Linear Differential Equations Driven by Delta Functions

H. Vic Dannon
vic0@comcast.net
May, 2014

Abstract We solve the linear differential equation $L(x)G(x,\xi) = \delta(x-\xi)$, for the Hyper-real Green function $G(x,\xi)$, where $L(x)$ is a linear differential operator, and $\delta(x-\xi)$ is the hyper-real Delta function.

Keywords: Sturm-Liouville Expansions, Infinitesimal, Infinite-Hyper-Real, Hyper-Real, infinite Hyper-real, Infinitesimal Calculus, Delta Function, Fourier Series, Laguerre Polynomials, Legendre Functions, Bessel Functions, Delta Function,

2000 Mathematics Subject Classification 26E35; 26E30; 26E15; 26E20; 26A06; 26A12; 03E10; 03E55; 03E17; 03H15; 46S20; 97I40; 97I30.

Contents

27. Green's Function Integral Expansion for Weber Equation

$$y''(x) + (\lambda - \tfrac{\nu^2 - \frac{1}{4}}{x^2})y(x) = 0\,,\ a < x < \infty,\ \ a > 0$$

28. Green's Function for Hankel's

$$y''(x) + (\lambda - \tfrac{\nu^2 - \frac{1}{4}}{x^2})y(x) = 0\,,\ 0 < x < \infty$$

29. Green's Function for Bessel's

$$y''(x) + (\lambda + x)y(x) = 0\,,\ \ -\infty < x < \infty$$

30. Green's Function for Bessel's

$$y''(x) + (\lambda + e^{2x})y(x) = 0\,,\ \ -\infty < x < \infty$$

31. Green's Function for Bessel's

$$y''(x) + (\lambda - e^{2x})y(x) = 0\,,\ \ -\infty < x < \infty$$

References

0.

Linear Differential Equation driven by Delta Function

A linear Differential Equation for the Hyper-real function $y(x)$, forced by the Hyper-real function $f(x)$ is

$$L(x, D)y(x) = f(x),$$

where the Differential Operator $L(x, D)$ is linear.

If there is a Hyper-real Green function,

$$G(x, \xi)$$

that satisfies the equation

$$L(x, D)G(x, \xi) = \delta(x, \xi),$$

with the same Boundary condition that $y(x)$ satisfies

then, $G(x, \xi)$ is the Kernel of an Integral Operator that transforms a function $f(x)$ into the solution $y(x)$ of the linear differential equation $L(x, D)y(x) = f(x)$.

Indeed, if

$$y(x) = \int_{\xi=a}^{\xi=b} G(x, \xi)f(\xi)d\xi,$$

then

$$L(x,D)y(x) = L(x,D)\int_{\xi=a}^{\xi=b} G(x,\xi)f(\xi)d\xi\,,$$

$$= L(x,D)\sum_{\xi=a}^{\xi=b} G(x,\xi)f(\xi)d\xi\,,$$

$$= \sum_{\xi=a}^{\xi=b} \underbrace{L(x,D)G(x,\xi)}_{\delta(x-\xi)} f(\xi)d\xi$$

$$= f(x).$$

We shall see that the expansion of $\delta(x-\xi)$ in series, or integrals guarantees the existence of a Green Function that can be expanded similarly.

The Delta Function can be defined only as a hyper-real function in infinitesimal Calculus.

We proceed with the definition of the Hyper-real line.

1.

Hyper-real Line

Each real number α can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of infinitesimal hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the infinite hyper-reals.
4. The infinite hyper-reals are greater than any real number, yet strictly smaller than infinity.
5. The infinite hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a non-constant hyper-real.
7. The Hyper-reals are the totality of constant hyper-reals, a family of infinitesimals, a family of infinitesimals with

negative sign, a family of infinite hyper-reals, a family of infinite hyper-reals with negative sign, and non-constant hyper-reals.

8. The hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.
9. That line includes the real numbers separated by the non-constant hyper-reals. Each real number is the center of an interval of hyper-reals, that includes no other real number.
10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.
11. Zero is not an infinitesimal, because zero is not strictly greater than zero.
12. We do not add infinity to the hyper-real line.
13. The infinitesimals, the infinitesimals with negative signs, the infinite hyper-reals, and the infinite hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.
14. The hyper-real line is embedded in $\mathbb{R}^{\infty}$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the hyper-real onto the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal hyper-reals, or to the infinite hyper-reals, or to the non-constant hyper-reals.

16. No neighbourhood of a hyper-real is homeomorphic to an $\mathbb{R}^n$ ball. Therefore, the hyper-real line is not a manifold.

17. The hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

2.

Hyper-real Function

2.1 Definition of a hyper-real function

$f(x)$ is a hyper-real function, iff it is from the hyper-reals into the hyper-reals.

This means that any number in the domain, or in the range of a hyper-real $f(x)$ is either one of the following

- real
- real + infinitesimal
- real – infinitesimal
- infinitesimal
- infinitesimal with negative sign
- infinite hyper-real
- infinite hyper-real with negative sign

Clearly,

2.2 *Every function from the reals into the reals is a hyper-real function.*

3.

Integral of a Hyper-real Function

In [Dan3], we defined the integral of a Hyper-real Function.

Let $f(x)$ be a hyper-real function on the interval $[a,b]$.

The interval may not be bounded.

$f(x)$ may take infinite hyper-real values, and need not be bounded.

At each

$$a \le x \le b,$$

there is a rectangle with base $[x - \frac{dx}{2}, x + \frac{dx}{2}]$, height $f(x)$, and area

$$f(x)dx.$$

We form the **Integration Sum** of all the areas for the x's that start at $x = a$, and end at $x = b$,

$$\sum_{x \in [a,b]} f(x)dx.$$

If for any infinitesimal dx, the Integration Sum has the same hyper-real value, then $f(x)$ is integrable over the interval $[a,b]$.

Then, we call the Integration Sum the integral of $f(x)$ from $x = a$, to $x = b$, and denote it by

$$\int_{x=a}^{x=b} f(x)dx\,.$$

If the hyper-real is infinite, then it is the integral over $[a,b]$,

If the hyper-real is finite,

$$\int_{x=a}^{x=b} f(x)dx = \text{ real part of the hyper-real}\,.\square$$

3.1 The countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers, $Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty.$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[a,b]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(x)dx$.

The Lower Integral is the Integration Sum where $f(x)$ is replaced by its lowest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

3.2
$$\sum_{x \in [a,b]} \left(\inf_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

The Upper Integral is the Integration Sum where $f(x)$ is replaced by its largest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

3.3
$$\sum_{x \in [a,b]} \left(\sup_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

If the integral is a finite hyper-real, we have

3.4 *A hyper-real function has a finite integral if and only if its upper integral and its lower integral are finite, and differ by an infinitesimal.*

4.

Delta Function

In [Dan5], we have defined the Delta Function, and established its properties

1. The Delta Function is a hyper-real function defined from the hyper-real line into the set of two hyper-reals $\left\{0, \frac{1}{dx}\right\}$. The hyper-real 0 is the sequence $\left\langle 0, 0, 0, ... \right\rangle$. The infinite hyper-real $\frac{1}{dx}$ depends on our choice of dx.
2. We will usually choose the family of infinitesimals that is spanned by the sequences $\left\langle \frac{1}{n} \right\rangle, \left\langle \frac{1}{n^2} \right\rangle, \left\langle \frac{1}{n^3} \right\rangle, \ldots$ It is a semigroup with respect to vector addition, and includes all the scalar multiples of the generating sequences that are non-zero. That is, the family includes infinitesimals with negative sign. Therefore, $\frac{1}{dx}$ will mean the sequence $\left\langle n \right\rangle$. Alternatively, we may choose the family spanned by the sequences $\left\langle \frac{1}{2^n} \right\rangle, \left\langle \frac{1}{3^n} \right\rangle, \left\langle \frac{1}{4^n} \right\rangle, \ldots$ Then, $\frac{1}{dx}$ will mean the

sequence $\langle 2^n \rangle$. Once we determined the basic infinitesimal dx, we will use it in the Infinite Riemann Sum that defines an Integral in Infinitesimal Calculus.

3. The Delta Function is strictly smaller than ∞

4. We define, $\delta(x) \equiv \frac{1}{dx}\chi_{[-\frac{dx}{2},\frac{dx}{2}]}(x)$,

$$\text{where} \quad \chi_{[-\frac{dx}{2},\frac{dx}{2}]}(x) = \begin{cases} 1, x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right] \\ 0, \text{otherwise} \end{cases}.$$

5. Hence,

   - for $x < 0$, $\delta(x) = 0$
   - at $x = -\frac{dx}{2}$, $\delta(x)$ jumps from 0 to $\frac{1}{dx}$,
   - for $x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right]$, $\delta(x) = \frac{1}{dx}$.
   - at $x = 0$, $\delta(0) = \frac{1}{dx}$
   - at $x = \frac{dx}{2}$, $\delta(x)$ drops from $\frac{1}{dx}$ to 0.
   - for $x > 0$, $\delta(x) = 0$.
   - $x\delta(x) = 0$

6. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle \chi_{[-\frac{1}{2},\frac{1}{2}]}(x), 2\chi_{[-\frac{1}{4},\frac{1}{4}]}(x), 3\chi_{[-\frac{1}{6},\frac{1}{6}]}(x) ... \right\rangle$

7. If $dx = \left\langle \frac{2}{n} \right\rangle$, $\delta(x) = \left\langle \frac{1}{2\cosh^2 x}, \frac{2}{2\cosh^2 2x}, \frac{3}{2\cosh^2 3x}, ... \right\rangle$

8. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle e^{-x}\chi_{[0,\infty)}, 2e^{-2x}\chi_{[0,\infty)}, 3e^{-3x}\chi_{[0,\infty)}, \ldots \right\rangle$

9. $$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1.$$

10. $$\delta(\xi - x) = \frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} e^{-ik(\xi - x)} dk$$

5.

Convergent Series

In [Dan8], we defined convergence of infinite series in Infinitesimal Calculus

5.1 Sequence Convergence to a finite hyper-real a

$$a_n \to a \;\; \textit{iff} \;\; \langle a_n \rangle - \langle a \rangle = \text{infinitesimal}.$$

5.2 Sequence Convergence to an infinite hyper-real A

$$a_n \to A \;\; \textit{iff} \;\; \langle a_n \rangle \text{ represents the infinite hyper-real } A.$$

5.3 Series Convergence to a finite hyper-real s

$$a_1 + a_2 + ... \to s \;\; \textit{iff} \;\; \langle a_1 + ... + a_n \rangle - \langle s \rangle = \text{infinitesimal}.$$

5.4 Series Convergence to an Infinite Hyper-real S

$a_1 + a_2 + ... \to S$ *iff*

$$\langle a_1 + ... + a_n \rangle \text{ represents the infinite hyper-real } S.$$

6.

Green's Function for

$y'(x) + P(x)y(x) = f(x)$, with $y(0) = 0$

6.1 $$\boxed{G(x,\xi) = e^{\int_{\theta=0}^{\theta=\xi} P(\theta)d\theta} e^{-\int_{\theta=0}^{\theta=x} P(\theta)d\theta} = e^{\int_{\theta=x}^{\theta=\xi} P(\theta)d\theta}}$$

<u>*Proof*</u>:

We solve the equation

$$\partial_x G(x,\xi) + P(x)G(x,\xi) = \delta(x-\xi), \quad \text{with } G(0,\xi) = 0$$

Multiplying both sides by the integrating factor

$$e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta}$$

$$e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} \partial_x G(x,\xi) + e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} P(x)G(x,\xi) = e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} \delta(x-\xi)$$

$$\partial_x \{e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} G(x,\xi)\} = e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} \delta(x-\xi)$$

Integrating with respect to x,

$$e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} G(x,\xi) = \int_{x=-\infty}^{x=\infty} e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} \delta(x-\xi)dx$$

$$= \sum_{x=-\infty}^{x=\infty} e^{\int\limits_{\theta=0}^{\theta=x} P(\theta)d\theta} \delta(x-\xi)dx$$

$$= e^{\int\limits_{\theta=0}^{\theta=\xi} P(\theta)d\theta}$$

Therefore,

$$G(x,\xi) = e^{\int\limits_{\theta=0}^{\theta=\xi} P(\theta)d\theta} e^{-\int\limits_{\theta=0}^{\theta=x} P(\theta)d\theta}$$

$$= e^{\int\limits_{\theta=x}^{\theta=\xi} P(\theta)d\theta}. \square$$

6.2 $$y(x) = \int\limits_{\xi=0}^{\xi=x} e^{\int\limits_{\theta=x}^{\theta=\xi} P(\theta)d\theta} f(x)dx$$

7.

