The Fourier-Bessel Integral of the Polar Delta of a Complex Variable and of an Analytic Function

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Abstract The Fourier-Bessel Transform applies to a function of two variables that has a polar symmetry

$$g(x, y) = f(\rho).$$

The Transform

$$\mathcal{F}_{Bess}\left\{f(\rho)\right\} = 2\pi \int_{\rho=0}^{\rho=\infty} f(\rho) J_0(2\pi\nu\rho)\rho d\rho$$

is the integration sum over $0 \le \rho < \infty$ of the infinitesimal projections $2\pi f(\rho)\rho d\rho$ on the Polarly symmetric Bessel functions

$$J_0(2\pi\nu\rho).$$

The Fourier-Bessel Transform, and Integral would be particularly applicable to functions $f(z)$ of Complex variable $z = \rho e^{i\theta}$

But No attempt was made to define it.

We define here the Fourier-Bessel Transform of the Complex Polar Delta Function, and the Fourier-Bessel Transform of a hyper-complex Analytic function $f(z)$, along a closed path γ_ζ in the complex plane ζ.

Then,

1. the Fourier-Bessel Integral of Hyper-complex Polar Delta is

$$\frac{\delta(z-\zeta)}{\zeta} = (2\pi)^2 \oint_{|\nu|=dr} J_0(2\pi\nu z) J_0(2\pi\nu\zeta)\nu d\nu \,.$$

2. the Cauchy Integral Formula

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma_\zeta} \frac{f(\zeta)}{\zeta - z} d\zeta \,,$$

yields the Fourier-Bessel Integral Theorem

$$f(z) = 2\pi \oint_{|\nu|=1} \left(2\pi \oint_{|\zeta - z|=1} f(\zeta) J_0(2\pi\nu\zeta)\zeta d\zeta \right) J_0(2\pi\nu z)\nu d\nu \,.$$

3. The convergence of the Fourier Integral implies the existence of the Fourier Transform of $f(z)$, and its inverse transform:

If $f(z)$ is a hyper-complex analytic function on a hyper-complex domain that includes the circle $|\zeta - z| = 1$,

Then,

1) *the hyper-complex integral*

$$2\pi \oint_{|\zeta - z|=1} f(\zeta) J_0(2\pi\nu\zeta)\zeta d\zeta$$

converges to

$$F(\nu) \equiv \left(\mathcal{F}_{Bessel}\{f(\zeta)\} \right)(\nu)$$

1) *the hyper-complex integral*

$$2\pi \oint_{|\nu|=1} F(\nu) J_0(2\pi\nu z)\nu d\nu$$

converges to

$$f(z) \equiv \left(\mathcal{F}_{Bessel}^{-1}\{F(\nu)\} \right)(z)$$

Keywords: Infinitesimal, Infinite-Hyper-Real, Hyper-Real, Cardinal, Infinity. Non-Archimedean, Non-Standard Analysis, Calculus, Limit, Continuity, Derivative, Integral, Complex Variable, Complex Analysis, Analytic Functions, Holomorphic, Cauchy Integral Theorem, Cauchy Integral Formula, Contour Integral.

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Contents

Introduction

0.1 The Fourier-Bessel Transform

The Fourier-Bessel Transform applies to a function of two variables that has a polar symmetry

$$g(x,y) = f(\rho).$$

The Transform

$$\mathcal{F}_{Bess}\left\{f(\rho)\right\} = 2\pi \int_{\rho=0}^{\rho=\infty} f(\rho) J_0(2\pi\nu\rho)\rho d\rho$$

is the integration sum over $0 \le \rho < \infty$ of the infinitesimal projections $2\pi f(\rho)\rho d\rho$ on the Polarly symmetric Bessel functions

$$J_0(2\pi\nu\rho).$$

Calculus of Limits Conditions for the Fourier-Bessel Integral Theorem stated in [Watson, p. 458], require the existence of

$$\lim_{\lambda\to\infty} \int_{\rho=0}^{\rho=\infty} f(\rho)\left(\int_{\omega=0}^{\omega=\lambda} J_0(\omega\sigma)J_0(\omega\rho)\omega d\omega\right)\rho d\rho$$

We have shown in [Dan10] that as $\lambda \to \infty$, the Bessel functions integral with respect to $\omega = 2\pi\nu$ is singular. Hence, in the Calculus of Limits, the Transform is not defined.

The Fourier-Bessel Transform would be particularly applicable to functions of Complex variable

$$z = \rho e^{i\theta}$$

But No attempt was made to define it.

The Fourier Transform of $f(z)$, as defined by Titchmarsh, does not suggest a path to follow:

0.2 The Fourier Transform of $f(z)$

The Fourier Transform of a function of Complex variable is defined in [Titchmarsh, p. 44], in

THEOREM 26.

Let *$f(z)$ be an analytic function, regular for*

$$-a < y < b,$$

where $a > 0$, $b > 0$.

In any strip interior to $-a < y < b$, and for any $\varepsilon > 0$, let

$$f(z) = \begin{cases} O(e^{-(\lambda-\varepsilon)x}), & (x \to \infty) \\ O(e^{(\mu-\varepsilon)x}), & (x \to -\infty) \end{cases},$$

where $\lambda > 0$, $\mu > 0$.

Then,

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\zeta=-\infty}^{\zeta=\infty} f(\zeta) e^{i\omega\zeta} d\zeta$$

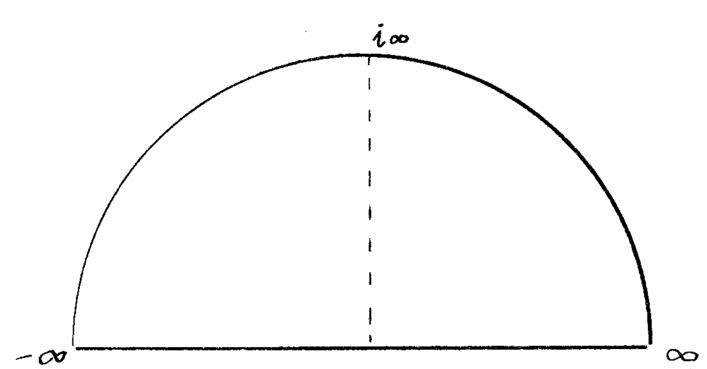
satisfies conditions similar to those imposed on $f(z)$, *with* a, b, λ, μ *replaced by* λ, μ, b, a*;*

and

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{\omega=-\infty}^{\omega=\infty} F(\omega) e^{-i\omega z} d\omega$$

In [Dan11], we noted that the Conditions of Theorem 26 above, limit the integration path to the real line.

Instead of integrating along a closed path such as



which may be homologous to a unit circle, the Conditions require the function to approximately vanish out of a compact interval, and limit the admissible functions that may have a Fourier Transform.

