The Fourier Series of a Singular Function of a Complex Variable

H. Vic Dannon
vic0@comcast.net
November, 2014
Revised January, 2015

Abstract To date, the Fourier Series of a function of a complex variable was never defined.

Following Real Analysis methods, the Fourier Transform of a function of a complex variable was integrated along the real line.

But the coefficients of the Fourier Series of a function of complex variable are Fourier Transforms integrated along closed paths in the Complex plane.

Recently, we defined the Fourier Transform along a closed path in the Complex Plane, and we proceed here to define, and obtain the properties of the Fourier Series of a function of a complex variable:

Let the Hyper-Complex, Path Integrable function $f(z)$ be defined on the circle

$$\zeta = z_0 + \rho e^{i\alpha},$$

where

$$\rho = \left| z - z_0 \right|,$$

$$\alpha = \mathrm{Arg}(\zeta - z_0).$$

Fixing ζ, fixes ρ, and

$$f(\zeta) = f(\rho e^{i\alpha}) \equiv \varphi(\alpha).$$

Denote

$$\frac{1}{2\pi} \int_{\alpha=-\pi}^{\alpha=\pi} \varphi(\alpha) e^{-in\alpha} d\alpha \equiv c_n$$

The Complex Fourier Series associated with $f(z)$ is

$$\mathcal{FS}\left\{ f(z) \right\} = .. + c_{-n} e^{i(-n)\theta} + .. + c_{-1} e^{i(-1)\theta} + c_0 + c_1 e^{i(1)\theta} + .. + c_n e^{i(n)\theta} + ..,$$

where

$$\theta = \mathrm{Arg}(z - z_0).$$

Then, we show that on the Hyper-Complex Pierced disk $0 < \left| z - z_0 \right| < r$, the Fourier Series of an Analytic Function is its Laurent Series.

Consequently,

For an Analytic Function on the Hyper-Complex Pierced disk $0 < \left| z - z_0 \right| < r$,
the Fourier Series Theorem $f(z) = \mathcal{FS}\left\{ f(z) \right\}$, *holds*

Keywords: Infinitesimal, Infinite-Hyper-Real, Hyper-Real, Cardinal, Infinity. Non-Archimedean, Non-Standard Analysis, Calculus, Limit, Continuity, Derivative, Integral, Complex Variable, Complex Analysis, Analytic Functions, Holomorphic, Cauchy Integral Theorem, Cauchy Integral Formula, Contour Integral.

2000 Mathematics Subject Classification 26E35; 26E30; 26E15; 26E20; 26A06; 26A12; 03E10; 03E55; 03E17; 03H15; 46S20; 97I40; 97I30.

Contents

Introduction

The Fourier Series of an Analytic function $f(z)$ was never defined. In [Titchmarsh, p.42], the Fourier Transform is defined by integration along the real line. Integration along closed paths in the Complex Domain is never considered.

Since the coefficients of the Fourier Series of an analytic function are Fourier Transforms integrated along a closed path, the Fourier Series of an analytic function could not be defined.

In [Dan9] we defined the Fourier Transform of an Analytic function $f(z)$, and the Fourier Integral of an Analytic Function, along closed paths in the Hyper-Complex plane.

That enables us to define here the Fourier Series associated with a Path-Integrable function $f(z)$, and we show that it is the Laurent Series of $f(z)$ on the pierced disk $0 < \left|z - z_0\right| < r$.

That is, for an Analytic Function on the Hyper-Complex Pierced disk $0 < \left|z - z_0\right| < r$, the Laurent Series, and the Fourier Series coincide:

In [Dan7], we applied Infinitesimal Complex Calculus to derive the Cauchy Integral Formula in an infinitesimal disk, and in

[Dan9] we introduced the Fourier Transform of an analytic function $f(z)$.

We start by recalling the Hyper-real Line, and the Hyper-Complex Plane.

1.

Hyper-real Line

Each real number α can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant Hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of infinitesimal Hyper-reals.

2. The infinitesimals are smaller than any real number, yet strictly greater than zero.

3. Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the infinite Hyper-reals.

4. The infinite Hyper-reals are greater than any real number, yet strictly smaller than infinity.

5. The infinite Hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.

6. The sum of a real number with an infinitesimal is a non-constant Hyper-real.

7. The Hyper-reals are the totality of constant Hyper-reals, a family of infinitesimals, a family of infinitesimals with negative sign, a family of infinite Hyper-reals, a family of infinite Hyper-reals with negative sign, and non-constant Hyper-reals.

8. The Hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.

9. That line includes the real numbers separated by the non-constant Hyper-reals. Each real number is the center of an interval of Hyper-reals, that includes no other real number.

10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.

11. Zero is not an infinitesimal, because zero is not strictly greater than zero.

12. We do not add infinity to the Hyper-real line.

13. The infinitesimals, the infinitesimals with negative signs, the infinite Hyper-reals, and the infinite Hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.

14. The Hyper-real line is embedded in $\mathbb{R}^{\infty}$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the Hyper-real onto the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal Hyper-reals, or to the infinite Hyper-reals, or to the non-constant Hyper-reals.

16. No neighbourhood of a Hyper-real is homeomorphic to an $\mathbb{R}^{n}$ ball. Therefore, the Hyper-real line is not a manifold.

17. The Hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

2.

Hyper-Complex Plane

Each complex number $\alpha + i\beta$ can be represented by a Cauchy sequence of rational complex numbers, $\left\langle r_1 + is_1, r_2 + is_2, r_3 + is_3 ... \right\rangle$ so that $r_n + is_n \to \alpha + i\beta$.

The constant sequence $(\alpha + i\beta, \alpha + i\beta, \alpha + i\beta, ...)$ is a Constant Hyper-Complex Number.

Following [Dan2] we claim that,

1. Any set of sequences $(\iota_1 + io_1, \iota_2 + io_2, \iota_3 + io_3, ...)$, where $(\iota_1, \iota_2, \iota_3, ...)$ belongs to one family of infinitesimal hyper reals, and $(o_1, o_2, o_3, ...)$ belongs to another family of infinitesimal hyper-reals, constitutes a family of infinitesimal hyper-complex numbers.

2. Each hyper-complex infinitesimal has a polar representation $dz = (dr)e^{i\phi} = o_* e^{i\phi}$, where $dr = o_*$ is an infinitesimal, and $\phi = \arg(dz)$.

3. The infinitesimal hyper-complex numbers are smaller in length, than any complex number, yet strictly greater than zero.

4. Their reciprocals $\left(\frac{1}{\iota_1+io_1},\frac{1}{\iota_2+io_2},\frac{1}{\iota_3+io_3},...\right)$ are the infinite hyper-complex numbers.

