi_n(\xi) = \sum_{n=0}^{n=\infty} \psi_n(x)\psi_n(\xi)$$

$$\underbrace{\hspace{10em}}_{[\lambda - \lambda_n]\psi_n(x)}$$

$$\sum_{n=0}^{n=\infty} a_n(\lambda)[\lambda - \lambda_n]\psi_n(x)\psi_n(\xi) = \sum_{n=0}^{n=\infty} \psi_n(x)\psi_n(\xi),$$

Equating the coefficients, for each $n = 0, 1, 2, 3, \dots$

$$a_n(\lambda)[\lambda - \lambda_n] = 1,$$

$$a_n(\lambda) = \frac{1}{\lambda - \lambda_n}.$$

17.1 *The Hyper-real Green Function is*

$$G(x, \xi, \lambda) = \sum_{n=0}^{n=\infty} \frac{1}{\lambda - \lambda_n} \psi_n(x) \psi_n(\xi).$$

That is,

17.2 *The Hyper-real Green Function is the infinite sequence*

$$\left\langle \frac{1}{\lambda - \lambda_0} \psi_0(x) \psi_0(\xi) + \frac{1}{\lambda - \lambda_1} \psi_1(x) \psi_1(\xi) + \dots + \frac{1}{\lambda - \lambda_n} \psi_n(x) \psi_n(\xi) \right\rangle_{n=0}^{n=\infty}.$$

17.3 *The Hyper-real $y(x, \lambda)$ for Orthonormal Eigen-functions*

$$y(x, \lambda) = \int_{\xi=a}^{\xi=b} G(x, \xi, \lambda) f(\xi) d\xi$$

Proof: 11.3. \square

18.

$G(x, \xi, \lambda)$ Analytic Continuation in λ into the Complex Plane

18.1 Fix x , and ξ , consider λ in the complex plane, and let

$$G(x, \xi, \zeta) = \sum_{n=1}^{n=\infty} \frac{1}{\zeta - \lambda_n} \psi_n(x) \psi_n(\xi)$$

be the Hyper-Complex Analytic Continuation of $G(x, \xi, \lambda)$ into the ζ Hyper-complex plane.

Let γ be a contour that encloses all the λ_n 's.

Then
$$\oint_{\gamma} G(x, \xi, \zeta) d\zeta = 2\pi i \delta(x - \xi)$$

First Proof: (that uses the Residue Theorem)

By Sturm-Liouville Oscillation Theorem [Inc. 10.6],

λ_n are the simple poles of $G(x, \xi, \zeta)$ on the positive real line.

Applying the Residue Theorem

$$\oint_{\gamma} G(x, \xi, \zeta) d\zeta = 2\pi i \sum_{n=1}^{n=\infty} \text{Res} \{ G(x, \xi, \zeta) \} \Big|_{\zeta=\lambda_n}.$$

Since the Residue of $G(x, \xi, \zeta)$ at each λ_n is $\psi_n(x) \psi_n(\xi)$,

$$= 2\pi i \sum_{n=1}^{n=\infty} \psi_n(x) \psi_n(\xi)$$

$$= 2\pi i \delta(x - \xi). \square$$

Since we do not supply here a proof of the Residue Theorem, we shall give a direct proof:

Second Proof (that uses the Circular Complex Delta)

$$\begin{aligned} \oint_{\gamma} G(x, \xi, \zeta) d\zeta &= \oint_{\gamma} \sum_{n=1}^{n=\infty} \frac{1}{\zeta - \lambda_n} \psi_n(x) \psi_n(\xi) d\zeta \\ &= \sum_{n=1}^{n=\infty} \psi_n(x) \psi_n(\xi) \oint_{\gamma} \frac{1}{\zeta - \lambda_n} d\zeta. \end{aligned}$$

Now, by [Dan10],

$$\frac{1}{2\pi i} \frac{1}{\zeta - \lambda_n} \text{ is the Hyper-complex Circular Delta Function,}$$

and

$$\oint_{\gamma} \frac{1}{2\pi i} \frac{1}{\zeta - \lambda_n} d\zeta = 1, \text{ for each } n = 1, 2, 3, \dots$$

Therefore,

$$\begin{aligned} \oint_{\gamma} G(x, \xi, \zeta) d\zeta &= \sum_{n=1}^{n=\infty} \psi_n(x) \psi_n(\xi) \underbrace{\oint_{\gamma} \frac{1}{\zeta - \lambda_n} d\zeta}_{2\pi i} \\ &= 2\pi i \sum_{n=1}^{n=\infty} \psi_n(x) \psi_n(\xi) \\ &= 2\pi i \delta(x - \xi). \square \end{aligned}$$

19.

Green Function for Hermit Sturm-

Liouville $u''(x) + (\lambda - x^2)u(x) = 0, x \text{ real}$

Put

$$u(x) = e^{-\frac{1}{2}x^2} y(x),$$

and obtain Hermit's equation

$$y''(x) + 2xy'(x) + (\lambda - 1)y(x) = 0,$$

Then, the eigen values are

$$\lambda_n = 2n + 1, \quad n = 0, 1, 2, 3, \dots$$

and the corresponding eigen functions are Hermit Polynomials of degree n ,

$$H_n(x).$$

Therefore,

$$e^{-\frac{1}{2}x^2} H_n(x) \text{ solve } u''(x) + (\lambda - x^2)u(x) = 0.$$

[Titchmarsh, p. 75] shows that

19.1 *The Normalized eigen functions are*

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!} \sqrt{\pi}} e^{-\frac{1}{2}x^2} H_n(x).$$