For an Analytic Function these Conditions are unnecessary, because Cauchy Theorem, Cauchy Integral Formula, and the Residue Theorem hold along paths in the complex plane.

The limitation is particularly awkward considering that an Analytic function is inseparable from its disk of convergence, and its derivative may be obtained in any radial direction.

Under the limitations, the Fourier Transform $F(\omega)$ is an analytic function in a strip parallel to the x axis.

At the borders of the strip, the Analytic $F(\omega)$ which is inseparable from its disk of convergence, will have one sided derivative.

Nevertheless, similar conditions that define the Fourier-Bessel Transform of $f(z)$, were not found.

0.3 Definition along a Closed Path

Here, we show that Cauchy Integral Formula yields the Fourier-Bessel Integral of an Analytic Function.

That is, for an Analytic function, the Cauchy Integral Formula, and the Fourier-Bessel Integral, coincide.

The Fourier-Bessel Integral defines the Fourier-Bessel Transform of an Analytic function $f(z)$ along a closed path.

We start by recalling the Hyper-real line, and the Hyper-Complex Plane.

1.

Hyper-real Line

Each real number α can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant Hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of infinitesimal Hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the infinite Hyper-reals.
4. The infinite Hyper-reals are greater than any real number, yet strictly smaller than infinity.
5. The infinite Hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a non-constant Hyper-real.

7. The Hyper-reals are the totality of constant Hyper-reals, a family of infinitesimals, a family of infinitesimals with negative sign, a family of infinite Hyper-reals, a family of infinite Hyper-reals with negative sign, and non-constant Hyper-reals.
8. The Hyper-reals are totally ordered, and aligned along a line: the <u>Hyper-real Line</u>.
9. That line includes the real numbers separated by the non-constant Hyper-reals. Each real number is the center of an interval of Hyper-reals, that includes no other real number.
10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.
11. Zero is not an infinitesimal, because zero is not strictly greater than zero.
12. We do not add infinity to the Hyper-real line.
13. The infinitesimals, the infinitesimals with negative signs, the infinite Hyper-reals, and the infinite Hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.

14. The Hyper-real line is embedded in $\mathbb{R}^{\infty}$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the Hyper-real onto the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal Hyper-reals, or to the infinite Hyper-reals, or to the non-constant Hyper-reals.

16. No neighbourhood of a Hyper-real is homeomorphic to an $\mathbb{R}^{n}$ ball. Therefore, the Hyper-real line is not a manifold.

17. The Hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

2.

Hyper-Complex Plane

Each complex number $\alpha + i\beta$ can be represented by a Cauchy sequence of rational complex numbers, $\left\langle r_1 + is_1, r_2 + is_2, r_3 + is_3 ... \right\rangle$ so that $r_n + is_n \to \alpha + i\beta$.

The constant sequence $(\alpha + i\beta, \alpha + i\beta, \alpha + i\beta, ...)$ is a Constant Hyper-Complex Number.

Following [Dan2] we claim that,

1. Any set of sequences $(\iota_1 + io_1, \iota_2 + io_2, \iota_3 + io_3, ...)$, where $(\iota_1, \iota_2, \iota_3, ...)$ belongs to one family of infinitesimal hyper reals, and $(o_1, o_2, o_3, ...)$ belongs to another family of infinitesimal hyper-reals, constitutes a family of infinitesimal hyper-complex numbers.
2. Each hyper-complex infinitesimal has a polar representation $dz = (dr)e^{i\phi} = o_* e^{i\phi}$, where $dr = o_*$ is an infinitesimal, and $\phi = \arg(dz)$.
3. The infinitesimal hyper-complex numbers are smaller in length, than any complex number, yet strictly greater than

zero.

4. Their reciprocals $\left(\frac{1}{\iota_1 + io_1}, \frac{1}{\iota_2 + io_2}, \frac{1}{\iota_3 + io_3}, ...\right)$ are the infinite hyper-complex numbers.
5. The infinite hyper-complex numbers are greater in length than any complex number, yet strictly smaller than infinity.
6. The sum of a complex number with an infinitesimal hyper-complex is a non-constant hyper-complex.
7. The Hyper-Complex Numbers are the totality of constant hyper-complex numbers, a family of hyper-complex infinitesimals, a family of infinite hyper-complex, and non-constant hyper-complex.
8. The Hyper-Complex Plane is the direct product of a Hyper-Real Line by an imaginary Hyper-Real Line.
9. In Cartesian Coordinates, the Hyper-Real Line serves as an x coordinate line, and the imaginary as an iy coordinate line.
10. In Polar Coordinates, the Hyper-Real Line serves as a Range r line, and the imaginary as an $i\theta$ coordinate. Radial symmetry leads to Polar Coordinates.
11. The Hyper-Complex Plane includes the complex numbers separated by the non-constant hyper-complex

numbers. Each complex number is the center of a disk of hyper-complex numbers, that includes no other complex number.

12. In particular, zero is separated from any complex number by a disk of complex infinitesimals.

13. Zero is not a complex infinitesimal, because the length of zero is not strictly greater than zero.

14. We do not add infinity to the hyper-complex plane.

15. The hyper-complex plane is embedded in $\mathbb{C}^{\infty}$, and is not homeomorphic to the Complex Plane $\mathbb{C}$. There is no bi-continuous one-one mapping from the hyper-complex Plane onto the Complex Plane.

16. In particular, there are no points in the Complex Plane that can be assigned uniquely to the hyper-complex infinitesimals, or to the infinite hyper-complex numbers, or to the non-constant hyper-complex numbers.

17. No neighbourhood of a hyper-complex number is homeomorphic to a $\mathbb{C}^{n}$ ball. Therefore, the Hyper-Complex Plane is not a manifold.

18. The Hyper-Complex Plane is not spanned by two elements, and is not two-dimensional.

3.

Hyper-Complex Function

3.1 Definition of a hyper-complex function

$f(z)$ *is a hyper-complex function, iff it is from the hyper-complex numbers into the hyper-complex numbers.*

This means that any number in the domain, or in the range of a hyper-complex $f(x)$ is either one of the following

- complex
- complex + infinitesimal
- infinitesimal
- infinite hyper-complex

3.2 *Every function from complex numbers into complex numbers is a hyper-complex function.*

3.3 $\frac{\sin(dz)}{dz}$ *has the constant hyper-complex value* 1

Proof: $\sin(dz) = dz - \frac{(dz)^3}{3!} + \frac{(dz)^5}{5!} - ...$

$$\frac{\sin(dz)}{dz} = 1 - \frac{(dz)^2}{3!} + \frac{(dz)^4}{5!} - ...$$

3.4 $\cos(dz)$ ***has the constant hyper-complex value*** 1

Proof: $\cos(dz) = 1 - \frac{(dz)^2}{2!} + \frac{(dz)^4}{4!} - ...$

3.5 e^{dz} ***has the constant hyper-complex value*** 1

Proof: $e^{dz} = 1 + dz + \frac{(dz)^2}{2!} + \frac{(dz)^3}{3!} + \frac{(dz)^4}{4!} + ...$

3.6 $e^{\frac{1}{dz}}$ **is an infinite hyper-complex, and** $\left|e^{\frac{1}{dz}}\right| = e^{\frac{1}{dr}\cos\phi}$.

