Periodic Delta Function, and Dirichlet Summation of Fourier Series

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Abstract The Fourier Series Theorem supplies the conditions under which the Fourier Series associated with a function equals that function.

It is believed to hold in the Calculus of Limits under the Dirichlet Conditions. In fact,

> *The Theorem cannot be proved in the Calculus of Limits under any conditions,*

because the summation of the Fourier Series requires integration of the singular Dirichlet Kernel.

In Infinitesimal Calculus, the Dirichlet Kernel

$$\tfrac{1}{2} + \cos\pi(\xi - x) + \cos 2\pi(\xi - x) + \cos 3\pi(\xi - x) + \ldots =$$

$$= \ldots + \tfrac{1}{2}e^{-i2\pi(\xi - x)} + \tfrac{1}{2}e^{-i\pi(\xi - x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi - x)} + \tfrac{1}{2}e^{i2\pi(\xi - x)} + \ldots$$

is the Periodic Delta Function,

$$\delta_{Periodic}(\xi - x) = \ldots + \delta(\xi - x + 2) + \delta(\xi - x) + \delta(\xi - x - 2) + \ldots.$$

The Dirichlet Kernel equals its Fourier Series ,

$\mathcal{FS}\left\{\delta_{Periodic}(\xi - x)\right\} =$

$$= ... + \tfrac{1}{2}e^{-i2\pi(\xi - x)} + \tfrac{1}{2}e^{-i\pi(\xi - x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi - x)} + \tfrac{1}{2}e^{i2\pi(\xi - x)} + ...$$

And the Fourier Series associated with any periodic hyper-real $f(x)$, equals $f(x)$

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Contents

The Origin of the Fourier Series Theorem

Trigonometric series originated from the solution of the equation of a vibrating string:

A uniform string is fixed at its endpoints

$$x = 0, \text{ and } x = l,$$

and vibrates in the XY plane.

The string amplitude at point x, at time t is

$$y(x,t),$$

where

$$y(0,t) = y(l,t) = 0.$$

We model the string vibrations with the equation

$$\frac{\partial^2 y}{\partial t^2} = \alpha^2 \frac{\partial^2 y}{\partial x^2},$$

where the propagation speed

$$\alpha = \frac{dx}{dt},$$

is assumed to be independent of x, and t.

0.1 Taylor

noted that

$$y(x,t) = \sin(n\frac{\pi}{l}x)\cos(n\frac{\pi}{l}\alpha t)$$

satisfies the equation with its boundary conditions for $n = 1,2,3,...$,

and suggested that

$n = 1$ determines the tone pitch produced by the whole string.

$n = 2$ determines the tone pitch produced by half length string.

$n = 3$ determines the tone pitch produced by third length string.

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0.2 D'Alembert

observed that the change of variables into

$$\xi = x + \alpha t\,, \text{ and } \quad \eta = x - \alpha t\,,$$

transforms the vibrations equation into

$$\frac{\partial^2 y}{\partial\eta\partial\xi} = 0\,,$$

and yields the solution

$$f(x + \alpha t) + \varphi(x - \alpha t)$$

0.3 Daniel Bernoulli

concluded from D'Alembert solution that the equation permits any of Taylor's tones, at the same time. Therefore, the general solution is

$$y(x,t) = A_1 \sin(\frac{\pi}{l}x)\cos(\frac{\pi}{l}\alpha[t - \beta_1]) + A_2 \sin(2\frac{\pi}{l}x)\cos(2\frac{\pi}{l}\alpha[t - \beta_2]) +$$

$$+A_3 \sin(3\frac{\pi}{l}x)\cos(3\frac{\pi}{l}\alpha[t - \beta_3]) + ...$$

0.4 Euler

introduced the trigonometric series into the analysis of the vibrating string.

Euler noted that Bernoulli's is the general solution if and only if

$$y(x,0) = \tfrac{1}{2}b_0 + a_1 \sin(\frac{\pi}{l}x) + b_1 \cos(\frac{\pi}{l}x) + a_2 \sin(2\frac{\pi}{l}x) + b_2 \cos(2\frac{\pi}{l}x) + \ldots,$$

for $0 \le x \le l$.

Euler further showed [Euler-1], that if

$$f(x) = \tfrac{1}{2}b_0 + b_1 \cos(x) + b_2 \cos(2x) + b_3 \cos(3x) + \ldots$$

in $0 \le x \le \pi$, then

$$\int_{x=0}^{x=\pi} f(x)\cos(kx)dx = b_k \int_{x=0}^{x=\pi} \cos^2(kx)dx = \tfrac{1}{2}b_k \int_{x=0}^{x=\pi} [1-\cos(2kx)]dx = \tfrac{\pi}{2}b_k .$$

Therefore,

$$b_k = \frac{2}{\pi}\int_{x=0}^{x=\pi} f(x)\cos(kx)dx .$$

Thus,

the "Fourier-Coefficients" are indeed Euler's Coefficients.

In [Euler2], Euler obtained formulas for the partial sums of the so called "Dirichlet Kernel".

Thus,

the "Dirichlet Kernel" is indeed Euler's Kernel.

0.5 Fourier

Let $f(x)$ be defined on the interval $[-1,1]$, so that $f(1) = f(-1)$.

For each $n = ..., -3, -2, -1, 0, 1, 2, 3, ...$, denote

$$c_n = \frac{1}{2}\int_{u=-1}^{u=1} f(u)e^{-in\pi u}du.$$

The Fourier Series associated with $f(x)$ is

$$... + c_{-n}e^{i(-n)\pi x} + ... + c_{-1}e^{i(-1)\pi x} + c_0 + c_1 e^{i(1)\pi x} + ... + c_n e^{i(n)\pi x} + ...$$

The equality of the associated series to the function is the Fourier Series Theorem, and the question is under which conditions does the Theorem hold.

Fourier claimed that any function equals its associated series.

0.6 Euler

obtained in [Euler3] the Trigonometric Series of an infinitely differentiable periodic function with period 1

Expanding the infinitely differentiable function

$$y = f(x)$$

in a Taylor Series about x,

$$f(x+1) = f(x) + f'(x) + \tfrac{1}{2!}f''(x) + \tfrac{1}{3!}f'''(x) + ...$$

Since $f(x)$ is periodic with period 1, $f(x+1) = f(x)$, and

$$0 = y' + \tfrac{1}{2!}y'' + \tfrac{1}{3!}y''' + ...$$

Substituting

$$y = A_m e^{mx},$$

where A_m is a constant that depends on m, we obtain

$$0 = m + \tfrac{1}{2!}m^2 + \tfrac{1}{3!}m^3 + ... = e^m - 1.$$

That is,

$$\left(1 + \frac{m}{n}\right)^n - 1^n \to 0, \text{ as } n \to \infty.$$

Clearly,

$$\left(1 + \frac{m}{n}\right) - 1 = \frac{m}{n},$$

is always a factor of $\left(1 + \frac{m}{n}\right)^n - 1^n$.