Green's Function for

$y''(x) + P(x)y'(x) + Q(x)y(x) = f(x)$, with $y(0) = 0$, and $y'(0) = 0$

7.1 $y''(x) + P(x)y'(x) + Q(x)y(x) = f(x)$, with $y(0) = 0$, $y'(0) = 0$

$u(x)$ solves $u''(x) + P(x)u'(x) + Q(x)u(x) = 0$, with $u(0) = 0$

$v(x)$ solves $v''(x) + P(x)v'(x) + Q(x)v(x) = 0$, with $v(0) = 0$

Then,

$$\boxed{G(x,\xi) = \frac{\begin{vmatrix} u(\xi) & v(\xi) \\ u(x) & v(x) \end{vmatrix}}{\begin{vmatrix} u(\xi) & v(\xi) \\ u'(\xi) & v'(\xi) \end{vmatrix}}}$$

Proof:

We solve the equation

$$\partial_x^2 G(x,\xi) + P(x)\partial_x G(x,\xi) + Q(x)G(x,\xi) = \delta(x-\xi),$$

$$\text{with } G(0,\xi) = 0, \text{ and } \partial_x G(x,\xi)\Big|_{x=0} = 0$$

Let $u(x)$ solve the equation

$$u''(x) + P(x)u'(x) + Q(x)u(x) = 0, \text{ with } u(0) = 0$$

Then,

$$uG'' + PuG' + QuG = u\delta(x-\xi),$$

$$Gu'' + PGu' + QGu = 0,$$

Subtracting,

$$\underbrace{uG'' - Gu''}_{(uG' - Gu')'} + P(uG' - Gu') = u\delta(x-\xi)$$

Multiplying by the integrating factor $e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta}$,

$$\partial_x \{ e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} [u(x)G'(x,\xi) - G(x,\xi)u'(x)] \} = e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} u(x)\delta(x-\xi).$$

Integrating,

$$e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} [u(x)G'(x,\xi) - G(x,\xi)u'(x)] = \int_{\tau=0}^{\tau=x} e^{\int_{\theta=0}^{\theta=\tau} P(\theta)d\theta} u(\tau)\delta(\tau-\xi)d\tau. \quad \text{(I)}$$

Similarly, for $v(x)$ that solves the equation

$$v''(x) + P(x)v'(x) + Q(x)v(x) = 0, \text{ with } v(0) = 0$$

$$e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} [v(x)G'(x,\xi) - G(x,\xi)v'(x)] = \int_{\tau=0}^{\tau=x} e^{\int_{\theta=0}^{\theta=\tau} P(\theta)d\theta} v(\tau)\delta(\tau-\xi)d\tau. \quad \text{(II)}$$

To eliminate G', subtract (I)$\times v(x)$ $-$(II) $\times u(x)$,

$$e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} \begin{vmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{vmatrix} G(x,\xi) = \int_{\tau=0}^{\tau=x} e^{\int_{\theta=0}^{\theta=\tau} P(\theta)d\theta} \begin{vmatrix} u(\tau) & v(\tau) \\ u(x) & v(x) \end{vmatrix} \delta(\tau-\xi)d\tau$$

Substituting, by Abel equality,

$$e^{\int_{\theta=0}^{\theta=x} P(\theta)d\theta} \begin{vmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{vmatrix} = \begin{vmatrix} u(0) & v(0) \\ u'(0) & v'(0) \end{vmatrix},$$

and

$$e^{\int_{\theta=0}^{\theta=\tau} P(\theta)d\theta} = \frac{\begin{vmatrix} u(0) & v(0) \\ u'(0) & v'(0) \end{vmatrix}}{\begin{vmatrix} u(\tau) & v(\tau) \\ u'(\tau) & v'(\tau) \end{vmatrix}}$$

$$\begin{vmatrix} u(0) & v(0) \\ u'(0) & v'(0) \end{vmatrix} G(x,\xi) = \int_{\tau=0}^{\tau=x} \frac{\begin{vmatrix} u(0) & v(0) \\ u'(0) & v'(0) \end{vmatrix}}{\begin{vmatrix} u(\tau) & v(\tau) \\ u'(\tau) & v'(\tau) \end{vmatrix}} \begin{vmatrix} [u(\tau) & v(\tau) \\ u(x) & v(x) \end{vmatrix} \delta(\tau - \xi) d\tau$$

$$G(x,\xi) = \int_{\tau=0}^{\tau=x} \frac{1}{\begin{vmatrix} u(\tau) & v(\tau) \\ u'(\tau) & v'(\tau) \end{vmatrix}} \begin{vmatrix} [u(\tau) & v(\tau) \\ u(x) & v(x) \end{vmatrix} \delta(\tau - \xi) d\tau$$

$$= \frac{\begin{vmatrix} [u(\xi) & v(\xi) \\ u(x) & v(x) \end{vmatrix}}{\begin{vmatrix} u(\xi) & v(\xi) \\ u'(\xi) & v'(\xi) \end{vmatrix}} . \square$$

8.

Green's Function for

$y''(x) + P(x)y'(x) + Q(x)y(x) = f(x)$, with $y(a) = 0$, and $y(b) = 0$

8.1 $y''(x) + P(x)y'(x) + Q(x)y(x) = f(x)$, with $y(a) = 0$, $y(b) = 0$,

$u''(x) + P(x)u'(x) + Q(x)u(x) = 0$, with $u(a) = 0$, $u(b) \neq 0$,

$v''(x) + P(x)v'(x) + Q(x)v(x) = 0$, with $v(a) \neq 0$, $v(b) = 0$

Then,
$$\boxed{G(x,\xi) = \frac{1}{\begin{vmatrix} u(\xi) & v(\xi) \\ u'(\xi) & v'(\xi) \end{vmatrix}} \begin{cases} u(\xi)v(x), & a \leq \xi \leq x \\ \\ u(x)u(x), & b \geq \xi \geq x \end{cases}}$$

<u>*Proof*</u>:

We solve the equation

$$y''(x) + P(x)y'(x) + Q(x)y(x) = f(x),$$

with $y(a) = 0$, and $y(b) = 0$.

Let $u(x)$ solve the equation

$$u''(x) + P(x)u'(x) + Q(x)u(x) = 0, \quad \text{with } u(a) = 0, \text{ and } u(b) \neq 0,$$

and let $v(x)$ solve the equation

$$v''(x) + P(x)v'(x) + Q(x)v(x) = 0, \quad \text{with } v(a) \neq 0, \text{ and } v(b) = 0.$$

By **7** a particular solution to the equation for $y(x)$ is

$$y_p(x) = \int_{\xi=a}^{\xi=x} \frac{1}{\begin{vmatrix} u(\xi) & v(\xi) \\ u'(\xi) & v'(\xi) \end{vmatrix}} \begin{vmatrix} [u(\xi) & v(\xi) \\ u(x) & v(x) \end{vmatrix} f(\xi) d\xi \,,$$

and the general solution is

$$y(x) = c_1 u(x) + c_2 v(x) + y_p(x) \,.$$

$$0 = y(a) = c_1 \underbrace{u(a)}_{0} + c_2 \underbrace{v(a)}_{\neq 0} + \underbrace{y_p(a)}_{0} \quad \Rightarrow \quad \underline{c_2 = 0}$$

$$0 = y(b) = c_1 \underbrace{u(b)}_{\neq 0} + c_2 \underbrace{v(b)}_{0} + y_p(b) \quad \Rightarrow$$

$$c_1 = -\frac{1}{u(b)} y_p(b)$$

$$= -\frac{1}{u(b)} \int_{\xi=a}^{\xi=b} \frac{1}{\begin{vmatrix} u(\xi) & v(\xi) \\ u'(\xi) & v'(\xi) \end{vmatrix}} \underbrace{\begin{vmatrix} [u(\xi) & v(\xi) \\ u(b) & v(b) \end{vmatrix}}_{u(\xi)\underbrace{v(b)}_{0} - u(b)v(\xi)} f(\xi) d\xi$$

$$= \int_{\xi=a}^{\xi=b} \frac{v(\xi)}{\begin{vmatrix} u(\xi) & v(\xi) \\ u'(\xi) & v'(\xi) \end{vmatrix}} f(\xi) d\xi$$

$$y(x) = u(x) \int_{\xi=a}^{\xi=b} \frac{v(\xi)}{\begin{vmatrix} u(\xi) & v(\xi) \\ u'(\xi) & v'(\xi) \end{vmatrix}} f(\xi) d\xi + \int_{\xi=a}^{\xi=x} \frac{u(\xi)v(x) - u(x)v(\xi)}{\begin{vmatrix} u(\xi) & v(\xi) \\ u'(\xi) & v'(\xi) \end{vmatrix}} f(\xi) d\xi$$

$$= \int_{\xi=x}^{\xi=b} \frac{u(x)v(\xi)}{\begin{vmatrix} u(\xi) & v(\xi) \\ u'(\xi) & v'(\xi) \end{vmatrix}} f(\xi)d\xi + \int_{\xi=a}^{\xi=x} \frac{u(\xi)v(x)}{\begin{vmatrix} u(\xi) & v(\xi) \\ u'(\xi) & v'(\xi) \end{vmatrix}} f(\xi)d\xi$$

Therefore,

$$G(x,\xi) = \frac{1}{\begin{vmatrix} u(\xi) & v(\xi) \\ u'(\xi) & v'(\xi) \end{vmatrix}} \begin{cases} u(\xi)v(x), & a \le \xi \le x \\ \\ u(x)u(x), & b \ge \xi \ge x \end{cases} . \Box$$

9.

Hyper-real Sturm-Liouville Problem

The Hyper-real Sturm-Liouville equation is the second order Hyper-real linear differential equation for the Hyper-real function $y(x)$,

$$-y''(x) + q(x) = \lambda y(x),$$

on an interval that may be bounded, or may be the whole Hyper-real line.

The hyper-real function $q(x)$ is (assumed in the literature to be) continuous on the interval, and bounded at its endpoints.

The choice of the number λ (which may be real or complex), allows the equation with boundary conditions, at the interval endpoints to become an eigen-value problem:

λ_n is an eigen-value, and $\psi_n(x)$ is the corresponding Hyper-real eigen-function iff

$$-\psi_n''(x)(x) + q(x)\psi_n(x) = \lambda_n \psi_n(x).$$

The eigen-functions are orthogonal, over the interval.

$$\int_{x=a}^{x=b} \psi_n(x)\psi_m(x)dx = 0, \text{ for } n \neq m.$$

10.

Delta Expansion in Non-Normalized Eigen-functions

As described in [Titchmarsh, Chapter I], given

numbers

$$\alpha\text{, and }\beta,$$

the Sturm-Liouville Problem on the interval with endpoints a, and b has solutions

$$\phi_\alpha(x,\lambda)\text{, with }\phi_\alpha(a,\lambda) = \sin\alpha\text{, and }\phi_\alpha{}'(a,\lambda) = -\cos\alpha,$$

$$\chi_\beta(x,\lambda)\text{, with }\chi_\beta(b,\lambda) = \sin\beta\text{, and }\chi_\beta{}'(b,\lambda) = -\cos\beta,,$$

which are entire functions of λ.