5. The infinite hyper-complex numbers are greater in length than any complex number, yet strictly smaller than infinity.

6. The sum of a complex number with an infinitesimal hyper-complex is a non-constant hyper-complex.

7. The Hyper-Complex Numbers are the totality of constant hyper-complex numbers, a family of hyper-complex infinitesimals, a family of infinite hyper-complex, and non-constant hyper-complex.

8. The Hyper-Complex Plane is the direct product of a Hyper-Real Line by an imaginary Hyper-Real Line.

9. In Cartesian Coordinates, the Hyper-Real Line serves as an x coordinate line, and the imaginary as an iy coordinate line.

10. In Polar Coordinates, the Hyper-Real Line serves as a Range r line, and the imaginary as an $i\theta$ coordinate. Radial symmetry leads to Polar Coordinates.

11. The Hyper-Complex Plane includes the complex numbers separated by the non-constant hyper-complex numbers. Each complex number is the center of a disk of hyper-complex numbers, that includes no other complex number.

12. In particular, zero is separated from any complex number by a disk of complex infinitesimals.

13. Zero is not a complex infinitesimal, because the length of zero is not strictly greater than zero.

14. We do not add infinity to the hyper-complex plane.

15. The hyper-complex plane is embedded in $\mathbb{C}^{\infty}$, and is not homeomorphic to the Complex Plane $\mathbb{C}$. There is no bi-continuous one-one mapping from the hyper-complex Plane onto the Complex Plane.

16. In particular, there are no points in the Complex Plane that can be assigned uniquely to the hyper-complex infinitesimals, or to the infinite hyper-complex numbers, or to the non-constant hyper-complex numbers.

17. No neighbourhood of a hyper-complex number is homeomorphic to a $\mathbb{C}^{n}$ ball. Therefore, the Hyper-Complex Plane is not a manifold.

18. The Hyper-Complex Plane is not spanned by two elements, and is not two-dimensional.

3.

Hyper-Complex Function

3.1 Definition of a hyper-complex function

$f(z)$ *is a hyper-complex function, iff it is from the hyper-complex numbers into the hyper-complex numbers.*

This means that any number in the domain, or in the range of a hyper-complex $f(x)$ is either one of the following

- complex
- complex + infinitesimal
- infinitesimal
- infinite hyper-complex

3.2 *Every function from complex numbers into complex numbers is a hyper-complex function.*

3.3 $\dfrac{\sin(dz)}{dz}$ *has the constant hyper-complex value* 1

Proof: $\sin(dz) = dz - \frac{(dz)^3}{3!} + \frac{(dz)^5}{5!} - ...$

$$\frac{\sin(dz)}{dz} = 1 - \frac{(dz)^2}{3!} + \frac{(dz)^4}{5!} - ...$$

3.4 $\cos(dz)$ *has the constant hyper-complex value* 1

Proof: $\cos(dz) = 1 - \frac{(dz)^2}{2!} + \frac{(dz)^4}{4!} - ...$

3.5 e^{dz} *has the constant hyper-complex value* 1

Proof: $e^{dz} = 1 + dz + \frac{(dz)^2}{2!} + \frac{(dz)^3}{3!} + \frac{(dz)^4}{4!} + ...$

3.6 $e^{\frac{1}{dz}}$ **is an infinite hyper-complex, and** $\left|e^{\frac{1}{dz}}\right| = e^{\frac{1}{dr}\cos\phi}$.

Proof: $\left|e^{\frac{1}{dz}}\right| = e^{\frac{1}{dr}\mathrm{Re}[e^{-i\phi}]} = e^{\frac{1}{dr}\cos\phi}$.

3.7 $\log(dz)$ **is an infinite hyper-complex, and** $\left|\log(dz)\right| > \frac{1}{dr}$

Proof: $\left|\log(dz)\right| = \sqrt{[\log(dr)]^2 + \phi^2} > \log(dr) > \frac{1}{dr}$

4.

Hyper-Complex Path Integral

Following the definition of the Hyper-real Integral in [Dan3], the Hyper-Complex Integral of $f(z)$ over a path $z(t)$, $t \in [\alpha, \beta]$, in its domain, is the sum of the areas $f(z)z'(t)dt = f(z)dz(t)$ of the rectangles with base $z'(t)dt = dz$, and height $f(z)$.

4.1 Hyper-Complex Path Integral Definition

Let $f(z)$ be hyper-complex function, defined on a domain in the Hyper-Complex Plane. The domain may not be bounded.

$f(z)$ may take infinite hyper-complex values, and need not be bounded.

Let $z(t)$, $t \in [\alpha, \beta]$, be a path, $\gamma(a,b)$, so that $dz = z'(t)dt$, and $z'(t)$ is continuous.

For each t, there is a hyper-complex rectangle with base $[z(t) - \frac{dz}{2}, z(t) + \frac{dz}{2}]$, height $f(z)$, and area $f(z(t))dz(t)$.

We form the **Integration Sum** of all the areas that start at $z(\alpha) = a$, and end at $z(\beta) = b$,

$$\sum_{t \in [\alpha, \beta]} f(z(t)) dz(t).$$

If for any infinitesimal $dz = z'(t)dt$, the Integration Sum equals the same hyper-complex number, then $f(z)$ is Hyper-Complex Integrable over the path $\gamma(a,b)$.

Then, we call the Integration Sum the Hyper-Complex Integral of $f(z)$ over the $\gamma(a,b)$, and denote it by $\int_{\gamma(a,b)} f(z)dz$.

If the hyper-complex number is an infinite hyper-complex, then it equals $\int_{\gamma(a,b)} f(z)dz$.

If the hyper-complex number is finite, then its constant part equals $\int_{\gamma(a,b)} f(z)dz$.□

The Integration Sum may take infinite hyper-complex values, such as $\frac{1}{dz}$, but may not equal to ∞.

The Hyper-Complex Integral of the function $f(z) = \frac{1}{|z|}$ over a path that goes through $z = 0$ diverges.

4.2 The Countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers, $Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty.$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[\alpha, \beta]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(z)dz$.

4.3 Continuous $f(z)$ is Path-Integrable

Hyper-Complex $f(z)$ *Continuous on* D *is Path-Integrable on* D

Proof:

Let $z(t)$, $t \in [\alpha, \beta]$, be a path, $\gamma(a,b)$, so that $dz = z'(t)dt$, and $z'(t)$ is continuous. Then,

$$f(z(t))z'(t) = \big(u(x(t), y(t)) + iv(x(t), y(t))\big)\big(x'(t) + iy'(t)\big)$$

$$= \underbrace{\left[u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t) \right]}_{U(t)} +$$

$$+ i \underbrace{\left[u(x(t), y(t))y'(t) + v(x(t), y(t))x'(t) \right]}_{V(t)}$$

$$= U(t) + iV(t),$$

where $U(t)$, and $V(t)$ are Hyper-Real Continuous on $[\alpha, \beta]$.