Otherwise, for any n, $\left(1 + \frac{m}{n}\right)^n - 1^n$ is the product of factors based

on the complex roots of 1,

$$e^{k\frac{2\pi}{n}}.$$

Namely, factors of the form

$$1 + \frac{m}{n} - e^{k\frac{2\pi}{n}}.$$

Combining the factors with conjugate roots $e^{k\frac{2\pi}{n}}$, and $e^{-k\frac{2\pi}{n}}$

$$\left\{\left(1 + \tfrac{m}{n}\right)^n - 1^n\right\} = \frac{m}{n}\prod_{k=1}^{k<\frac{n}{2}}(1 + \tfrac{m}{n} - e^{k\frac{2\pi}{n}})(1 + \tfrac{m}{n} - e^{-k\frac{2\pi}{n}})$$

Now,

$$(1+\tfrac{m}{n}-e^{k\frac{2\pi}{n}})(1+\tfrac{m}{n}-e^{-k\frac{2\pi}{n}}) = (1+\tfrac{m}{n})^2 - 2(1+\tfrac{m}{n})\cos(k\tfrac{2\pi}{n})+1$$

$$= 2(1+\tfrac{m}{n})(1-\cos(k\tfrac{2\pi}{n})) + \tfrac{m^2}{n^2}$$

$$= 2(1+\tfrac{m}{n})2\sin^2(\tfrac{k\pi}{n}) + \tfrac{m^2}{n^2}$$

$$= 4\sin^2(\tfrac{k\pi}{n})\left\{1+\tfrac{m}{n}+\frac{m^2}{4k^2\pi^2}\left(\frac{\frac{k\pi}{n}}{\sin(\frac{k\pi}{n})}\right)^2\right\}$$

As $n \to \infty$,

$$\frac{\frac{k\pi}{n}}{\sin(\frac{k\pi}{n})} \to 1\,,$$

$$\tfrac{m}{n} \to 0\,,$$

and we have,

$$\to 4\lim_{n\to\infty}\sin^2(\tfrac{k\pi}{n})\left\{\frac{4k^2\pi^2}{m^2}+1\right\}$$

Thus, for each $k = 1,2,3,...$

$$\frac{4k^2\pi^2}{m^2}+1 = 0$$

Therefore,

$$m = 2ik\pi\,, \text{ and } m = -2ik\pi\,.$$

Hence, the general solution is the sum of

$$A_k e^{2ik\pi x} + A_{-k}e^{-2ik\pi x} = \underbrace{i(A_k - A_{-k})}_{a_k}\sin(2k\pi x) + \underbrace{(A_k + A_{-k})}_{b_k}\cos(2k\pi x)$$

That is,

$$y = C + a_1 \sin(2\pi x) + b_1 \cos(2\pi x) + a_2 \sin(4\pi x) + b_2 \cos(4\pi x) + ...$$

At $x = 0$,

$$1 = y(0) = C + b_1 + b_2 + ...$$

$$C = 1 - b_1 - b_2 - ...$$

Hence,

$$y = 1 + a_1 \sin(2\pi x) + b_1[\cos(2\pi x) - 1] + a_2 \sin(4\pi x) + b_2[\cos(4\pi x) - 1] + ...\square$$

The Fourier coefficients $a_1, b_1, a_2, b_2, ...$ can be obtained from the formulas for them.

0.7 Dirichlet

gave the following three conditions for the Fourier Series to equal its function

1. Piecewise Continuity of $f(x)$, and $f'(x)$ in $[c - L, c + L]$, for arbitrary c.
2. $f(x)$ is periodic with period $T = 2L$
3. At a discontinuity point, $f(x)$ is replaced by $\frac{1}{2}\big(f(x+0) + f(x-0)\big)$.

0.8 Riemann

attempted to extend Dirichlet Conditions in his paper "*On the Representation of a Function by a Trigonometric Series*".

In [Dan6] we follow Riemann's derivation of necessary conditions for the equality of a function to its Fourier Series, and disprove his claim that these conditions are sufficient.

Riemann fails to represent the Trigonometric Series as a convolution of the second primitive of the function with the Dirichlet Kernel, because the infinite Series

$$\cos(x-t)+\cos 2(x-t)+\cos 3(x-t)+\ldots$$

diverges to infinity at $x=t$, and cannot be integrated.

Riemann fails to show divergence of the Fourier Coefficients of a function that has infinitely many maxima or minima on any interval.

Riemann's examples of Fourier Series expansions of marginally legitimate functions, suggest that Fourier's claim that any function equals its Fourier Series remains undisputed.

1.

The Divergence of the Dirichlet Kernel in the Calculus of Limits

Dirichlet Conditions reflect the belief that a smooth enough function equals its Fourier Series.

In fact, in the Calculus of Limits, no smoothness of the function guarantees even the convergence of the Fourier Series.

1.1 The Dirichlet Kernel is either singular or zero

In the Calculus of Limits, the Fourier Series is the limit of the sequence of Partial Sums

$$\mathcal{S}_n\left\{f(x)\right\} = c_n e^{in\pi x} + ... + c_1 e^{i\pi x} + c_0 + c_{-1}e^{-i\pi x} + ...c_{-n}e^{-in\pi x}$$

$$= \left(\tfrac{1}{2}\int_{\xi=-1}^{\xi=1} f(\xi)e^{-in\pi\xi}d\xi\right)e^{in\pi x} + ... + \left(\tfrac{1}{2}\int_{\xi=-1}^{\xi=1} f(\xi)d\xi\right) + ... + \left(\tfrac{1}{2}\int_{\xi=-1}^{\xi=1} f(\xi)e^{in\pi\xi}d\xi\right)e^{-in\pi x}$$

$$= \int_{\xi=-1}^{\xi=1} f(\xi)\left\{\tfrac{1}{2}e^{-in\pi(\xi-x)} + ... + \tfrac{1}{2}e^{-i\pi(\xi-x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi-x)} + ... + \tfrac{1}{2}e^{in\pi(\xi-x)}\right\}d\xi\,.$$

As $n \to \infty$, the Dirichlet Sequence

$$D_n(\xi - x) = \tfrac{1}{2}e^{-in\pi(\xi-x)} + ... + \tfrac{1}{2}e^{-i\pi(\xi-x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi-x)} + ... + \tfrac{1}{2}e^{in\pi(\xi-x)}$$

becomes the Dirichlet Kernel, the infinite series

$$... + \tfrac{1}{2}e^{-in\pi(\xi - x)} + ... + \tfrac{1}{2}e^{-i\pi(\xi - x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi - x)} + ... + \tfrac{1}{2}e^{in\pi(\xi - x)} + ...$$

Clearly, the Dirichlet Kernel is singular at any even $\xi - x$.

In particular,

$$x = \xi \Rightarrow e^{in\pi(x-\xi)} = 1,$$

and the Dirichlet Kernel diverges to

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + ... = \infty.$$

Therefore, while the partial sums of the Fourier Series exist, their limit does not. That is, due to the singularity at $x = \xi$, the Fourier Series does not converge in the Calculus of Limits.

Avoiding the singularity at $\xi = x$, by using the Cauchy Principal Value of the integral does not recover the Fourier Series Theorem, because at any uneven $\xi - x$, the Dirichlet Kernel is known to vanish, and the integral is identically zero, for any function $f(x)$.

