Periodic Delta Function, and Fejer-Cesaro Summation of Fourier Series

H. Vic Dannon
vic0@comcast.net
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Abstract The Fejer Summation Theorem supplies the conditions under which the Fejer-Cesaro Summation of Fourier Series, associated with a function $f(x)$ equals $f(x)$.

It is believed to hold in the Calculus of Limits. In fact,

The Theorem cannot be proved in the Calculus of Limits under any conditions,

because the Fejer Summation requires integration of the singular Fejer Kernel.

In Infinitesimal Calculus, the Fejer Kernel is the Periodic Delta Function,

$$\delta_{periodic}(x) = ... + \delta(x+4) + \delta(x+2) + \delta(x) + \delta(x-2) + \delta(x-4) + ...$$

This function violates the Calculus of Limits Conditions

- *The Hyper-real* $\delta(x)$*, is not defined in the Calculus of Limits, and* $|\delta(x)|$ *is not integrable in any bounded interval.*

- $\frac{1}{2}\Big(\delta(x+0)+\delta(x-0)\Big)=0$ ***does not replace*** $\delta(x)$ ***at its discontinuity point,*** $x=0$.

But $\delta_{Periodic}(x)$ equals its Fejer Summation, and the Fejer Summation associated with any periodic hyper-real $f(x)$, equals $f(x)$.

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Contents

The Origin of the Fejer-Cesaro Summation Theorem

Let $f(x)$ be a function defined on $[-1,1]$, so that $f(1) = f(-1)$.

The Fourier Coefficients of $f(x)$ are

$$\frac{1}{2}\int_{\xi=-1}^{\xi=1} f(\xi)e^{-in\pi\xi}d\xi \equiv c_n\,,\ n = ...,-2,-1,0,1,2,...,$$

The Fourier Series partial sums

$$\mathcal{S}_n\left\{f(x)\right\} = \int_{\xi=-1}^{\xi=1} f(\xi)\underbrace{\left\{\tfrac{1}{2}e^{-in\pi(\xi-x)} + ... + \tfrac{1}{2}e^{-i\pi(\xi-x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi-x)} + ... + \tfrac{1}{2}e^{in\pi(\xi-x)}\right\}}_{\text{Dirichlet Sequence}}d\xi\,,$$

give rise to the Dirichlet Sequence

$$D_n(x) = \tfrac{1}{2}e^{-in\pi x} + ... + \tfrac{1}{2}e^{-i\pi x} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi x} + ... + \tfrac{1}{2}e^{in\pi x}$$

$$= \tfrac{1}{2} + \cos\pi x + \cos 2\pi x + ... + \cos n\pi x$$

$$= \frac{\sin(n+\frac{1}{2})\pi x}{2\sin\frac{1}{2}\pi x}, \quad n = 0,1,2,..$$

0.1 Cesaro

To assign a numerical value to the divergent series

$$1 - 1 + 1 - 1 + 1 - 1 + ...,$$

Cesaro suggested to consider the convergence of the Arithmetic Means of its Partial Sums

$$\sigma_0 = s_0 = 1,$$

$$\sigma_1 = \frac{s_0 + s_1}{2} = \frac{1 + (1-1)}{2} = \tfrac{1}{2},$$

$$\sigma_2 = \frac{s_0 + s_1 + s_2}{3} = \frac{1 + (1-1) + (1-1+1)}{3} = \tfrac{2}{3},$$

$$\sigma_3 = \frac{s_0 + s_1 + s_2 + s_3}{4} = \frac{1 + (1-1) + (1-1+1) + (1-1+1-1)}{4} = \tfrac{1}{2},$$

....

Thus,

$$\sigma_{2k+1} = \tfrac{1}{2},$$

$$\sigma_{2k} = \tfrac{k+1}{2k+1} \to \tfrac{1}{2}$$

and the series converges to $\frac{1}{2}$.

we conclude that

the infinite series $1 - 1 + 1 - 1 + 1 - 1 + \ldots$ *has Cesaro Sum of* $\frac{1}{2}$

For any series

$$a_0 + a_1 + a_2 + a_3 + \ldots, \textit{ with partial sums } s_0, s_1, s_2, \ldots$$

$$\textit{If } \frac{s_0 + s_1 + \ldots + s_m}{m+1} \to \sigma$$

Then σ *is the Cesaro Sum of* $a_0 + a_1 + a_2 + a_3 + \ldots$

0.2 Fejer

applied Cesaro summation to Fourier Series.

The Fejer Summation partial sums are the Arithmetic Means

$$\mathcal{F}_{ej}\mathcal{S}_n\left\{f(x)\right\} = \frac{\mathcal{S}_0\left\{f(x)\right\} + \mathcal{S}_1\left\{f(x)\right\} + ... + \mathcal{S}_n\left\{f(x)\right\}}{n+1}$$

$$= \int_{\xi=-1}^{\xi=1} f(\xi) \underbrace{\frac{1}{n+1}\{(n+1)\tfrac{1}{2} + n\cos[\pi(\xi - x)] + ... + \cos[\pi n(\xi - x)]\}}_{\text{Fejer Sequence}} d\xi\,.$$

The Fejer Summation associated with the function $f(x)$ is clearly different from the Fourier Series associated with $f(x)$, but it may nevertheless converge to $f(x)$.

The equality of the Fejer Summation associated with $f(x)$, to $f(x)$ is the Fejer Summation Theorem.

The question is under which conditions does the Theorem hold.

1.

The Divergence of the Fejer Kernel in the Calculus of Limits

The Fejer Summation is believed to converge to $f(x)$ provided that

1. $\left|f(x)\right|$ is integrable on $[-1,1]$
2. $f(x)$ is periodic with period $T = 2$
3. $\frac{1}{2}\left(f(x+0)+f(x-0)\right)$ replaces $f(x)$ at a discontinuity point.

These Conditions reflect the belief that the equality depends only on the function, regardless of the singularity of the Fejer Kernel.

The Fejer Summation is not an infinite series, where $S_{n+1} = S_n + a_{n+1}$. It has a singular Kernel, and it raises the question whether it equals $f(x)$.