Therefore, by [Dan3, 12.4], $U(t)$, and $V(t)$ are integrable on $[\alpha, \beta]$.

Hence, $f(z(t))z'(t)$ is integrable on $[\alpha, \beta]$.

Since

$$\int_{t=\alpha}^{t=\beta} f(z(t))z'(t)dt = \int_{\gamma(a,b)} f(z)dz,$$

$f(z)$ is Path-Integrable on $\gamma(a, b)$.$\square$

5.

Hyper-real Delta Function

In [Dan5], we defined the Hyper-real Delta Function, and established its properties

1. The Delta Function is a Hyper-real function defined from the Hyper-real line into the set of two Hyper-reals $\left\{0, \frac{1}{dx}\right\}$. The Hyper-real 0 is the sequence $\left\langle 0, 0, 0, ... \right\rangle$. The infinite Hyper-real $\frac{1}{dx}$ depends on our choice of dx.

2. We will usually choose the family of infinitesimals that is spanned by the sequences $\left\langle \frac{1}{n} \right\rangle, \left\langle \frac{1}{n^2} \right\rangle, \left\langle \frac{1}{n^3} \right\rangle, ...$ It is a semigroup with respect to vector addition, and includes all the scalar multiples of the generating sequences that are non-zero. That is, the family includes infinitesimals with negative sign. Therefore, $\frac{1}{dx}$ will mean the sequence $\left\langle n \right\rangle$. Alternatively, we may choose the family spanned by the sequences $\left\langle \frac{1}{2^n} \right\rangle, \left\langle \frac{1}{3^n} \right\rangle, \left\langle \frac{1}{4^n} \right\rangle, \ldots$ Then, $\frac{1}{dx}$ will mean the sequence $\left\langle 2^n \right\rangle$. Once we determined the basic infinitesimal dx, we will use it in the Infinite Riemann Sum that defines an Integral in Infinitesimal Calculus.

3. The Delta Function is strictly smaller than ∞

4. We define, $\delta(x) \equiv \frac{1}{dx} \chi_{\left[-\frac{dx}{2}, \frac{dx}{2}\right]}(x)$,

where $\chi_{\left[-\frac{dx}{2}, \frac{dx}{2}\right]}(x) = \begin{cases} 1, x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right] \\ 0, \text{otherwise} \end{cases}$.

5. Hence,

- for $x < 0$, $\delta(x) = 0$
- at $x = -\frac{dx}{2}$, $\delta(x)$ jumps from 0 to $\frac{1}{dx}$,
- for $x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right]$, $\delta(x) = \frac{1}{dx}$.
- at $x = 0$, $\delta(0) = \frac{1}{dx}$
- at $x = \frac{dx}{2}$, $\delta(x)$ drops from $\frac{1}{dx}$ to 0.
- for $x > 0$, $\delta(x) = 0$.
- $x\delta(x) = 0$

6. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle \chi_{[-\frac{1}{2},\frac{1}{2}]}(x), 2\chi_{[-\frac{1}{4},\frac{1}{4}]}(x), 3\chi_{[-\frac{1}{6},\frac{1}{6}]}(x)... \right\rangle$

7. If $dx = \left\langle \frac{2}{n} \right\rangle$, $\delta(x) = \left\langle \frac{1}{2\cosh^2 x}, \frac{2}{2\cosh^2 2x}, \frac{3}{2\cosh^2 3x}, ... \right\rangle$

8. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle e^{-x}\chi_{[0,\infty)}, 2e^{-2x}\chi_{[0,\infty)}, 3e^{-3x}\chi_{[0,\infty)}, ... \right\rangle$

9. $$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1.$$

6.

Hyper-Complex Delta Function $\delta(z)$

In [Dan9], we introduced the Hyper-Complex Delta Function of a Complex Variable $\delta(z)$:

1) The Hyper-Complex Delta Function $\delta(z)$ is defined from the Hyper-Complex plane into the set of two hyper-complex numbers, $\left\{0, \dfrac{1}{2\pi i dz}\right\}$.

The hyper-complex 0 is the sequence $\langle 0, 0, 0, ... \rangle$.

The infinite hyper-complex $\dfrac{1}{2\pi i}\dfrac{1}{dz} = \dfrac{1}{2\pi i}\dfrac{1}{dr}e^{-i\phi}$ depends on

$\operatorname{Arg} z = \phi$. $\dfrac{1}{dr}$ will mean the sequence $\langle n \rangle$.

2) $\delta(z)$ *is an infinite hyper-complex on the infinitesimal hyper-complex disk* $|z| \le dr$. *In particular,* $\boxed{\delta(z) < \infty}$

3) $$\boxed{\delta(z-z_0)=\frac{1}{2\pi i}\frac{1}{dr}e^{-i\,\mathrm{Arg}(z-z_0)}\chi_{\{|z-z_0|\le dr\}}(z)},$$

where $\chi_{\{|z-z_0|\le dr\}}(z)=\begin{cases}0, |z-z_0|>dr\\ 1, |z-z_0|\le dr\end{cases}.$

- **on the disk,** $|z-z_0|\le dr$, $\delta(z-z_0)=\frac{1}{2\pi i}\frac{1}{dz}$.
- **off the disk, for** $|z-z_0|>dr$, $\delta(z-z_0)=0$.

4) $$\boxed{(\delta(z))^n=\frac{1}{(2\pi i)^n}\frac{1}{(dr)^n}e^{-in\phi}\chi_{\{|z|\le dr\}}(z)},\quad n=2,3,\ldots$$

5) $$\boxed{\delta(\zeta-z)=\frac{d}{dz}\frac{1}{2\pi i}\big(\mathrm{Log}(\zeta-z)\big)\chi_{\{|\zeta-z|\le dr\}}(\zeta)}$$

6) $$\boxed{\frac{d}{dz}\delta(\zeta-z)=\frac{1}{2\pi i}\frac{1}{(\zeta-z)^2}\chi_{\{|\zeta-z|\le dr\}}(z)}$$

- **in the disk** $|\zeta-z|\le dr$, $\frac{d}{dz}\delta(\zeta-z)=\frac{1}{2\pi i}\frac{1}{(dr)^2}e^{-2i\theta}$.
- **off the disk, in** $|\zeta-z|>dr$, $\frac{d}{dz}\delta(\zeta-z)=0$.