Proof: $\left|e^{\frac{1}{dz}}\right| = e^{\frac{1}{dr}\mathrm{Re}[e^{-i\phi}]} = e^{\frac{1}{dr}\cos\phi}$.

3.7 $\log(dz)$ **is an infinite hyper-complex, and** $\left|\log(dz)\right| > \frac{1}{dr}$

Proof: $\left|\log(dz)\right| = \sqrt{[\log(dr)]^2 + \phi^2} > \log(dr) > \frac{1}{dr}$

4.

Hyper-Complex Path Integral

Following the definition of the Hyper-real Integral in [Dan3], the Hyper-Complex Integral of $f(z)$ over a path $z(t)$, $t \in [\alpha, \beta]$, in its domain, is the sum of the areas $f(z)z'(t)dt = f(z)dz(t)$ of the rectangles with base $z'(t)dt = dz$, and height $f(z)$.

4.1 Hyper-Complex Path Integral Definition

Let $f(z)$ be hyper-complex function, defined on a domain in the Hyper-Complex Plane. The domain may not be bounded.

$f(z)$ may take infinite hyper-complex values, and need not be bounded.

Let $z(t)$, $t \in [\alpha, \beta]$, be a path, $\gamma(a,b)$, so that $dz = z'(t)dt$, and $z'(t)$ is continuous.

For each t, there is a hyper-complex rectangle with base $[z(t) - \frac{dz}{2}, z(t) + \frac{dz}{2}]$, height $f(z)$, and area $f(z(t))dz(t)$.

We form the **Integration Sum** of all the areas that start at $z(\alpha) = a$, and end at $z(\beta) = b$,

$$\sum_{t\in[\alpha,\beta]} f(z(t))dz(t).$$

If for any infinitesimal $dz = z'(t)dt$, the Integration Sum equals the same hyper-complex number, then $f(z)$ is Hyper-Complex Integrable over the path $\gamma(a,b)$.

Then, we call the Integration Sum the Hyper-Complex Integral of $f(z)$ over the $\gamma(a,b)$, and denote it by $\int\limits_{\gamma(a,b)} f(z)dz$.

If the hyper-complex number is an infinite hyper-complex, then it equals $\int\limits_{\gamma(a,b)} f(z)dz$.

If the hyper-complex number is finite, then its constant part equals $\int\limits_{\gamma(a,b)} f(z)dz$.□

The Integration Sum may take infinite hyper-complex values, such as $\frac{1}{dz}$, but may not equal to ∞.

The Hyper-Complex Integral of the function $f(z) = \frac{1}{|z|}$ over a path that goes through $z = 0$ diverges.

4.2 The Countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:
We proved that the number of the Natural Numbers, $Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty.$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[\alpha, \beta]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(z)dz$.

4.3 Continuous $f(z)$ is Path-Integrable

Hyper-Complex $f(z)$ *Continuous on* D *is Path-Integrable on* D

Proof:

Let $z(t)$, $t \in [\alpha, \beta]$, be a path, $\gamma(a,b)$, so that $dz = z'(t)dt$, and $z'(t)$ is continuous. Then,

$$f(z(t))z'(t) = \left(u(x(t), y(t)) + iv(x(t), y(t))\right)\left(x'(t) + iy'(t)\right)$$

$$= \underbrace{\Big[u(x(t),y(t))x'(t) - v(x(t),y(t))y'(t) \Big]}_{U(t)} +$$

$$+i \underbrace{\Big[u(x(t),y(t))y'(t) + v(x(t),y(t))x'(t) \Big]}_{V(t)}$$

$$= U(t) + iV(t),$$

where $U(t)$, and $V(t)$ are Hyper-Real Continuous on $[\alpha, \beta]$.

Therefore, by [Dan3, 12.4], $U(t)$, and $V(t)$ are integrable on $[\alpha, \beta]$.

Hence, $f(z(t))z'(t)$ is integrable on $[\alpha, \beta]$.

Since

$$\int_{t=\alpha}^{t=\beta} f(z(t))z'(t)dt = \int_{\gamma(a,b)} f(z)dz,$$

$f(z)$ is Path-Integrable on $\gamma(a,b)$.□

5.

Hyper-real Delta Function

In [Dan5], we defined the Hyper-real Delta Function, and established its properties

1. The Delta Function is a Hyper-real function defined from the Hyper-real line into the set of two Hyper-reals $\left\{0, \frac{1}{dx}\right\}$. The Hyper-real 0 is the sequence $\langle 0, 0, 0, ...\rangle$. The infinite Hyper-real $\frac{1}{dx}$ depends on our choice of dx.
2. We will usually choose the family of infinitesimals that is spanned by the sequences $\left\langle \frac{1}{n} \right\rangle, \left\langle \frac{1}{n^2} \right\rangle, \left\langle \frac{1}{n^3} \right\rangle, \ldots$ It is a semigroup with respect to vector addition, and includes all the scalar multiples of the generating sequences that are non-zero. That is, the family includes infinitesimals with negative sign. Therefore, $\frac{1}{dx}$ will mean the sequence $\langle n \rangle$. Alternatively, we may choose the family spanned by the

sequences $\left\langle \frac{1}{2^n} \right\rangle, \left\langle \frac{1}{3^n} \right\rangle, \left\langle \frac{1}{4^n} \right\rangle, \ldots$ Then, $\frac{1}{dx}$ will mean the sequence $\left\langle 2^n \right\rangle$. Once we determined the basic infinitesimal dx, we will use it in the Infinite Riemann Sum that defines an Integral in Infinitesimal Calculus.

3. The Delta Function is strictly smaller than ∞

4. We define, $\delta(x) \equiv \frac{1}{dx} \chi_{\left[-\frac{dx}{2}, \frac{dx}{2}\right]}(x)$,

$$\text{where} \quad \chi_{\left[-\frac{dx}{2}, \frac{dx}{2}\right]}(x) = \begin{cases} 1, x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right] \\ 0, \text{otherwise} \end{cases}.$$

5. Hence,

❖ for $x < 0$, $\delta(x) = 0$

❖ at $x = -\frac{dx}{2}$, $\delta(x)$ jumps from 0 to $\frac{1}{dx}$,

❖ for $x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right]$, $\delta(x) = \frac{1}{dx}$.

❖ at $x = 0$, $\delta(0) = \frac{1}{dx}$

❖ at $x = \frac{dx}{2}$, $\delta(x)$ drops from $\frac{1}{dx}$ to 0.

❖ for $x > 0$, $\delta(x) = 0$.