In the Calculus of Limits, no smoothness of the function guarantees the convergence of the Fejer Summation.

1.1 The Fejer Kernel is either singular or zero

In the Calculus of Limits, the Fejer Summation is the limit of the

$$\mathcal{F}_{ej}\mathcal{S}_n\left\{f(x)\right\} = \tfrac{1}{n}c_{-n}e^{-in\pi x} + \ldots + \tfrac{n-1}{n}c_{-1}e^{-i\pi x} + c_0 + \tfrac{n-1}{n}c_1e^{i\pi x} + \ldots + \tfrac{1}{n}c_ne^{in\pi x}$$

$$= \left(\frac{1}{2n} \int_{\xi=-1}^{\xi=1} f(\xi) e^{-in\pi\xi} d\xi \right) e^{in\pi x} + ... + \left(\frac{1}{2} \int_{\xi=-1}^{\xi=1} f(\xi) d\xi \right) + ... + \left(\frac{1}{2n} \int_{\xi=-1}^{\xi=1} f(\xi) e^{in\pi\xi} d\xi \right) e^{-in\pi x}$$

$$= \int_{\xi=-1}^{\xi=1} f(\xi) \underbrace{\left\{ \tfrac{1}{2n} e^{-in\pi(\xi-x)} + ... + \tfrac{n-1}{2n} e^{-i\pi(\xi-x)} + \tfrac{1}{2} + \tfrac{n-1}{2n} e^{i\pi(\xi-x)} + ... + \tfrac{1}{2n} e^{in\pi(\xi-x)} \right\}}_{\text{Fejer Sequence}} d\xi \,.$$

As $n \to \infty$, the Fejer Sequence becomes the Fejer Kernel, which is singular, and diverges at any $\xi - x = 2k$.

Thus, the Fejer Summation does not converge in the Calculus of Limits.

Avoiding the singularity at $\xi = x$, by using the Cauchy Principal Value of the integral does not recover the Theorem, because at any $\xi - x \neq 2k$, the Fejer Kernel vanishes, and the integral is identically zero, for any function $f(x)$.

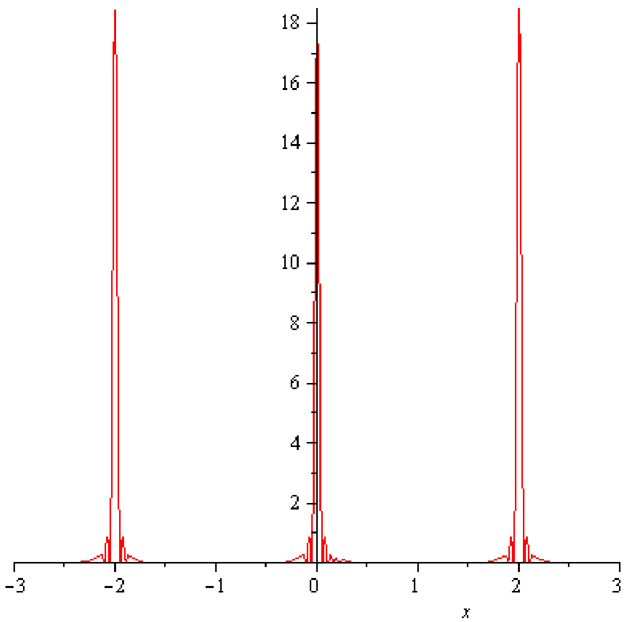
Plots of the Fejer sequence confirm that

In the Calculus of Limits,

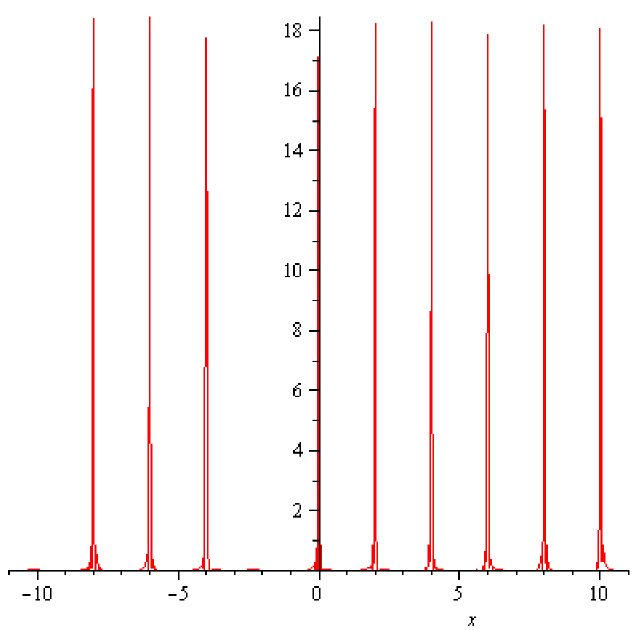
the Fejer Kernel is either singular or zero

1.2 Plots of Fejer Sequence

$plot\left(\dfrac{\sin^2\left(\pi \frac{37x}{2}\right)}{2 \cdot 37 \, \sin^2\left(\pi \frac{x}{2}\right)}, x = -3..3 \right)$ plots the spikes at $x = 0$, $x = -2$, $x = 2$



$$plot\left(\frac{\sin^2\left(\pi\frac{37\,x}{2}\right)}{2\cdot 37\ \sin^2\left(\pi\frac{x}{2}\right)},\ x=-11\,..11\right)$$ gives 9 spikes



Thus, the Fejer Summation Theorem does not hold in the Calculus of Limits.

1.3 Infinitesimal Calculus Solution

By resolving the problem of the infinitesimals [Dan2], we obtained the Infinite Hyper-reals that are strictly smaller than ∞, and constitute the value of the Delta Function at the singularity.

The controversy surrounding the Leibnitz Infinitesimals derailed the development of the Infinitesimal Calculus, and the Delta Function could not be defined and investigated properly.

In Infinitesimal Calculus, [Dan3], we can differentiate over jump discontinuities, and integrate over singularities.