Therefore,

$$\mathbf{19.2} \quad \boxed{G(x, \xi, \lambda) = \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}\xi^2} \sum_{n=0}^{n=\infty} \frac{1}{(\lambda - 2n - 1)2^n n!} H_n(x) H_n(\xi)}$$

Proof: $G(x, \xi, \lambda) = \sum_{n=0}^{n=\infty} \frac{1}{\lambda - \lambda_n} \psi_n(x) \psi_n(\xi). \square$

Namely,

19.3 *The Hyper-real Green Function in real x , and ξ is the infinite sequence*

$$\frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}\xi^2} \left\langle \frac{1}{(\lambda - 1)} H_0(x) H_0(\xi) + \dots + \frac{1}{(\lambda - 2n - 1)2^n n!} H_n(x) H_n(\xi) \right\rangle_{n=0}^{n=\infty}$$

20.

Green Function for Legendre

Sturm-Liouville $u''(\theta) + [\lambda + \frac{1}{4} \frac{1}{\cos^2 \theta}]u(\theta) = 0,$

$$-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$$

Legendre's equation with $m = 0$ is

$$(1 - x^2)y''(x) + 2xy'(x) + (\lambda - \frac{1}{4})y(x) = 0.$$

The eigen values are

$$\lambda_n = (n + \frac{1}{2})^2, \quad n = 0, 1, 2, 3, \dots$$

and the corresponding eigen functions are Legendre Polynomials of degree n ,

$$P_n(x).$$

Put

$$x = \sin \theta,$$

to obtain

$$y''(\theta) - y'(\theta) \tan \theta + (\lambda - \frac{1}{4})y(\theta) = 0.$$

Put

$$y(\theta) = \frac{1}{\sqrt{\cos \theta}} u(\theta),$$

to obtain

$$u''(\theta) + [\lambda + \frac{1}{4}(1 + \tan^2 \theta)]u(\theta) = 0.$$

Substituting $\frac{1}{\cos^2 \theta} = 1 + \tan^2 \theta,$

$$u''(\theta) + \left[\lambda + \frac{1}{4} \frac{1}{\cos^2 \theta}\right] u(\theta) = 0.$$

Therefore,

$$\sqrt{\cos \theta} P_n(\sin \theta) \text{ solve } u''(\theta) + \left(\lambda + \frac{1}{4 \cos^2 \theta}\right) u(\theta) = 0.$$

[Titchmarsh, p. 79] shows that

20.1 *The Normalized eigen functions are*

$$\psi_n(\theta) = \sqrt{n + \frac{1}{2}} \sqrt{\cos \theta} P_n(\sin \theta).$$

Therefore,

20.2 *for* $-\frac{1}{2}\pi < \theta$ *and* $\varphi < \frac{1}{2}\pi$

$$G(\theta, \varphi, \lambda) = \sqrt{\cos \theta} \sqrt{\cos \varphi} \sum_{n=0}^{n=\infty} \frac{1}{\lambda - (n + \frac{1}{2})^2} (n + \frac{1}{2}) P_n(\sin \theta) P_n(\sin \varphi)$$

Namely,

20.3 *The Hyper-real Green Function in* $-\frac{1}{2}\pi < \theta, \varphi < \frac{1}{2}\pi$ *is*

the infinite sequence

$$\sqrt{\cos \theta} \sqrt{\cos \varphi} \left\langle \frac{1}{\lambda - \frac{1}{4}} P_0(\sin \theta) P_0(\sin \varphi) + \dots + \frac{1}{\lambda - (n + \frac{1}{2})^2} (n + \frac{1}{2}) P_n(\sin \theta) P_n(\sin \varphi) \right\rangle_{n=0}^{n=\infty}$$

21.

Green Function for Legendre Sturm-Liouville

$$u''(\theta) + \left[\lambda + \left(\frac{1}{4} - m^2\right) \frac{1}{\cos^2 \theta}\right] u(\theta) = 0, \quad -\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$$

Legendre's equation with for $m = 0, 1, 2, 3, \dots$, is

$$(1 - x^2)y''(x) + 2xy'(x) + \left[\lambda - \frac{1}{4} - \frac{1}{1-x^2} m^2\right] y(x) = 0.$$

The eigen values are

$$\lambda_n^m = \left(n - m + \frac{1}{2}\right)^2, \quad n = 2m, 2m + 1, \dots$$

and the corresponding eigen functions are Legendre Functions

$$P_n^m(x).$$

Put

$$x = \sin \theta,$$

to obtain

$$y''(\theta) - y'(\theta) \tan \theta + \left[\lambda - \frac{1}{4} - m^2 \frac{1}{\cos^2 \theta}\right] y(\theta) = 0.$$

Put

$$y(\theta) = \frac{1}{\sqrt{\cos \theta}} u(\theta),$$

to obtain

$$u''(\theta) + \left[\lambda + \frac{1}{4}(1 + \tan^2 \theta) - \frac{1}{4} \frac{1}{\cos^2 \theta} m^2\right] u(\theta) = 0.$$

Substituting $\frac{1}{\cos^2 \theta} = 1 + \tan^2 \theta$,

$$u''(\theta) + \left[\lambda + \left(\frac{1}{4} - m^2\right) \frac{1}{\cos^2 \theta}\right] u(\theta) = 0.$$