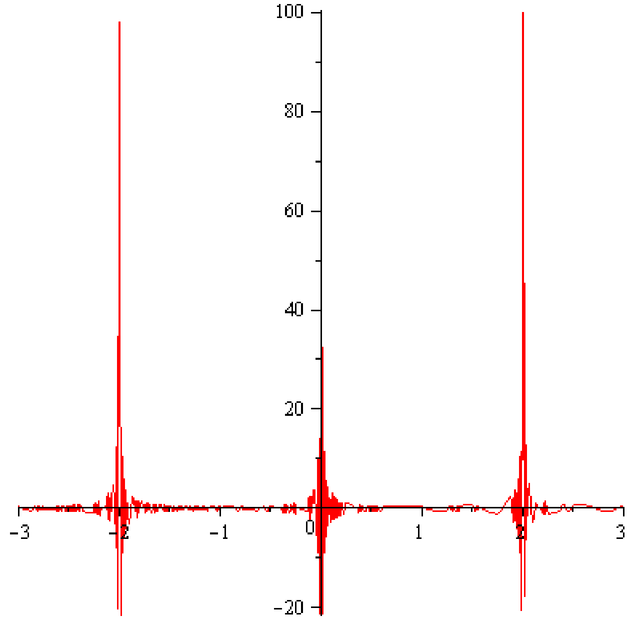
Plots of the Dirichlet Sequence confirm that

In the Calculus of Limits,

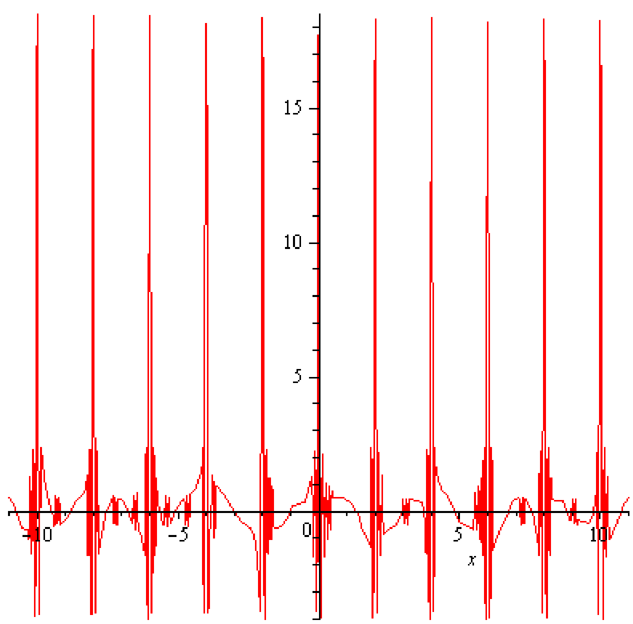
the Dirichlet Kernel is either singular or zero

1.2 Plots of Dirichlet Sequence

$$plot\left(\frac{\sin\left(\pi\frac{201\,x}{2}\right)}{2\ \sin\left(\pi\frac{x}{2}\right)}, x = -3..3\right)$$ plots the spikes at $x = 0$, $x = -2$, $x = 2$.



$$plot\left(\frac{\sin\left(\pi\frac{37\,x}{2}\right)}{2\ \sin\left(\pi\frac{x}{2}\right)},\ x=-11\,..11\right)$$ gives 11 spikes



Thus, the Fourier Series Theorem does not hold in the Calculus of Limits.

1.3 Infinitesimal Calculus Solution

By resolving the problem of the infinitesimals [Dan2], we obtained the Infinite Hyper-reals that are strictly smaller than ∞, and constitute the value of the Delta Function at the singularity.

The controversy surrounding the Leibnitz Infinitesimals derailed the development of the Infinitesimal Calculus, and the Delta Function could not be defined and investigated properly.

In Infinitesimal Calculus, [Dan3], we can differentiate over jump discontinuities, and integrate over singularities.

The Delta Function, the idealization of an impulse in Radar circuits, is a Discontinuous Hyper-Real function which definition requires Infinite Hyper-reals, and which analysis requires Infinitesimal Calculus.

In [Dan5], we show that in infinitesimal Calculus, the hyper-real

$$\delta(x) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x} d\omega$$

is zero for any $x \neq 0$,

it spikes at $x = 0$, so that its Infinitesimal Calculus integral is $\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1$,

and $\delta(0) = \frac{1}{dx} < \infty$.

Here, we show that in Infinitesimal calculus, the Dirichlet Kernel is a periodic hyper-real Delta Function: A periodic train of Delta Functions.

And the Fourier Series $\mathcal{FS}\left\{f(x)\right\}$ associated with a Hyper-real periodic function $f(x)$, equals $f(x)$.

2.

Hyper-real Line

The minimal domain and range, needed for the definition and analysis of a hyper-real function, is the hyper-real line.

Each real number α can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of infinitesimal hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the infinite hyper-reals.
4. The infinite hyper-reals are greater than any real number, yet strictly smaller than infinity.
5. The infinite hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a non-constant hyper-real.

7. The Hyper-reals are the totality of constant hyper-reals, a family of infinitesimals, a family of infinitesimals with negative sign, a family of infinite hyper-reals, a family of infinite hyper-reals with negative sign, and non-constant hyper-reals.
8. The hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.
9. That line includes the real numbers separated by the non-constant hyper-reals. Each real number is the center of an interval of hyper-reals, that includes no other real number.
10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.
11. Zero is not an infinitesimal, because zero is not strictly greater than zero.
12. We do not add infinity to the hyper-real line.
13. The infinitesimals, the infinitesimals with negative signs, the infinite hyper-reals, and the infinite hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.

14. The hyper-real line is embedded in $\mathbb{R}^{\infty}$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the hyper-real onto the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal hyper-reals, or to the infinite hyper-reals, or to the non-constant hyper-reals.

16. No neighbourhood of a hyper-real is homeomorphic to an $\mathbb{R}^{n}$ ball. Therefore, the hyper-real line is not a manifold.

17. The hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

3.

Integral of a Hyper-real Function

In [Dan3], we defined the integral of a Hyper-real Function.

Let $f(x)$ be a hyper-real function on the interval $[a,b]$.

The interval may not be bounded.

$f(x)$ may take infinite hyper-real values, and need not be bounded.

At each

$$a \le x \le b,$$

there is a rectangle with base $[x - \frac{dx}{2}, x + \frac{dx}{2}]$, height $f(x)$, and area

$$f(x)dx.$$

We form the **Integration Sum** of all the areas for the x's that start at $x = a$, and end at $x = b$,

$$\sum_{x \in [a,b]} f(x)dx.$$

If for any infinitesimal dx, the Integration Sum has the same hyper-real value, then $f(x)$ is integrable over the interval $[a,b]$.

Then, we call the Integration Sum the integral of $f(x)$ from $x = a$, to $x = b$, and denote it by

$$\int_{x=a}^{x=b} f(x)dx \,.$$

If the hyper-real is infinite, then it is the integral over $[a,b]$,

If the hyper-real is finite,

$$\int_{x=a}^{x=b} f(x)dx = \text{ real part of the hyper-real}\,.\square$$

3.1 The countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers, $Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty \,.$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[a,b]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(x)dx$.

The Lower Integral is the Integration Sum where $f(x)$ is replaced by its lowest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

3.2
$$\sum_{x \in [a,b]} \left(\inf_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

The Upper Integral is the Integration Sum where $f(x)$ is replaced by its largest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

3.3
$$\sum_{x \in [a,b]} \left(\sup_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

If the integral is a finite hyper-real, we have

3.4 *A hyper-real function has a finite integral if and only if its upper integral and its lower integral are finite, and differ by an infinitesimal.*

4.