7) $$\boxed{\frac{d^k}{dz^k}\delta(\zeta - z) = \frac{1}{2\pi i}\frac{k!}{(\zeta - z)^{k+1}}\chi_{\{|z|\le dr\}}(z)}$$

- **in the disk** $\left|\zeta - z\right| \le dr$, $\frac{d^k}{dz^k}\delta(\zeta - z) = \frac{k!}{2\pi i}\frac{1}{(dr)^{k+1}}e^{-i(k+1)\theta}$,

- **off the disk, in** $\left|\zeta - z\right| > dr$, $\frac{d^k}{dz^k}\delta(\zeta - z) = 0$.

8) $$\boxed{\delta(az) = \frac{1}{a}\delta(z)}$$

9) z_1 = **only zero of** $f(z)$, $f'(z_1) \ne 0 \Rightarrow$

$$\Rightarrow \boxed{\delta(f(z)) = \frac{1}{f'(z_1)}\delta(z - z_1)}$$

10) z_1, z_2 **are the only zeros of** $f(z); f'(z_1), f'(z_2) \ne 0 \Rightarrow$

$$\Rightarrow \boxed{\delta(f(z)) = \frac{1}{f'(z_1)}\delta(z - z_1) + \frac{1}{f'(z_2)}\delta(z - z_2)}$$

11) $$\boxed{\delta(z^2 - a^2) = \frac{1}{2a}\delta(z - a) + \frac{1}{2a}\delta(z + a)}$$

12) $$\boxed{\delta\big((z-a)(z-b)\big) = \frac{1}{a-b}\delta(z-a) + \frac{1}{b-a}\delta(z-b)}$$

13) $z_1, ...z_n$ **are the only zeros of** $f(z)\,; f'(z_1), .., f'(z_n) \neq 0 \Rightarrow$

$$\boxed{\delta(f(z)) = \frac{1}{f'(z_1)}\delta(z-z_1) + ... + \frac{1}{f'(z_n)}\delta(z-z_n)}$$

14) $z_1, z_2, ...$ **are zeros of** $f(z)$, $f'(z_1), f'(z_2), ... \neq 0 \Rightarrow$

$$\boxed{\delta(f(z)) = \frac{1}{f'(z_1)}\delta(z-z_1) + \frac{1}{f'(z_n)}\delta(z-z_n) + ...}$$

15)

$$\boxed{\delta(\sin z) = .. + \delta(z+2\pi) - \delta(z+\pi) + +\delta(z) - \delta(z-\pi) + \delta(z-2\pi) + ..}$$

16) $$\boxed{\oint_{|\zeta - z| = dr} \delta(\zeta - z)d\zeta = 1}$$

17) ***If*** $f(z)$ ***is Hyper-Complex Differentiable function at*** z

Then, $$\boxed{\oint_{|\zeta - z| = dr} f(\zeta)\delta(\zeta - z)d\zeta = f(z)}$$

18)

$$\boxed{\frac{d}{dz}f(z) = \oint_{|\zeta - z| = dr} f(\zeta)\frac{d}{dz}\delta(\zeta - z)d\zeta}$$

19)

$$\boxed{\frac{d^k}{dz^k}f(z) = \oint_{|\zeta - z| = dr} f(\zeta)\frac{d^k}{dz^k}\delta(\zeta - z)d\zeta}$$

7.

Cauchy Integral Formula

7.1 Cauchy Integral Formula

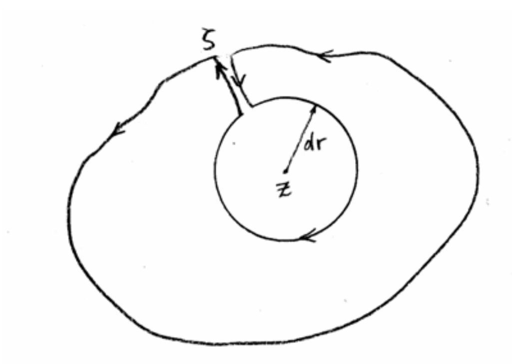
If $f(z)$ is Hyper-Complex Differentiable function on a Hyper-Complex Simply-Connected Domain D.

Then,

$$\boxed{f(z) = \frac{1}{2\pi i}\oint_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta},$$

for any loop γ, and any point z in its interior.

Proof: The Hyper-Complex function $\dfrac{f(\zeta)}{\zeta - z}$ is Differentiable on the Hyper-Complex Simply-Connected domain D, and on a path that includes γ and an infinitesimal circle about z.



Then, the integrals on the lines between γ and the circle have opposite signs and cancel each other.

The integral over the circle has a negative sign because its direction is clockwise, and by Cauchy Integral Theorem,

$$\oint_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta - \oint_{|\zeta - z| = dr} \frac{f(\zeta)}{\zeta - z} d\zeta = 0 .$$

Therefore,

$$\oint_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = \oint_{|\zeta - z| = dr} \frac{f(\zeta)}{\zeta - z} d\zeta$$

$$= 2\pi i \underbrace{\oint_{|\zeta - z| = dr} f(\zeta) \underbrace{\frac{1}{2\pi i} \frac{1}{\zeta - z}}_{\delta(\zeta - z)} d\zeta}_{f(z)} . \square$$

8.

Laurent Series

8.1 Laurent Series of a Singular $f(z)$

If $f(z)$ *is Hyper-Complex Differentiable function on a Hyper-Complex pierced disk* $0 < \left| z - z_0 \right| < r$

Then,

$$f(z) = ... + a_{-3}(z - z_0)^{-3} + a_{-2}(z - z_0)^{-2} + a_{-1}(z - z_0)^{-1} +$$

$$+a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ...$$

where

$$a_{-k} = \frac{1}{2\pi i} \oint_{\gamma} f(\zeta)(\zeta - z_0)^{k-1} d\zeta, \quad k = 1, 2, ... ,$$

$$a_{-3} = \frac{1}{2\pi i} \oint_{\gamma} f(\zeta)(\zeta - z_0)^2 d\zeta$$

$$a_{-2} = \frac{1}{2\pi i} \oint_{\gamma} f(\zeta)(\zeta - z_0) d\zeta,$$

$$a_{-1} = \frac{1}{2\pi i} \oint_{\gamma} f(\zeta) d\zeta,$$

$$a_0 = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\zeta)}{\zeta - z_0} d\zeta,$$