Then,

$$\frac{d}{dx}\underbrace{\begin{vmatrix} \phi_\alpha(x,\lambda) & \chi_\beta(x,\lambda) \\ \phi_\alpha{}'(x,\lambda) & \chi_\beta{}'(x,\lambda) \end{vmatrix}}_{W[\phi_\alpha,\chi_\beta]} = \underbrace{\begin{vmatrix} \phi_\alpha{}'(x,\lambda) & \chi_\beta{}'(x,\lambda) \\ \phi_\alpha{}'(x,\lambda) & \chi_\beta{}'(x,\lambda) \end{vmatrix}}_{0} + \begin{vmatrix} \phi_\alpha(x,\lambda) & \chi_\beta(x,\lambda) \\ \phi_\alpha{}''(x,\lambda) & \chi_\beta{}''(x,\lambda) \end{vmatrix}$$

$$= \phi_\alpha\chi_\beta{}'' - \chi_\beta\phi_\alpha{}''$$

$$= \phi_\alpha(q-\lambda)\chi_\beta - \chi_\beta(q-\lambda)\phi_\alpha$$

$$= 0.$$

Hence, the Wronskian $W[\phi_\alpha,\chi_\beta]$ is a function of λ alone:

$$W[\phi_\alpha,\chi_\beta] = \omega(\lambda).$$

Now, if the only zeros of $\omega(\lambda)$ are the simple zeros

$$\lambda_0,\ \lambda_1,\ \lambda_2,\ \lambda_3,\ldots,$$

Then, for each $n = 0,1,2,3,...$

$$0 = \omega(\lambda_n) = \begin{vmatrix} \phi_\alpha(x,\lambda_n) & \chi_\beta(x,\lambda_n) \\ \phi_\alpha{}'(x,\lambda_n) & \chi_\beta{}'(x,\lambda_n) \end{vmatrix}.$$

That is, for each $n = 0,1,2,3,...$ there is a number k_n so that

$$\chi_\beta(x,\lambda_n) = k_n\phi_\alpha(x,\lambda_n).$$

Titchmarsh applied the Residue Theorem to obtain the coefficients in the Sturm-Liouville expansion of $f(x)$.

Following Titchmarsh, we conclude that the Hyper-real Sturm-Liouville expansion of a Hyper-real function $f(x)$ in the Hyper-real eigen-functions $\phi_\alpha(x,\lambda_n)$ is

$$f(x) = \sum_{n=0}^{n=\infty}\left(\frac{k_n}{\omega'(\lambda_n)}\int_{\xi=a}^{\xi=b} f(\xi)\phi_\alpha(\xi,\lambda_n)d\xi\right)\phi_\alpha(x,\lambda_n)$$

$$= \sum_{n=0}^{n=\infty}\left(\frac{k_n}{\omega'(\lambda_n)}\sum_{\xi=a}^{\xi=b} f(\xi)\phi_\alpha(\xi,\lambda_n)d\xi\right)\phi_\alpha(x,\lambda_n)$$

Exchanging summation order

$$= \sum_{\xi=a}^{\xi=b}\sum_{n=0}^{n=\infty}\frac{k_n}{\omega'(\lambda_n)}\phi_\alpha(x,\lambda_n)\phi_\alpha(\xi,\lambda_n)f(\xi)d\xi$$

$$= \int_{\xi=a}^{\xi=b}\underbrace{\sum_{n=0}^{n=\infty}\frac{k_n}{\omega'(\lambda_n)}\phi_\alpha(x,\lambda_n)\phi_\alpha(\xi,\lambda_n)}_{\delta(x-\xi)}f(\xi)d\xi$$

Therefore,

10.1 *The Hyper-real Delta Function Expansion in non-normalized eigen-functions is*

$$\boxed{\delta(x-\xi) = \sum_{n=0}^{n=\infty} \frac{k_n}{\omega'(\lambda_n)} \phi_\alpha(x,\lambda_n)\phi_\alpha(\xi,\lambda_n)}$$

That is,

10.2 *The Hyper-real Delta Function is the infinite sequence*

$$\left\langle \frac{k_0}{\omega'(\lambda_0)} \phi_\alpha(x,\lambda_0)\phi_\alpha(\xi,\lambda_0) + ... + \frac{k_n}{\omega'(\lambda_n)} \phi_\alpha(x,\lambda_n)\phi_\alpha(\xi,\lambda_n) \right\rangle.$$

11.

Green's Function Expansion in Non-Normalized Eigen-functions

Green's Function is defined by

$$\boxed{D_x^2 G(x,\xi,\lambda) + [\lambda - q(x)]G(x,\xi,\lambda) = \delta(x-\xi)}$$

with the Boundary conditions at the intervals endpoints.

Substituting Delta from **10.1**,

$$D_x^2 G(x,\xi,\lambda) + [\lambda - q(x)]G(x,\xi,\lambda) = \sum_{n=0}^{n=\infty} \frac{k_n}{\omega'(\lambda_n)} \phi_\alpha(x,\lambda_n)\phi_\alpha(\xi,\lambda_n)$$

Thus, Green's Function must have an expansion in $\phi_\alpha(x,\lambda_n)\phi_\alpha(\xi,\lambda_n)$,

$$G(x,\xi,\lambda) = \sum_{n=0}^{n=\infty} a_n(\lambda)\phi_\alpha(x,\lambda_n)\phi_\alpha(\xi,\lambda_n).$$

To determine the $a_n(\lambda)$, we substitute the expansion into the Sturm-Liouville equation for $G(x,\xi,\lambda)$,

$$\sum_{n=0}^{n=\infty} a_n(\lambda)\underbrace{\{\underbrace{\phi_\alpha''(x,\lambda_n)}_{-[\lambda_n - q(x)]\phi_\alpha(x,\lambda_n)} + [\lambda - q(x)]\phi_\alpha(x,\lambda_n)\}}_{[\lambda-\lambda_n]\phi_\alpha(x,\lambda_n)}\phi_\alpha(\xi,\lambda_n) = \sum_{n=0}^{n=\infty} \frac{k_n}{\omega'(\lambda_n)}\phi_\alpha(x,\lambda_n)\phi_\alpha(\xi,\lambda_n)$$

$$\sum_{n=0}^{n=\infty} a_n(\lambda)[\lambda - \lambda_n]\phi_\alpha(x,\lambda_n)\phi_\alpha(\xi,\lambda_n) = \sum_{n=0}^{n=\infty} \frac{k_n}{\omega'(\lambda_n)}\phi_\alpha(x,\lambda_n)\phi_\alpha(\xi,\lambda_n),$$

Equating the coefficients, for each $n = 0,1,2,3,...$

$$a_n(\lambda)[\lambda - \lambda_n] = \frac{k_n}{\omega'(\lambda_n)},$$

$$a_n(\lambda) = \frac{1}{\lambda - \lambda_n} \frac{k_n}{\omega'(\lambda_n)}.$$

11.1 *The Hyper-real Green Function is*

$$\boxed{G(x,\xi,\lambda) = \sum_{n=0}^{n=\infty} \frac{1}{\lambda - \lambda_n} \frac{k_n}{\omega'(\lambda_n)} \phi_\alpha(x,\lambda_n)\phi_\alpha(\xi,\lambda_n)}.$$

That is,

11.2 *The Hyper-real Green Function is the infinite sequence*

$$\left\langle \frac{1}{\lambda - \lambda_0} \frac{k_0}{\omega'(\lambda_0)} \phi_\alpha(x,\lambda_0)\phi_\alpha(\xi,\lambda_0) + ... + \frac{1}{\lambda - \lambda_n} \frac{k_n}{\omega'(\lambda_n)} \phi_\alpha(x,\lambda_n)\phi_\alpha(\xi,\lambda_n) \right\rangle_{n=0}^{n=\infty}.$$

11.3 *The Hyper-real* $y(x,\lambda)$ *for Non-Normalized Eigen-functions*

$$y(x,\lambda) = \int_{\xi=a}^{\xi=b} G(x,\xi,\lambda) f(\xi) d\xi$$

__Proof__: We substitute it into the Sturm-Liouville equation

$$y''(x,\lambda) = \int_{\xi=a}^{\xi=b} D_x^2 G(x,\xi,\lambda) f(\xi) d\xi,$$

$$y''(x,\lambda) + [\lambda - q(x)]y(x,\lambda) = \int_{\xi=a}^{\xi=b} \{\underbrace{D_x^2 G(x,\xi,\lambda) + [\lambda - q(x)]G(x,\xi,\lambda)}_{\delta(x-\xi)}\} f(\xi) d\xi$$

$$= f(x). \square$$

12.

Green Function for Sine Sturm-Liuoville $y''(x) + \lambda y(x) = 0$ & $\alpha = \beta = 0$

Two independent solutions are

$$\cos\sqrt{\lambda}x\,, \text{ and } \quad \sin\sqrt{\lambda}x\,.$$

For $\underline{\alpha = 0}$,

$$\phi_0(x,\lambda) = -\tfrac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}(x-a)$$

satisfies the Boundary conditions

$$\phi_0(a,\lambda) = \sin 0 = 0\,,$$

and

$$\phi_0\,'(a,\lambda) = -\cos 0 = -1\,.$$

For $\underline{\beta = 0}$,

$$\chi_0(x,\lambda) = \tfrac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}(b-x)$$

satisfies the Boundary conditions

$$\chi_0(b,\lambda) = \sin 0 = 0\,,$$

and

$$\chi_0\,'(b,\lambda) = -\cos 0 = -1\,,$$

Therefore,

$$\omega(\lambda) = \begin{vmatrix} -\dfrac{\sin\sqrt{\lambda}(x-a)}{\sqrt{\lambda}} & \dfrac{\sin\sqrt{\lambda}(b-x)}{\sqrt{\lambda}} \\ -\cos\sqrt{\lambda}(x-a) & -\cos\sqrt{\lambda}(b-x) \end{vmatrix}$$

$$= \tfrac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}(b-a).$$

Hence, the zeros of $\omega(\lambda)$ are

$$\lambda_n = \left(\frac{n\pi}{b-a}\right)^2, \quad n = 1,2,3,...$$

$$\omega'(\lambda) = \frac{d}{d\lambda}\left\{\tfrac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}(b-a)\right\}$$

$$= \tfrac{1}{\lambda}\left\{\left[\cos\sqrt{\lambda}(b-a)\right]\tfrac{b-a}{2\sqrt{\lambda}}\sqrt{\lambda} - \tfrac{1}{2\sqrt{\lambda}}\sin\sqrt{\lambda}(b-a)\right\}.$$

$$\omega'(\lambda_n) = \tfrac{1}{2\lambda_n}(b-a)\cos\underbrace{\sqrt{\lambda_n}(b-a)}_{n\pi}$$

$$= \tfrac{1}{2\lambda_n}(b-a)(-1)^n.$$

$$k_n = \frac{\chi_0(x,\lambda_n)}{\phi_0(x,\lambda_n)}$$

$$= -\frac{\sin\sqrt{\lambda_n}(b-x)}{\sin\sqrt{\lambda_n}(x-a)}$$

$$= -\frac{\sin\frac{b-x}{b-a}n\pi}{\sin\frac{x-a}{b-a}n\pi}$$

$$= -\frac{\sin[n\pi - \frac{x-a}{b-a}n\pi]}{\sin\frac{x-a}{b-a}n\pi}$$

$$= \frac{1}{\sin \frac{x-a}{b-a} n\pi} \{ -\underbrace{\sin(n\pi)}_{0} \cos \tfrac{x-a}{b-a} n\pi + \underbrace{\cos(n\pi)}_{(-1)^n} \sin \tfrac{x-a}{b-a} n\pi \}$$

$$= (-1)^n .$$

Therefore,

12.1 *The Hyper-real Green Function in* $[a,b]$

$$\boxed{G(x,\xi,\lambda) = \frac{2}{b-a} \sum_{n=1}^{n=\infty} \frac{1}{\lambda - \left(\frac{n\pi}{b-a}\right)^2} \sin\left(n\pi \frac{x-a}{b-a}\right) \sin\left(n\pi \frac{\xi-a}{b-a}\right)}$$

Proof: $G(x,\xi,\lambda) = \sum_{n=1}^{n=\infty} \frac{1}{\lambda - \lambda_n} \frac{k_n}{\omega'(\lambda_n)} \phi_\alpha(\xi,\lambda_n) \phi_\alpha(x,\lambda_n)$

$$= \sum_{n=1}^{n=\infty} \frac{1}{\lambda - \left(\frac{n\pi}{b-a}\right)^2} \frac{(-1)^n}{\frac{b-a}{2\lambda_n}(-1)^n} \frac{\sin\sqrt{\lambda_n}(x-a)}{\sqrt{\lambda_n}} \frac{\sin\sqrt{\lambda_n}(\xi-a)}{\sqrt{\lambda_n}}$$

$$= \sum_{n=1}^{n=\infty} \frac{1}{\lambda - \left(\frac{n\pi}{b-a}\right)^2} \frac{2}{b-a} \sin\left(n\pi \frac{x-a}{b-a}\right) \sin\left(n\pi \frac{\xi-a}{b-a}\right). \square$$

Namely,

12.2 *The Hyper-real Green Function in* $[a,b]$ *is the infinite sequence*

$$\frac{2}{b-a} \left\langle \frac{1}{\lambda - \left(\frac{\pi}{b-a}\right)^2} \sin\left(\pi \frac{x-a}{b-a}\right) \sin\left(\pi \frac{\xi-a}{b-a}\right) + \ldots \right.$$

$$\left. + \frac{1}{\lambda - \left(\frac{n\pi}{b-a}\right)^2} \sin\left(n\pi \frac{x-a}{b-a}\right) \sin\left(n\pi \frac{\xi-a}{b-a}\right) \right\rangle_{n=1}^{n=\infty}$$

13.