❖ $x\delta(x) = 0$

6. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle \chi_{[-\frac{1}{2},\frac{1}{2}]}(x), 2\chi_{[-\frac{1}{4},\frac{1}{4}]}(x), 3\chi_{[-\frac{1}{6},\frac{1}{6}]}(x)... \right\rangle$

7. If $dx = \left\langle \frac{2}{n} \right\rangle$, $\delta(x) = \left\langle \frac{1}{2\cosh^2 x}, \frac{2}{2\cosh^2 2x}, \frac{3}{2\cosh^2 3x}, ... \right\rangle$

8. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle e^{-x}\chi_{[0,\infty)}, 2e^{-2x}\chi_{[0,\infty)}, 3e^{-3x}\chi_{[0,\infty)}, ... \right\rangle$

9. $$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1.$$

6.

Hyper-Complex Delta Function $\delta(z)$

In [Dan9], we introduced the Complex Delta Function:

1) The Hyper-Complex Delta Function is defined from the Hyper-Complex plane into the set of two hyper-complex numbers, $\left\{0, \frac{1}{2\pi i dz}\right\}$.

The hyper-complex 0 is the sequence $\langle 0, 0, 0, ... \rangle$.

The infinite hyper-complex $\frac{1}{2\pi i}\frac{1}{dz} = \frac{1}{2\pi i}\frac{1}{dr}e^{-i\phi}$ depends on $\operatorname{Arg} z = \phi$. $\frac{1}{dr}$ will mean the sequence $\langle n \rangle$.

2) *$\delta(z)$ is an infinite hyper-complex on the infinitesimal hyper-complex disk $|z| \le dr$. In particular,* $\boxed{\delta(z) < \infty}$

3) For any infinitesimal dz,

- on the disk, $|z - z_0| \le dr$, $\delta(z - z_0) = \frac{1}{2\pi i}\frac{1}{dz}$.

- off the disk, for $\left|z - z_0\right| > dr$, $\delta(z - z_0) = 0$.

4) $$\boxed{\delta(z - z_0) = \frac{1}{2\pi i}\frac{1}{dr}e^{-i(\phi - \phi_0)}\chi_{\{|z - z_0| \le dr\}}(z)},$$

where $\phi = \arg z$, $\phi_0 = \arg z_0$

$$\chi_{\{|z - z_0| \le dr\}}(z) = \begin{cases} 0, \left|z - z_0\right| > dr \\ 1, \left|z - z_0\right| \le dr \end{cases}.$$

5) $$\boxed{(\delta(z))^n = \frac{1}{(2\pi i)^n}\frac{1}{(dr)^n}e^{-in\phi}\chi_{\{|z| \le dr\}}(z)}, \quad n = 2, 3, \ldots$$

6) $$\boxed{\delta(\zeta - z) = \frac{d}{dz}\frac{1}{2\pi i}\left(\mathrm{Log}(\zeta - z)\right)\chi_{\{|\zeta - z| \le dr\}}(\zeta)}$$

7) $$\boxed{\frac{d}{dz}\delta(\zeta - z) = \frac{1}{2\pi i}\frac{1}{(\zeta - z)^2}\chi_{\{|\zeta - z| \le dr\}}(z)}$$

- in the disk $\left|\zeta - z\right| \le dr$, $\dfrac{d}{dz}\delta(\zeta - z) = \dfrac{1}{2\pi i}\dfrac{1}{(dr)^2}e^{-2i\theta}$.

- off the disk, in $\left|\zeta - z\right| > dr$, $\dfrac{d}{dz}\delta(\zeta - z) = 0$.

8) $$\boxed{\frac{d^k}{dz^k}\delta(\zeta - z) = \frac{1}{2\pi i}\frac{k!}{(\zeta - z)^{k+1}}\chi_{\{|z| \le dr\}}(z)}$$

- in the disk $\left|\zeta - z\right| \le dr$, $\dfrac{d^k}{dz^k}\delta(\zeta - z) = \dfrac{k!}{2\pi i}\dfrac{1}{(dr)^{k+1}}e^{-i(k+1)\theta}$,

- off the disk, in $\left|\zeta - z\right| > dr$, $\frac{d^k}{dz^k}\delta(\zeta - z) = 0$.

9) $\boxed{\delta(az) = \frac{1}{a}\delta(z)}$

10) $z_1 =$ only zero of $f(z)$, $f'(z_1) \neq 0 \Rightarrow$

$$\Rightarrow \boxed{\delta(f(z)) = \frac{1}{f'(z_1)}\delta(z - z_1)}$$

11) z_1, z_2 are the only zeros of $f(z); f'(z_1), f'(z_2) \neq 0 \Rightarrow$

$$\Rightarrow \boxed{\delta(f(z)) = \frac{1}{f'(z_1)}\delta(z - z_1) + \frac{1}{f'(z_2)}\delta(z - z_2)}$$

12) $\boxed{\delta(z^2 - a^2) = \frac{1}{2a}\delta(z - a) + \frac{1}{2a}\delta(z + a)}$

13) $\boxed{\delta\big((z - a)(z - b)\big) = \frac{1}{a - b}\delta(z - a) + \frac{1}{b - a}\delta(z - b)}$

14) $z_1, ...z_n$ are the only zeros of $f(z); f'(z_1), .., f'(z_n) \neq 0 \Rightarrow$

$$\boxed{\delta(f(z)) = \frac{1}{f'(z_1)}\delta(z - z_1) + ... + \frac{1}{f'(z_n)}\delta(z - z_n)}$$

15) $z_1, z_2, \ldots$ **are zeros of** $f(z)$, $f'(z_1), f'(z_2), \ldots \neq 0 \Rightarrow$

$$\boxed{\delta(f(z)) = \frac{1}{f'(z_1)}\delta(z - z_1) + \frac{1}{f'(z_n)}\delta(z - z_n) + \ldots}$$

16)

$$\boxed{\delta(\sin z) = .. + \delta(z + 2\pi) - \delta(z + \pi) + +\delta(z) - \delta(z - \pi) + \delta(z - 2\pi) + ..}$$

17)
$$\boxed{\oint_{|\zeta - z| = dr} \delta(\zeta - z)d\zeta = 1}$$

18) ***If*** $f(z)$ ***is Hyper-Complex Differentiable function at*** z

Then,
$$\boxed{\oint_{|\zeta - z| = dr} f(\zeta)\delta(\zeta - z)d\zeta = f(z)}$$

19)
$$\boxed{\frac{d}{dz}f(z) = \oint_{|\zeta - z| = dr} f(\zeta)\frac{d}{dz}\delta(\zeta - z)d\zeta}$$

20)
$$\boxed{\frac{d^k}{dz^k}f(z) = \oint_{|\zeta - z| = dr} f(\zeta)\frac{d^k}{dz^k}\delta(\zeta - z)d\zeta}$$

7.