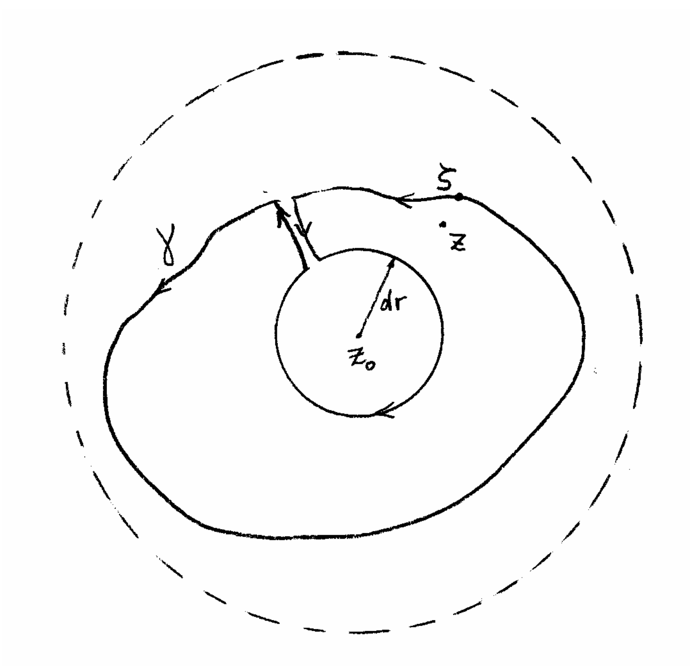
$$a_1 = \frac{1}{2\pi i}\oint_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^2} d\zeta\,,$$

$$a_2 = \frac{1}{2\pi i}\oint_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^3} d\zeta$$

$$a_k = \frac{1}{2\pi i}\oint_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta\,, \quad k = 0,1,2,...$$

for any loop γ, *and for any point* $z \neq z_0$ *in its interior*.

Proof: The Hyper-Complex Differentiable function $f(z)$ satisfies Cauchy Integral Formula in the Hyper-Complex domain D, bounded by a path that includes γ and an infinitesimal circle about z_0



Then, the integrals on the lines between γ and the circle have opposite signs and cancel each other.

The integral over the circle has a negative sign because its direction is clockwise, and by Cauchy Integral Formula,

$$f(z) = \frac{1}{2\pi i}\left(\oint_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta - \oint_{\left|\zeta - z\right| = dr} \frac{f(\zeta)}{\zeta - z} d\zeta \right).$$

For ζ along γ,

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \frac{1}{\zeta - z_0}\left(1 + \frac{z - z_0}{\zeta - z_0} + \left(\frac{z - z_0}{\zeta - z_0}\right)^2 + ...\right).$$

Then,

$$\oint_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = \underbrace{\oint_{\gamma} \frac{f(\zeta)}{\zeta - z_0} d\zeta}_{2\pi i a_0(z_0)} + \underbrace{\oint_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^2} d\zeta}_{2\pi i a_1(z_0)} (z - z_0) + \underbrace{\oint_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^3} d\zeta}_{2\pi i a_2(z_0)} (z - z_0)^2 + ...$$

For ζ along the circle $\left|\zeta - z\right| = dr$,

$$-\frac{1}{\zeta - z} = \frac{1}{z - z_0} \frac{1}{1 - \frac{\zeta - z_0}{z - z_0}} = \frac{1}{z - z_0}\left(1 + \frac{\zeta - z_0}{z - z_0} + \left(\frac{\zeta - z_0}{z - z_0}\right)^2 + ...\right).$$

Then,

$$\oint \frac{-f(\zeta)}{\zeta - z} d\zeta = \underbrace{\oint f(\zeta) d\zeta}_{2\pi i a_{-1}(z_0)} \frac{1}{z - z_0} + \underbrace{\oint \frac{f(\zeta)}{\zeta - z_0} d\zeta}_{2\pi i a_{-2}(z_0)} \frac{1}{(z - z_0)^2} + \underbrace{\oint \frac{f(\zeta)}{(\zeta - z_0)^2} d\zeta}_{2\pi i a_{-3}(z_0)} \frac{1}{(z - z_0)^3} + ...$$

Note that by the Cauchy Integral Theorem the integrals of a_{-1}, a_{-2}, a_{-3},... can be taken along γ.□

9.

Hyper-real Fourier Transform

In [Dan6], we defined the Fourier Transform and established its properties

1. $\mathcal{F}\left\{\delta(x)\right\} = 1$

2. $\delta(x)$ $=$ *the inverse Fourier Transform of the unit function* 1

$$= \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x}d\omega$$

$$= \int_{\nu=-\infty}^{\nu=\infty} e^{2\pi i x}d\nu, \quad \omega = 2\pi\nu$$

3. $\left.\frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x}d\omega\right|_{x=0} = \frac{1}{dx} =$ an infinite Hyper-real

$$\left.\int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x}d\omega\right|_{x\neq 0} = 0$$

4. Fourier Integral Theorem

$$f(x) = \frac{1}{2\pi}\int_{k=-\infty}^{k=\infty}\left(\int_{\xi=-\infty}^{\xi=\infty} f(\xi)e^{-ik\xi}d\xi\right)e^{ikx}dk$$

does not hold in the Calculus of Limits, under any conditions.

5. Fourier Integral Theorem in Infinitesimal Calculus

If $f(x)$ *is a Hyper-real function,*

Then,

- *the Fourier Integral Theorem holds.*
- $\int\limits_{x=-\infty}^{x=\infty} f(x)e^{-i\alpha x}dx$ ***converges to*** $F(\alpha)$
- $\frac{1}{2\pi}\int\limits_{\alpha=-\infty}^{\alpha=\infty} F(\alpha)e^{-i\alpha x}d\alpha$ ***converges to*** $f(x)$

6. 2-Dimesional Fourier Transform

$$\mathcal{F}\left\{f(x,y)\right\} = \int\limits_{y=-\infty}^{y=\infty}\int\limits_{x=-\infty}^{x=\infty} f(x,y)e^{-i\omega_x x - i\omega_y y}dxdy$$

$$= \int\limits_{y=-\infty}^{y=\infty}\int\limits_{x=-\infty}^{x=\infty} f(x,y)e^{-2\pi i(\nu_x x+\nu_y y)}dxdy\,, \quad \begin{aligned}\omega_x &= 2\pi\nu_x \\ \omega_y &= 2\pi\nu_y\end{aligned}$$

7. 2-Dimesional Inverse Fourier Transform

$$\mathcal{F}^{-1}\left\{F(\omega_x,\omega_y)\right\} = \frac{1}{(2\pi)^2}\int\limits_{\omega_y=-\infty}^{\omega_y=\infty}\int\limits_{\omega_x=-\infty}^{\omega_x=\infty} F(\omega_x,\omega_y)e^{i(\omega_x x+\omega_y y)}d\omega_x d\omega_y$$

$$= \int_{\nu_y=-\infty}^{\nu_y=\infty} \int_{\nu_x=-\infty}^{\nu_x=\infty} F(2\pi\nu_x, 2\pi\nu_y) e^{2\pi i(\nu_x x + \nu_y y)} d\nu_x d\nu_y \,, \quad \begin{aligned} \omega_x &= 2\pi\nu_x \\ \omega_y &= 2\pi\nu_y \end{aligned}$$