The Delta Function, the idealization of an impulse in Radar circuits, is a Discontinuous Hyper-Real function which definition requires Infinite Hyper-reals, and which analysis requires Infinitesimal Calculus.

In [Dan5], we show that in infinitesimal Calculus, the hyper-real

$$\delta(x) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x} d\omega$$

is zero for any $x \neq 0$,

it spikes at $x = 0$, so that its Infinitesimal Calculus integral is $\int\limits_{x=-\infty}^{x=\infty} \delta(x)dx = 1$,

and $\delta(0) = \frac{1}{dx} < \infty$.

Here, we show that in Infinitesimal calculus, the Fejer Kernel is the periodic hyper-real Delta Function: A periodic train of Delta Functions.

And the Fejer Summation $\mathcal{F}_{ej}\mathcal{S}\left\{f(x)\right\}$ associated with a Hyper-real periodic function $f(x)$, equals $f(x)$.

2.

Hyper-real Line

Each real number α can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of infinitesimal hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the infinite hyper-reals.
4. The infinite hyper-reals are greater than any real number, yet strictly smaller than infinity.
5. The infinite hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a non-constant hyper-real.
7. The Hyper-reals are the totality of constant hyper-reals, a family of infinitesimals, a family of infinitesimals with

negative sign, a family of infinite hyper-reals, a family of infinite hyper-reals with negative sign, and non-constant hyper-reals.

8. The hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.
9. That line includes the real numbers separated by the non-constant hyper-reals. Each real number is the center of an interval of hyper-reals, that includes no other real number.
10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.
11. Zero is not an infinitesimal, because zero is not strictly greater than zero.
12. We do not add infinity to the hyper-real line.
13. The infinitesimals, the infinitesimals with negative signs, the infinite hyper-reals, and the infinite hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.
14. The hyper-real line is embedded in $\mathbb{R}^{\infty}$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the hyper-real onto the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal hyper-reals, or to the infinite hyper-reals, or to the non-constant hyper-reals.

16. No neighbourhood of a hyper-real is homeomorphic to an $\mathbb{R}^n$ ball. Therefore, the hyper-real line is not a manifold.

17. The hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

3.

Integral of a Hyper-real Function

In [Dan3], we defined the integral of a Hyper-real Function.

Let $f(x)$ be a hyper-real function on the interval $[a,b]$.

The interval may not be bounded.

$f(x)$ may take infinite hyper-real values, and need not be bounded.

At each

$$a \le x \le b,$$

there is a rectangle with base $[x - \frac{dx}{2}, x + \frac{dx}{2}]$, height $f(x)$, and area

$$f(x)dx.$$

We form the **Integration Sum** of all the areas for the x's that start at $x = a$, and end at $x = b$,

$$\sum_{x \in [a,b]} f(x)dx.$$

If for any infinitesimal dx, the Integration Sum has the same hyper-real value, then $f(x)$ is integrable over the interval $[a,b]$.

Then, we call the Integration Sum the integral of $f(x)$ from $x = a$, to $x = b$, and denote it by

$$\int_{x=a}^{x=b} f(x)dx\,.$$

If the hyper-real is infinite, then it is the integral over $[a,b]$,

If the hyper-real is finite,

$$\int_{x=a}^{x=b} f(x)dx = \text{ real part of the hyper-real}\,.\square$$

3.1 The countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers, $Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty\,.$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[a,b]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(x)dx$.

The Lower Integral is the Integration Sum where $f(x)$ is replaced by its lowest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

3.2 $$\sum_{x \in [a,b]} \left(\inf_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

The Upper Integral is the Integration Sum where $f(x)$ is replaced by its largest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

3.3 $$\sum_{x \in [a,b]} \left(\sup_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

If the integral is a finite hyper-real, we have

3.4 *A hyper-real function has a finite integral if and only if its upper integral and its lower integral are finite, and differ by an infinitesimal.*

4.

Delta Function

In [Dan5], we have defined the Delta Function, and established its properties

1. The Delta Function is a hyper-real function defined from the hyper-real line into the set of two hyper-reals $\left\{0, \frac{1}{dx}\right\}$. The hyper-real 0 is the sequence $\langle 0, 0, 0, ... \rangle$. The infinite hyper-real $\frac{1}{dx}$ depends on our choice of dx.
2. We will usually choose the family of infinitesimals that is spanned by the sequences $\left\langle \frac{1}{n} \right\rangle, \left\langle \frac{1}{n^2} \right\rangle, \left\langle \frac{1}{n^3} \right\rangle, \ldots$ It is a semigroup with respect to vector addition, and includes all the scalar multiples of the generating sequences that are non-zero. That is, the family includes infinitesimals with negative sign. Therefore, $\frac{1}{dx}$ will mean the sequence $\langle n \rangle$. Alternatively, we may choose the family spanned by the sequences $\left\langle \frac{1}{2^n} \right\rangle, \left\langle \frac{1}{3^n} \right\rangle, \left\langle \frac{1}{4^n} \right\rangle, \ldots$ Then, $\frac{1}{dx}$ will mean the

sequence $\langle 2^n \rangle$. Once we determined the basic infinitesimal dx, we will use it in the Infinite Riemann Sum that defines an Integral in Infinitesimal Calculus.

3. The Delta Function is strictly smaller than ∞

4. We define, $\delta(x) \equiv \frac{1}{dx}\chi_{[-\frac{dx}{2},\frac{dx}{2}]}(x)$,

$$\text{where} \quad \chi_{[-\frac{dx}{2},\frac{dx}{2}]}(x) = \begin{cases} 1, x \in \left[-\frac{dx}{2},\frac{dx}{2}\right] \\ 0, \text{otherwise} \end{cases}.$$

5. Hence,

❖ for $x < 0$, $\delta(x) = 0$

❖ at $x = -\frac{dx}{2}$, $\delta(x)$ jumps from 0 to $\frac{1}{dx}$,

❖ for $x \in \left[-\frac{dx}{2},\frac{dx}{2}\right]$, $\delta(x) = \frac{1}{dx}$.