Delta Function

In [Dan5], we have defined the Delta Function, and established its properties

1. The Delta Function is a hyper-real function defined from the hyper-real line into the set of two hyper-reals $\left\{0, \frac{1}{dx}\right\}$. The hyper-real 0 is the sequence $\langle 0, 0, 0, ... \rangle$. The infinite hyper-real $\frac{1}{dx}$ depends on our choice of dx.
2. We will usually choose the family of infinitesimals that is spanned by the sequences $\left\langle \frac{1}{n} \right\rangle, \left\langle \frac{1}{n^2} \right\rangle, \left\langle \frac{1}{n^3} \right\rangle, \ldots$ It is a semigroup with respect to vector addition, and includes all the scalar multiples of the generating sequences that are non-zero. That is, the family includes infinitesimals with negative sign. Therefore, $\frac{1}{dx}$ will mean the sequence $\langle n \rangle$. Alternatively, we may choose the family spanned by the sequences $\left\langle \frac{1}{2^n} \right\rangle, \left\langle \frac{1}{3^n} \right\rangle, \left\langle \frac{1}{4^n} \right\rangle, \ldots$ Then, $\frac{1}{dx}$ will mean the

sequence $\langle 2^n \rangle$. Once we determined the basic infinitesimal dx, we will use it in the Infinite Riemann Sum that defines an Integral in Infinitesimal Calculus.

3. The Delta Function is strictly smaller than ∞

4. We define, $\delta(x) \equiv \frac{1}{dx} \chi_{[-\frac{dx}{2}, \frac{dx}{2}]}(x)$,

where $\chi_{[-\frac{dx}{2}, \frac{dx}{2}]}(x) = \begin{cases} 1, x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right] \\ 0, \text{otherwise} \end{cases}$.

5. Hence,

❖ for $x < 0$, $\delta(x) = 0$

❖ at $x = -\frac{dx}{2}$, $\delta(x)$ jumps from 0 to $\frac{1}{dx}$,

❖ for $x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right]$, $\delta(x) = \frac{1}{dx}$.

❖ at $x = 0$, $\delta(0) = \frac{1}{dx}$

❖ at $x = \frac{dx}{2}$, $\delta(x)$ drops from $\frac{1}{dx}$ to 0.

❖ for $x > 0$, $\delta(x) = 0$.

❖ $x\delta(x) = 0$

6. If $dx = \langle \frac{1}{n} \rangle$, $\delta(x) = \left\langle \chi_{[-\frac{1}{2},\frac{1}{2}]}(x), 2\chi_{[-\frac{1}{4},\frac{1}{4}]}(x), 3\chi_{[-\frac{1}{6},\frac{1}{6}]}(x) ... \right\rangle$

7. If $dx = \langle \frac{2}{n} \rangle$, $\delta(x) = \left\langle \frac{1}{2\cosh^2 x}, \frac{2}{2\cosh^2 2x}, \frac{3}{2\cosh^2 3x}, ... \right\rangle$

8. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle e^{-x}\chi_{[0,\infty)}, 2e^{-2x}\chi_{[0,\infty)}, 3e^{-3x}\chi_{[0,\infty)}, \ldots \right\rangle$

9. $$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1.$$

10. $$\delta(\xi - x) = \frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} e^{-ik(\xi - x)} dk$$

5.

Periodic Delta Function $\delta_{Periodic}(\xi - x)$

5.1 Periodic Delta Function Definition

$$\delta_{Periodic}(\xi - x) = ... + \delta(\xi - x + 2) + \delta(\xi - x) + \delta(\xi - x - 2) + ...$$

is a periodic hyper-real Delta function, with period $T = 2$.

5.2 Fourier Transform of $\delta_{Periodic}(x)$

$$\mathcal{F}\left\{\delta_{Periodic}(x)\right\} = ... + e^{-i4\pi\nu} + 1 + e^{i4\pi\nu} + ...$$

Proof: $\mathcal{F}\left\{\delta_{Periodic}(x)\right\} = ... + \mathcal{F}\left\{\delta(x+2)\right\} + \mathcal{F}\left\{\delta(x)\right\} + \mathcal{F}\left\{\delta(x-2)\right\} + ...$

$$= ... + \int_{x=-\infty}^{x=\infty} \delta(x+2)e^{-i2\pi\nu x}dx + \int_{x=-\infty}^{x=\infty} \delta(x)e^{-i2\pi\nu x}dx + \int_{x=-\infty}^{x=\infty} \delta(x-2)e^{-i2\pi\nu x}dx + ...$$

$$= ... + e^{2\pi i 2\nu} + 1 + e^{-2\pi i 2\nu} + ... \square$$

5.3 Fourier Integral Theorem for $\delta_{Periodic}(x)$

$$\mathcal{F}^{-1}\mathcal{F}\left\{\delta_{Periodic}(x)\right\} = \delta_{Periodic}(x)$$

Proof: $\mathcal{F}^{-1}\mathcal{F}\left\{\delta_{Periodic}(x)\right\} = ... + \mathcal{F}^{-1}\left\{e^{2\pi i 2\nu}\right\} + \mathcal{F}^{-1}\left\{1\right\} + \mathcal{F}^{-1}\left\{e^{-2\pi i 2\nu}\right\} + ...$

$$= ... + \int_{\nu=-\infty}^{\nu=\infty} e^{i2\pi\nu 2}e^{i2\pi\nu x}d\nu + \int_{\nu=-\infty}^{\nu=\infty} e^{i2\pi\nu x}d\nu + \int_{\nu=-\infty}^{\nu=\infty} e^{-i2\pi\nu 2}e^{i2\pi\nu x}d\nu + ...$$

$$= ... + \delta(x+2) + \delta(x) + \delta(x-2) + ... \square$$

6.

Convergent Series

In [Dan8], we defined convergence of infinite series in Infinitesimal Calculus

6.1 Sequence Convergence to a finite hyper-real a

$$a_n \to a \;\; \textit{iff} \;\; \langle a_n \rangle - \langle a \rangle = \text{infinitesimal}.$$

6.2 Sequence Convergence to an infinite hyper-real A

$$a_n \to A \;\; \textit{iff} \;\; \langle a_n \rangle \text{ represents the infinite hyper-real } A.$$

6.3 Series Convergence to a finite hyper-real s

$$a_1 + a_2 + ... \to s \;\; \textit{iff} \;\; \langle a_1 + ... + a_n \rangle - \langle s \rangle = \text{infinitesimal}.$$

6.4 Series Convergence to an Infinite Hyper-real S

$$a_1 + a_2 + ... \to S \;\; \textit{iff}$$

$$\langle a_1 + ... + a_n \rangle \text{ represents the infinite hyper-real } S.$$

7.