8. 2-Dimesional Fourier Integral Theorem

$$f(x,y) = \frac{1}{(2\pi)^2} \int_{\omega_y=-\infty}^{\omega_y=\infty} \int_{\omega_x=-\infty}^{\omega_x=\infty} \left(\int_{\eta=-\infty}^{\eta=\infty} \int_{\xi=-\infty}^{\xi=\infty} f(\xi,\eta) e^{-i\omega_x \xi - i\omega_y \eta} d\xi d\eta \right) e^{i(\omega_x x + \omega_y y)} d\omega_x d\omega_y$$

$$= \int_{\eta=-\infty}^{\eta=\infty} \int_{\xi=-\infty}^{\xi=\infty} f(\xi,\eta) \left(\frac{1}{2\pi} \int_{\omega_x=-\infty}^{\omega_x=\infty} e^{i\omega_x(x-\xi)} d\omega_x \right) d\xi \left(\frac{1}{2\pi} \int_{\omega_y=-\infty}^{\omega_y=\infty} e^{i\omega_y(y-\eta)} d\omega_y \right) d\eta$$

$$= \int_{\eta=-\infty}^{\eta=\infty} \int_{\xi=-\infty}^{\xi=\infty} f(\xi,\eta) \left(\int_{\nu_x=-\infty}^{\nu_x=\infty} e^{2\pi i\nu_x(x-\xi)} d\nu_x \right) d\xi \left(\int_{\nu_y=-\infty}^{\nu_y=\infty} e^{2\pi i\nu_y(y-\eta)} d\nu_y \right) d\eta \,, \quad \begin{aligned} \omega_x &= 2\pi\nu_x \\ \omega_y &= 2\pi\nu_y \end{aligned}$$

9. 2-Dimesional Delta Function

$$\delta(x,y) = \left(\frac{1}{2\pi} \int_{\omega_x=-\infty}^{\omega_x=\infty} e^{i\omega_x x} d\omega_x \right) \left(\frac{1}{2\pi} \int_{\omega_y=-\infty}^{\omega_y=\infty} e^{i\omega_y y} d\omega_y \right)$$

$$= \left(\int_{\nu_x=-\infty}^{\nu_x=\infty} e^{2\pi i\nu_x x} d\nu_x \right) \left(\int_{\nu_y=-\infty}^{\nu_y=\infty} e^{2\pi i\nu_y y} d\nu_y \right), \quad \begin{aligned} \omega_x &= 2\pi\nu_x \\ \omega_y &= 2\pi\nu_y \end{aligned}$$

10.

Fourier Transform of $\delta(z)$, and of an Analytic $f(z)$

In [Dan11], we defined the Fourier Transform of the Hyper-Complex Delta Function, and of a Hyper-complex Analytic Function

1) The Fourier Transform of the function $f(\theta) = u(\theta) + iv(\theta)$ of a hyper-real θ is the Integration Sum over the infinitesimal projections of $f(\theta)d\theta$ on $e^{-i\omega\theta}$

$$\sum_{\theta=-\infty}^{\theta=\infty} f(\theta)e^{-i\omega\theta}d\theta .$$

2) In the complex plane, an integration path from $-\infty$, to ∞ is a closed path through ∞.

Therefore, we define the Fourier Transform of a hyper-complex function $f(z)$ along a closed path γ by the Integration Sum

$$\mathcal{F}\{f(z)\} = \sum_{z\in\gamma} f(z)e^{-i\omega z}dz ,$$

where γ may be the unit circle $\zeta = e^{i\phi}$.

3) Fourier Transform of $\delta(z)$

$$\mathcal{F}\{\delta(z)\} = \oint_{|z|=1} \delta(z) e^{-i\omega z} dz,$$

$$= 1.$$

4) Fourier Integral of $\delta(z-\zeta)$

$$\boxed{\delta(z-\zeta) = \frac{1}{2\pi} \oint_{|\omega|=1} e^{i\omega(z-\zeta)} d\omega}$$

5) Fourier Integral Theorem for $f(z)$ on Infinitesimal Circles

If $f(z)$ *is a hyper-complex analytic function,*

Then, the Complex Fourier Integral Theorem holds:

$$\boxed{f(z) = \frac{1}{2\pi} \oint_{|\omega|=\eta} \left(\oint_{|\zeta-z|=\varepsilon} f(\zeta) e^{-i\omega\zeta} d\zeta \right) e^{iz\omega} d\omega},$$

where ε, *and* η *are infinitesimals*

6) Fourier Integral Theorem for $f(z)$ on Unit Circles

If $f(z)$ *is a hyper-complex analytic function, in a hyper-complex domain that includes the unit circle* $|\zeta - z| = 1$

Then, the Complex Fourier Integral Theorem holds.

$$\boxed{f(z) = \frac{1}{2\pi} \oint_{|\omega|=1} \left(\oint_{|\zeta - z|=1} f(\zeta) e^{-i\omega\zeta} d\zeta \right) e^{iz\omega} d\omega}$$

7) ***The Fourier Integral of an Analytic Hyper-Complex*** $f(z)$ ***is the Cauchy Integral Formula for*** $f(z)$

8) **Existence of the Fourier Transform of** $f(z)$

If $f(z)$ *is a hyper-complex analytic function on a hyper-complex domain that includes the circle* $|\zeta - z| = 1$*,*

Then,

1) *the hyper-complex integral*

$$\oint_{|\zeta - z|=1} f(\zeta) e^{-i\omega\zeta} d\zeta$$

converges to $\hat{f}(\omega)$

2) *the hyper-complex integral*

$$\frac{1}{2\pi} \oint_{|\omega|=1} \hat{f}(\omega) e^{iz\omega} d\omega$$

converges to $f(z)$

11.

Fourier Series of Hyper-Complex Path-Integrable Function

In [Dan10], we presented the Fourier Series for a hyper-real $f(x)$ integrable on $[-\pi, \pi]$, so that $f(-\pi) = f(\pi)$: The Fourier Series associated with $f(x)$ is

$$\mathcal{FS}\left\{f(x)\right\} = ... + c_{-n}e^{i(-n)x} + ... + c_{-1}e^{i(-1)x} + c_0 + c_1 e^{i(1)x} + ... + c_n e^{i(n)x} + ...$$

where for each $n = ..., -3, -2, -1, 0, 1, 2, 3, ...$, the integrals

$$\frac{1}{2\pi}\int_{u=-\pi}^{u=\pi} f(u)e^{-inu}du \equiv c_n$$

exist, with finite, or infinite hyper-real values. The c_n are the Fourier Coefficients of $f(x)$.