❖ at $x = 0$, $\delta(0) = \frac{1}{dx}$

❖ at $x = \frac{dx}{2}$, $\delta(x)$ drops from $\frac{1}{dx}$ to 0.

❖ for $x > 0$, $\delta(x) = 0$.

❖ $x\delta(x) = 0$

6. If $dx = \langle \frac{1}{n} \rangle$, $\delta(x) = \left\langle \chi_{[-\frac{1}{2},\frac{1}{2}]}(x), 2\chi_{[-\frac{1}{4},\frac{1}{4}]}(x), 3\chi_{[-\frac{1}{6},\frac{1}{6}]}(x)... \right\rangle$

7. If $dx = \langle \frac{2}{n} \rangle$, $\delta(x) = \left\langle \frac{1}{2\cosh^2 x}, \frac{2}{2\cosh^2 2x}, \frac{3}{2\cosh^2 3x}, ... \right\rangle$

8. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle e^{-x}\chi_{[0,\infty)}, 2e^{-2x}\chi_{[0,\infty)}, 3e^{-3x}\chi_{[0,\infty)}, ... \right\rangle$

9. $$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1.$$

In [Dan6], we obtained

10. $$\delta(\xi - x) = \frac{1}{2\pi}\int_{k=-\infty}^{k=\infty} e^{-ik(\xi - x)}dk$$

In [Dan8], we defined the Periodic Delta Function, and obtained

11.

$$\delta_{Periodic}(x) = ... + \delta(x+4) + \delta(x+2) + \delta(x) + \delta(x-2) + \delta(x-4) + ...$$

$$= ... + \tfrac{1}{2}e^{-in\pi x} + ... + \tfrac{1}{2}e^{-i\pi x} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi x} + ... + \tfrac{1}{2}e^{in\pi x} + ...$$

5.

Periodic Delta Function $\delta_{periodic}(\xi - x)$

5.1 Periodic Delta Function

$$\delta_{Periodic}(\xi - x) = ... + \delta(\xi - x + 2) + \delta(\xi - x) + \delta(\xi - x - 2) + ...$$

is a periodic hyper-real Delta function, with period $T = 2$.

In [Dan8], we obtained

$$\delta_{Periodic}(\xi - x) = .. + \tfrac{1}{2}e^{-in\pi(\xi - x)} + .. + \tfrac{1}{2}e^{-i\pi(\xi - x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi - x)} + .. + \tfrac{1}{2}e^{in\pi(\xi - x)} + ..$$

6.

Convergent Series

In [Dan10], we defined convergence of infinite series in Infinitesimal Calculus

6.1 Sequence Convergence to a finite hyper-real a

$$a_n \to a \;\; \textit{iff} \;\; \langle a_n \rangle - \langle a \rangle = \text{infinitesimal}.$$

6.2 Sequence Convergence to an infinite hyper-real A

$$a_n \to A \;\; \textit{iff} \;\; \langle a_n \rangle \text{ represents the infinite hyper-real } A.$$

6.3 Series Convergence to a finite hyper-real s

$$a_1 + a_2 + \ldots \to s \;\; \textit{iff} \;\; \langle a_1 + \ldots + a_n \rangle - \langle s \rangle = \text{infinitesimal}.$$

6.4 Series Convergence to an Infinite Hyper-real S

$$a_1 + a_2 + \ldots \to S \;\; \textit{iff}$$

$$\langle a_1 + \ldots + a_n \rangle \text{ represents the infinite hyper-real } S.$$

7.

Fejer Sequence and $\delta_{periodic}(\xi - x)$

7.1 Fejer Sequence Definition

Let $f(x)$ be an integrable function on $[-1,1]$.

Then, for each $n = ...,-3,-2,-1,0,1,2,3,...$, the integrals

$$\frac{1}{2}\int_{\xi=-1}^{\xi=1} f(\xi)e^{-in\pi\xi}d\xi \equiv c_n$$

are the Fourier Coefficients of $f(x)$.

The Fourier Series partial sums

$$\mathcal{S}_n\left\{f(x)\right\} = \int_{\xi=-1}^{\xi=1} f(\xi)\underbrace{\left\{\tfrac{1}{2}e^{-in\pi(\xi-x)} + ... + \tfrac{1}{2}e^{-i\pi(\xi-x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi-x)} + ... + \tfrac{1}{2}e^{in\pi(\xi-x)}\right\}}_{\text{Dirichlet Sequence}}d\xi\,,$$

give rise to the Dirichlet Sequence

$$D_n(x) = \tfrac{1}{2}e^{-in\pi x} + ... + \tfrac{1}{2}e^{-i\pi x} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi x} + ... + \tfrac{1}{2}e^{in\pi x}$$

$$= \tfrac{1}{2} + \cos\pi x + \cos 2\pi x + ... + \cos n\pi x$$

$$= \frac{\sin(n+\frac{1}{2})\pi x}{2\sin\frac{1}{2}\pi x}, \quad n = 0,1,2,..$$

The Fejer Summation partial sums are the Arithmetic Means

$$\mathcal{F}_{ej}\mathcal{S}_n\left\{f(x)\right\} = \frac{\mathcal{S}_0\left\{f(x)\right\} + \mathcal{S}_1\left\{f(x)\right\} + ... + \mathcal{S}_n\left\{f(x)\right\}}{n+1}$$

$$= \int_{\xi=-1}^{\xi=1} f(\xi) \underbrace{\frac{1}{n+1}\{(n+1)\tfrac{1}{2} + n\cos[\pi(\xi - x)] + ... + \cos[\pi n(\xi - x)]\}}_{\text{Fejer Sequence}} d\xi \,.$$

They give rise to the Fejer Sequence

$$F_n(x) = \tfrac{1}{2} + \tfrac{n}{n+1}\cos[\pi(\xi - x)] + ... + \tfrac{1}{n+1}\cos[\pi n(\xi - x)]$$