Therefore,

$$\sqrt{\cos \theta} P_n^m(\sin \theta) \text{ solve } u''(\theta) + \left[\lambda + \left(\frac{1}{4} - m^2\right) \frac{1}{\cos^2 \theta}\right] u(\theta) = 0.$$

[Titchmarsh, p. 80] shows that

21.1 *The Normalized eigen functions are*

$$\psi_n(\theta) = \sqrt{\frac{(n-2m)!}{n!}} \sqrt{n-m+\frac{1}{2}} \sqrt{\cos \theta} P_n^m(\sin \theta).$$

Therefore,

21.2 *For $-\frac{1}{2}\pi < \theta, \varphi < \frac{1}{2}\pi$, and for $m = 0, 1, 2, 3, \dots$,*

$$G(\theta, \varphi, \lambda) = \sqrt{\cos \theta} \sqrt{\cos \varphi} \times \\ \times \sum_{n=2m}^{n=\infty} \frac{1}{\lambda - (n-m+\frac{1}{2})^2} \frac{(n-2m)!}{n!} (n-m+\frac{1}{2}) P_n^m(\sin \theta) P_n^m(\sin \varphi)$$

Namely,

21.3 *The Hyper-real Green Function in $-\frac{1}{2}\pi < \theta, \varphi < \frac{1}{2}\pi$ is the infinite sequence*

$$\sqrt{\cos \theta} \sqrt{\cos \varphi} \times \\ \times \left\langle \sum_{k=2m}^{k=n} \frac{1}{\lambda - (n-m+\frac{1}{2})^2} \frac{(k-2m)!}{k!} (k-m+\frac{1}{2}) P_k^m(\sin \theta) P_k^m(\sin \varphi) \right\rangle_{n=m}^{n=\infty}$$

22.

Green Function for Bessel Sturm-Liouville

$$y''(x) + \left[\lambda - \frac{1}{x^2}(\nu^2 - \frac{1}{4})\right]y(x) = 0, \quad 0 < x < b$$

Two independent solutions are

$$\sqrt{x}J_\nu(x\sqrt{\lambda}), \quad \text{and} \quad \sqrt{x}Y_\nu(x\sqrt{\lambda})$$

By [Titchmarsh, p. 85],

$$y(x, \lambda) = \frac{1}{2}\pi \frac{1}{J_\nu(b\sqrt{\lambda})} \sqrt{x} \begin{vmatrix} J_\nu(b\sqrt{\lambda}) & J_\nu(x\sqrt{\lambda}) \\ Y_\nu(b\sqrt{\lambda}) & Y_\nu(x\sqrt{\lambda}) \end{vmatrix} \int_{\xi=0}^{\xi=x} \sqrt{\xi} J_\nu(\xi\sqrt{\lambda}) f(\xi) d\xi$$

$$+ \frac{1}{2}\pi \frac{1}{J_\nu(b\sqrt{\lambda})} \sqrt{x} J_\nu(x\sqrt{\lambda}) \int_{\xi=x}^{\xi=b} \sqrt{\xi} \begin{vmatrix} J_\nu(b\sqrt{\lambda}) & J_\nu(\xi\sqrt{\lambda}) \\ Y_\nu(b\sqrt{\lambda}) & Y_\nu(\xi\sqrt{\lambda}) \end{vmatrix} f(\xi) d\xi$$

Therefore,

22.1

$$G(x, \xi, \lambda) = \frac{1}{2}\pi \frac{1}{J_\nu(b\sqrt{\lambda})} \begin{cases} \sqrt{x} \begin{vmatrix} J_\nu(b\sqrt{\lambda}) & J_\nu(x\sqrt{\lambda}) \\ Y_\nu(b\sqrt{\lambda}) & Y_\nu(x\sqrt{\lambda}) \end{vmatrix} \sqrt{\xi} J_\nu(\xi\sqrt{\lambda}), & \xi \leq x \\ \sqrt{\xi} \begin{vmatrix} J_\nu(b\sqrt{\lambda}) & J_\nu(\xi\sqrt{\lambda}) \\ Y_\nu(b\sqrt{\lambda}) & Y_\nu(\xi\sqrt{\lambda}) \end{vmatrix} \sqrt{x} J_\nu(x\sqrt{\lambda}), & \xi \geq x \end{cases}$$

Alternatively, we may expand $G(x, \xi, \lambda)$ in eigen functions:

For $\boxed{\nu \geq 1}$

By [Titchmarsh, p. 82],

22.2 The eigen-values

λ_n are the zeros of $J_\nu(b\sqrt{\lambda})$, $n = 1, 2, 3, \dots$

and the normalized eigen-functions are

$$\frac{\sqrt{2}}{bJ'_\nu(b\sqrt{\lambda_n})} J_\nu(x\sqrt{\lambda_n}).$$

Therefore,

22.3 For $\nu \geq 1$, in $0 < x < b$

$$\boxed{G(x, \xi, \lambda) = \frac{2}{b^2} \sqrt{x\xi} \sum_{n=1}^{n=\infty} \frac{1}{(\lambda - \lambda_n) J'_\nu(b\sqrt{\lambda_n})} J_\nu(x\sqrt{\lambda_n}) J_\nu(\xi\sqrt{\lambda_n}), \quad \nu \geq 1}$$

Namely,

22.4 For $\nu \geq 1$ the Hyper-real Green Function in $0 < x, \xi < b$ is
the infinite sequence

$$\frac{2}{b^2} \sqrt{x\xi} \left\langle \sum_{k=1}^{k=n} \frac{1}{(\lambda - \lambda_k) J'_\nu(b\sqrt{\lambda_k})} J_\nu(x\sqrt{\lambda_k}) J_\nu(\xi\sqrt{\lambda_k}) \right\rangle_{n=1}^{n=\infty}$$