For each x, $\mathcal{FS}\left\{f(x)\right\}$ may assume finite or infinite hyper-real values.

We show in [Dan10] that the Calculus of limits does not supply relevant conditions to the equality of the Fourier Series the function, while in Infinitesimal Calculus the function does equal its Fourier Series

Let the Hyper-Complex function, $f(z)$ be Path-Integrable, hence, analytic on a Hyper-Complex domain that (without loss of generality) includes the unit circle

$$\zeta = e^{i\alpha}.$$

Then,

$$f(\zeta) = f(e^{i\alpha}) = \varphi(\alpha).$$

The Path Integral

$$\frac{1}{2\pi}\int_{\alpha=-\pi}^{\alpha=\pi} \varphi(\alpha)e^{-in\alpha}d\alpha$$

exists as a Fourier Transform of an analytic function:

In fact, it equals

$$\frac{1}{2\pi i}\oint_{|\zeta|=1} \underbrace{\underbrace{\varphi(\alpha)}_{f(\zeta)}\underbrace{e^{-i\alpha}}_{\zeta^{-1}}}_{g(\zeta)} e^{-i\underbrace{n\alpha e^{-i\alpha}}_{\omega}\underbrace{e^{i\alpha}}_{\zeta}} \underbrace{ie^{i\alpha}d\alpha}_{d\zeta}$$

which is the Fourier Transform of the function

$$g(\zeta) = \frac{f(\zeta)}{\zeta},$$

at

$$\omega = \frac{n\alpha}{\zeta}.$$

Thus, denote

$$c_n \equiv \frac{1}{2\pi} \int_{\alpha=-\pi}^{\alpha=\pi} \varphi(\alpha) e^{-in\alpha} d\alpha \,,$$

That is,

………………………………….

$$c_{-3} = \frac{1}{2\pi} \int_{\alpha=-\pi}^{\alpha=\pi} \varphi(\alpha) e^{3i\alpha} d\alpha$$

$$c_{-2} = \frac{1}{2\pi} \int_{\alpha=-\pi}^{\alpha=\pi} \varphi(\alpha) e^{2i\alpha} d\alpha$$

$$c_{-1} = \frac{1}{2\pi} \int_{\alpha=-\pi}^{\alpha=\pi} \varphi(\alpha) e^{i\alpha} d\alpha$$

$$c_0 = \frac{1}{2\pi} \int_{\alpha=-\pi}^{\alpha=\pi} \varphi(\alpha) d\alpha$$

$$c_1 = \frac{1}{2\pi} \int_{\alpha=-\pi}^{\alpha=\pi} \varphi(\alpha) e^{-i\alpha} d\alpha$$

$$c_2 = \frac{1}{2\pi} \int_{\alpha=-\pi}^{\alpha=\pi} \varphi(\alpha) e^{-2i\alpha} d\alpha$$

$$c_3 = \frac{1}{2\pi} \int_{\alpha=-\pi}^{\alpha=\pi} \varphi(\alpha) e^{-3i\alpha} d\alpha$$

…………………………………….

The Complex Fourier Series associated with $f(z)$ is

$$\mathcal{FS}\left\{f(z)\right\} = .. + c_{-n}e^{i(-n)\theta} + .. + c_{-1}e^{i(-1)\theta} + c_0 + c_1e^{i(1)\theta} + .. + c_ne^{i(n)\theta} + ..,$$

where

$$\theta = \mathrm{Arg}(z - z_0).$$

We proceed to show that this Series satisfies the Fourier Series Theorem for an analytic Function on the Hyper-Complex pierced disk $0 < \left|z - z_0\right| < r$.

12.

Fourier Series of a singular Function

12.1 The Fourier Series of an analytic Function on the Hyper-Complex Pierced disk $0 < \left|z - z_0\right| < r$ is its Laurent Series

<u>*Proof*</u>:

By 8.1, *If* $f(z)$ *is Hyper-Complex Differentiable function on a Hyper-Complex pierced disk* $0 < \left|z - z_0\right| < r$

Then,

$$f(z) = ... + a_{-3}(z - z_0)^{-3} + a_{-2}(z - z_0)^{-2} + a_{-1}(z - z_0)^{-1} +$$

$$+a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ...$$

where

$$a_{-k} = \frac{1}{2\pi i}\oint_{\gamma} f(\zeta)(\zeta - z_0)^{k-1}d\zeta\,, \quad k = 1, 2, ... ,$$

$$a_{-3} = \frac{1}{2\pi i}\oint_{\gamma} f(\zeta)(\zeta - z_0)^{2}d\zeta$$

$$a_{-2} = \frac{1}{2\pi i}\oint_{\gamma} f(\zeta)(\zeta - z_0)d\zeta\,,$$

$$a_{-1} = \frac{1}{2\pi i}\oint_{\gamma} f(\zeta)d\zeta\,,$$

$$a_0 = \frac{1}{2\pi i}\oint_{\gamma} \frac{f(\zeta)}{\zeta - z_0}d\zeta\,,$$

$$a_1 = \frac{1}{2\pi i}\oint_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^2}d\zeta\,,$$

$$a_2 = \frac{1}{2\pi i}\oint_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^3}d\zeta$$

$$a_k = \frac{1}{2\pi i}\oint_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}d\zeta\,, \quad k = 0,1,2,...$$

for any loop γ, *and for any point* $z \neq z_0$ *in its interior.*

Let γ be the circle

$$\zeta = z_0 + \rho e^{i\alpha},$$

where

$$\rho = \left| z - z_0 \right|,$$

$$\alpha = \mathrm{Arg}(\zeta - z_0).$$

Then, fixing z, fixes ρ. Thus,

$$f(\zeta) = f(\rho e^{i\alpha}) = \varphi(\alpha),$$

$$d\zeta = \rho i e^{i\alpha} d\alpha ,$$

and

$$\theta = \operatorname{Arg}(z - z_0).$$

Therefore,

$$a_{-3}(z-z_0)^{-3} = \left(\frac{1}{2\pi i} \oint_{|z-z_0|=\rho} f(\zeta)(\zeta - z_0)^2 d\zeta \right) \frac{1}{(z-z_0)^3},$$