7.2 $$\boxed{F_{m-1}(x) = \tfrac{1}{2} + \tfrac{m-1}{m}\cos\pi x + ... + \tfrac{m-(m-1)}{m}\cos(m-1)\pi x},$$

$$= \frac{D_0(x) + D_1(x) + ...D_{m-1}(x)}{m},$$

$$= \frac{1}{2m}\frac{\sin^2(\tfrac{1}{2}m\pi x)}{\sin^2(\tfrac{1}{2}\pi x)}, \qquad m = 1, 2, ..$$

Proof:

$$F_{m-1}(x) = \frac{D_0(x) + D_1(x) + ...D_{m-1}(x)}{m}$$

$$= \frac{1}{m}\left\{\frac{\sin\frac{1}{2}\pi x}{2\sin\frac{1}{2}\pi x} + \frac{\sin\frac{3}{2}\pi x}{2\sin\frac{1}{2}\pi x} + ... + \frac{\sin(m-\frac{1}{2})\pi x}{2\sin\frac{1}{2}\pi x}\right\}$$

$$= \frac{1}{2m\sin\frac{1}{2}\pi x}\left\{\sin\tfrac{1}{2}\pi x + \sin\tfrac{3}{2}\pi x + ... + \sin(m - \tfrac{1}{2})\pi x\right\}$$

$$= \frac{1}{2m\sin\frac{1}{2}\pi x}\left\{\tfrac{\cos 0 - \cos\pi x}{2\sin\frac{1}{2}\pi x} + \tfrac{\cos\pi x - \cos 2\pi x}{2\sin\frac{1}{2}\pi x} + ... + \tfrac{\cos(m-1)\pi x - \cos m\pi x}{2\sin\frac{1}{2}\pi x}\right\}$$

$$= \frac{1}{4m\sin^2\frac{1}{2}\pi x}\left\{1 - \cos m\pi x\right\}$$

$$= \frac{1}{2m}\frac{\sin^2(\frac{1}{2}m\pi x)}{\sin^2(\frac{1}{2}\pi x)}.\square$$

7.3 Fejer Sequence is a Periodic Delta Sequence, and represents a Periodic Delta Function

Each $F_{m-1}(x) = \frac{1}{2m}\frac{\sin^2(\frac{1}{2}m\pi x)}{\sin^2(\frac{1}{2}\pi x)}$, $m = 1, 2, 3, \ldots$

1. *has the sifting property on each interval*,

$$\ldots \int_{x=-5}^{x=-3} F_{m-1}(x)dx = 1; \quad \int_{x=-3}^{x=-1} F_{m-1}(x)dx = 1; \quad \int_{x=-1}^{x=1} F_{m-1}(x)dx = 1 \ldots$$

2. is a continuous function

3. peaks on each of these intervals to $\lim_{x \to 2k} F_m(x) = \frac{1}{2}m$.

<u>*Proof of (1)*</u>

$$\int_{x=-1}^{x=1} F_{m-1}(x)dx = \int_{x=-1}^{x=1} \left[\tfrac{1}{2} + \tfrac{m-1}{m}\cos\pi x + \ldots + \tfrac{1}{m}\cos(m-1)\pi x\right]dx$$

$$= \left[\tfrac{1}{2}x + \tfrac{m-1}{m\pi}\sin\pi x + \ldots + \tfrac{1}{m(m-1)\pi}\sin(m-1)\pi x\right]_{x=-1}^{x=1}$$

$$= 1.\square$$

<u>*Proof of (3)*</u>

As $x \to 0$, $\frac{1}{2m}\frac{\sin^2(\frac{1}{2}m\pi x)}{\sin^2(\frac{1}{2}\pi x)} \to \frac{0}{0}$

Applying Bernoulli's rule,

$$\frac{1}{2m}\frac{\left[\sin^2\frac{1}{2}m\pi x\right]'}{\left[\sin^2\frac{1}{2}\pi x\right]'} \underset{x\to 0}{\to} \frac{1}{2m}\frac{(2\sin\frac{1}{2}m\pi x)\cos\frac{1}{2}m\pi x(\frac{1}{2}m\pi)}{(2\sin\frac{1}{2}\pi x)\cos\frac{1}{2}\pi x(\frac{1}{2}\pi)}\Bigg|_{x=0}$$

$$= \frac{1}{2}\frac{\sin m\pi x}{\sin \pi x}\Bigg|_{x=0} = \frac{0}{0}$$

Applying Bernoulli's rule to $\frac{1}{2}\frac{\sin m\pi x}{\sin \pi x}$,

$$\frac{1}{2}\frac{[\sin m\pi x]'}{[\sin \pi x]'} \underset{x\to 0}{\to} \frac{1}{2}\frac{\pi m\cos m\pi x}{\pi\cos \pi x}\Bigg|_{x=0} = \tfrac{1}{2}m\,.\square$$

7.4 *Fejer Sequence Represents a Periodic Delta Function*

$$\delta_{periodic}(\xi - x) = \left\langle \frac{1}{2m}\left(\frac{\sin\frac{m}{2}\pi(\xi - x)}{\sin\frac{1}{2}\pi(\xi - x)}\right)^2 \right\rangle$$

8.

Fejer Kernel and $\delta_{periodic}(\xi - x)$

8.1 Fejer Kernel in the Calculus of Limits

$$F_{ejer}(\xi - x) = \lim_{m\to\infty} \frac{1}{2m}\frac{\sin^2 \frac{1}{2}m\pi(\xi - x)}{\sin^2 \frac{1}{2}\pi(\xi - x)}$$

$$= \lim_{m\to\infty}\left\{\tfrac{1}{2} + \tfrac{m-1}{m}\cos\pi(\xi - x) + ... + \tfrac{m-(m-1)}{m}\cos(m-1)\pi(\xi - x)\right\}$$

8.2 *In the Calculus of Limits, the Fejer Kernel does not have the sifting property*

<u>*Proof*</u>:

By 4.3, as $\xi - x \to 2k$,

$$\frac{1}{2m}\frac{\sin^2 \frac{1}{2}m\pi(\xi - x)}{\sin^2 \frac{1}{2}\pi(\xi - x)} \to \tfrac{1}{2}m.$$