Dirichlet Sequence and $\delta_{Periodic}(\xi - x)$

7.1 Dirichlet Sequence Definition

The Fourier Series partial sums

$$\mathcal{S}_n\left\{f(x)\right\} = \int_{\xi=-1}^{\xi=1} f(\xi)\underbrace{\left\{\tfrac{1}{2}e^{-in\pi(\xi-x)} + ... + \tfrac{1}{2}e^{-i\pi(\xi-x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi-x)} + ... + \tfrac{1}{2}e^{in\pi(\xi-x)}\right\}}_{\text{Dirichlet Sequence}} d\xi .$$

give rise to the Dirichlet Sequence

$$D_n(\xi - x) = \tfrac{1}{2}e^{-in\pi(\xi-x)} + ... + \tfrac{1}{2}e^{-i\pi(\xi-x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi-x)} + ... + \tfrac{1}{2}e^{in\pi(\xi-x)}$$

$$= \tfrac{1}{2} + \cos\pi(\xi - x) + \cos 2\pi(\xi - x) + ... + \cos n\pi(\xi - x)$$

$$= \frac{\sin(n + \frac{1}{2})\pi(\xi - x)}{2\sin\frac{1}{2}\pi(\xi - x)}, \quad n = 0,1,2,..$$

7.2 Dirichlet Sequence is a Periodic Delta Sequence

Each $D_n(x) = \dfrac{\sin(n + \frac{1}{2})\pi x}{2\sin\frac{1}{2}\pi x}$, $n = 0,1,2,3,...$

1. *has the sifting property on each interval*,

$$..\int_{x=-5}^{x=-3} D_n(x)dx = 1; \quad \int_{x=-3}^{x=-1} D_n(x)dx = 1; \quad \int_{x=-1}^{x=1} D_n(x)dx = 1..$$

2. *is a continuous function*

3. *peaks on each of these intervals to* $\lim\limits_{x \to 2m} D_n(x) = n + \frac{1}{2}$.

Proof of (1)

$$\int_{x=-1}^{x=1} D_n(x)dx = \int_{x=-1}^{x=1} (\tfrac{1}{2} + \cos \pi x + \cos 2\pi x + ... + \cos n\pi x)dx$$

$$= \left(\tfrac{1}{2} x + \tfrac{1}{\pi} \sin \pi x + \tfrac{1}{2\pi} \sin 2\pi x + ... + \tfrac{1}{n\pi} \sin n\pi x \right)_{x=-1}^{x=1}$$

$$= 1. \square$$

Proof of (3)

As $x \to 0$, $\frac{1}{2} + \cos \pi x + \cos 2\pi x + ... + \cos n\pi x \to n + \frac{1}{2}. \square$

7.3 *Dirichlet Sequence Represents* $\delta_{Periodic}(\xi - x)$

$$\delta_{Periodic}(\xi - x) = \left\langle \frac{\sin \frac{2n+1}{2} \pi(\xi - x)}{2 \sin \frac{1}{2} \pi(\xi - x)} \right\rangle$$

$$= \left\langle \tfrac{1}{2} + \cos \pi(\xi - x) + \cos 2\pi(\xi - x) + ... + \cos n\pi(\xi - x) \right\rangle$$

$$= \left\langle \tfrac{1}{2} e^{-in\pi(\xi - x)} + ... + \tfrac{1}{2} e^{-i\pi(\xi - x)} + \tfrac{1}{2} + \tfrac{1}{2} e^{i\pi(\xi - x)} + ... + \tfrac{1}{2} e^{in\pi(\xi - x)} \right\rangle . \square$$

8.

Dirichlet Kernel and $\delta_{periodic}(\xi - x)$

8.1 Dirichlet Kernel in the Calculus of Limits

The Fourier Series partial sums

$$\mathcal{S}_n\left\{f(x)\right\} = \int_{\xi=-1}^{\xi=1} f(\xi)\underbrace{\left\{\tfrac{1}{2}e^{-in\pi(\xi-x)} + ... + \tfrac{1}{2}e^{-i\pi(\xi-x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi-x)} + ... + \tfrac{1}{2}e^{in\pi(\xi-x)}\right\}}_{\text{Dirichlet Sequence}} d\xi\,,$$

give rise to the Dirichlet Sequence.

The limit of the Dirichlet Sequence is the Dirichlet Kernel

$$D_{irichlet}(\xi - x) = ... + \tfrac{1}{2}e^{-i2\pi(\xi-x)} + \tfrac{1}{2}e^{-i\pi(\xi-x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi-x)} + \tfrac{1}{2}e^{i2\pi(\xi-x)} + ...$$

$$= \frac{1}{2} + \cos\pi(x-\xi) + \cos 2\pi(x-\xi) + ...$$

$$= \lim_{n\to\infty}\frac{\sin(n+\frac{1}{2})\pi(x-\xi)}{2\sin\frac{1}{2}\pi(x-\xi)}.$$

8.2 *In the Calculus of Limits, the Dirichlet Kernel does not have the sifting property*

<u>*Proof*</u>: As $\xi \to x$,

$$D_{irichlet}(\xi - x) = \tfrac{1}{2} + \cos\pi(x-\xi) + \cos 2\pi(x-\xi) + ... + \cos n\pi(x-\xi) + ...$$

$$\to \tfrac{1}{2} + +1 + 1 + ... = \infty\,.\square$$

8.3 Hyper-real Dirichlet Kernel in Infinitesimal Calculus

$$D_{irichlet}(\xi - x) = \begin{cases} \left\langle \frac{1}{2} + n \right\rangle, & \xi - x = 2m \\ 0, & \xi - x \neq 2m \end{cases}$$

Proof: at any $\xi - x = 2m$, $m = ... - 3, -2, -1, 0, 1, 2, 3, ...,$

$$D_{irichlet}(\xi - x) = \tfrac{1}{2} + \cos 2m\pi + \cos 2m2\pi + \cos 2m3\pi + ...$$

$$= \left\langle \tfrac{1}{2} + n \right\rangle . \square$$

For $\xi - x \neq 2m$, we follow an Euler type argument [Hardy,p.2].

We have

$$s = 1 + e^{i\pi(\xi - x)} + e^{i2\pi(\xi - x)} + e^{i3\pi(\xi - x)} + ...$$

$$= 1 + e^{i\pi(\xi - x)} \underbrace{\left[1 + e^{i\pi(\xi - x)} + e^{i2\pi(\xi - x)} + ...\right]}_{s}$$

At any $\xi - x \neq 2m$, we have

$$e^{in\pi(\xi - x)} \neq e^{in\pi 2m} = 1 .$$

Thus, $s \neq 1 + 1 + 1 + ...$, and

$$s = \frac{1}{1 - e^{i\pi(\xi - x)}}$$

$$= \frac{e^{-\frac{1}{2}i\pi(\xi - x)}}{e^{-\frac{1}{2}i\pi(\xi - x)} - e^{\frac{1}{2}i\pi(\xi - x)}}$$

$$= \frac{\cos(\frac{1}{2}\pi(\xi - x)) - i\sin(\frac{1}{2}\pi(\xi - x))}{-2i\sin(\frac{1}{2}\pi(\xi - x))}$$

$$= i\tfrac{1}{2}\cot(\tfrac{1}{2}\pi(\xi - x)) + \tfrac{1}{2}$$

Therefore,

$$\mathrm{Re}[s] = \tfrac{1}{2},$$

$$\mathrm{Re}\Big(1 + e^{i\pi(\xi - x)} + e^{i2\pi(\xi - x)} + e^{i3\pi(\xi - x)} + ...\Big) = \tfrac{1}{2},$$

$$\tfrac{1}{2} + \cos\pi(\xi - x) + \cos 2\pi(\xi - x) + \cos 3\pi(\xi - x) + ... = 0 . \square$$