Green Function for Cosine Sturm-Liuoville $y''(x) + \lambda y(x) = 0$ & $\alpha = \beta = \frac{1}{2}\pi$

Similarly to the former section, for $\alpha = \beta = \frac{1}{2}\pi$, we obtain

13.1 *The Hyper-real Green Function in* $[a,b]$

$$\boxed{G(x,\xi,\lambda) = \frac{1}{b-a}\left\{\frac{1}{\lambda} + 2\sum_{n=1}^{n=\infty}\frac{1}{\lambda - \left(\frac{n\pi}{b-a}\right)^2}\cos\left(n\pi\frac{x-a}{b-a}\right)\cos\left(n\pi\frac{\xi-a}{b-a}\right)\right\}}$$

Namely,

13.2 *The Hyper-real Green Function in* $[a,b]$ *is the infinite sequence*

$$\frac{1}{b-a}\left\langle\frac{1}{\lambda} + \ldots + 2\frac{1}{\lambda - \left(\frac{n\pi}{b-a}\right)^2}\cos\left(n\pi\frac{x-a}{b-a}\right)\cos\left(n\pi\frac{\xi-a}{b-a}\right)\right\rangle_{n=1}^{n=\infty}$$

14.

Green's Function for Bessel Sturm-Liuoville $y''(x) + \lambda y(x) = 0 \,\&\, \alpha = 0$

For $\underline{\alpha = 0}$,

$$\phi_0(x,\lambda) = -\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}$$

satisfies the Boundary conditions

$$\phi_0(a,\lambda) = \sin 0 = 0\,,$$

and

$$\phi_0{}'(a,\lambda) = -\cos 0 = -1\,.$$

The solution

$$\chi_\beta(x,\lambda) = \tfrac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}(b-x)\cos\beta + \cos[\sqrt{\lambda}(b-x)]\sin\beta$$

satisfies the Boundary conditions

$$\chi_\beta(b,\lambda) = \sin\beta\,,$$

and

$$\chi_\beta{}'(b,\lambda) = -\cos\beta\,,$$

Therefore, at $x = b$,

$$\omega(\lambda) = \begin{vmatrix} -\tfrac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}b & \sin\beta \\ -\cos\sqrt{\lambda}b & -\cos\beta \end{vmatrix}$$

Hence, the zeros of $\omega(\lambda)$ are the roots of

$$\tfrac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}b\cos\beta = -\cos\sqrt{\lambda}b\sin\beta .$$

That is,

$$\tan\sqrt{\lambda_n}b = -\sqrt{\lambda_n}\tan\beta .$$

[Titchmarsh, p. 17] obtains

$$\omega'(\lambda_n) = \tfrac{1}{2\lambda_n} b\cos\beta\cos(b\sqrt{\lambda_n})\{1+\lambda_n\tan^2\beta + \tfrac{1}{b}\tan\beta\}$$

$$k_n = \cos\beta\cos(b\sqrt{\lambda_n})\{1+\lambda_n\tan^2\beta\}$$

Therefore,

14.1 *The Green Function expansion in* $[a,b]$ *is*

$$\boxed{G(x,\xi,\lambda) = \frac{2}{b}\sum_{n=0}^{n=\infty}\frac{1}{\lambda-\lambda_n}\frac{1+\lambda_n\tan^2\beta}{1+\lambda_n\tan^2\beta+\frac{1}{b}\tan\beta}\sin(x\sqrt{\lambda_n})\sin(\xi\sqrt{\lambda_n})}$$

Proof: $G(x,\xi,\lambda) = \displaystyle\sum_{n=0}^{n=\infty}\frac{1}{\lambda-\lambda_n}\frac{k_n}{\omega'(\lambda_n)}\phi_\alpha(\xi,\lambda_n)\phi_\alpha(x,\lambda_n)$.□

Namely,

14.2 *The Hyper-real Green Function in* $[a,b]$ *is the infinite sequence*

$$\frac{2}{b}\left\langle\sum_{j=0}^{j=n}\frac{1}{\lambda-\lambda_n}\frac{1+\lambda_j\tan^2\beta}{1+\lambda_j\tan^2\beta+\frac{1}{b}\tan\beta}\sin(x\sqrt{\lambda_j})\sin(\xi\sqrt{\lambda_j})\right\rangle_{n=0}^{n=\infty}$$

15.

Green's Function for Bessel Sturm Liuoville $u''(x) + (\lambda - \frac{\nu^2 - \frac{1}{4}}{x^2})u(x) = 0$ & $\alpha = \beta = 0$

Put

$$u(x) = \sqrt{x}y(x),$$

to obtain Bessel's

$$y''(x) + \frac{1}{x}y'(x) + (\lambda - \frac{\nu^2}{x^2})y(x) = 0.$$

Two independent solutions to Bessel's equation are

$$J_\nu(x\sqrt{\lambda}), \text{ and } Y_\nu(x\sqrt{\lambda}).$$

Two independent solutions to $u''(x) + (\lambda - \frac{\nu^2 - \frac{1}{4}}{x^2})u(x) = 0$ are

$$\sqrt{x}J_\nu(x\sqrt{\lambda}), \text{ and } \sqrt{x}Y_\nu(x\sqrt{\lambda}).$$

For $\underline{\alpha = 0}$,

$$\phi_0(x,\lambda) = \tfrac{1}{2}\pi\sqrt{ax}\{J_\nu(x\sqrt{\lambda})Y_\nu(a\sqrt{\lambda}) - Y_\nu(x\sqrt{\lambda})J_\nu(a\sqrt{\lambda})\}$$

satisfies the Boundary conditions

$$\phi_0(a,\lambda) = \sin 0 = 0,$$

and

$$\phi_0'(a,\lambda) = -\cos 0 = -1.$$

For $\underline{\beta = 0}$,

$$\chi_0(x,\lambda) = \tfrac{1}{2}\pi\sqrt{bx}\{J_\nu(x\sqrt{\lambda})Y_\nu(b\sqrt{\lambda}) - Y_\nu(x\sqrt{\lambda})J_\nu(b\sqrt{\lambda})\}$$

satisfies the Boundary conditions

$$\chi_0(b,\lambda) = \sin 0 = 0,$$

and

$$\chi_0'(b,\lambda) = -\cos 0 = -1,$$

[Titchmarsh, p.18] obtains

$$\omega(\lambda) = \begin{vmatrix} \phi_0(x,\lambda) & \chi_0(x,\lambda) \\ \partial_x\phi_0(x,\lambda) & \partial_x\chi_0(x,\lambda) \end{vmatrix}$$

$$= \tfrac{1}{2}\pi\sqrt{ab}\{J_\nu(a\sqrt{\lambda})Y_\nu(b\sqrt{\lambda}) - Y_\nu(a\sqrt{\lambda})J_\nu(b\sqrt{\lambda})\}.$$

$$\omega'(\lambda) = -\left\{\frac{a}{2\sqrt{\lambda}}\frac{J_\nu'(a\sqrt{\lambda})}{J_\nu(b\sqrt{\lambda})} + \frac{b}{2\sqrt{\lambda}}\frac{J_\nu'(b\sqrt{\lambda})}{J_\nu(b\sqrt{\lambda})}\right\}\omega(\lambda) - \frac{\sqrt{ab}}{2\lambda}\left\{\frac{J_\nu(b\sqrt{\lambda})}{J_\nu(a\sqrt{\lambda})} - \frac{J_\nu(a\sqrt{\lambda})}{J_\nu(b\sqrt{\lambda})}\right\}$$

$$\omega'(\lambda_n) = -\frac{\sqrt{ab}}{2\lambda_n}\left\{\frac{J_\nu^2(b\sqrt{\lambda_n}) - J_\nu^2(a\sqrt{\lambda_n})}{J_\nu(a\sqrt{\lambda_n})J_\nu(b\sqrt{\lambda_n})}\right\}$$

$$k_n = \sqrt{\frac{b}{a}}\frac{J_\nu(b\sqrt{\lambda_n})}{J_\nu(a\sqrt{\lambda_n})},$$

λ_n are the zeros of $\omega(\lambda)$.

15.1 *The Hyper-real Green Function in* $[a,b]$ *is*

$$\boxed{\begin{aligned} G(x,\xi,\lambda) = \frac{\pi^2}{2}\sqrt{x\xi}\sum_{n=0}^{n=\infty}\frac{1}{\lambda-\lambda_n}\frac{\lambda_n J_\nu^2(b\sqrt{\lambda_n})}{J_\nu^2(a\sqrt{\lambda_n}) - J_\nu^2(b\sqrt{\lambda_n})}\times \\ \times\left\{J_\nu(x\sqrt{\lambda_n})Y_\nu(a\sqrt{\lambda_n}) - Y_\nu(x\sqrt{\lambda_n})J_\nu(a\sqrt{\lambda_n})\right\}\times \\ \times\left\{J_\nu(\xi\sqrt{\lambda_n})Y_\nu(a\sqrt{\lambda_n}) - Y_\nu(\xi\sqrt{\lambda_n})J_\nu(a\sqrt{\lambda_n})\right\} \end{aligned}}$$

Proof: $G(x,\xi,\lambda) = \sum_{n=0}^{n=\infty} \frac{1}{\lambda - \lambda_n} \frac{k_n}{\omega'(\lambda_n)} \phi_\alpha(\xi,\lambda_n)\phi_\alpha(x,\lambda_n)$.□

Namely,

15.2 *The Hyper-real Green Function in* $[a,b]$ *is the infinite sequence*

$$\frac{\pi^2}{2}\sqrt{x\xi}\left\langle \sum_{j=0}^{j=n} \frac{1}{\lambda - \lambda_n} \frac{\lambda_j J_\nu^2(b\sqrt{\lambda_j})}{J_\nu^2(a\sqrt{\lambda_j}) - J_\nu^2(b\sqrt{\lambda_j})} \times \right.$$

$$\times\left\{ J_\nu(x\sqrt{\lambda_j})Y_\nu(a\sqrt{\lambda_j}) - Y_\nu(x\sqrt{\lambda_j})J_\nu(a\sqrt{\lambda_j}) \right\} \times$$

$$\left. \times\left\{ J_\nu(\xi\sqrt{\lambda_j})Y_\nu(a\sqrt{\lambda_j}) - Y_\nu(\xi\sqrt{\lambda_j})J_\nu(a\sqrt{\lambda_j}) \right\} \right\rangle_{n=0}^{n=\infty}$$

16.

Delta Expansion in Orthonormal Eigen-functions

The Hyper-real eigen-functions of a Hyper-real Sturm-Liouville problem over the interval with endpoints a, and b,

$$\psi_0(x),\ \ \psi_1(x),\ \ \psi_2(x),\ \ldots$$

can be normalized so that

$$\int\limits_{x=a}^{x=b} \psi_n(x)\psi_m(x)dx = \delta_{nm}.$$

Then, a hyper-real function $f(x)$ may be expanded in them by

$$f(x) = \sum_{n=0}^{n=\infty}\left(\int\limits_{\xi=a}^{\xi=b} f(\xi)\psi_n(\xi)d\xi\right)\psi_n(x).$$

$$= \sum_{n=0}^{n=\infty}\sum_{\xi=a}^{\xi=b} f(\xi)\psi_n(\xi)d\xi\psi_n(x)$$

Exchanging summation order,

$$= \sum_{\xi=a}^{\xi=b}\left\{\sum_{n=0}^{n=\infty}\psi_n(x)\psi_n(\xi)\right\}f(\xi)d\xi$$

$$= \int\limits_{\xi=a}^{\xi=b}\underbrace{\sum_{n=0}^{n=\infty}\psi_n(x)\psi_n(\xi)}_{\delta(x-\xi)}f(\xi)d\xi.$$

Therefore,

16.1 *The Hyper-real Delta Function expansion in orthonormal Sturm-Liuoville eigen-functions is*

$$\boxed{\delta(x-\xi) = \sum_{n=0}^{n=\infty} \psi_n(x)\psi_n(\xi)}.$$

That is,

16.2 *The Hyper-real Delta Function is the infinite sequence*

$$\left\langle \psi_0(x)\psi_0(\xi) + \psi_1(x)\psi_1(\xi) + ... + \psi_n(x)\psi_n(\xi) \right\rangle.$$

17.