Hence,

$$\lim_{m\to\infty}\lim_{\xi - x\to 2k}\frac{1}{2m}\frac{\sin^2 \frac{1}{2}m\pi(\xi - x)}{\sin^2 \frac{1}{2}\pi(\xi - x)} \to \lim_{m\to\infty}\tfrac{1}{2}m = \infty.\square$$

8.3 Hyper-real Fejer Kernel in Infinitesimal Calculus

$$F_{ejer}(\xi - x) = \begin{cases} \left\langle \frac{1}{2}n \right\rangle, \xi - x = 2k \\ 0, \quad \xi - x \neq 2k \end{cases}.$$

Proof: at any $\xi - x = 2k$,

$$\frac{1}{2m}\frac{\sin^2 \frac{1}{2}m\pi(\xi - x)}{\sin^2 \frac{1}{2}\pi(\xi - x)}\Bigg|_{\xi - x = 2k} = \left\langle \tfrac{1}{2}m \right\rangle . \square$$

For $\xi - x \neq 2k$, and for any m, $\frac{\sin^2 \frac{1}{2}m\pi(\xi - x)}{\sin^2 \frac{1}{2}\pi(\xi - x)}$ is bounded by $M(x)$. Therefore,

$$0 \leq \frac{1}{2m}\frac{\sin^2 \frac{1}{2}m\pi(\xi - x)}{\sin^2 \frac{1}{2}\pi(\xi - x)} \leq \tfrac{1}{2m}M(\xi - x)$$

Hence, for $\xi - x \neq 2k$,

$$\frac{1}{2m}\frac{\sin^2 \frac{1}{2}m\pi(\xi - x)}{\sin^2 \frac{1}{2}\pi(\xi - x)} = \text{infinitesimal} . \square$$

8.4 ***Let $\frac{1}{2}N = \frac{1}{dx}$ be an infinite Hyper-real. Then,***

$$F_{ejer}(\xi - x) = \frac{1}{2N}\frac{\sin^2 \frac{1}{2}N\pi(\xi - x)}{\sin^2 \frac{1}{2}\pi(\xi - x)}$$

$$= \tfrac{1}{2} + \tfrac{N-1}{N}\cos \pi(\xi - x) + ... + \tfrac{N-(N-1)}{N}\cos(N-1)\pi(\xi - x)$$

$$= ... + \delta(\xi - x + 2) + \delta(\xi - x) + \delta(\xi - x - 2) + ...$$

$$= \delta_{periodic}(\xi - x)$$

Proof:

$$F_{ejer}(\xi - x) = \tfrac{1}{2} + \tfrac{N-1}{N}\cos \pi(\xi - x) + ... + \tfrac{N-(N-1)}{N}\cos(N-1)\pi(\xi - x)$$

By 8.3,

$$= ... + \begin{cases} \left\langle \frac{N}{2} \right\rangle, \xi - x = -2 \\ 0, \xi - x \neq -2 \end{cases} + \begin{cases} \left\langle \frac{N}{2} \right\rangle, \xi = x \\ 0, \xi \neq x \end{cases} + \begin{cases} \left\langle \frac{N}{2} \right\rangle, \xi - x = 2 \\ 0, \xi - x \neq 2 \end{cases} + ...$$

$$= ... + \begin{cases} \frac{1}{dx}, \xi - x = -2 \\ 0, \xi - x \neq -2 \end{cases} + \begin{cases} \frac{1}{dx}, \xi = x \\ 0, \xi \neq x \end{cases} + \begin{cases} \frac{1}{dx}, \xi - x = 2 \\ 0, \xi - x \neq 2 \end{cases} + ...$$

$$= ... + \delta(\xi - x + 2) + \delta(\xi - x) + \delta(\xi - x - 2) + ...$$

$$= \delta_{Periodic}(\xi - x). \ \square$$

9.

Fejer Summation and $\delta_{periodic}(\xi - x)$

9.1 Fejer Summation of a Hyper-real Function

Let $f(x)$ be a hyper-real function integrable on $[-1,1]$.

Then, for each $n = ..., -3, -2, -1, 0, 1, 2, 3, ...$, the integrals

$$\frac{1}{2}\int_{\xi=-1}^{\xi=1} f(\xi)e^{-in\pi\xi}d\xi \equiv c_n$$

exist, with finite, or infinite hyper-real values. The c_n are the Fourier Coefficients of $f(x)$.

The Fourier Series partial sums

$$\mathcal{S}_n\left\{f(x)\right\} = \int_{\xi=-1}^{\xi=1} f(\xi)\underbrace{\left\{\tfrac{1}{2}e^{-in\pi(\xi-x)} + ... + \tfrac{1}{2}e^{-i\pi(\xi-x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi-x)} + ... + \tfrac{1}{2}e^{in\pi(\xi-x)}\right\}}_{\text{Dirichlet Sequence}}d\xi,$$

give rise to the Dirichlet Sequence

$$D_n(x) = \tfrac{1}{2}e^{-in\pi x} + ... + \tfrac{1}{2}e^{-i\pi x} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi x} + ... + \tfrac{1}{2}e^{in\pi x}$$

$$= \tfrac{1}{2} + \cos\pi x + \cos 2\pi x + ... + \cos n\pi x$$

$$= \frac{\sin(n+\frac{1}{2})\pi x}{2\sin\frac{1}{2}\pi x}, \quad n = 0, 1, 2, ..$$

The Fejer Summation partial sums are the Arithmetic Means

$$\mathcal{F}_{ej}\mathcal{S}_n\left\{f(x)\right\} = \frac{\mathcal{S}_0\left\{f(x)\right\} + \mathcal{S}_1\left\{f(x)\right\} + ... + \mathcal{S}_n\left\{f(x)\right\}}{n+1}$$

$$= \int_{\xi=-1}^{\xi=1} f(\xi) \underbrace{\frac{1}{n+1}\{(n+1)\tfrac{1}{2} + n\cos[\pi(\xi - x)] + ... + \cos[\pi n(\xi - x)]\}}_{\text{Fejer Sequence}} d\xi \,.$$

They give rise to the Fejer Sequence

$$F_n(x) = \tfrac{1}{2} + \tfrac{n}{n+1}\cos[\pi(\xi - x)] + ... + \tfrac{1}{n+1}\cos[\pi n(\xi - x)]$$

By 4.2, for $m = 1, 2, ..,$

$$F_{m-1}(x) = \tfrac{1}{2} + \tfrac{m-1}{m}\cos\pi x + ... + \tfrac{1}{m}\cos(m-1)\pi x\,,$$

$$= \frac{D_0(x) + D_1(x) + ...D_{m-1}(x)}{m}\,,$$

$$= \frac{1}{2m}\frac{\sin^2(\frac{1}{2}m\pi x)}{\sin^2(\frac{1}{2}\pi x)}.$$

Let $\frac{1}{2}N = \frac{1}{dx}$ be an infinite Hyper-real.