8.4 ***Let*** $N = \frac{1}{dx}$ ***be an infinite Hyper-real. Then***

$$D_{irichlet}(\xi - x) = \frac{\sin(N + \frac{1}{2})\pi(\xi - x)}{2\sin\frac{1}{2}\pi(\xi - x)}$$

$$= \tfrac{1}{2} + \cos\pi(\xi - x) + \cos 2\pi(\xi - x) + ... + \cos N\pi(\xi - x)$$

$$= \tfrac{1}{2}e^{-iN\pi(\xi - x)} + .. + \tfrac{1}{2}e^{-i\pi(\xi - x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi - x)} + .. + \tfrac{1}{2}e^{iN\pi(\xi - x)}$$

$$= \delta(\xi - x + 2N) + ... + \delta(\xi - x + 2) + \delta(\xi - x) + \delta(\xi - x - 2) + ... + \delta(\xi - x - 2N)$$

$$= \delta_{Periodic}(\xi - x)$$

__Proof__:

$$D_{irichlet}(\xi - x) = \tfrac{1}{2} + \cos\pi(\xi - x) + \cos 2\pi(\xi - x) + ... + \cos N\pi(\xi - x)$$

By 8.3,

$$= ... + \begin{cases} N, \xi - x = -2 \\ 0, \xi - x \neq -2 \end{cases} + \begin{cases} N, \xi = x \\ 0, \xi \neq x \end{cases} + \begin{cases} N, \xi - x = 2 \\ 0, \xi - x \neq 2 \end{cases} + ...$$

$$= ... + \begin{cases} \frac{1}{dx}, \xi - x = -2 \\ 0, \xi - x \neq -2 \end{cases} + \begin{cases} \frac{1}{dx}, \xi = x \\ 0, \xi \neq x \end{cases} + \begin{cases} \frac{1}{dx}, \xi - x = 2 \\ 0, \xi - x \neq 2 \end{cases} + ...$$

$$= ... + \delta(\xi - x + 2) + \delta(\xi - x) + \delta(\xi - x - 2) + ...$$

$$= \delta_{Periodic}(\xi - x). \ \square$$

8.5 $... + \delta(\xi - x + 2) + \delta(\xi - x) + \delta(\xi - x - 2) + ... =$

$$= .. + \tfrac{1}{2}e^{-i2\pi(\xi - x)} + \tfrac{1}{2}e^{-i\pi(\xi - x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi - x)} + \tfrac{1}{2}e^{i2\pi(\xi - x)} + ..$$

$$= \tfrac{1}{2} + \cos\pi(\xi - x) + \cos 2\pi(\xi - x) + \cos 3\pi(\xi - x) + ...$$

8.6 $... + \delta(\theta - \phi + 2\pi) + \delta(\theta - \phi) + \delta(\theta - \phi - 2\pi) + ... =$

$$= .. + \tfrac{1}{2}e^{-i2(\theta - \phi)} + \tfrac{1}{2}e^{-i(\theta - \phi)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i(\theta - \phi)} + \tfrac{1}{2}e^{i2(\theta - \phi)} + ..$$

$$= \tfrac{1}{2} + \cos(\theta - \phi) + \cos 2(\theta - \phi) + \cos 3(\theta - \phi) + ...$$

8.7 $D_{irichlet}(x)$ ***is discontinuous at*** $x = 0$

Proof: Since $N = \frac{1}{dx}$, we have

$$D_{irichlet}(x) = \begin{cases} \dfrac{\sin(\frac{1}{dx} + \frac{1}{2})\pi x}{2\sin\frac{1}{2}\pi x}, & x \neq 2m \\ \dfrac{1}{dx}, & x = 2m \end{cases}.$$

Therefore,

$$\sup_{[-\frac{dx}{2}, \frac{dx}{2}]} \delta(t) - \inf_{[-\frac{dx}{2}, \frac{dx}{2}]} \delta(t) = \frac{1}{dx} - \inf_{[-\frac{dx}{2}, \frac{dx}{2}]} \frac{\sin\left(\frac{1}{dx} + \frac{1}{2}\right)\pi t}{2\sin\frac{1}{2}\pi t}$$

The $\sup$, and the $\inf$ apply to the largest and smallest vectors over the infinitesimal interval $[-\frac{dx}{2}, \frac{dx}{2}]$ in the total order defined on the hyper-real line.

$$\frac{\sin\left(\frac{1}{dx}+\frac{1}{2}\right)\pi t}{2\sin\frac{1}{2}\pi t} = \frac{\sin(\frac{1}{dx}\pi t)\cos(\frac{1}{2}\pi t) + \cos(\frac{1}{dx}\pi t)\sin(\frac{1}{2}\pi t)}{2\sin(\frac{1}{2}\pi t)}$$

$$= \tfrac{1}{2}\sin(\tfrac{1}{dx}\pi t)\cot(\tfrac{1}{2}\pi t) + \tfrac{1}{2}\cos(\tfrac{1}{dx}\pi t)$$

Since $\cos(\frac{1}{dx}\pi t)$ is bounded by 1, the last expression depends on

$$\sin(\tfrac{1}{dx}\pi t)\cot(\tfrac{1}{2}\pi t).$$

$\cot(\frac{1}{2}\pi t)$, is decreasing to $-\infty$ in $[-\frac{dx}{2},0]$, and decreasing from ∞ in $[0,\frac{dx}{2}]$, but for $0 < \left|\frac{1}{2}\pi t\right| < \frac{dx}{2}$,

$$\sin(\tfrac{1}{dx}\pi t)\cot(\tfrac{1}{2}\pi t) > 0\,,$$

and is least where the spike is least, at $t = -\frac{dx}{2}$, and at $t = \frac{dx}{2}$.

Taking $t = \frac{dx}{2}$,

$$\inf_{[-\frac{dx}{2},\frac{dx}{2}]}\frac{\sin\left(\frac{1}{dx}+\frac{1}{2}\right)\pi t}{2\sin\frac{1}{2}\pi t} = \tfrac{1}{2}\sin(\tfrac{1}{dx}\pi\tfrac{dx}{2})\cot(\tfrac{1}{2}\pi\tfrac{dx}{2}) + \tfrac{1}{2}\cos(\tfrac{1}{dx}\pi\tfrac{dx}{2})$$

$$= \tfrac{1}{2}\sin(\tfrac{\pi}{2})\cot(\tfrac{\pi}{4}dx) + \tfrac{1}{2}\cos(\tfrac{\pi}{2})$$

Therefore,

$$\sup_{[-\frac{dx}{2},\frac{dx}{2}]}\delta(t) - \inf_{[-\frac{dx}{2},\frac{dx}{2}]}\delta(t) = \frac{1}{dx} - \frac{1}{2}\cot(\tfrac{\pi}{4}dx)$$

$$= \frac{1}{dx}\left(1 - \tfrac{1}{2}\cot(\tfrac{\pi}{4}dx)dx\right)$$

Since $\cot(\frac{\pi dx}{4}) = \frac{4}{\pi dx} - \frac{\pi dx}{12} - \frac{(\pi dx)^3}{180} - ....,$

$$= \frac{1}{dx}\Big(1 - \tfrac{1}{2}\big[\tfrac{4}{\pi dx} - \tfrac{\pi dx}{12} - \tfrac{(\pi dx)^3}{180} - ...\big]dx\Big)$$

$$> \frac{1}{dx}\Big(1 - \tfrac{2}{\pi}\Big)$$

$$= \frac{\pi - 2}{\pi}\frac{1}{dx} \neq \text{infinitesimal}. \square$$

9.