Green's Function Expansion in Orthonormal Eigen-functions

Green's Function is defined as the solution to Sturm-Liuoville problem. Namely, the Sturm-Liouville equation

$$\boxed{D_x^2 G(x,\xi,\lambda) + [\lambda - q(x)]G(x,\xi,\lambda) = \delta(x-\xi)}$$

with the Boundary conditions at the intervals endpoints.

That is,

$$D_x^2 G(x,\xi,\lambda) + [\lambda - q(x)]G(x,\xi,\lambda) = \sum_{n=0}^{n=\infty} \psi_n(x)\psi_n(\xi)$$

Hence, Green's Function must have an expansion in $\psi_n(x)\psi_n(\xi)$,

$$G(x,\xi,\lambda) = \sum_{n=0}^{n=\infty} a_n(\lambda)\psi_n(x)\psi_n(\xi).$$

To determine the $a_n(\lambda)$, we substitute into the Sturm-Liouville equation for $G(x,\xi,\lambda)$,

$$\sum_{n=0}^{n=\infty} a_n(\lambda)\{\underbrace{\underbrace{\psi_n''(x)}_{-[\lambda_n - q(x)]\psi_n(x)} + [\lambda - q(x)]\psi_n(x)}_{[\lambda-\lambda_n]\psi_n(x)}\}\psi_n(\xi) = \sum_{n=0}^{n=\infty} \psi_n(x)\psi_n(\xi)$$

$$\sum_{n=0}^{n=\infty} a_n(\lambda)[\lambda - \lambda_n]\psi_n(x)\psi_n(\xi) = \sum_{n=0}^{n=\infty} \psi_n(x)\psi_n(\xi),$$

Equating the coefficients, for each $n = 0, 1, 2, 3, ...$

$$a_n(\lambda)[\lambda - \lambda_n] = 1,$$

$$a_n(\lambda) = \frac{1}{\lambda - \lambda_n}.$$

17.1 *The Hyper-real Green Function is*

$$\boxed{G(x, \xi, \lambda) = \sum_{n=0}^{n=\infty} \frac{1}{\lambda - \lambda_n} \psi_n(x)\psi_n(\xi)}.$$

That is,

17.2 *The Hyper-real Green Function is the infinite sequence*

$$\left\langle \tfrac{1}{\lambda-\lambda_0} \psi_0(x)\psi_0(\xi) + \tfrac{1}{\lambda-\lambda_1} \psi_1(x)\psi_1(\xi) + ... + \tfrac{1}{\lambda-\lambda_n} \psi_n(x)\psi_n(\xi) \right\rangle_{n=0}^{n=\infty}.$$

17.3 *The Hyper-real* $y(x, \lambda)$ *for Orthonormal Eigen-functions*

$$y(x, \lambda) = \int_{\xi=a}^{\xi=b} G(x, \xi, \lambda) f(\xi) d\xi$$

Proof: 11.3.□

18.

$G(x,\xi,\lambda)$ Analytic Continuation in λ into the Complex Plane

18.1 *Fix x, and ξ, consider λ in the complex plane, and let*

$$G(x,\xi,\zeta) = \sum_{n=1}^{n=\infty} \frac{1}{\zeta - \lambda_n} \psi_n(x)\psi_n(\xi)$$

be the Hyper-Complex Analytic Continuation of $G(x,\xi,\lambda)$ into the ζ Hyper-complex plane.

Let γ be a contour that encloses all the λ_n's.

<u>Then</u>

$$\oint_{\gamma} G(x,\xi,\zeta)d\zeta = 2\pi i\delta(x-\xi)$$

<u>First Proof</u>: (that uses the Residue Theorem)

By Sturm-Liuoville Oscillation Theorem [Inc. 10.6],

λ_n are the simple poles of $G(x,\xi,\zeta)$ on the positive real line.

Applying the Residue Theorem

$$\oint_{\gamma} G(x,\xi,\zeta)d\zeta = 2\pi i \sum_{n=1}^{n=\infty} \text{Res}\left\{G(x,\xi,\zeta)\right\}\Big|_{\zeta=\lambda_n}.$$

Since the Residue of $G(x,\xi,\zeta)$ at each λ_n is $\psi_n(x)\psi_n(\xi)$,

$$= 2\pi i \sum_{n=1}^{n=\infty} \psi_n(x)\psi_n(\xi)$$

$$= 2\pi i\delta(x-\xi).\square$$

Since we do not supply here a proof of the Residue Theorem, we shall give a direct proof:

<u>*Second Proof*</u> (that uses the Circular Complex Delta)

$$\oint_\gamma G(x,\xi,\zeta)d\zeta = \oint_\gamma \sum_{n=1}^{n=\infty} \frac{1}{\zeta-\lambda_n}\psi_n(x)\psi_n(\xi)d\zeta$$

$$= \sum_{n=1}^{n=\infty} \psi_n(x)\psi_n(\xi)\oint_\gamma \frac{1}{\zeta-\lambda_n}d\zeta\,.$$

Now, by [Dan10],

$\dfrac{1}{2\pi i}\dfrac{1}{\zeta-\lambda_n}$ is the Hyper-complex Circular Delta Function,

and

$$\oint_\gamma \frac{1}{2\pi i}\frac{1}{\zeta-\lambda_n}d\zeta = 1, \text{ for each } n = 1,2,3,\ldots$$

Therefore,

$$\oint_\gamma G(x,\xi,\zeta)d\zeta = \sum_{n=1}^{n=\infty} \psi_n(x)\psi_n(\xi)\underbrace{\oint_\gamma \frac{1}{\zeta-\lambda_n}d\zeta}_{2\pi i}$$

$$= 2\pi i\sum_{n=1}^{n=\infty} \psi_n(x)\psi_n(\xi)$$

$$= 2\pi i\delta(x-\xi).\square$$

19.

Green Function for Hermit Sturm-Liouville $u''(x) + (\lambda - x^2)u(x) = 0$, x real

Put

$$u(x) = e^{-\frac{1}{2}x^2} y(x),$$

and obtain Hermit's equation

$$y''(x) + 2xy'(x) + (\lambda - 1)y(x) = 0,$$

Then, the eigen values are

$$\lambda_n = 2n + 1, \quad n = 0,1,2,3,..$$

and the corresponding eigen functions are Hermit Polynomials of degree n,

$$H_n(x).$$

Therefore,

$$e^{-\frac{1}{2}x^2} H_n(x) \textit{ solve } u''(x) + (\lambda - x^2)u(x) = 0.$$

[Titchmarsh, p. 75] shows that

19.1 *The Normalized eigen functions are*

$$\psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-\frac{1}{2}x^2} H_n(x).$$

Therefore,

19.2 $$\boxed{G(x,\xi,\lambda) = \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}\xi^2} \sum_{n=0}^{n=\infty} \frac{1}{(\lambda - 2n - 1)2^n n!} H_n(x) H_n(\xi)}$$

Proof: $G(x,\xi,\lambda) = \sum_{n=0}^{n=\infty} \frac{1}{\lambda - \lambda_n} \psi_n(x)\psi_n(\xi)$. $\square$

Namely,

19.3 *The Hyper-real Green Function in real x, and ξ is the infinite sequence*

$$\frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}\xi^2} \left\langle \frac{1}{(\lambda - 1)} H_0(x) H_0(\xi) + \dots + \frac{1}{(\lambda - 2n - 1)2^n n!} H_n(x) H_n(\xi) \right\rangle_{n=0}^{n=\infty}$$

20.

Green Function for Legendre Sturm-Liouville $u''(\theta)+[\lambda+\frac{1}{4}\frac{1}{\cos^2\theta}]u(\theta)=0$, $-\frac{1}{2}\pi<\theta<\frac{1}{2}\pi$

Legendre's equation with $m=0$ is

$$(1-x^2)y''(x)+2xy'(x)+(\lambda-\tfrac{1}{4})y(x)=0.$$

The eigen values are

$$\lambda_n=(n+\tfrac{1}{2})^2,\ \ n=0,1,2,3,...$$

and the corresponding eigen functions are Legendre Polynomials of degree n,

$$P_n(x).$$

Put

$$x=\sin\theta,$$

to obtain

$$y''(\theta)-y'(\theta)\tan\theta+(\lambda-\tfrac{1}{4})y(\theta)=0.$$

Put

$$y(\theta)=\frac{1}{\sqrt{\cos\theta}}u(\theta),$$

to obtain

$$u''(\theta)+[\lambda+\tfrac{1}{4}(1+\tan^2\theta)]u(\theta)=0.$$

Substituting $\frac{1}{\cos^2\theta}=1+\tan^2\theta$,

$$u''(\theta) + [\lambda + \tfrac{1}{4}\tfrac{1}{\cos^2\theta}]u(\theta) = 0\,.$$

Therefore,

$$\sqrt{\cos\theta}P_n(\sin\theta) \ \textbf{ solve } \ u''(\theta) + (\lambda + \tfrac{1}{4\cos^2\theta})u(\theta) = 0\,.$$

[Titchmarsh, p. 79] shows that

20.1 *The Normalized eigen functions are*

$$\psi_n(\theta) = \sqrt{n + \tfrac{1}{2}}\sqrt{\cos\theta}P_n(\sin\theta)\,.$$

Therefore,

20.2 *for* $-\frac{1}{2}\pi < \theta$ and $\varphi < \frac{1}{2}\pi$

$$\boxed{G(\theta,\varphi,\lambda) = \sqrt{\cos\theta}\sqrt{\cos\varphi}\sum_{n=0}^{n=\infty}\frac{1}{\lambda - (n+\frac{1}{2})^2}(n+\tfrac{1}{2})P_n(\sin\theta)P_n(\sin\varphi)}$$

Namely,

20.3 *The Hyper-real Green Function in* $-\frac{1}{2}\pi < \theta, \varphi < \frac{1}{2}\pi$ *is*

the infinite sequence

$$\sqrt{\cos\theta}\sqrt{\cos\varphi}\left\langle \frac{1}{\lambda - \frac{1}{4}}\tfrac{1}{2}P_0(\sin\theta)P_0(\sin\varphi) + ... + \frac{1}{\lambda - (n+\frac{1}{2})^2}(n+\tfrac{1}{2})P_n(\sin\theta)P_n(\sin\varphi)\right\rangle_{n=0}^{n=\infty}$$

21.