The Hyper-real Fejer Kernel is

$$F_{ejer}(x) = \tfrac{1}{2} + \tfrac{N-1}{N}\cos\pi x + ... + \tfrac{1}{N}\cos(N-1)\pi x$$

$$= ... + \delta(x+4) + \delta(x+2) + \delta(x) + \delta(x-2) + \delta(x+4)...$$

The Fejer Summation associated with $f(x)$ is

$$\mathcal{F}_{ejer}\mathcal{S}\left\{f(x)\right\} = \tfrac{1}{N}c_{-N}e^{-Ni\pi x} + ... + \tfrac{N-1}{N}c_{-1}e^{-i\pi x} + c_0 + \tfrac{N-1}{N}c_1e^{i\pi x} + ... + \tfrac{1}{N}c_Ne^{Ni\pi x}$$

For each x, it may assume finite or infinite hyper-real values.

9.2 $$\mathcal{F}_{ejer}\mathcal{S}\left\{\delta_{Periodic}(\xi - x)\right\} = \delta_{Periodic}(\xi - x)$$

Proof: Let N be an infinite hyper-real.

$$\mathcal{F}_{ejer}\mathcal{S}\left\{\delta_{Periodic}(\xi - x)\right\} = \tfrac{1}{N}c_{-N}e^{-Ni\pi(\xi-x)} + .. + \tfrac{N-1}{N}c_{-1}e^{-i\pi(\xi-x)}$$

$$+c_0 + \tfrac{N-1}{N}c_1 e^{i\pi(\xi-x)} + .. + \tfrac{1}{N}c_N e^{Ni\pi(\xi-x)},$$

where

- $\frac{N-n}{N}c_n = \frac{1}{2}\int\limits_{u=-1}^{u=1}\delta_{Periodic}(u)e^{-in\pi u}du$,
- $T = 2$ is the period,
- and $c = 0$.

For $\delta_{Periodic}(u)$ with $T = 2$,

$$\tfrac{N-n}{N}c_n = \tfrac{1}{2}\int\limits_{u=-1}^{u=1}\left[... + \delta(u+2) + \delta(u) + \delta(u-2) + ...\right]e^{-in\pi u}du$$

$$= \tfrac{1}{2}\int\limits_{u=-1}^{u=1}\delta(u)e^{-in\pi u}du = \tfrac{1}{2}.$$

Therefore,

$$\mathcal{F}_{ejer}\mathcal{S}\left\{\delta_{Periodic}(\xi - x)\right\} = \tfrac{1}{2}e^{-iN\pi(\xi-x)} + ... + \tfrac{1}{2}e^{-i\pi(\xi-x)}$$

$$+\tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi-x)} + ... + \tfrac{1}{2}e^{iN\pi(\xi-x)},$$

$$= \delta_{Periodic}(\xi - x),$$

by **5**.□

10.

Fejer Summation Theorem

The Fejer Summation Theorem for a hyper-real function, $f(x)$, is the Fundamental Theorem of Fejer Summation.

It supplies the conditions under which the Fejer Summation associated with $f(x)$ equals $f(x)$.

It is believed to hold in the Calculus of Limits under some Conditions. In fact,

The Theorem cannot be proved in the Calculus of Limits under any conditions,

because the Fejer Summation requires integration of the singular Fejer Kernel.

10.1 Fejer Summation Theorem cannot be proved in the Calculus of Limits

Proof: Take $L = 1$, and $c = 0$.

In the Calculus of Limits, the Fejer Summation is the limit of

$$\mathcal{F}_{ej}\mathcal{S}_n\left\{f(x)\right\} = \tfrac{1}{n}c_{-n}e^{-in\pi x} + ... + \tfrac{n-1}{n}c_{-1}e^{-i\pi x}$$

$$+c_0 + \tfrac{n-1}{n}c_1 e^{i\pi x} + ... + \tfrac{1}{n}c_n e^{in\pi x},$$

$$= \left(\frac{1}{2n} \int_{\xi=-1}^{\xi=1} f(\xi) e^{in\pi\xi} d\xi \right) e^{-in\pi x} + ... + \left(\frac{n-1}{2n} \int_{\xi=-1}^{\xi=1} f(\xi) e^{i\pi\xi} d\xi \right) e^{-i\pi x}$$

$$+ \left(\frac{1}{2} \int_{\xi=-1}^{\xi=1} f(\xi) d\xi \right) + \left(\frac{n-1}{2n} \int_{\xi=-1}^{\xi=1} f(\xi) e^{-i\pi\xi} d\xi \right) e^{i\pi x} + ... + \left(\frac{1}{2n} \int_{\xi=-1}^{\xi=1} f(\xi) e^{-in\pi\xi} d\xi \right) e^{in\pi x}$$

$$= \int_{\xi=-1}^{\xi=1} f(\xi) \Big\{ \frac{1}{2n} e^{in\pi(\xi-x)} + ... + \frac{n-1}{2n} e^{i\pi(\xi-x)} + \frac{1}{2} +$$

$$+ \frac{n-1}{2n} e^{-i\pi(\xi-x)} + ... + \frac{1}{2n} e^{-in\pi(\xi-x)} \Big\} d\xi .$$