For $\boxed{0 < \nu < 1, \quad \nu \neq \frac{1}{2}}$

By [Titchmarsh, p. 83],

22.5 The eigen-values

λ_n are the zeros of $cJ_\nu(b\sqrt{\lambda}) - \lambda^\nu J_{-\nu}(b\sqrt{\lambda})$,

where $c = \text{const.}$

$$r_n = -\text{Res} \left\{ \sqrt{\lambda} \frac{c\sqrt{\lambda^{-\nu}} J_\nu'(b\sqrt{\lambda}) - \sqrt{\lambda^\nu} J_{-\nu}'(b\sqrt{\lambda})}{c\sqrt{\lambda^{-\nu}} J_\nu(b\sqrt{\lambda}) - \sqrt{\lambda^\nu} J_{-\nu}(b\sqrt{\lambda})} + \frac{1}{2b} \right\}_{\lambda=\lambda_n}$$

and the normalized eigen-functions are

$$\sqrt{|r_n|} \frac{\pi\sqrt{b}}{2\sin\nu\pi} \sqrt{x} J_\nu(b\sqrt{\lambda_n}) \left\{ c\lambda_n^{-\nu} J_\nu(x\sqrt{\lambda_n}) - J_{-\nu}(x\sqrt{\lambda_n}) \right\}.$$

Therefore,

22.6 If $0 < \nu < 1$, $\nu \neq \frac{1}{2}$, then in $0 < x < b$

$$G(x, \xi, \lambda) = \frac{\pi^2 b}{4 \sin^2 \nu \pi} \sqrt{x \xi} \sum_{n=1}^{n=\infty} \frac{1}{\lambda - \lambda_n} |r_n| J_\nu^2(b\sqrt{\lambda_n}) \times \\ \times \left\{ c\lambda_n^{-\nu} J_\nu(x\sqrt{\lambda_n}) - J_{-\nu}(x\sqrt{\lambda_n}) \right\} \left\{ c\lambda_n^{-\nu} J_\nu(\xi\sqrt{\lambda_n}) - J_{-\nu}(\xi\sqrt{\lambda_n}) \right\}$$

Namely,

22.7 If $0 < \nu < 1$, $\nu \neq \frac{1}{2}$, then the Hyper-real Green Function in

$0 < x, \xi < b$ is the infinite sequence

$$\frac{\pi^2 b}{4 \sin^2 \nu \pi} \sqrt{x \xi} \left\langle \sum_{k=1}^{k=n} \frac{1}{\lambda - \lambda_k} |r_k| J_\nu^2(b\sqrt{\lambda_k}) \left\{ c\lambda_k^{-\nu} J_\nu(x\sqrt{\lambda_k}) - J_{-\nu}(x\sqrt{\lambda_k}) \right\} \times \right.$$

$$\times \left\{ c\lambda_k^{-\nu} J_\nu(\xi\sqrt{\lambda_k}) - J_{-\nu}(\xi\sqrt{\lambda_k}) \right\}_{n=1}^{n=\infty}$$

22.8 If $0 < \nu < 1$, $\nu \neq \frac{1}{2}$, then

For $c = \text{infinite hyper-real}$, the expansion is in J_ν :

$$G(x, \xi, \lambda) = \frac{\pi^2 b}{4 \sin^2 \nu \pi} c^2 \sqrt{x\xi} \sum_{n=1}^{n=\infty} \frac{1}{\lambda - \lambda_n} |r_n| J_\nu^2(b\sqrt{\lambda_n}) \lambda_n^{-2\nu} J_\nu(x\sqrt{\lambda_n}) J_\nu(\xi\sqrt{\lambda_n})$$

For $c = \text{infinitesimal}$, the expansion is in $J_{-\nu}$:

$$G(x, \xi, \lambda) = \frac{\pi^2 b}{4 \sin^2 \nu \pi} \sqrt{x\xi} \sum_{n=1}^{n=\infty} \frac{1}{\lambda - \lambda_n} |r_n| J_\nu^2(b\sqrt{\lambda_n}) J_{-\nu}(x\sqrt{\lambda_n}) J_{-\nu}(\xi\sqrt{\lambda_n})$$

23.

Green Function for Bessel Sturm

Liouville $y''(x) + [\lambda - x]y(x) = 0, \quad 0 < x < \infty,$

with $\alpha = 0$

By [Titchmarsh, p. 91],

23.1 The eigen-values

λ_n are the zeros of $J_{\frac{1}{3}}(\frac{2}{3}\lambda^{\frac{3}{2}}) + J_{-\frac{1}{3}}(\frac{2}{3}\lambda^{\frac{3}{2}}), \quad n = 1, 2, 3, \dots$

and the normalized eigen-functions are

$$\frac{-1}{\lambda_n \left[J_{\frac{2}{3}}(\frac{2}{3}\lambda_n^{\frac{3}{2}}) - J_{-\frac{2}{3}}(\frac{2}{3}\lambda_n^{\frac{3}{2}}) \right]} \sqrt{\lambda_n - x} \left\{ J_{\frac{1}{3}}(\frac{2}{3}[\lambda_n - x]^{\frac{3}{2}}) + J_{-\frac{1}{3}}(\frac{2}{3}[\lambda_n - x]^{\frac{3}{2}}) \right\}$$