Green Function for Legendre Sturm-Liouville

$$u''(\theta) + [\lambda + (\tfrac{1}{4} - m^2)\tfrac{1}{\cos^2\theta}]u(\theta) = 0, \ -\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi$$

Legendre's equation with for $m = 0,1,2,3...$, is

$$(1-x^2)y''(x) + 2xy'(x) + [\lambda - \tfrac{1}{4} - \tfrac{1}{1-x^2}m^2]y(x) = 0.$$

The eigen values are

$$\lambda_n^m = (n - m + \tfrac{1}{2})^2, \quad n = 2m, 2m+1, ...$$

and the corresponding eigen functions are Legendre Functions

$$P_n^m(x).$$

Put

$$x = \sin\theta,$$

to obtain

$$y''(\theta) - y'(\theta)\tan\theta + [\lambda - \tfrac{1}{4} - m^2\tfrac{1}{\cos^2\theta}]y(\theta) = 0.$$

Put

$$y(\theta) = \frac{1}{\sqrt{\cos\theta}}u(\theta),$$

to obtain

$$u''(\theta) + [\lambda + \tfrac{1}{4}(1+\tan^2\theta) - \tfrac{1}{4}\tfrac{1}{\cos^2\theta}m^2]u(\theta) = 0.$$

Substituting $\frac{1}{\cos^2\theta} = 1 + \tan^2\theta$,

$$u''(\theta)+[\lambda+(\tfrac{1}{4}-m^2)\tfrac{1}{\cos^2\theta}]u(\theta)=0.$$

Therefore,

$$\sqrt{\cos\theta}P_n^m(\sin\theta) \ \textbf{ solve } \ u''(\theta)+[\lambda+(\tfrac{1}{4}-m^2)\tfrac{1}{\cos^2\theta}]u(\theta)=0.$$

[Titchmarsh, p. 80] shows that

21.1 *The Normalized eigen functions are*

$$\psi_n(\theta)=\sqrt{\frac{(n-2m)!}{n!}}\sqrt{n-m+\tfrac{1}{2}}\sqrt{\cos\theta}P_n^m(\sin\theta).$$

Therefore,

21.2 *For* $-\frac{1}{2}\pi<\theta,\varphi<\frac{1}{2}\pi$, *and for* $m=0,1,2,3...$,

$$\boxed{\begin{aligned}&G(\theta,\varphi,\lambda)=\sqrt{\cos\theta}\sqrt{\cos\varphi}\times\\&\qquad\times\sum_{n=2m}^{n=\infty}\frac{1}{\lambda-(n-m+\frac{1}{2})^2}\frac{(n-2m)!}{n!}(n-m+\tfrac{1}{2})P_n^m(\sin\theta)P_n^m(\sin\varphi)\end{aligned}}$$

Namely,

21.3 *The Hyper-real Green Function in* $-\frac{1}{2}\pi<\theta,\varphi<\frac{1}{2}\pi$ *is the infinite sequence*

$$\sqrt{\cos\theta}\sqrt{\cos\varphi}\times$$

$$\times\left\langle\sum_{k=2m}^{k=n}\frac{1}{\lambda-(n-m+\frac{1}{2})^2}\frac{(k-2m)!}{k!}(k-m+\tfrac{1}{2})P_k^m(\sin\theta)P_k^m(\sin\varphi)\right\rangle_{n=m}^{n=\infty}$$

22.

Green Function for Bessel Sturm-Liouville

$$y''(x)+[\lambda-\tfrac{1}{x^2}(\nu^2-\tfrac{1}{4})]y(x)=0,\ 0<x<b$$

Two independent solutions are

$$\sqrt{x}J_\nu(x\sqrt{\lambda}),\ \text{and}\ \sqrt{x}Y_\nu(x\sqrt{\lambda})$$

By [Titchmarsh, p. 85],

$$y(x,\lambda)=\tfrac{1}{2}\pi\frac{1}{J_\nu(b\sqrt{\lambda})}\sqrt{x}\begin{vmatrix}J_\nu(b\sqrt{\lambda}) & J_\nu(x\sqrt{\lambda})\\ Y_\nu(b\sqrt{\lambda}) & Y_\nu(x\sqrt{\lambda})\end{vmatrix}\int_{\xi=0}^{\xi=x}\sqrt{\xi}J_\nu(\xi\sqrt{\lambda})f(\xi)d\xi$$

$$+\tfrac{1}{2}\pi\frac{1}{J_\nu(b\sqrt{\lambda})}\sqrt{x}J_\nu(x\sqrt{\lambda})\int_{\xi=x}^{\xi=b}\sqrt{\xi}\begin{vmatrix}J_\nu(b\sqrt{\lambda}) & J_\nu(\xi\sqrt{\lambda})\\ Y_\nu(b\sqrt{\lambda}) & Y_\nu(\xi\sqrt{\lambda})\end{vmatrix}f(\xi)d\xi$$

Therefore,

22.1

$$\boxed{G(x,\xi,\lambda)=\tfrac{1}{2}\pi\frac{1}{J_\nu(b\sqrt{\lambda})}\begin{cases}\sqrt{x}\begin{vmatrix}J_\nu(b\sqrt{\lambda}) & J_\nu(x\sqrt{\lambda})\\ Y_\nu(b\sqrt{\lambda}) & Y_\nu(x\sqrt{\lambda})\end{vmatrix}\sqrt{\xi}J_\nu(\xi\sqrt{\lambda}), & \xi\le x\\[2ex] \sqrt{\xi}\begin{vmatrix}J_\nu(b\sqrt{\lambda}) & J_\nu(\xi\sqrt{\lambda})\\ Y_\nu(b\sqrt{\lambda}) & Y_\nu(\xi\sqrt{\lambda})\end{vmatrix}\sqrt{x}J_\nu(x\sqrt{\lambda}), & \xi\ge x\end{cases}}$$

Alternatively, we may expand $G(x, \xi, \lambda)$ in eigen functions:

For $\boxed{\nu \geq 1}$

By [Titchmarsh, p. 82],

22.2 *The eigen-values*

$$\lambda_n \textit{ are the zeros of } J_\nu(b\sqrt{\lambda}), \quad n = 1, 2, 3, ...$$

and the normalized eigen-functions are

$$\frac{\sqrt{2}}{bJ_\nu{}'(b\sqrt{\lambda_n})} J_\nu(x\sqrt{\lambda_n}).$$

Therefore,

22.3 For $\nu \geq 1$, *in* $0 < x < b$

$$\boxed{G(x, \xi, \lambda) = \frac{2}{b^2}\sqrt{x\xi}\sum_{n=1}^{n=\infty}\frac{1}{(\lambda - \lambda_n)J_\nu{}'(b\sqrt{\lambda_n})} J_\nu(x\sqrt{\lambda_n})J_\nu(\xi\sqrt{\lambda_n}), \qquad \nu \geq 1}$$

Namely,

22.4 *For* $\nu \geq 1$ *the Hyper-real Green Function in* $0 < x, \xi < b$ *is the infinite sequence*

$$\frac{2}{b^2}\sqrt{x\xi}\left\langle \sum_{k=1}^{k=n}\frac{1}{(\lambda - \lambda_n)J_\nu{}'(b\sqrt{\lambda_k})} J_\nu(x\sqrt{\lambda_k})J_\nu(\xi\sqrt{\lambda_k})\right\rangle_{n=1}^{n=\infty}$$

For $\boxed{0 < \nu < 1, \quad \nu \neq \frac{1}{2}}$

By [Titchmarsh, p. 83],

22.5 *The eigen-values*

λ_n *are the zeros of* $cJ_\nu(b\sqrt{\lambda}) - \lambda^\nu J_{-\nu}(b\sqrt{\lambda})$,

where $c = const$.

$$r_n = -\operatorname{Res}\left\{\sqrt{\lambda}\,\frac{c\sqrt{\lambda^{-\nu}}J_\nu{}'(b\sqrt{\lambda}) - \sqrt{\lambda^\nu}J_{-\nu}{}'(b\sqrt{\lambda})}{c\sqrt{\lambda^{-\nu}}J_\nu(b\sqrt{\lambda}) - \sqrt{\lambda^\nu}J_{-\nu}(b\sqrt{\lambda})} + \frac{1}{2b}\right\}_{\lambda=\lambda_n}$$

and the normalized eigen-functions are

$$\sqrt{|r_n|}\,\frac{\pi\sqrt{b}}{2\sin\nu\pi}\sqrt{x}J_\nu(b\sqrt{\lambda_n})\left\{c\lambda_n^{-\nu}J_\nu(x\sqrt{\lambda_n}) - J_{-\nu}(x\sqrt{\lambda_n})\right\}.$$

Therefore,

22.6 *If* $0 < \nu < 1$, $\nu \neq \frac{1}{2}$, *then in* $0 < x < b$

$$\boxed{\begin{aligned} G(x,\xi,\lambda) = {} & \frac{\pi^2 b}{4\sin^2\nu\pi}\sqrt{x\xi}\sum_{n=1}^{n=\infty}\frac{1}{\lambda-\lambda_n}|r_n|J_\nu^2(b\sqrt{\lambda_n})\times \\ & \times\left\{c\lambda_n^{-\nu}J_\nu(x\sqrt{\lambda_n}) - J_{-\nu}(x\sqrt{\lambda_n})\right\}\left\{c\lambda_n^{-\nu}J_\nu(\xi\sqrt{\lambda_n}) - J_{-\nu}(\xi\sqrt{\lambda_n})\right\}\end{aligned}}$$

Namely,

22.7 If $0 < \nu < 1$, $\nu \neq \dfrac{1}{2}$, *then the Hyper-real Green Function in*

$0 < x, \xi < b$ *is the infinite sequence*

$$\frac{\pi^2 b}{4\sin^2\nu\pi}\sqrt{x\xi}\left\langle\sum_{k=1}^{k=n}\frac{1}{\lambda-\lambda_n}|r_k|J_\nu^2(b\sqrt{\lambda_k})\left\{c\lambda_k^{-\nu}J_\nu(x\sqrt{\lambda_k}) - J_{-\nu}(x\sqrt{\lambda_k})\right\}\times\right.$$

$$\times\left\{c\lambda_k^{-\nu}J_\nu(\xi\sqrt{\lambda_k}) - J_{-\nu}(\xi\sqrt{\lambda_k})\right\}\Big\rangle_{n=1}^{n=\infty}$$

22.8 If $0 < \nu < 1$, $\nu \neq \dfrac{1}{2}$, then

For $c =$*infinite hyper-real, the expansion is in* J_ν:

$$G(x,\xi,\lambda) = \frac{\pi^2 b}{4\sin^2\nu\pi}c^2\sqrt{x\xi}\sum_{n=1}^{n=\infty}\frac{1}{\lambda-\lambda_n}\left|r_n\right|J_\nu^2(b\sqrt{\lambda_n})\lambda_n^{-2\nu}J_\nu(x\sqrt{\lambda_n})J_\nu(\xi\sqrt{\lambda_n})$$

For $c =$*infinitesimal, the expansion is in* $J_{-\nu}$:

$$G(x,\xi,\lambda) = \frac{\pi^2 b}{4\sin^2\nu\pi}\sqrt{x\xi}\sum_{n=1}^{n=\infty}\frac{1}{\lambda-\lambda_n}\left|r_n\right|J_\nu^2(b\sqrt{\lambda_n})J_{-\nu}(x\sqrt{\lambda_n})J_{-\nu}(\xi\sqrt{\lambda_n})$$

23.

Green Function for Bessel Sturm Liouville $y''(x) + [\lambda - x]y(x) = 0, \;\; 0 < x < \infty$, with $\alpha = 0$

By [Titchmarsh, p. 91],

23.1 *The eigen-values*

$$\lambda_n \textit{ are the zeros of } J_{\frac{1}{3}}(\tfrac{2}{3}\lambda^{\frac{3}{2}}) + J_{-\frac{1}{3}}(\tfrac{2}{3}\lambda^{\frac{3}{2}}), \;\; n = 1,2,3,...$$

and the normalized eigen-functions are

$$\frac{-1}{\lambda_n\left[J_{\frac{2}{3}}(\frac{2}{3}\lambda_n^{\frac{3}{2}}) - J_{-\frac{2}{3}}(\frac{2}{3}\lambda_n^{\frac{3}{2}})\right]}\sqrt{\lambda_n - x}\left\{J_{\frac{1}{3}}(\tfrac{2}{3}[\lambda_n - x]^{\frac{3}{2}}) + J_{-\frac{1}{3}}(\tfrac{2}{3}[\lambda_n - x]^{\frac{3}{2}})\right\}$$

Therefore,

23.2 *The Hyper-real Green Function in* $0 < x < \infty$ is

$$\boxed{\begin{aligned} G(x,\xi,\lambda) = \sum_{n=1}^{n=\infty}\frac{1}{\lambda - \lambda_n}&\frac{1}{\lambda_n^2\left[J_{\frac{2}{3}}(\frac{2}{3}\lambda_n^{\frac{3}{2}}) - J_{-\frac{2}{3}}(\frac{2}{3}\lambda_n^{\frac{3}{2}})\right]^2}\sqrt{\lambda_n - x}\sqrt{\lambda_n - \xi} \\ &\times\left\{J_{\frac{1}{3}}(\tfrac{2}{3}[\lambda_n - x]^{\frac{3}{2}}) + J_{-\frac{1}{3}}(\tfrac{2}{3}[\lambda_n - x]^{\frac{3}{2}})\right\} \\ &\times\left\{J_{\frac{1}{3}}(\tfrac{2}{3}[\lambda_n - \xi]^{\frac{3}{2}}) + J_{-\frac{1}{3}}(\tfrac{2}{3}[\lambda_n - \xi]^{\frac{3}{2}})\right\}\end{aligned}}$$

Namely,

23.3 *The Hyper-real Green Function in* $0 < x, \xi < \infty$ *is the infinite sequence*

$$\left\langle \sum_{k=1}^{k=n} \frac{1}{\lambda - \lambda_n} \frac{1}{\lambda_n^2 \left[J_{\frac{2}{3}}(\frac{2}{3}\lambda_k^{\frac{3}{2}}) - J_{-\frac{2}{3}}(\frac{2}{3}\lambda_k^{\frac{3}{2}}) \right]^2} \sqrt{\lambda_k - x}\sqrt{\lambda_k - \xi} \times \right.$$

$$\times \left\{ J_{\frac{1}{3}}(\tfrac{2}{3}[\lambda_k - x]^{\frac{3}{2}}) + J_{-\frac{1}{3}}(\tfrac{2}{3}[\lambda_k - x]^{\frac{3}{2}}) \right\} \times$$

$$\left. \times \left\{ J_{\frac{1}{3}}(\tfrac{2}{3}[\lambda_k - \xi]^{\frac{3}{2}}) + J_{-\frac{1}{3}}(\tfrac{2}{3}[\lambda_k - \xi]^{\frac{3}{2}}) \right\} \right\rangle_{n=1}^{n=\infty}$$

24.