As $n \to \infty$, the Fejer Sequence

$$F_n(\xi - x) = \frac{1}{2n} e^{in\pi(\xi-x)} + ... + \frac{n-1}{2n} e^{i\pi(\xi-x)}$$

$$+ \frac{1}{2} + \frac{n-1}{2n} e^{-i\pi(\xi-x)} + ... + \frac{1}{2n} e^{-in\pi(\xi-x)}$$

becomes the Fejer Kernel, the infinite series

$$... + \frac{1}{2n} e^{in\pi(\xi-x)} + ... + \frac{n-1}{2n} e^{i\pi(\xi-x)}$$

$$+ \frac{1}{2} + \frac{1}{2n} e^{-i\pi(\xi-x)} + ... + \frac{n-1}{2n} e^{-in\pi(\xi-x)} + ...,$$

By 8.2, The Fejer Kernel is singular whenever $\xi - x = 2k$, and the Fejer Summation diverges in the Calculus of Limits.

Avoiding the singularity at $\xi - x = 2k$, by using the Cauchy Principal Value of the integral does not recover the Theorem, because at any $\xi - x \neq 2k$, the Fejer Kernel is zero, and the integral is identically zero, for any function $f(x)$.

Thus, the Fejer Summation Theorem cannot be proved the Calculus of Limits.□

10.2 Calculus of Limits Conditions are irrelevant to Fejer Summation Theorem

Proof: The Fejer Conditions are

1. $\left|f(x)\right|$ is integrable on $[c-L, c+L]$
2. $f(x)$ is periodic with period $T = 2L$
3. $\frac{1}{2}\big(f(x+0)+f(x-0)\big)$ replaces $f(x)$ at a discontinuity point.

It is clear from 10.1 that the Fejer conditions on $f(x)$ do not resolve the singularity of the Fejer kernel, and are not sufficient for the Fejer Summation Theorem.□

In Infinitesimal Calculus, by 8.4, the Fejer Kernel is the Periodic Delta Function, and by 9.2, it equals its Fejer Summation.

Then, the Fejer Summation Theorem holds for any periodic integrable Hyper-Real Function:

10.3 Fejer Summation Theorem for Hyper-real $f(x)$

If $f(x)$ *is hyper-real function integrable on* $[c-L, c+L]$*, so that*

$f(c-L) = f(c+L)$

Then,
$$f(x) = \mathcal{F}_{ejer}\mathcal{S}\left\{f(x)\right\}$$

Proof: Take $L = 1$, $c = 0$, and $\frac{1}{2}N = \frac{1}{dx}$ an infinite Hyper- real.

$$f(x) = \int_{\xi=-1}^{\xi=1} f(\xi)\underbrace{\left\{... + \delta(\xi - x + 2) + \delta(\xi - x) + \delta(\xi - x - 2) + ...\right\}}_{\delta_{Periodic}(\xi - x),\ \text{where the period of Delta is T=2}} d\xi$$

By 8.4, $\delta_{Periodic}(\xi - x) = \mathcal{F}_{ej}(\xi - x)$

$$= \int_{\xi=-1}^{\xi=1} f(\xi)\left\{\tfrac{1}{2N}e^{iN\pi(\xi-x)} + ... + \tfrac{N-1}{2N}e^{i\pi(\xi-x)} + \tfrac{1}{2} + \right.$$

$$\left. + \tfrac{N-1}{2N}e^{-i\pi(\xi-x)} + ... + \tfrac{1}{2N}e^{-iN\pi(\xi-x)}\right\}d\xi\,.$$

In [Dan6], we established that the Fourier Transform,

$$\int_{\xi=-\infty}^{\xi=\infty} f(\xi)e^{-i2\pi\nu\xi}d\xi\,,$$

exists for any Hyper-real function $f(x)$. That is, the summation

$$\sum_{\xi=-\infty}^{\xi=\infty} f(\xi)e^{-i2\pi\nu\xi}d\xi$$

exists for any Hyper-real function $f(x)$. Consequently, the summations over intervals exist, and we may write the integral as the sum of integrals over intervals

$$= \tfrac{1}{N}\underbrace{\left(\tfrac{1}{2}\int_{\xi=-1}^{\xi=1} f(\xi)e^{iN\pi\xi}d\xi\right)}_{c_{-N}}e^{-iN\pi x} + ... + \tfrac{N-1}{2N}\underbrace{\left(\tfrac{1}{2}\int_{\xi=-1}^{\xi=1} f(\xi)e^{i\pi\xi}d\xi\right)}_{c_{-1}}e^{-i\pi x} + \underbrace{\left(\tfrac{1}{2}\int_{\xi=-1}^{\xi=1} f(\xi)d\xi\right)}_{c_0} +$$

$$+\frac{N-1}{2N}\underbrace{\left(\frac{1}{2}\int_{\xi=-1}^{\xi=1} f(\xi)e^{-i\pi\xi}d\xi\right)}_{c_1}e^{i\pi x}+...+\frac{1}{2N}\underbrace{\left(\frac{1}{2}\int_{\xi=-1}^{\xi=1} f(\xi)e^{-iN\pi\xi}d\xi\right)}_{c_N}e^{iN\pi x}$$

$$=\frac{1}{N}c_{-N}e^{-iN\pi x}+...+\frac{N-1}{N}c_{-1}e^{-i\pi x}+c_0+\frac{N-1}{N}c_1e^{i\pi x}+...+\frac{1}{N}c_Ne^{iN\pi x}$$

$$=\mathcal{F}_{ej}\mathcal{S}\left\{f(x)\right\}.\square$$

In particular, the Periodic Delta Function violates the Fejer Conditions

- *The Hyper-real* $\delta(x)$, *is not defined in the Calculus of Limits, and* $\left|\delta(x)\right|$ *is not integrable in any bounded interval.*
- $\frac{1}{2}\Big(\delta(x+0)+\delta(x-0)\Big)=0$ *does not replace* $\delta(x)$ *at its discontinuity point,* $x=0$.

But by 9.2, $\delta_{Periodic}(x)$ satisfies the Fejer Summation Theorem.

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