Cauchy Integral Formula

7.1 Cauchy Integral Formula

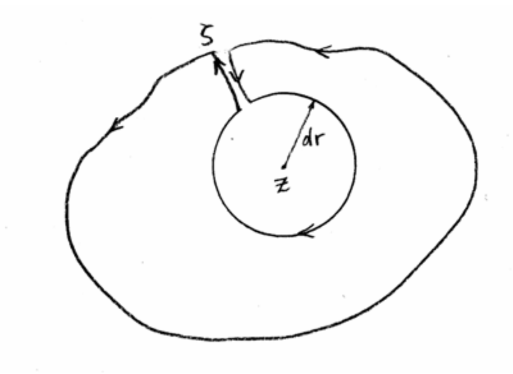
If $f(z)$ *is Hyper-Complex Differentiable function on a Hyper-Complex Simply-Connected Domain* D.

Then,

$$\boxed{f(z) = \frac{1}{2\pi i}\oint_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta},$$

for any loop γ, *and any point* z *in its interior*.

<u>*Proof*</u>: The Hyper-Complex function $\frac{f(\zeta)}{\zeta - z}$ is Differentiable on the Hyper-Complex Simply-Connected domain D, and on a path that includes γ and an infinitesimal circle about z.



Then, the integrals on the lines between γ and the circle have opposite signs and cancel each other.

The integral over the circle has a negative sign because its direction is clockwise, and by Cauchy Integral Theorem,

$$\oint_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta - \oint_{|\zeta - z| = dr} \frac{f(\zeta)}{\zeta - z} d\zeta = 0\,.$$

Therefore,

$$\oint_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = \oint_{|\zeta - z| = dr} \frac{f(\zeta)}{\zeta - z} d\zeta$$

$$= 2\pi i \underbrace{\oint_{|\zeta - z| = dr} f(\zeta) \underbrace{\frac{1}{2\pi i} \frac{1}{\zeta - z}}_{\delta(\zeta - z)} d\zeta}_{f(z)}\,.\square$$

8.

Hyper-real Fourier Transform

In [Dan6], we defined the Fourier Transform and established its properties

1. $\mathcal{F}\left\{\delta(x)\right\} = 1$

2. $\delta(x)$ $=$ *the inverse Fourier Transform of the unit function* 1

$$= \frac{1}{2\pi}\int\limits_{\omega=-\infty}^{\omega=\infty} e^{i\omega x}d\omega$$

$$= \int\limits_{\nu=-\infty}^{\nu=\infty} e^{2\pi i x}d\nu\,, \quad \omega = 2\pi\nu$$

3. $\left.\frac{1}{2\pi}\int\limits_{\omega=-\infty}^{\omega=\infty} e^{i\omega x}d\omega\right|_{x=0} = \frac{1}{dx} =$ an infinite Hyper-real

$$\left.\int\limits_{\omega=-\infty}^{\omega=\infty} e^{i\omega x}d\omega\right|_{x\neq 0} = 0$$

4. Fourier Integral Theorem

$$f(x) = \frac{1}{2\pi}\int\limits_{k=-\infty}^{k=\infty}\left(\int\limits_{\xi=-\infty}^{\xi=\infty} f(\xi)e^{-ik\xi}d\xi\right)e^{ikx}dk$$

does not hold in the Calculus of Limits, under any conditions.

5. Fourier Integral Theorem in Infinitesimal Calculus

If $f(x)$ *is a Hyper-real function,*

Then,

- *the Fourier Integral Theorem holds.*
- $\int_{x=-\infty}^{x=\infty} f(x)e^{-i\alpha x}dx$ ***converges to*** $F(\alpha)$
- $\frac{1}{2\pi}\int_{\alpha=-\infty}^{\alpha=\infty} F(\alpha)e^{-i\alpha x}d\alpha$ ***converges to*** $f(x)$

6. 2-Dimesional Fourier Transform

$$\mathcal{F}\left\{f(x,y)\right\} = \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} f(x,y)e^{-i\omega_x x - i\omega_y y}dxdy$$

$$= \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} f(x,y)e^{-2\pi i(\nu_x x+\nu_y y)}dxdy\,, \quad \begin{aligned}\omega_x &= 2\pi\nu_x\\ \omega_y &= 2\pi\nu_y\end{aligned}$$

7. 2-Dimesional Inverse Fourier Transform

$$\mathcal{F}^{-1}\left\{F(\omega_x,\omega_y)\right\} = \frac{1}{(2\pi)^2}\int_{\omega_y=-\infty}^{\omega_y=\infty}\int_{\omega_x=-\infty}^{\omega_x=\infty} F(\omega_x,\omega_y)e^{i(\omega_x x+\omega_y y)}d\omega_x d\omega_y$$

$$= \int_{\nu_y=-\infty}^{\nu_y=\infty} \int_{\nu_x=-\infty}^{\nu_x=\infty} F(2\pi\nu_x, 2\pi\nu_y) e^{2\pi i(\nu_x x+\nu_y y)} d\nu_x d\nu_y\,, \quad \begin{aligned} \omega_x &= 2\pi\nu_x \\ \omega_y &= 2\pi\nu_y \end{aligned}$$

8. 2-Dimesional Fourier Integral Theorem

$$f(x,y) = \frac{1}{(2\pi)^2} \int_{\omega_y=-\infty}^{\omega_y=\infty} \int_{\omega_x=-\infty}^{\omega_x=\infty} \left(\int_{\eta=-\infty}^{\eta=\infty} \int_{\xi=-\infty}^{\xi=\infty} f(\xi,\eta) e^{-i\omega_x\xi - i\omega_y\eta} d\xi d\eta \right) e^{i(\omega_x x+\omega_y y)} d\omega_x d\omega_y$$

$$= \int_{\eta=-\infty}^{\eta=\infty} \int_{\xi=-\infty}^{\xi=\infty} f(\xi,\eta) \left(\frac{1}{2\pi} \int_{\omega_x=-\infty}^{\omega_x=\infty} e^{i\omega_x(x-\xi)} d\omega_x \right) d\xi \left(\frac{1}{2\pi} \int_{\omega_y=-\infty}^{\omega_y=\infty} e^{i\omega_y(y-\eta)} d\omega_y \right) d\eta$$

$$= \int_{\eta=-\infty}^{\eta=\infty} \int_{\xi=-\infty}^{\xi=\infty} f(\xi,\eta) \left(\int_{\nu_x=-\infty}^{\nu_x=\infty} e^{2\pi i\nu_x(x-\xi)} d\nu_x \right) d\xi \left(\int_{\nu_y=-\infty}^{\nu_y=\infty} e^{2\pi i\nu_y(y-\eta)} d\nu_y \right) d\eta\,, \quad \begin{aligned} \omega_x &= 2\pi\nu_x \\ \omega_y &= 2\pi\nu_y \end{aligned}$$

9. 2-Dimesional Delta Function

$$\delta\left(x,y\right) = \left(\frac{1}{2\pi} \int_{\omega_x=-\infty}^{\omega_x=\infty} e^{i\omega_x x} d\omega_x \right) \left(\frac{1}{2\pi} \int_{\omega_y=-\infty}^{\omega_y=\infty} e^{i\omega_y y} d\omega_y \right)$$

$$= \left(\int_{\nu_x=-\infty}^{\nu_x=\infty} e^{2\pi i\nu_x x} d\nu_x \right) \left(\int_{\nu_y=-\infty}^{\nu_y=\infty} e^{2\pi i\nu_y y} d\nu_y \right), \quad \begin{aligned} \omega_x &= 2\pi\nu_x \\ \omega_y &= 2\pi\nu_y \end{aligned}$$

9.