Green Function for Bessel Sturm-Liouville $y''(x)+[\mu - x]y(x)=0$, $0<x<\infty$ with $\alpha = \frac{1}{2}\pi$

By [Titchmarsh, p. 92],

24.1 *The eigen-values*

$$\mu_n \text{ are the zeros of } J_{\frac{2}{3}}(\tfrac{2}{3}\mu^{\frac{3}{2}}) - J_{-\frac{2}{3}}(\tfrac{2}{3}\mu^{\frac{3}{2}}),\quad n=1,2,3,...$$

and the normalized eigen-functions are

$$\frac{-1}{\mu_n\left[J_{\frac{1}{3}}(\frac{2}{3}\mu_n^{\frac{3}{2}})+J_{-\frac{1}{3}}(\frac{2}{3}\mu_n^{\frac{3}{2}})\right]}\sqrt{\mu_n - x}\left\{J_{\frac{1}{3}}(\tfrac{2}{3}[\mu_n - x]^{\frac{3}{2}})+J_{-\frac{1}{3}}(\tfrac{2}{3}[\mu_n - x]^{\frac{3}{2}})\right\}$$

Therefore,

24.2 *the Hyper-real Green Function in* $0<x<\infty$ *is*

$$\boxed{\begin{aligned} G(x,\xi,\mu) = \sum_{n=1}^{n=\infty}\frac{1}{\mu-\mu_n}\frac{1}{\mu_n^2\left[J_{\frac{1}{3}}(\frac{2}{3}\mu_n^{\frac{3}{2}})+J_{-\frac{1}{3}}(\frac{2}{3}\mu_n^{\frac{3}{2}})\right]^2}\sqrt{\mu_n - x}\sqrt{\mu_n-\xi}\,\times \\ \times\left\{J_{\frac{1}{3}}(\tfrac{2}{3}[\mu_n - x]^{\frac{3}{2}})+J_{-\frac{1}{3}}(\tfrac{2}{3}[\mu_n - x]^{\frac{3}{2}})\right\}\times \\ \times\left\{J_{\frac{1}{3}}(\tfrac{2}{3}[\mu_n - \xi]^{\frac{3}{2}})+J_{-\frac{1}{3}}(\tfrac{2}{3}[\mu_n - \xi]^{\frac{3}{2}})\right\}\end{aligned}}$$

Namely,

24.3 *the Hyper-real Green Function in* $0 < x, \xi < \infty$ *is the infinite sequence*

$$\left\langle \sum_{k=1}^{k=n} \frac{1}{\mu - \mu_n} \frac{1}{\mu_n^2 \left[J_{\frac{1}{3}}(\frac{2}{3}\mu_k^{\frac{3}{2}}) + J_{-\frac{1}{3}}(\frac{2}{3}\mu_k^{\frac{3}{2}}) \right]^2} \sqrt{\mu_k - x}\sqrt{\mu_k - \xi} \times \right.$$

$$\times \left\{ J_{\frac{1}{3}}(\tfrac{2}{3}[\mu_k - x]^{\frac{3}{2}}) + J_{-\frac{1}{3}}(\tfrac{2}{3}[\mu_k - x]^{\frac{3}{2}}) \right\} \times$$

$$\left. \times \left\{ J_{\frac{1}{3}}(\tfrac{2}{3}[\mu_k - \xi]^{\frac{3}{2}}) + J_{-\frac{1}{3}}(\tfrac{2}{3}[\mu_k - \xi]^{\frac{3}{2}}) \right\} \right\rangle_{n=1}^{n=\infty}$$

25.

Green's Function Integral Expansion in Eigen-functions

If the eigen-values are only infinitesimally separated from each other, the series summation of eigen-functions over discrete eigen values is replaced by Hyper-real integration over the continuum of eigen-values. Then, the expansion is represented by an integral.

Titchmarsh applied the Residue Theorem to obtain the projections of a function $f(x)$ on the eigen-functions of Sturm-Liouville problems, with continuous spectrum of eigen-values.

This yields the Integral expansion of the Delta Function in Sturm-Liuoville eigen-functions.

The Delta Function Expansion determines the Green's Function integral expansion in Sturm-Liuoville eigen-functions.

26.

Green's Function Integral Expansion in Cosine Functions of $y''(x) + \lambda y(x) = 0,\ 0 < x < \infty$

We may solve Green's Equation

$$D_x^2 G(x,\xi,\lambda) + \lambda G(x,\xi,\lambda) = \delta(x-\xi)$$

by Fourier Transforming it

$$\underbrace{D_x^2 G(x,\xi,\lambda)}_{\downarrow} + \lambda \underbrace{G(x,\xi,\lambda)}_{\downarrow} = \underbrace{\delta(x-\xi)}_{\downarrow}$$

$$(i\omega)^2 \hat{G}(\omega,\xi,\lambda) \quad \hat{G}(\omega,\xi,\lambda) \qquad \int_{x=-\infty}^{x=\infty} \delta(x-\xi) e^{-i\omega x} dx = e^{-i\omega\xi}$$

$$(i\omega)^2 \hat{G}(\omega,\xi,\lambda) + \lambda \hat{G}(\omega,\xi,\lambda) = e^{-i\omega\xi},$$

$$\hat{G}(\omega,\xi,\lambda) = \frac{1}{\lambda - \omega^2} e^{-i\omega\xi},$$

Then, Inverse Transforming it,

$$G(x,\xi,\lambda) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} \left\{ \frac{1}{\lambda-\omega^2} e^{-i\omega\xi} \right\} e^{i\omega x} d\omega$$

$$= \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} \frac{1}{\lambda - \omega^2} e^{i\omega(x-\xi)} d\omega \, .$$

That is,

26.1 *The Hyper-real Green Function is*

$$\boxed{G(x, \xi, \lambda) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} \frac{1}{\lambda - \omega^2} e^{i\omega(x-\xi)} d\omega}$$

Since Fourier Transforming applies only to Linear Differential Equations with constant coefficients, we demonstrate a method that applies to equations with variable coefficients:

The eigen-values

$$\lambda \equiv \omega^2 ,$$

are the interval of hyper-real positive numbers $(0, \infty)$.

By [Titchmarsh, p. 72], the Hyper-real function $f(x)$ is given for any hyper-real x by

$$f(x) = \frac{1}{\pi} \int_{\xi=-\infty}^{\xi=\infty} \int_{\lambda=0}^{\lambda=\infty} \cos(x\sqrt{\lambda}) \cos(\xi\sqrt{\lambda}) \tfrac{1}{2\sqrt{\lambda}} d\lambda f(\xi) d\xi \, +$$

$$+ \frac{1}{\pi} \int_{\xi=-\infty}^{\xi=\infty} \int_{\lambda=0}^{\lambda=\infty} \sin(x\sqrt{\lambda}) \sin(\xi\sqrt{\lambda}) \tfrac{1}{2\sqrt{\lambda}} d\lambda f(\xi) d\xi$$

$$= \frac{1}{\pi} \int_{\xi=-\infty}^{\xi=\infty} \left\{ \int_{\lambda=0}^{\lambda=\infty} \{ \underbrace{\cos(x\sqrt{\lambda}) \cos(\xi\sqrt{\lambda}) + \sin(x\sqrt{\lambda}) \sin(\xi\sqrt{\lambda})}_{\cos[(x-\xi)\sqrt{\lambda}]} \} \tfrac{1}{2\sqrt{\lambda}} d\lambda \right\} f(\xi) d\xi$$

$$= \int_{\xi=-\infty}^{\xi=\infty} \underbrace{\frac{1}{\pi} \int_{\omega=0}^{\omega=\infty} \cos\omega(x-\xi) d\omega}_{\delta(x-\xi)} f(\xi) d\xi$$

Hence, the Hyper-real Delta in $-\infty < x, \xi < \infty$ is

$$\delta(x-\xi) = \frac{1}{\pi} \int_{\omega=0}^{\omega=\infty} \underbrace{\cos\omega(x-\xi)}_{\frac{1}{2}(e^{i\omega(x-\xi)}+e^{-i\omega(x-\xi)})} d\omega$$

$$= \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} e^{i\omega(x-\xi)} d\omega$$

$$= \frac{1}{2\pi} \sum_{\omega=-\infty}^{\omega=\infty} e^{i\omega(x-\xi)} d\omega \,.$$

Since Green's Function satisfies

$$D_x^2 G(x, \xi, \lambda) + \lambda G(x, \xi, \lambda) = \delta(x-\xi),$$

the Hyper-real Green Function must have an expansion in $e^{i\omega(x-\xi)}$:

$$G(x, \xi, \lambda) = \frac{1}{2\pi} \sum_{\omega=-\infty}^{\omega=\infty} a(\omega) e^{i\omega(x-\xi)} d\omega \,.$$

Substituting in the Green's Function Differential Equation

$$\frac{1}{2\pi} \sum_{\omega=-\infty}^{\omega=\infty} a(\omega)\{-\omega^2 + \lambda\} e^{i\omega(x-\xi)} d\omega = \frac{1}{2\pi} \sum_{\omega=-\infty}^{\omega=\infty} e^{i\omega(x-\xi)} d\omega \,.$$

Equating each term,

$$a(\omega)\{-\omega^2 + \lambda\} = 1,$$

$$a(\omega) = \frac{1}{\lambda - \omega^2}$$

$$G(x,\xi,\lambda) = \frac{1}{2\pi}\sum_{\omega=-\infty}^{\omega=\infty}\frac{1}{\lambda-\omega^2}\,e^{i\omega(x-\xi)}d\omega\,,$$

$$= \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty}\frac{1}{\lambda-\omega^2}\,e^{i\omega(x-\xi)}d\omega\,.$$

27.

Green's Function Integral Expansion for Weber Equation

$y''(x) + [\lambda - \frac{1}{x^2}(\nu^2 - \frac{1}{4})]y(x) = 0$, $a < x < \infty$

$a > 0$

The eigen-values

$$-\lambda \equiv s^2,$$

are the interval of hyper-real negative numbers $(-\infty, 0)$.