Fourier Series and $\delta_{Periodic}(\xi - x)$

9.1 Fourier Series of a Hyper-real Function

Let $f(x)$ be a hyper-real function integrable on $[c-L, c+L]$,so that $f(c-L) = f(c+L)$

Then, for each $n = ..., -3, -2, -1, 0, 1, 2, 3, ...$, the integrals

$$\frac{1}{2L}\int_{u=c-L}^{u=c+L} f(u)e^{-in\frac{\pi}{L}u}du \equiv c_n$$

exist, with finite, or infinite hyper-real values. The c_n are the Fourier Coefficients of $f(x)$.

The Fourier Series associated with $f(x)$ is

$$\mathcal{FS}\left\{f(x)\right\} = ... + c_{-n}e^{i(-n)\frac{\pi}{L}x} + ... + c_{-1}e^{i(-1)\frac{\pi}{L}x} + c_0 + c_1e^{i(1)\frac{\pi}{L}x} + ... + c_ne^{i(n)\frac{\pi}{L}x} + ...$$

For each x, it may assume finite or infinite hyper-real values.

9.2 $\mathcal{FS}\left\{\delta_{Periodic}(\xi - x)\right\} = \delta_{Periodic}(\xi - x)$

Proof: Take $c = 0$, and $T = 2$.

$\mathcal{FS}\left\{\delta_{Periodic}(\xi - x)\right\} =$

$$= .. + c_{-n}e^{-ni\pi x} + .. + c_{-1}e^{-i\pi x} + c_0 + c_1e^{i\pi x} + .. + c_ne^{ni\pi x} + ..,$$

where

$$c_n = \frac{1}{2}\int_{x=-1}^{x=1} \delta_{Periodic}(\xi - x)e^{-in\pi x}dx .$$

Substituting from 8.4

$$\delta_{Periodic}(\xi - x) = .. + \tfrac{1}{2}e^{-im\pi(\xi-x)} + ... + \tfrac{1}{2}e^{-i\pi(\xi-x)} +$$
$$+ \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi-x)} + .. + \tfrac{1}{2}e^{im\pi(\xi-x)} + ...$$

$$c_n = \frac{1}{2}\int_{x=-1}^{x=1} \left\{ .. + \tfrac{1}{2}e^{-im\pi(\xi-x)} + ... + \tfrac{1}{2} + .. + \tfrac{1}{2}e^{im\pi(\xi-x)} + ... \right\} e^{-in\pi x}dx$$

$$= .. + \tfrac{1}{2}e^{-im\pi\xi} \underbrace{\int_{x=-1}^{x=1} \tfrac{1}{2}e^{im\pi x}e^{-in\pi x}dx}_{\delta_{mn}} + .. + \tfrac{1}{2}e^{im\pi\xi} \underbrace{\int_{x=-1}^{x=1} \tfrac{1}{2}e^{-im\pi x}e^{-in\pi x}dx}_{\delta_{-mn}} + ..$$

$$= \tfrac{1}{2}e^{-in\pi\xi} .$$

Hence, $c_{-n} = \tfrac{1}{2}e^{in\pi\xi}$

Therefore,

$$\mathcal{FS}\left\{\delta_{Periodic}(\xi - x)\right\} =$$

$$= .. + \tfrac{1}{2}e^{in\pi\xi}e^{-ni\pi x} + .. + \tfrac{1}{2}e^{i\pi\xi}e^{-i\pi x} + \tfrac{1}{2} +$$
$$+ \tfrac{1}{2}e^{-i\pi\xi}e^{i\pi x} + .. + \tfrac{1}{2}e^{-in\pi\xi}e^{ni\pi x} + ..$$

$$= .. + \tfrac{1}{2}e^{in\pi(\xi-x)} + ... + \tfrac{1}{2}e^{i\pi(\xi-x)} + \tfrac{1}{2} +$$
$$+ \tfrac{1}{2}e^{-i\pi(\xi-x)} + .. + \tfrac{1}{2}e^{-in\pi(\xi-x)} + ...$$

$$= \delta_{Periodic}(\xi - x). \square$$

10.

Fourier Series Theorem

The Fourier Series Theorem for a hyper-real function, $f(x)$, is the Fundamental Theorem of Fourier Series.

It supplies the conditions under which the Fourier Series associated with $f(x)$ equals $f(x)$.

It is believed to hold in the Calculus of Limits under the Dirichlet Conditions. In fact,

The Theorem cannot be proved in the Calculus of Limits under any conditions,

because integration over the singularity of the Dirichlet Kernel, is impossible in the Calculus of Limits.

10.1 Fourier Series Theorem cannot be proved in the Calculus of Limits

Proof: Take $L = 1$, and $c = 0$.

In the Calculus of Limits, the Fourier Series is the limit of

$$\mathcal{S}_n\left\{f(x)\right\} = c_n e^{in\pi x} + ... + c_1 e^{i\pi x} + c_0 + c_{-1}e^{-i\pi x} + ...c_{-n}e^{-in\pi x}$$

$$= \left(\frac{1}{2}\int\limits_{\xi=-1}^{\xi=1} f(\xi)e^{-in\pi\xi}d\xi\right)e^{in\pi x} + ... + \left(\frac{1}{2}\int\limits_{\xi=-1}^{\xi=1} f(\xi)d\xi\right) + ... + \left(\frac{1}{2}\int\limits_{\xi=-1}^{\xi=1} f(\xi)e^{in\pi\xi}d\xi\right)e^{-in\pi x}$$

$$= \int_{\xi=-1}^{\xi=1} f(\xi)\left\{\tfrac{1}{2}e^{-in\pi(\xi-x)} + ... + \tfrac{1}{2}e^{-i\pi(\xi-x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi-x)} + ... + \tfrac{1}{2}e^{in\pi(\xi-x)}\right\}d\xi\,.$$

As $n \to \infty$, the Dirichlet Sequence

$$D_n(\xi - x) = \tfrac{1}{2}e^{-in\pi(\xi-x)} + ... + \tfrac{1}{2}e^{-i\pi(\xi-x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi-x)} + ... + \tfrac{1}{2}e^{in\pi(\xi-x)}$$

becomes the Dirichlet Kernel, the infinite series

$$... + \tfrac{1}{2}e^{-in\pi(\xi-x)} + ... + \tfrac{1}{2}e^{-i\pi(\xi-x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi-x)} + ... + \tfrac{1}{2}e^{in\pi(\xi-x)} + ...$$

By 8.3, The Dirichlet Kernel is singular at any even $\xi - x$, and vanishes otherwise.

In particular,

$$x = \xi \Rightarrow e^{in\pi(x-\xi)} = 1\,,$$

and the Dirichlet Kernel diverges to

$$... + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + ... = \infty\,.$$

Therefore, while the partial sums of the Fourier Series exist, their limit does not. That is, due to the singularity at $x = \xi$, the Fourier Series can be shown to diverge in the Calculus of Limits.