Hyper-real Polar Delta $\dfrac{\delta(\rho-\sigma)}{\sigma}$

In [Dan10], we defined the Hyper-real Polar Delta Function, and established its properties:

Denoting

$$\delta(\rho-\sigma)=\frac{1}{d\rho}\chi_{[\sigma-\frac{d\rho}{2},\sigma+\frac{d\rho}{2}]}(\rho),\ \ \rho-\sigma\geq 0\,,$$

$$\delta(\phi-\theta)=\frac{1}{d\phi}\chi_{[\theta-\frac{d\phi}{2},\theta+\frac{d\phi}{2}]}(\phi),\ \ 0\leq\phi-\phi_0\leq 2\pi\,,$$

1) Transforming between Polar and Cartesian Coordinates

$$\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \end{aligned},\qquad \begin{aligned} \xi &= \sigma\cos\theta \\ \eta &= \sigma\sin\theta \end{aligned},\qquad \sigma>0\,,$$

$$\delta(x-\xi)\delta(y-\eta)=\frac{1}{\sigma}\delta(\rho-\sigma)\delta(\phi-\theta),$$

2) $$\boxed{2\pi\delta(x-\xi)\delta(y-\eta)=\frac{\delta(\rho-\sigma)}{\sigma}}$$

3)
$$\frac{\delta(\rho-\sigma)}{\sigma} = 2\pi\delta(x-\xi)\delta(y-\eta)$$

is the Polarly Symmetric Delta Function,

4) Transforming between Polar and Cartesian Coordinates

$$\begin{matrix} x = \rho\cos\theta & \xi = \sigma\cos\phi & \omega_x = \omega\cos\gamma \\ y = \rho\sin\theta & \eta = \sigma\sin\phi & \omega_y = \omega\sin\gamma \end{matrix},$$

$$\boxed{\delta(x-\xi)\delta(y-\eta) = (2\pi)^2 \int_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\rho)J_0(2\pi\nu\sigma)\nu d\nu}$$

5)
$$\boxed{\begin{aligned} \frac{\delta\left(\rho-\sigma\right)}{\sigma} &= (2\pi)^3 \int_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\rho)J_0(2\pi\nu\sigma)\nu d\nu \\ &= 2\pi \int_{\omega=0}^{\omega=\infty} J_0(\omega\rho)J_0(\omega\sigma)\omega d\omega \end{aligned}}$$

6) Sifting by $\delta_{\text{Polar}}(\rho-\sigma)$

$$\boxed{\int_{\sigma=0}^{\sigma=\infty} \frac{\delta\left(\rho-\sigma\right)}{\sigma}\sigma d\sigma = 1}$$

7)

$$\boxed{\begin{aligned}\delta(\rho-\sigma) &= 2\pi\sigma\int_{\omega=0}^{\omega=\infty} J_0(\omega\rho)J_0(\omega\sigma)\omega d\omega \\ &= (2\pi)^3\sigma\int_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\rho)J_0(2\pi\nu\sigma)\nu d\nu\end{aligned}}$$

10.

Hyper-real 2-D Fourier-Bessel Transform

In [Dan10], we defined the Hyper-real 2-D Fourier-Bessel Transform, established its properties, and showed that the Fourier-Bessel Integral holds:

1) The 2-D Fourier-Bessel Transform

$$\mathcal{F}_{Bess}\left\{f(\rho)\right\} = 2\pi \int_{\rho=0}^{\rho=\infty} f(\rho)J_0(\rho\omega)\rho d\rho$$

$$= 2\pi \int_{\rho=0}^{\rho=\infty} f(\rho)J_0(2\pi\nu\rho)\rho d\rho\,, \qquad \omega = 2\pi\nu$$

2) The 2-D Inverse Fourier-Bessel Transform

$$\mathcal{F}_{Bess}^{-1}\left\{F(\omega)\right\} = \frac{1}{2\pi} \int_{\omega=0}^{\omega=\infty} F(\omega)J_0(\rho\omega)\omega d\omega\,, \qquad \omega = \sqrt{\omega_x^2 + \omega_y^2}$$

$$= 2\pi \int_{\nu=0}^{\nu=\infty} F(2\pi\nu)J_0(2\pi\nu\rho)\nu d\nu\,, \quad \omega = 2\pi\nu$$

3) Fourier-Bessel Integral Theorem

If $f(\rho)$ *is Hyper-real function,*

Then, the Fourier-Bessel Integral Theorem holds.

$$f(\rho) = 2\pi \int_{\nu=0}^{\nu=\infty} \left(2\pi \int_{\sigma=0}^{\sigma=\infty} f(\sigma) J_0(2\pi\nu\sigma)\sigma d\sigma \right) J_0(2\pi\nu\rho)\nu d\nu$$

$$= \int_{\sigma=0}^{\sigma=\infty} f(\sigma) \left((2\pi)^2 \int_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\sigma) J_0(2\pi\nu\rho)\nu d\nu \right) \sigma d\sigma .$$

4) Existence of the Transform, and its Inverse

If $f(\rho)$ *is Hyper-real function,*

Then,

- ❖ $2\pi \int_{\sigma=0}^{\sigma=\infty} f(\sigma) J_0(2\pi\nu\sigma)\sigma d\sigma$ *converges to* $F(2\pi\nu)$
- ❖ $2\pi \int_{\nu=0}^{\nu=\infty} F(2\pi\nu) J_0(2\pi\nu\rho)\nu d\nu$ *converges to* $f(\rho)$

11.

Hyper-complex Bessel Function

$J_0(z)$

for any complex number z in $\mathbb{C}$,

$$z = x + iy = \rho e^{i\phi},$$

The Hyper-Complex Bessel Function $J_0(z)$ is defined in [Abramowitz] as the sum of its Taylor Series,

$$J_0(z) = 1 - \frac{\frac{1}{4}z^2}{(1!)^2} + \frac{(\frac{1}{4}z^2)^2}{(2!)^2} - \frac{(\frac{1}{4}z^2)^3}{(3!)^2} + ...$$

and in [Watson, p.16],

$$= 1 - \frac{z^2}{2^2} + \frac{z^4}{2^2 4^2} - \frac{z^6}{2^2 4^2 6^2} + ...$$

$$= J_0(-z)$$

which is even in z.