Therefore,

23.2 The Hyper-real Green Function in $0 < x < \infty$ is

$$G(x, \xi, \lambda) = \sum_{n=1}^{n=\infty} \frac{1}{\lambda - \lambda_n} \frac{1}{\lambda_n^2 \left[J_{\frac{2}{3}}(\frac{2}{3}\lambda_n^{\frac{3}{2}}) - J_{-\frac{2}{3}}(\frac{2}{3}\lambda_n^{\frac{3}{2}}) \right]^2} \sqrt{\lambda_n - x} \sqrt{\lambda_n - \xi} \\ \times \left\{ J_{\frac{1}{3}}(\frac{2}{3}[\lambda_n - x]^{\frac{3}{2}}) + J_{-\frac{1}{3}}(\frac{2}{3}[\lambda_n - x]^{\frac{3}{2}}) \right\} \\ \times \left\{ J_{\frac{1}{3}}(\frac{2}{3}[\lambda_n - \xi]^{\frac{3}{2}}) + J_{-\frac{1}{3}}(\frac{2}{3}[\lambda_n - \xi]^{\frac{3}{2}}) \right\}$$

Namely,

23.3 *The Hyper-real Green Function in $0 < x, \xi < \infty$ is the infinite sequence*

$$\left\langle \sum_{k=1}^{k=n} \frac{1}{\lambda - \lambda_n} \frac{1}{\lambda_n^2 \left[J_{\frac{2}{3}}\left(\frac{2}{3}\lambda_k^{\frac{3}{2}}\right) - J_{-\frac{2}{3}}\left(\frac{2}{3}\lambda_k^{\frac{3}{2}}\right) \right]^2} \sqrt{\lambda_k - x} \sqrt{\lambda_k - \xi} \times \right.$$

$$\times \left\{ J_{\frac{1}{3}}\left(\frac{2}{3}[\lambda_k - x]^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3}[\lambda_k - x]^{\frac{3}{2}}\right) \right\} \times$$

$$\times \left. \left\{ J_{\frac{1}{3}}\left(\frac{2}{3}[\lambda_k - \xi]^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3}[\lambda_k - \xi]^{\frac{3}{2}}\right) \right\} \right\rangle_{n=1}^{n=\infty}$$

24.

Green Function for Bessel Sturm-

Liouville $y''(x) + [\mu - x]y(x) = 0, 0 < x < \infty$

with $\alpha = \frac{1}{2} \pi$

By [Titchmarsh, p. 92],

24.1 The eigen-values

μ_n are the zeros of $J_{\frac{2}{3}}(\frac{2}{3}\mu^{\frac{3}{2}}) - J_{-\frac{2}{3}}(\frac{2}{3}\mu^{\frac{3}{2}}), n = 1, 2, 3, \dots$

and the normalized eigen-functions are

$$\frac{-1}{\mu_n \left[J_{\frac{1}{3}}(\frac{2}{3}\mu_n^{\frac{3}{2}}) + J_{-\frac{1}{3}}(\frac{2}{3}\mu_n^{\frac{3}{2}}) \right]} \sqrt{\mu_n - x} \left\{ J_{\frac{1}{3}}(\frac{2}{3}[\mu_n - x]^{\frac{3}{2}}) + J_{-\frac{1}{3}}(\frac{2}{3}[\mu_n - x]^{\frac{3}{2}}) \right\}$$

Therefore,

24.2 the Hyper-real Green Function in $0 < x < \infty$ is

$$G(x, \xi, \mu) = \sum_{n=1}^{n=\infty} \frac{1}{\mu - \mu_n} \frac{1}{\mu_n^2 \left[J_{\frac{1}{3}}(\frac{2}{3}\mu_n^{\frac{3}{2}}) + J_{-\frac{1}{3}}(\frac{2}{3}\mu_n^{\frac{3}{2}}) \right]^2} \sqrt{\mu_n - x} \sqrt{\mu_n - \xi} \times$$

$$\times \left\{ J_{\frac{1}{3}}(\frac{2}{3}[\mu_n - x]^{\frac{3}{2}}) + J_{-\frac{1}{3}}(\frac{2}{3}[\mu_n - x]^{\frac{3}{2}}) \right\} \times$$

$$\times \left\{ J_{\frac{1}{3}}(\frac{2}{3}[\mu_n - \xi]^{\frac{3}{2}}) + J_{-\frac{1}{3}}(\frac{2}{3}[\mu_n - \xi]^{\frac{3}{2}}) \right\}$$

Namely,

24.3 *the Hyper-real Green Function in $0 < x, \xi < \infty$ is the infinite sequence*

$$\left\langle \sum_{k=1}^{k=n} \frac{1}{\mu - \mu_n} \frac{1}{\mu_n^2 \left[J_{\frac{1}{3}}\left(\frac{2}{3} \mu_k^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3} \mu_k^{\frac{3}{2}}\right) \right]^2} \sqrt{\mu_k - x} \sqrt{\mu_k - \xi} \times \right.$$

$$\times \left\{ J_{\frac{1}{3}}\left(\frac{2}{3} [\mu_k - x]^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3} [\mu_k - x]^{\frac{3}{2}}\right) \right\} \times$$

$$\times \left. \left\{ J_{\frac{1}{3}}\left(\frac{2}{3} [\mu_k - \xi]^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3} [\mu_k - \xi]^{\frac{3}{2}}\right) \right\} \right\rangle_{n=1}^{n=\infty}$$

25.

