The Fourier Series of a Delta Function

H. Vic Dannon
vic0user@gmail.com
November, 2025

Abstract: For any $x \neq 0$, the infinite series

$$\frac{1}{2} + \cos x + \cos 2x + \cos 3x + \ldots, \quad \text{has no sum.}$$

Therefore, for any $x \neq 0$, the Fourier Series of Delta Function $\delta(x)$,

$$\frac{1}{\pi}\left(\frac{1}{2} + \cos 0 + \cos(2 \cdot 0) + \cos(3 \cdot 0) + \ldots\right) \text{ has no sum.}$$

Since

$$\text{for any } x \neq 0, \ \delta(x) = 0,$$

the Delta Function $\delta(x)$ does not equal its Fourier Series.

KeyWords: Delta Function, Fourier Series, Series with no sum,

The Fourier Series of the Delta Function $\delta(x)$ on the interval $[-\pi, \pi]$ is

$$\frac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \ldots,$$

where

$$a_0 = \frac{1}{\pi}\int_{x=-\pi}^{x=\pi} \delta(x)\cos(0\cdot x)dx = \frac{1}{\pi}\cos(0) = \frac{1}{\pi},$$

$$a_n = \frac{1}{\pi}\int_{x=-\pi}^{x=\pi} \delta(x)\cos(nx)dx = \frac{1}{\pi}\cos(0) = \frac{1}{\pi},$$

$$b_n = \frac{1}{\pi}\int_{x=-\pi}^{x=\pi} \delta(x)\sin(nx)dx = \frac{1}{\pi}\sin(0) = 0.$$

Therefore, the Fourier Series of $\delta(x)$ is

$$\frac{1}{\pi}\left(\frac{1}{2} + \cos x + \cos 2x + \cos 3x + ...\right).$$

For $x = 0$, this Fourier Series equals the Delta Function:

$$\frac{1}{\pi}\left(\frac{1}{2} + \cos 0 + \cos(2\cdot 0) + \cos(3\cdot 0) + ...\right)$$

$$= \frac{1}{\pi}\left(\frac{1}{2} + 1 + 1 + 1 + ...\right)$$

$$= \text{infinite hyper-real}$$

$$= \delta(0).$$

But for $x \neq 0$,

$$\frac{1}{\pi}\cos nx \text{ does not decrease to zero as } n \to \infty.$$

Therefore, <u>the series does not converge.</u>

Neither does it diverge to ∞.

In fact, <u>the series has no sum</u>:

The partial sums of that Fourier Series are

$$\frac{1}{\pi}\left(\frac{1}{2}+\cos x+\cos 2x+...+\cos nx\right)$$

$$=\frac{1}{\pi}\frac{\cos nx-\cos(n+1)x}{2(1-\cos x)}$$

$$=\frac{\sin(n+\frac{1}{2})x}{2\pi\sin(\frac{1}{2}x)}\ ,$$

which does not converge as $n \to \infty$.□

References

Murray Spiegel, "Mathematics Handbook", Schaum's Outline