The Series converge, and $J_0(z)$ is Analytic for any z in $\mathbb{C}$.

12.

Hyper-Complex Polar Delta Function $\dfrac{\delta(z-\zeta)}{\zeta}$

Denote

$$z = x + iy = \rho e^{i\phi},$$

$$\zeta = \xi + i\eta = \sigma e^{i\theta},$$

12.1 $$\frac{\delta(z-\zeta)}{\zeta} = \frac{1}{i\sigma e^{i\theta}}(2\pi)^2 \int_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\rho)J_0(2\pi\nu\sigma)\nu d\nu,$$

$$= \frac{1}{2\pi i}\frac{\delta(\rho-\sigma)}{\sigma}e^{-i\,\mathrm{Arg}(\zeta)}$$

<u>Proof</u>: $$\frac{\delta(z-\zeta)}{\zeta} = \frac{1}{\sigma e^{i\phi}}\delta(x-\xi+i[y-\eta])$$

$$= \frac{1}{\sigma e^{i\phi}}\delta(x-\xi)\delta(i[y-\eta]).$$

Since $\delta(i[y-\eta]) = \frac{1}{i}\delta(y-\eta)$, $\dfrac{\delta(z-\zeta)}{\zeta}$ is

$$= \frac{1}{i\sigma e^{i\phi}}\delta(x-\xi)\delta(y-\eta)$$

Since by 9.4, $\delta(x-\xi)\delta(y-\eta) = (2\pi)^2 \int\limits_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\rho)J_0(2\pi\nu\sigma)\nu d\nu$,

$$\frac{\delta(z-\zeta)}{\zeta} = \frac{1}{i\sigma e^{i\phi}}(2\pi)^2 \int\limits_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\rho)J_0(2\pi\nu\sigma)\nu d\nu$$

$$= \frac{1}{2\pi i\sigma e^{i\phi}}(2\pi)^3 \int\limits_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\rho)J_0(2\pi\nu\sigma)\nu d\nu$$

Since by 9.7, $\delta(\rho-\sigma) = (2\pi)^3 \int\limits_{\nu=0}^{\nu=\infty} J_0(2\pi\nu\rho)J_0(2\pi\nu\sigma)\nu d\nu$,

$$\frac{\delta(z-\zeta)}{\zeta} = \frac{1}{2\pi i}\frac{\delta(\rho-\sigma)}{\sigma} e^{-i\,\mathrm{Arg}(\zeta)}.\square$$

13.

Fourier-Bessel Transform of $\dfrac{\delta(z-\zeta)}{\zeta}$

13.1 The Fourier-Bessel Transform of hyper-complex $f(\zeta)$

The Fourier-Bessel Transform of the hyper-complex function $f(\rho) = u(\rho) + iv(\rho)$ of a hyper-real ρ is the Integration Sum

$$2\pi \sum_{\rho=0}^{\rho=\infty} f(\rho) J_0(2\pi\nu\rho)\rho d\rho .$$

As ρ varies, the infinitesimal projections of $2\pi f(\rho) d\rho$ on $J_0(2\pi\nu\rho)$, namely, $2\pi f(\rho) J_0(2\pi\nu\rho) d\rho$, sum up to the Fourier-Bessel Transform of $f(\rho)$.

We define the Fourier-Bessel Transform of a hyper-complex function $f(\zeta)$

$$\left(\mathcal{F}_{Bessel}\{f(\zeta)\}\right)(\nu)$$

along a closed path $\gamma(\zeta)$ by the Integration Sum

$$2\pi \sum_{\zeta \in \gamma} f(\zeta) J_0(2\pi\nu\zeta)\zeta d\zeta .$$

13.2 Fourier-Bessel Transform of $\dfrac{\delta(z-\zeta)}{\zeta}$

$$\mathcal{F}_{Bessel}\{\frac{\delta(z-\zeta)}{\zeta}\} = 2\pi \sum_{z \in \gamma} \delta(z-\zeta) J_0(2\pi\nu\zeta) d\zeta \,.$$

Without loss of generality, γ may be the unit circle $\zeta = e^{i\phi}$.

Therefore,

13.3
$$\boxed{\mathcal{F}_{Bessel}\{\frac{\delta(z-\zeta)}{\zeta}\} = 2\pi J_0(2\pi\nu z)}$$

Proof:
$$\mathcal{F}_{Bessel}\{\frac{\delta(z-\zeta)}{\zeta}\} = 2\pi \oint_{|\zeta|=1} \frac{\delta(z-\zeta)}{\zeta} J_0(2\pi\nu\zeta)\zeta d\zeta$$

$$= 2\pi \oint_{|\zeta|=1} \delta(z-\zeta) J_0(2\pi\nu\zeta) d\zeta$$

By Cauchy Theorem,

$$= 2\pi \frac{1}{2\pi i} \oint_{|z|=dr} \frac{1}{z-\zeta} J_0(2\pi\nu\zeta) d\zeta$$

By Cauchy Integral Theorem,

$$= 2\pi J_0(2\pi\nu\zeta)\Big|_{\zeta=z}$$

$$= 2\pi J_0(2\pi\nu z) \,. \square$$

14.

Fourier-Bessel Integral of $\dfrac{\delta(z-\zeta)}{\zeta}$

14.1 Inverse Complex Fourier-Bessel Transform of Hyper-Complex $F(\nu)$

We define the Inverse Fourier-Bessel Transform of a hyper-complex function $F(\nu)$

$$\left(\mathcal{F}^{-1}_{Bessel}\{F(\nu)\}\right)(z)$$

along a closed path $\gamma(\nu)$ by the Integration Sum

$$2\pi\sum_{\nu\in\gamma} F(\nu)J_0(2\pi\nu\zeta)\nu d\nu\,.$$

14.2 Fourier-Bessel Integral of $\dfrac{\delta(z-\zeta)}{\zeta}$

$$\boxed{\frac{\delta(z-\zeta)}{\zeta} = (2\pi)^2 \oint_{|\nu|=dr} J_0(2\pi\nu z)J_0(2\pi\nu\zeta)\nu d\nu}$$

<u>*Proof*</u>: Since

$$\mathcal{F}_{Bessel}\{\frac{\delta(z-\zeta)}{\zeta}\} = 2\pi J_0(2\pi\nu z),$$

the inverse Transform of $2\pi J_0(2\pi\nu z)$ is $\dfrac{\delta(z-\zeta)}{\zeta}$.