Green's Function Integral Expansion in Eigen-functions

If the eigen-values are only infinitesimally separated from each other, the series summation of eigen-functions over discrete eigen values is replaced by Hyper-real integration over the continuum of eigen-values. Then, the expansion is represented by an integral.

Titchmarsh applied the Residue Theorem to obtain the projections of a function $f(x)$ on the eigen-functions of Sturm-Liouville problems, with continuous spectrum of eigen-values.

This yields the Integral expansion of the Delta Function in Sturm-Liouville eigen-functions.

The Delta Function Expansion determines the Green's Function integral expansion in Sturm-Liouville eigen-functions.

26.

Green's Function Integral Expansion in Cosine Functions of

$$y''(x) + \lambda y(x) = 0, \quad 0 < x < \infty$$

We may solve Green's Equation

$$D_x^2 G(x, \xi, \lambda) + \lambda G(x, \xi, \lambda) = \delta(x - \xi)$$

by Fourier Transforming it

$$\begin{array}{ccc} \underbrace{D_x^2 G(x, \xi, \lambda)}_{\downarrow} + \lambda \underbrace{G(x, \xi, \lambda)}_{\downarrow} = \underbrace{\delta(x - \xi)}_{\downarrow} \\ (i\omega)^2 \hat{G}(\omega, \xi, \lambda) + \lambda \hat{G}(\omega, \xi, \lambda) = \int_{x=-\infty}^{x=\infty} \delta(x - \xi) e^{-i\omega x} dx = e^{-i\omega \xi} \end{array}$$

$$(i\omega)^2 \hat{G}(\omega, \xi, \lambda) + \lambda \hat{G}(\omega, \xi, \lambda) = e^{-i\omega \xi},$$

$$\hat{G}(\omega, \xi, \lambda) = \frac{1}{\lambda - \omega^2} e^{-i\omega \xi},$$

Then, Inverse Transforming it,

$$G(x, \xi, \lambda) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} \left\{ \frac{1}{\lambda - \omega^2} e^{-i\omega \xi} \right\} e^{i\omega x} d\omega$$

$$= \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} \frac{1}{\lambda - \omega^2} e^{i\omega(x-\xi)} d\omega.$$

That is,

26.1 *The Hyper-real Green Function is*

$$G(x, \xi, \lambda) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} \frac{1}{\lambda - \omega^2} e^{i\omega(x-\xi)} d\omega$$

Since Fourier Transforming applies only to Linear Differential Equations with constant coefficients, we demonstrate a method that applies to equations with variable coefficients:

The eigen-values

$$\lambda \equiv \omega^2,$$

are the interval of hyper-real positive numbers $(0, \infty)$.

By [Titchmarsh, p. 72], the Hyper-real function $f(x)$ is given for any hyper-real x by

$$\begin{aligned} f(x) &= \frac{1}{\pi} \int_{\xi=-\infty}^{\xi=\infty} \int_{\lambda=0}^{\lambda=\infty} \cos(x\sqrt{\lambda}) \cos(\xi\sqrt{\lambda}) \frac{1}{2\sqrt{\lambda}} d\lambda f(\xi) d\xi + \\ &\quad + \frac{1}{\pi} \int_{\xi=-\infty}^{\xi=\infty} \int_{\lambda=0}^{\lambda=\infty} \sin(x\sqrt{\lambda}) \sin(\xi\sqrt{\lambda}) \frac{1}{2\sqrt{\lambda}} d\lambda f(\xi) d\xi \\ &= \frac{1}{\pi} \int_{\xi=-\infty}^{\xi=\infty} \left\{ \int_{\lambda=0}^{\lambda=\infty} \underbrace{\{\cos(x\sqrt{\lambda}) \cos(\xi\sqrt{\lambda}) + \sin(x\sqrt{\lambda}) \sin(\xi\sqrt{\lambda})\}}_{\cos[(x-\xi)\sqrt{\lambda}]} \frac{1}{2\sqrt{\lambda}} d\lambda \right\} f(\xi) d\xi \end{aligned}$$

$$= \int_{\xi=-\infty}^{\xi=\infty} \underbrace{\frac{1}{\pi} \int_{\omega=0}^{\omega=\infty} \cos \omega(x - \xi) d\omega}_{\delta(x-\xi)} f(\xi) d\xi$$

Hence, the Hyper-real Delta in $-\infty < x, \xi < \infty$ is

$$\begin{aligned} \delta(x - \xi) &= \frac{1}{\pi} \int_{\omega=0}^{\omega=\infty} \underbrace{\cos \omega(x - \xi)}_{\frac{1}{2}(e^{i\omega(x-\xi)} + e^{-i\omega(x-\xi)})} d\omega \\ &= \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} e^{i\omega(x-\xi)} d\omega \\ &= \frac{1}{2\pi} \sum_{\omega=-\infty}^{\omega=\infty} e^{i\omega(x-\xi)} d\omega. \end{aligned}$$

Since Green's Function satisfies

$$D_x^2 G(x, \xi, \lambda) + \lambda G(x, \xi, \lambda) = \delta(x - \xi),$$

the Hyper-real Green Function must have an expansion in $e^{i\omega(x-\xi)}$:

$$G(x, \xi, \lambda) = \frac{1}{2\pi} \sum_{\omega=-\infty}^{\omega=\infty} a(\omega) e^{i\omega(x-\xi)} d\omega.$$

Substituting in the Green's Function Differential Equation

$$\frac{1}{2\pi} \sum_{\omega=-\infty}^{\omega=\infty} a(\omega) \{-\omega^2 + \lambda\} e^{i\omega(x-\xi)} d\omega = \frac{1}{2\pi} \sum_{\omega=-\infty}^{\omega=\infty} e^{i\omega(x-\xi)} d\omega.$$

Equating each term,

$$a(\omega) \{-\omega^2 + \lambda\} = 1,$$

$$a(\omega) = \frac{1}{\lambda - \omega^2}$$

$$\begin{aligned} G(x, \xi, \lambda) &= \frac{1}{2\pi} \sum_{\omega=-\infty}^{\omega=\infty} \frac{1}{\lambda - \omega^2} e^{i\omega(x-\xi)} d\omega, \\ &= \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} \frac{1}{\lambda - \omega^2} e^{i\omega(x-\xi)} d\omega. \end{aligned}$$

27.