By [Titchmarsh, p. 87], the Hyper-real function $f(x)$ is given for hyper-real $a < x < \infty$ by

$$f(x) = \sqrt{x} \int_{\xi=a}^{\xi=\infty} \int_{s=0}^{s=\infty} \frac{1}{J_\nu^2(as) + Y_\nu^2(as)} \left\{ J_\nu(xs)Y_\nu(as) - J_\nu(as)Y_\nu(xs) \right\} \times$$

$$\times \left\{ J_\nu(\xi s)Y_\nu(as) - J_\nu(as)Y_\nu(\xi s) \right\} s ds \sqrt{\xi} f(\xi) d\xi$$

Therefore, the Hyper-real Delta in $a < x, \xi < \infty$ is

$$\delta(x - \xi) = \int_{s=0}^{s=\infty} \sqrt{\xi} \frac{J_\nu(\xi s)Y_\nu(as) - J_\nu(as)Y_\nu(\xi s)}{J_\nu^2(as) + Y_\nu^2(as)} \sqrt{x} \left\{ J_\nu(xs)Y_\nu(as) - J_\nu(as)Y_\nu(xs) \right\} s ds$$

Since Green's Function satisfies

$$\partial_x^2 G(x,\xi,\lambda) + [\lambda - \frac{\nu^2 - \frac{1}{4}}{x^2}]G(x,\xi,\lambda) = \delta(x-\xi),$$

the Hyper-real Green Function must have an expansion in

$$\psi_\nu(x,\xi,s) = \frac{\sqrt{\xi}J_\nu(\xi s)Y_\nu(as) - J_\nu(as)\sqrt{\xi}Y_\nu(\xi s)}{J_\nu^2(as) + Y_\nu^2(as)}\left\{\sqrt{x}J_\nu(xs)Y_\nu(as) - J_\nu(as)\sqrt{x}Y_\nu(xs)\right\}.$$

Substituting

$$G(x,\xi,\lambda) = \sum_{s=0}^{s=\infty} a(s)\{\psi_\nu(x,\xi,s)\}sds$$

in the Green's Function Differential Equation,

$$\sum_{s=0}^{s=\infty} a(s)\{\partial_x^2\psi_\nu + [\lambda - \frac{\nu^2 - \frac{1}{4}}{x^2}]\psi_\nu\}sds = \sum_{s=0}^{s=\infty}\psi_\nu(x,\xi,s)sds.$$

Now, s is the eigen-value associated with $\sqrt{x}J_\nu(xs)$, with $\sqrt{x}Y_\nu(xs)$, with $\sqrt{x}J_\nu(xs)Y_\nu(as) - J_\nu(as)\sqrt{x}Y_\nu(xs)$, and thus with $\psi_\nu(x,\xi,s)$.

Hence,

$$\partial_x^2\psi_\nu = -[s - \frac{\nu^2 - \frac{1}{4}}{x^2}]\psi_\nu.$$

Substituting in the Green's Function differential Equation

$$\sum_{s=0}^{s=\infty} a(s)\{\lambda - s\}\psi_\nu sds = \sum_{s=0}^{s=\infty}\psi_\nu(x,\xi,s)sds$$

Equating each term,

$$a(s)\{\lambda - s\} = 1,$$

$$a(s) = \frac{1}{\lambda - s}$$

$$G(x,\xi,\lambda) = \sum_{s=0}^{s=\infty} \frac{1}{\lambda - s} \psi_{\nu}(x,\xi,s) s ds \, .$$

27.1

$$\boxed{G(x,\xi,\lambda) = \int_{s=0}^{s=\infty} \frac{1}{\lambda - s} \sqrt{\xi} \, \frac{J_{\nu}(\xi s) Y_{\nu}(as) - J_{\nu}(as) Y_{\nu}(\xi s)}{J_{\nu}^{2}(as) + Y_{\nu}^{2}(as)} \sqrt{x} \left\{ J_{\nu}(xs) Y_{\nu}(as) - J_{\nu}(as) Y_{\nu}(xs) \right\} s ds}$$

28.

Green's Function for Hankel's

$y''(x)+[\lambda-\frac{1}{x^2}(\nu^2-\frac{1}{4})]y(x)=0,\ 0<x<\infty$

By [Titchmarsh, p. 88],

$$y(x)=-\tfrac{1}{2}i\pi\sqrt{x}H_\nu^{(1)}(x\sqrt{\lambda})\int_{\xi=0}^{\xi=x}\sqrt{\xi}J_\nu(\xi\sqrt{\lambda})f(\xi)d\xi$$

$$-\tfrac{1}{2}i\pi\sqrt{x}J_\nu(x\sqrt{\lambda})\int_{\xi=x}^{\xi=\infty}\sqrt{\xi}H_\nu^{(1)}(\xi\sqrt{\lambda})f(\xi)d\xi$$

Therefore,

28.1
$$\boxed{G(x,\xi,\lambda)=-\frac{i\pi}{2}\sqrt{x\xi}\begin{cases}H_\nu^{(1)}(x\sqrt{\lambda})J_\nu(\xi\sqrt{\lambda}), & \xi\le x\\ H_\nu^{(1)}(\xi\sqrt{\lambda})J_\nu(x\sqrt{\lambda}), & \xi\ge x\end{cases}}$$

29.

Green's Function for Bessel's

$y''(x) + [\lambda + x)]y(x) = 0$, $-\infty < x < \infty$

By [Titchmarsh, p. 93],

$$y(x,\lambda) = -\tfrac{1}{6}i\pi\sqrt{x+\lambda}H^{(1)}_{\frac{1}{3}}(\tfrac{2}{3}[x+\lambda]^{\frac{3}{2}})\int_{\xi=-\infty}^{\xi=x}\sqrt{\xi+\lambda}H^{(2)}_{\frac{1}{3}}(\tfrac{2}{3}[\xi+\lambda]^{\frac{3}{2}})f(\xi)d\xi$$

$$-\tfrac{1}{6}i\pi\sqrt{x+\lambda}H^{(2)}_{\frac{1}{3}}(\tfrac{2}{3}[x+\lambda]^{\frac{3}{2}})\int_{\xi=x}^{\xi=\infty}\sqrt{\xi+\lambda}H^{(1)}_{\frac{1}{3}}(\tfrac{2}{3}[\xi+\lambda]^{\frac{3}{2}})f(\xi)d\xi$$

Therefore,

29.1

$$\boxed{G(x,\xi,\lambda) = -\frac{i\pi}{6}\sqrt{x+\lambda}\sqrt{\xi+\lambda}\begin{cases} H^{(1)}_{\frac{1}{3}}(\tfrac{2}{3}[x+\lambda]^{\frac{3}{2}})H^{(2)}_{\frac{1}{3}}(\tfrac{2}{3}[\xi+\lambda]^{\frac{3}{2}}), \ \xi \le x \\ \\ H^{(1)}_{\frac{1}{3}}(\tfrac{2}{3}[\xi+\lambda]^{\frac{3}{2}})H^{(2)}_{\frac{1}{3}}(\tfrac{2}{3}[x+\lambda]^{\frac{3}{2}}), \ \xi \ge x \end{cases}}$$

30.

Green's Function for Bessel's

$y''(x) + [\lambda + e^{2x})]y(x) = 0, \ -\infty < x < \infty$

By [Titchmarsh, p. 95],

$$y(x,\lambda) = -\frac{i\pi}{2\sinh(\pi\sqrt{\lambda})}\{J_{i\sqrt{\lambda}}(e^x) + J_{-i\sqrt{\lambda}}(e^x)\}\int_{\xi=-\infty}^{\xi=x} J_{-i\sqrt{\lambda}}(e^\xi)f(\xi)d\xi$$

$$-\frac{i\pi}{2\sinh(\pi\sqrt{\lambda})}J_{-i\sqrt{\lambda}}(e^x)\int_{\xi=-\infty}^{\xi=x}\{J_{i\sqrt{\lambda}}(e^\xi) + J_{-i\sqrt{\lambda}}(e^\xi)\}f(\xi)d\xi$$

Therefore,

30.1 $$\boxed{G(x,\xi,\lambda) = -\frac{i\pi}{2\sinh(\pi\sqrt{\lambda})}\begin{cases}\{J_{i\sqrt{\lambda}}(e^x) + J_{-i\sqrt{\lambda}}(e^x)\}J_{-i\sqrt{\lambda}}(e^\xi), & \xi \le x \\ \\ \{J_{i\sqrt{\lambda}}(e^\xi) + J_{-i\sqrt{\lambda}}(e^\xi)\}J_{-i\sqrt{\lambda}}(e^x), & \xi \ge x\end{cases}}$$

31.

Green's Function for Bessel's $y''(x) + [\lambda - e^{2x})]y(x) = 0,\ -\infty < x < \infty$

By [Titchmarsh, p. 96],

$$y(x,\lambda) = -K_{i\sqrt{\lambda}}(e^x)\int_{\xi=-\infty}^{\xi=x} I_{-i\sqrt{\lambda}}(e^\xi)f(\xi)d\xi - I_{-i\sqrt{\lambda}}(e^x)\int_{\xi=-\infty}^{\xi=x} K_{i\sqrt{\lambda}}(e^\xi)f(\xi)d\xi$$

Therefore,

31.1

$$\boxed{G(x,\xi,\lambda) = -\begin{cases} K_{i\sqrt{\lambda}}(e^x)I_{-i\sqrt{\lambda}}(e^\xi), & \xi \le x \\ \\ K_{i\sqrt{\lambda}}(e^\xi)I_{-i\sqrt{\lambda}}(e^x), & \xi \ge x \end{cases}}$$

References

[Abramowitz] Abramowitz, M., and Stegun, I., "*Handbook of Mathematical Functions with Formulas Graphs and Mathematical Tables*", U.S. Department of Commerce, National Bureau of Standards, 1964.

[Dan1] Dannon, H. Vic, "*Well-Ordering of the Reals, Equality of all Infinities, and the Continuum Hypothesis*" in Gauge Institute Journal Vol. 6 No. 2, May 2010;

Well-Ordering of the Reals, Equality of all Infinities, and the Continuum Hypothesis

[Dan2] Dannon, H. Vic, "*Infinitesimals*" in Gauge Institute Journal Vol.6 No. 4, November 2010;

Infinitesimals

[Dan3] Dannon, H. Vic, "*Infinitesimal Calculus*" in Gauge Institute Journal Vol. 7 No. 4, November 2011;

Infinitesimal Calculus

[Dan4] Dannon, H. Vic, "*Riemann's Zeta Function: the Riemann Hypothesis Origin, the Factorization Error, and the Count of the Primes*", in Gauge Institute Journal of Math and Physics, Vol. 5, No. 4, November 2009.

Riemann Zeta Function: the Riemann Hypothesis Origin, the Factorization Error, and the Count of the Primes

[Dan5] Dannon, H. Vic, "*The Delta Function*" in Gauge Institute Journal Vol. 8, No. 1, February, 2012;

The Delta Function

[Dan6] Dannon, H. Vic, "*Riemannian Trigonometric Series*", Gauge Institute Journal, Volume 7, No. 3, August 2011.

Riemannian Trigonometric Series

[Dan7] Dannon, H. Vic, "*Delta Function the Fourier Transform, and the Fourier Integral Theorem*" in Gauge Institute Journal Vol. 8, No. 2, May, 2012;

Delta Function, the Fourier Transform, and Fourier Integral Theorem

[Dan8] Dannon, H. Vic, "*Infinite Series with Infinite Hyper-real Sum* " in Gauge Institute Journal Vol. 8, No. 3, August, 2012;

Infinite Series with Infinite Hyper-real Sum

[Dan9] Dannon, H. Vic, "*Sturm-Liuoville Expansions of the Delta Function* "

Sturm-Liouville Expansions of the Delta Function

[Dan10] H. Vic Dannon, "*Complex Delta Function, Complex Fourier Transform, and Fourier Integral Theorem*".

Complex Delta Function, Complex Fourier Transform, and Fourier Integral Theorem

[Ferrers] Ferrers, N., M., "*An Elementary treatment on Spherical Harmonics*", Macmillan, 1877.

[Gradshteyn] Gradshteyn, I., S., and Ryzhik, I., M., "*Tables of Integrals Series and Products*", 7^{th} Edition, edited by Allan Jeffery, and Daniel Zwillinger, Academic Press, 2007

[Hardy] Hardy, G. H., *Divergent Series*, Chelsea 1991.

[Jackson] Jackson, Dunham, "*Fourier Series and Orthogonal Polynomials*", Mathematical association of America, 1941.

[Magnus] Magnus, W., Oberhettinger, F., Sony, R., P., "*Formulas and Theorems for the Special Functions of Mathematical Physics*" Third Edition, Springer-Verlag, 1966.

[Sansone] Sansone, Giovanni, "*Orthogonal Functions*", Revised Edition, Krieger, 1977.

[Spiegel] Spiegel, Murray, "Mathematical Handbook of formulas and tables" Schaum's Outline Series, McGraw Hill, 1968.

[Spanier] Spanier, Jerome, and Oldham, Keith, "*An Atlas of Functions*", Hemisphere, 1987.

[Szego2] Szego, Gabor, "*Orthogonal Polynomials*" Revised Edition, American Mathematical Society,1959.

[Szego4] Szego, Gabor, "*Orthogonal Polynomials*" Fourth Edition, American Mathematical Society,1975.

[Titchmarsh], E. C. Titchmarsh, "*Eigenfunction Expansions Associated with Second-order Differential Equations, Part I*", Second Edition, Oxford, 1962.

[Weisstein], Weisstein, Eric, W., "CRC Encyclopedia of Mathematics", Third Edition, CRC Press, 2009.