$$= \left(\frac{1}{2\pi i} \oint_{|z-z_0|=\rho} \varphi(\alpha)(\rho e^{i\alpha})^2 i\rho e^{i\alpha} d\alpha \right) \frac{1}{(\rho e^{i\theta})^3}$$

$$= \left(\frac{1}{2\pi} \oint_{|z-z_0|=\rho} \varphi(\alpha) e^{3i\alpha} d\alpha \right) e^{-3i\theta}$$

$$= \underbrace{\left(\frac{1}{2\pi} \int_{\alpha=-\pi}^{\alpha=\pi} \varphi(\alpha) e^{3i\alpha} d\alpha \right)}_{c_{-3}} e^{-3i\theta}$$

$$a_{-2}(z-z_0)^{-2} = \left(\frac{1}{2\pi i} \oint_{|z-z_0|=\rho} f(\zeta)(\zeta - z_0) d\zeta \right) \frac{1}{(z-z_0)^2},$$

$$= \left(\frac{1}{2\pi i} \oint_{|z-z_0|=\rho} \varphi(\alpha)(\rho e^{i\alpha}) i\rho e^{i\alpha} d\alpha \right) \frac{1}{(\rho e^{i\theta})^2}$$

$$= \underbrace{\left(\frac{1}{2\pi} \int_{\alpha=-\pi}^{\alpha=\pi} \varphi(\alpha) e^{2i\alpha} d\alpha \right)}_{c_{-2}} e^{-2i\theta}$$

$$a_{-1}(z - z_0)^{-1} = \left(\frac{1}{2\pi i} \oint_{|z-z_0|=\rho} f(\zeta) d\zeta \right) \frac{1}{z - z_0},$$

$$= \left(\frac{1}{2\pi i} \oint_{|z-z_0|=\rho} \varphi(\alpha) i\rho e^{i\alpha} d\alpha \right) \frac{1}{(\rho e^{i\theta})^2}$$

$$= \underbrace{\left(\frac{1}{2\pi} \int_{\alpha=-\pi}^{\alpha=\pi} \varphi(\alpha) e^{i\alpha} d\alpha \right)}_{c_{-1}} e^{-i\theta}$$

$$a_0 = \frac{1}{2\pi i} \oint_{|z-z_0|=\rho} f(\zeta) \frac{1}{\zeta - z_0} d\zeta,$$

$$= \frac{1}{2\pi i} \oint_{|z-z_0|=\rho} \varphi(\alpha) \frac{1}{\rho e^{i\alpha}} i\rho e^{i\alpha} d\alpha$$

$$= \underbrace{\left(\frac{1}{2\pi} \int_{\alpha=-\pi}^{\alpha=\pi} \varphi(\alpha) d\alpha \right)}_{c_0}$$

$$a_1(z - z_0) = \left(\frac{1}{2\pi i} \oint_{|z-z_0|=\rho} f(\zeta) \frac{1}{(\zeta - z_0)^2} d\zeta \right) (z - z_0)$$

$$= \left(\frac{1}{2\pi i} \oint_{|z-z_0|=\rho} \varphi(\alpha) \frac{1}{(\rho e^{i\alpha})^2} i\rho e^{i\alpha} d\alpha \right) \rho e^{i\theta}$$

$$= \underbrace{\left(\frac{1}{2\pi} \int_{\alpha=-\pi}^{\alpha=\pi} \varphi(\alpha) e^{-i\alpha} d\alpha \right)}_{c_1} e^{i\theta}$$

$$a_2(z-z_0)^2 = \left(\frac{1}{2\pi i} \oint_{|z-z_0|=\rho} f(\zeta) \frac{1}{(\zeta - z_0)^3} d\zeta \right) (z-z_0)^2$$

$$= \left(\frac{1}{2\pi i} \oint_{|z-z_0|=\rho} \varphi(\alpha) \frac{1}{(\rho e^{i\alpha})^3} i\rho e^{i\alpha} d\alpha \right) \rho e^{2i\theta}$$

$$= \underbrace{\left(\frac{1}{2\pi} \int_{\alpha=-\pi}^{\alpha=\pi} \varphi(\alpha) e^{-2i\alpha} d\alpha \right)}_{c_2} e^{2i\theta}. \square$$

Consequently,

12.2 ***For an Analytic Function on the Hyper-Complex Pierced disk*** $0 < |z - z_0| < r$***, the Fourier Series Theorem*** $f(z) = \mathcal{FS}\{f(z)\}$***, holds***

<u>*Proof*</u>: The singular $f(z)$ equals its Laurent Series, which, by 12.1 equals its Fourier Series.$\square$

References

[Bremermann] Hans Bremermann, "*Distributions, Complex Variables, and Fourier Transforms*" Addison-Wesley, 1965

[Dan1] Dannon, H. Vic, "*Well-Ordering of the Reals, Equality of all Infinities, and the Continuum Hypothesis*" in Gauge Institute Journal Vol.6 No 2, May 2010;

[Dan2] Dannon, H. Vic, "*Infinitesimals*" in Gauge Institute Journal Vol.6 No 4, November 2010;

[Dan3] Dannon, H. Vic, "*Infinitesimal Calculus*" in Gauge Institute Journal Vol.7 No 4, November 2011;

[Dan4] Dannon, H. Vic, "*The Delta Function*" in Gauge Institute Journal Vol.8 No 1, February 2012;

[Dan5] Dannon, H. Vic, "*Infinitesimal Vector Calculus*" in Gauge Institute Journal

[Dan6] Dannon, H. Vic, "*Circular and Spherical Delta Functions*" in Gauge Institute Journal

[Dan7] Dannon, H. Vic, "*Infinitesimal Complex Calculus*" in Gauge Institute Journal, Vol. 10 No. 4, November 2014.

[Dan8] H. Vic Dannon, "*Delta Function, the Fourier Transform, and Fourier Integral Theorem*", in Gauge Institute Journal Vol.8 No 2, May 2012.

[Dan9] "*Complex Delta Function*" in Gauge Institute Journal

[Dan10] "*Periodic Delta Function and Dirichlet Summation of Fourier Series*" in Gauge Institute Journal

[Dan11] *"The Fourier Integral of Delta Function of a Complex Variable, and of an Analytic Function"* In Gauge Institute Journal

[Needham] Tristan Needham, "Visual Complex Analysis" Oxford U. Press, 1998 (with corrections)

[Paley, Wiener] Raymond Paley, and Norbert Wiener, "*Fourier Transforms in the Complex Plane*" American Mathematical Society, 1934

[Sneddon] Ian Sneddon, "*Fourier Transforms*", McGraw-Hill, 1959.

[Titchmarsh] E. C. Titchmarsh "*Introduction to the theory of Fourier Integrals*", Third Edition, Chelsea, 1986.