Green's Function Integral

Expansion for Weber Equation

$$y''(x) + \left[\lambda - \frac{1}{x^2} \left(\nu^2 - \frac{1}{4}\right)\right]y(x) = 0, \quad a < x < \infty$$

$$a > 0$$

The eigen-values

$$-\lambda \equiv s^2,$$

are the interval of hyper-real negative numbers $(-\infty, 0)$.

By [Titchmarsh, p. 87], the Hyper-real function $f(x)$ is given for hyper-real $a < x < \infty$ by

$$f(x) = \sqrt{x} \int_{\xi=a}^{\xi=\infty} \int_{s=0}^{s=\infty} \frac{1}{J_\nu^2(as) + Y_\nu^2(as)} \left\{ J_\nu(xs)Y_\nu(as) - J_\nu(as)Y_\nu(xs) \right\} \times \\ \times \left\{ J_\nu(\xi s)Y_\nu(as) - J_\nu(as)Y_\nu(\xi s) \right\} s ds \sqrt{\xi} f(\xi) d\xi$$

Therefore, the Hyper-real Delta in $a < x, \xi < \infty$ is

$$\delta(x - \xi) = \int_{s=0}^{s=\infty} \sqrt{\xi} \frac{J_\nu(\xi s)Y_\nu(as) - J_\nu(as)Y_\nu(\xi s)}{J_\nu^2(as) + Y_\nu^2(as)} \sqrt{x} \left\{ J_\nu(xs)Y_\nu(as) - J_\nu(as)Y_\nu(xs) \right\} s ds$$

Since Green's Function satisfies

$$\partial_x^2 G(x, \xi, \lambda) + \left[\lambda - \frac{\nu^2 - \frac{1}{4}}{x^2} \right] G(x, \xi, \lambda) = \delta(x - \xi),$$

the Hyper-real Green Function must have an expansion in

$$\psi_\nu(x, \xi, s) = \frac{\sqrt{\xi} J_\nu(\xi s) Y_\nu(as) - J_\nu(as) \sqrt{\xi} Y_\nu(\xi s)}{J_\nu^2(as) + Y_\nu^2(as)} \left\{ \sqrt{x} J_\nu(xs) Y_\nu(as) - J_\nu(as) \sqrt{x} Y_\nu(xs) \right\}.$$

Substituting

$$G(x, \xi, \lambda) = \sum_{s=0}^{s=\infty} a(s) \{ \psi_\nu(x, \xi, s) \} s ds$$

in the Green's Function Differential Equation,

$$\sum_{s=0}^{s=\infty} a(s) \left\{ \partial_x^2 \psi_\nu + \left[\lambda - \frac{\nu^2 - \frac{1}{4}}{x^2} \right] \psi_\nu \right\} s ds = \sum_{s=0}^{s=\infty} \psi_\nu(x, \xi, s) s ds.$$

Now, s is the eigen-value associated with $\sqrt{x} J_\nu(xs)$, with $\sqrt{x} Y_\nu(xs)$, with $\sqrt{x} J_\nu(xs) Y_\nu(as) - J_\nu(as) \sqrt{x} Y_\nu(xs)$, and thus with $\psi_\nu(x, \xi, s)$.

Hence,

$$\partial_x^2 \psi_\nu = - \left[s - \frac{\nu^2 - \frac{1}{4}}{x^2} \right] \psi_\nu.$$

Substituting in the Green's Function differential Equation

$$\sum_{s=0}^{s=\infty} a(s) \{ \lambda - s \} \psi_\nu s ds = \sum_{s=0}^{s=\infty} \psi_\nu(x, \xi, s) s ds$$

Equating each term,

$$a(s) \{ \lambda - s \} = 1,$$

$$a(s) = \frac{1}{\lambda - s}$$

$$G(x, \xi, \lambda) = \sum_{s=0}^{s=\infty} \frac{1}{\lambda - s} \psi_{\nu}(x, \xi, s) s ds .$$

27.1

$$G(x, \xi, \lambda) = \int_{s=0}^{s=\infty} \frac{1}{\lambda - s} \sqrt{\xi} \frac{J_{\nu}(\xi s) Y_{\nu}(as) - J_{\nu}(as) Y_{\nu}(\xi s)}{J_{\nu}^2(as) + Y_{\nu}^2(as)} \sqrt{x} \{ J_{\nu}(xs) Y_{\nu}(as) - J_{\nu}(as) Y_{\nu}(xs) \} s ds$$

28.