That is,

$$\mathcal{F}^{-1}_{Bessel}\{2\pi J_0(2\pi\nu z)\} = \frac{\delta(z-\zeta)}{\zeta}.$$

By 14.1, the Inverse Transform along a closed path $\gamma(\nu)$ is the Integration Sum

$$2\pi\sum_{\nu\in\gamma}\{2\pi J_0(2\pi\nu z)\}J_0(2\pi\nu\zeta)\nu d\nu.$$

Therefore,

$$\frac{\delta(z-\zeta)}{\zeta} = 2\pi\sum_{\nu\in\gamma}\{2\pi J_0(2\pi\nu z)\}J_0(2\pi\nu\zeta)\nu d\nu$$

Taking $\gamma(\nu)$ to be the infinitesimal circle $|\nu| = dr$

$$= 2\pi\oint_{|\nu|=dr}\{2\pi J_0(2\pi\nu z)\}J_0(2\pi\nu\zeta)\nu d\nu.\square$$

By Cauchy Integral Theorem, the integration path may be along the unit circle.

Hence,

14.3 Fourier-Bessel representation of $\delta(z-\zeta)$

$$\boxed{\delta(z-\zeta) = (2\pi)^2\zeta\oint_{|\nu|=1} J_0(2\pi\nu z)J_0(2\pi\nu\zeta)\nu d\nu}$$

15.

Fourier-Bessel Integral of an Analytic $f(z)$

15.1 Fourier-Bessel Integral Theorem for hyper-complex Analytic $f(z)$, along infinitesimals paths

If $f(z)$ *is a hyper-complex analytic function,*

Then, the Complex Fourier-Bessel Integral Theorem holds:

$$f(z) = 2\pi \oint_{|\nu|=1} \left(2\pi \oint_{|\zeta - z|=\varepsilon} f(\zeta) J_0(2\pi\nu\zeta)\zeta d\zeta \right) J_0(2\pi\nu z)\nu d\nu \,,$$

where ε *is infinitesimal*

<u>*Proof*</u>:

By the Cauchy Integral Formula, 5.1,

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta \,,$$

By Cauchy Integral Theorem,

$$= \frac{1}{2\pi i} \oint_{|\zeta - z|=dr} \frac{f(\zeta)}{\zeta - z} d\zeta \,,$$

By the definition of $\delta(z)$,

$$= \oint_{|\zeta - z| = \varepsilon} f(\zeta)\delta(z-\zeta)d\zeta\,,$$

Substituting from 14.3, $\delta(z-\zeta) = (2\pi)^2 \zeta \oint_{|\nu|=1} J_0(2\pi\nu z)J_0(2\pi\nu\zeta)\nu d\nu$,

$$f(z) = \oint_{|\zeta - z| = \varepsilon} f(\zeta)\left((2\pi)^2 \zeta \oint_{|\nu|=1} J_0(2\pi\nu z)J_0(2\pi\nu\zeta)\nu d\nu\right)d\zeta\,.$$

By changing the Summation order,

$$f(z) = 2\pi \oint_{|\nu|=1}\left(2\pi \oint_{|\zeta - z| = \varepsilon} f(\zeta)J_0(2\pi\nu\zeta)\zeta d\zeta\right)J_0(2\pi\nu z)\nu d\nu\,.\square$$

Similarly, we obtain

15.2 Fourier Integral Theorem for $f(z)$ on Unit Circles

If $f(z)$ is a hyper-complex analytic function, in a hyper-complex domain that contains the unit circle $|\zeta - z| = 1$

Then, the Complex Fourier-Bessel Integral Theorem holds.

$$\boxed{f(z) = 2\pi \oint_{|\nu|=1}\left(2\pi \oint_{|\zeta - z| = 1} f(\zeta)J_0(2\pi\nu\zeta)\zeta d\zeta\right)J_0(2\pi\nu z)\nu d\nu}$$

Proof:

By the Cauchy Theorem, the Fourier-Bessel summation

$$2\pi \oint_{|\zeta - z| = \varepsilon} f(\zeta) J_0(2\pi\nu\zeta)\zeta d\zeta\,,$$

along the infinitesimal circle $|\zeta - z| = \varepsilon$, can be done along the unit circle. Hence it equals

$$2\pi \oint_{|\zeta - z| = 1} f(\zeta) J_0(2\pi\nu\zeta)\zeta d\zeta\,.$$

Therefore,

$$f(z) = 2\pi \oint_{|\nu|=1} \left(2\pi \oint_{|\zeta - z| = \varepsilon} f(\zeta) J_0(2\pi\nu\zeta)\zeta d\zeta \right) J_0(2\pi\nu z)\nu d\nu$$

$$= 2\pi \oint_{|\nu|=1} \left(2\pi \oint_{|\zeta - z| = 1} f(\zeta) J_0(2\pi\nu\zeta)\zeta d\zeta \right) J_0(2\pi\nu z)\nu d\nu\,.\square$$

It follows that due to 14.3, the Fourier-Bessel representation of $\delta(\zeta - z)$,

15.3 ***For an Analytic Hyper-Complex Function*** $f(z)$***, the Fourier-Bessel Integral is the Cauchy Integral Formula for*** $f(z)$

16.

Fourier-Bessel Transform of an Analytic $f(z)$

The convergence of the Fourier Integral of a hyper-complex analytic function $f(z)$, implies the existence of the Fourier Transform of $f(z)$, and its inverse transform

16.1

If $f(z)$ is a hyper-complex analytic function on a hyper-complex domain that includes the circle $|\zeta - z| = 1$,

Then,

2) the hyper-complex integral

$$2\pi \oint_{|\zeta - z|=1} f(\zeta) J_0(2\pi\nu\zeta)\zeta d\zeta$$

converges to

$$F(\nu) \equiv \left(\mathcal{F}_{Bessel}\{f(\zeta)\}\right)(\nu)$$

3) the hyper-complex integral

$$2\pi \oint_{|\nu|=1} F(\nu) J_0(2\pi\nu z)\nu d\nu$$

converges to

$$f(z) \equiv \left(\mathcal{F}_{Bessel}^{-1}\{F(\nu)\}\right)(z)$$

Proof:

The convergence

$$= 2\pi \oint_{|\nu|=1} \left(2\pi \oint_{|\zeta - z|=1} f(\zeta) J_0(2\pi\nu\zeta)\zeta d\zeta \right) J_0(2\pi\nu z)\nu d\nu = f(z)$$

mandates that

1) The Complex Fourier-Bessel Transform of $f(z)$,

$$2\pi \oint_{|\zeta - z|=1} f(\zeta) J_0(2\pi\nu\zeta)\zeta d\zeta,$$

converges to a hyper-complex function $F(\nu)$, some of its values may be infinite hyper-complex, like the complex Delta Function.

2) The Inverse Complex Fourier-Bessel Transform of $F(\nu)$

$$2\pi \oint_{|\nu|=1} F(\nu) J_0(2\pi\nu z)\nu d\nu$$

converges to the hyper-complex function $f(z)$.□

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