Avoiding the singularity at $\xi = x$, by using the Cauchy Principal Value of the integral does not recover the Theorem, because at any $\xi \neq x$, the Dirichlet Kernel is zero, and the integral is identically zero, for any function $f(x)$.

Thus, the Fourier Series Theorem cannot be proved the Calculus of Limits.□

10.2 Dirichlet Conditions are irrelevant to Fourier Series Theorem in the Calculus of Limits

Proof: The Dirichlet Conditions are

1. Piecewise Continuity of $f(x)$, and $f\,'(x)$ in $[c-L, c+L]$
2. $f(x)$ is periodic with period $T = 2L$
3. $\frac{1}{2}\big(f(x+0)+f(x-0)\big)$ replaces $f(x)$ at a discontinuity point.

It is clear from 10.1 that the Dirichlet conditions for $f(x)$ do not resolve the singularity of the Dirichlet kernel, and are not sufficient for the Fourier Series Theorem.$\square$

In Infinitesimal Calculus, by 8.4, the Dirichlet Kernel is the Periodic Delta Function, and by 9.2, it equals its Fourier Series.

Then, the Fourier Series Theorem holds for any integrable periodic Hyper-Real Function:

10.3 Fourier Series Theorem for periodic Hyper-real $f(x)$

If $f(x)$ *is hyper-real function integrable on* $[c-L, c+L]$*, so that*

$f(c-L) = f(c+L)$

Then,
$$f(x) = \mathcal{FS}\big\{f(x)\big\}$$

Proof: Take $L = 1$, and $c = 0$.

$$f(x) = \int_{\xi=-1}^{\xi=1} f(\xi) \underbrace{\left\{ ... + \delta(\xi - x + 2) + \delta(\xi - x) + \delta(\xi - x - 2) + ... \right\}}_{\delta_{Periodic}(\xi - x),\text{ where the period of Delta is T=2}} d\xi \, .$$

By 8.4, $\delta_{Periodic}(x - \xi) = D_{irichlet}(x - \xi)$

$$= \int_{\xi=-1}^{\xi=1} f(\xi) \left\{ ... + \tfrac{1}{2} e^{-in\pi(\xi - x)} + ... + \tfrac{1}{2} + ... + \tfrac{1}{2} e^{in\pi(\xi - x)} + ... \right\} d\xi$$

This Hyper-real Integral is the summation,

$$\sum_{\xi=-1}^{\xi=1} f(\xi) \{ ... + \tfrac{1}{2} e^{-in\pi(\xi - x)} + ... + \tfrac{1}{2} + ... + \tfrac{1}{2} e^{in\pi(\xi - x)} + ... \} d\xi$$

which amounts to the hyper-real function $f(x)$,and is well-defined. Hence, the summation of each term in the integrand exists, and we may write the integral as the sum

$$= .. + \underbrace{\left(\tfrac{1}{2} \int_{\xi=-1}^{\xi=1} f(\xi) e^{-in\pi\xi} d\xi \right)}_{c_{-n}} e^{in\pi x} + .. + \underbrace{\left(\tfrac{1}{2} \int_{\xi=-1}^{\xi=1} f(\xi) d\xi \right)}_{c_0} + .. + \underbrace{\left(\tfrac{1}{2} \int_{\xi=-1}^{\xi=1} f(\xi) e^{in\pi\xi} d\xi \right)}_{c_n} e^{-in\pi x} + ..$$

$$= ... + c_{-n} e^{in\pi x} + ... + c_{-1} e^{i\pi x} + c_0 + c_1 e^{-i\pi x} + ... c_n e^{-in\pi x} + ...$$

$$= \mathcal{FS}\left\{ f(x) \right\} . \square$$

10.4 Dirichlet Conditions are irrelevant to Fourier Series Theorem in Infinitesimal Calculus

Proof: The periodic Delta violates the Dirichlet Conditions

- ❖ The Hyper-real $\delta(x)$, and $\delta'(x)$ are not defined in the Calculus of Limits, and are not Piecewise Continuous in any bounded interval.
- ❖ At its discontinuity point, $x = 0$, the Hyper-real $\delta(x)$ is not replaced by $\frac{1}{2}\Big(\delta(x+0)+\delta(x-0)\Big) = 0$.

But by 9.2, $\delta_{Periodic}(\xi - x)$ satisfies the Fourier Series Theorem.□

References

[Dan1] Dannon, H. Vic, "*Well-Ordering of the Reals, Equality of all Infinities, and the Continuum Hypothesis*" in Gauge Institute Journal Vol. 6 No. 2, May 2010;

[Dan2] Dannon, H. Vic, "*Infinitesimals*" in Gauge Institute Journal Vol.6 No. 4, November 2010;

[Dan3] Dannon, H. Vic, "*Infinitesimal Calculus*" in Gauge Institute Journal Vol. 7 No. 4, November 2011;

[Dan4] Dannon, H. Vic, "*Riemann's Zeta Function: the Riemann Hypothesis Origin, the Factorization Error, and the Count of the Primes*", in Gauge Institute Journal of Math and Physics, Vol. 5, No. 4, November 2009.

[Dan5] Dannon, H. Vic, "*The Delta Function*" in Gauge Institute Journal Vol. 8, No. 1, February, 2012;

[Dan6] Dannon, H. Vic, "*Riemannian Trigonometric Series*", Gauge Institute Journal, Volume 7, No. 3, August 2011.

[Dan7] Dannon, H. Vic, "*Delta Function the Fourier Transform, and the Fourier Integral Theorem*" in Gauge Institute Journal Vol. 8, No. 2, May, 2012;

[Dan8] Dannon, H. Vic, "*Infinite Series with Infinite Hyper-real Sum* " in Gauge Institute Journal Vol. 8, No. 3, August, 2012;

[Euler1] Euler, Leonhard, "*Disquisitio Ulterior Super Seriebus Secundum Multipla Cuiusdam Anguli Progredientibus*", Leonhardi Euleri Opera Omnia, Series Prima, Volume XVI, pp.33-355. May 29, 1777.

[Euler2] Euler, Leonhard, "*Methodus Facilis Inveniendi Series per Sinus Cosinusve Angulorum Multiplorum Procedentes Quarum Usus In Universa*

Theoria Asronomiae Est Amplissimus". in Leonhardi Euleri Opera Omnia, Series Prima, Volume XVI, pp.311-332, May 26, 1777.

[Euler3] Euler, Leonhard, "*De Serierum Determinantione Sev Nova Methodus Inveniendi Terminos Generales Serierum*", Leonhardi Euleri Opera Omnia, Series Prima, Volume XIV, pp.463-515. 1753.

[Hardy] Hardy, G. H., *Divergent Series*, Chelsea 1991.

[Riemann] Riemann, Bernhard, "*On the Representation of a Function by a Trigonometric Series*".

(1) In "*Collected Papers, Bernhard Riemann*", translated from the 1892 edition by Roger Baker, Charles Christenson, and Henry Orde, Paper XII, Part 5, Conditions for the existence of a definite integral, pages 231-232, Part 6, Special Cases, pages 232-234. Kendrick press, 2004

(2) In "*God Created the Integers*" Edited by Stephen Hawking, Part 5, and Part 6, pages 836-840, Running Press, 2005.