Green's Function for Hankel's

$$y''(x) + \left[\lambda - \frac{1}{x^2} \left(\nu^2 - \frac{1}{4}\right)\right]y(x) = 0, \quad 0 < x < \infty$$

By [Titchmarsh, p. 88],

$$y(x) = -\frac{1}{2}i\pi\sqrt{x}H_\nu^{(1)}(x\sqrt{\lambda}) \int_{\xi=0}^{\xi=x} \sqrt{\xi}J_\nu(\xi\sqrt{\lambda})f(\xi)d\xi \\ - \frac{1}{2}i\pi\sqrt{x}J_\nu(x\sqrt{\lambda}) \int_{\xi=x}^{\xi=\infty} \sqrt{\xi}H_\nu^{(1)}(\xi\sqrt{\lambda})f(\xi)d\xi$$

Therefore,

$$\mathbf{28.1} \quad G(x, \xi, \lambda) = -\frac{i\pi}{2}\sqrt{x\xi} \begin{cases} H_\nu^{(1)}(x\sqrt{\lambda})J_\nu(\xi\sqrt{\lambda}), & \xi \leq x \\ H_\nu^{(1)}(\xi\sqrt{\lambda})J_\nu(x\sqrt{\lambda}), & \xi \geq x \end{cases}$$

29.

Green's Function for Bessel's

$$y''(x) + [\lambda + x]y(x) = 0, \quad -\infty < x < \infty$$

By [Titchmarsh, p. 93],

$$y(x, \lambda) = -\frac{1}{6}i\pi\sqrt{x + \lambda}H_{\frac{1}{3}}^{(1)}\left(\frac{2}{3}[x + \lambda]^{\frac{3}{2}}\right) \int_{\xi=-\infty}^{\xi=x} \sqrt{\xi + \lambda}H_{\frac{1}{3}}^{(2)}\left(\frac{2}{3}[\xi + \lambda]^{\frac{3}{2}}\right)f(\xi)d\xi$$

$$-\frac{1}{6}i\pi\sqrt{x + \lambda}H_{\frac{1}{3}}^{(2)}\left(\frac{2}{3}[x + \lambda]^{\frac{3}{2}}\right) \int_{\xi=x}^{\xi=\infty} \sqrt{\xi + \lambda}H_{\frac{1}{3}}^{(1)}\left(\frac{2}{3}[\xi + \lambda]^{\frac{3}{2}}\right)f(\xi)d\xi$$

Therefore,

29.1

$$G(x, \xi, \lambda) = -\frac{i\pi}{6}\sqrt{x + \lambda}\sqrt{\xi + \lambda} \begin{cases} H_{\frac{1}{3}}^{(1)}\left(\frac{2}{3}[x + \lambda]^{\frac{3}{2}}\right)H_{\frac{1}{3}}^{(2)}\left(\frac{2}{3}[\xi + \lambda]^{\frac{3}{2}}\right), & \xi \leq x \\ H_{\frac{1}{3}}^{(1)}\left(\frac{2}{3}[\xi + \lambda]^{\frac{3}{2}}\right)H_{\frac{1}{3}}^{(2)}\left(\frac{2}{3}[x + \lambda]^{\frac{3}{2}}\right), & \xi \geq x \end{cases}$$

30.**Green's Function for Bessel's**

$$y''(x) + [\lambda + e^{2x}]y(x) = 0, \quad -\infty < x < \infty$$

By [Titchmarsh, p. 95],

$$y(x, \lambda) = -\frac{i\pi}{2 \sinh(\pi\sqrt{\lambda})} \{J_{i\sqrt{\lambda}}(e^x) + J_{-i\sqrt{\lambda}}(e^x)\} \int_{\xi=-\infty}^{\xi=x} J_{-i\sqrt{\lambda}}(e^\xi) f(\xi) d\xi$$

$$- \frac{i\pi}{2 \sinh(\pi\sqrt{\lambda})} J_{-i\sqrt{\lambda}}(e^x) \int_{\xi=-\infty}^{\xi=x} \{J_{i\sqrt{\lambda}}(e^\xi) + J_{-i\sqrt{\lambda}}(e^\xi)\} f(\xi) d\xi$$

Therefore,

$$\mathbf{30.1} \quad G(x, \xi, \lambda) = -\frac{i\pi}{2 \sinh(\pi\sqrt{\lambda})} \begin{cases} \{J_{i\sqrt{\lambda}}(e^x) + J_{-i\sqrt{\lambda}}(e^x)\} J_{-i\sqrt{\lambda}}(e^\xi), & \xi \leq x \\ \{J_{i\sqrt{\lambda}}(e^\xi) + J_{-i\sqrt{\lambda}}(e^\xi)\} J_{-i\sqrt{\lambda}}(e^x), & \xi \geq x \end{cases}$$

31.**Green's Function for Bessel's**

$$y''(x) + [\lambda - e^{2x}]y(x) = 0, \quad -\infty < x < \infty$$

By [Titchmarsh, p. 96],

$$y(x, \lambda) = -K_{i\sqrt{\lambda}}(e^x) \int_{\xi=-\infty}^{\xi=x} I_{-i\sqrt{\lambda}}(e^\xi) f(\xi) d\xi - I_{-i\sqrt{\lambda}}(e^x) \int_{\xi=-\infty}^{\xi=x} K_{i\sqrt{\lambda}}(e^\xi) f(\xi) d\xi$$

Therefore,

$$\mathbf{31.1} \quad G(x, \xi, \lambda) = - \begin{cases} K_{i\sqrt{\lambda}}(e^x) I_{-i\sqrt{\lambda}}(e^\xi), & \xi \leq x \\ K_{i\sqrt{\lambda}}(e^\xi) I_{-i\sqrt{\lambda}}(e^x), & \xi \geq x \end{cases}$$

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