Circular and Spherical Delta Functions

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Abstract The Circular Delta function is a Plane Delta Function that spikes to $\frac{1}{2\pi d\rho}$ on the Infinitesimal Circle $\left|\vec{\rho} - \vec{\rho}_0\right| = d\rho$.

Unlike the Hyper-Real Linear Delta function [Dan4], the integration path does not cross the disk. Instead, the path encircles the singularity.

The Spherical Delta function is a 3-Space Delta Function that spikes to $\frac{1}{4\pi(dr)^2}$ on the Infinitesimal Sphere $\left|\vec{r} - \vec{r}_0\right| = dr$.

Unlike the Hyper-Real Linear Delta function [Dan4], the integration Surface does not contain the singularity at the Ball's center. Instead, the Surface enwraps the singularity.

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Contents

Introduction

The product of two Delta functions on the line, $\delta(x - x_0)\delta(y - y_0)$ is a Plane Delta Function that spikes at (x_0, y_0) to the infinite Hyper-real number $\frac{1}{dxdy}$.

But the Hyper-real plane allows encircling a singular point of a different Delta Function, along an infinitesimal circle centered at $\vec{\rho} = \vec{\rho}_0$, in order to pick a function value at $\vec{\rho} = \vec{\rho}_0$.

That Delta Function is the Circular Delta function that is defined on the Hyper-real Plane.

The Circular Delta Function spikes on the Infinitesimal Circle $\left|\vec{\rho} - \vec{\rho}_0\right| = d\rho$ to the hyper-real number $\frac{1}{2\pi d\rho}$, and vanishes elsewhere, including the singular point $\vec{\rho} = \vec{\rho}_0$.

Unlike the Hyper-Real Linear Delta function [Dan4], the integration path does not cross the Infinitesimal disk. Instead, the path encircles the singularity.

We use the Circular Delta with the 2nd Green's identity to represent a symmetrically radial function $f(\rho)$ in the Hyper-real plane.

In 3-Space, the product of three Delta functions on the line, $\delta(x - x_0)\delta(y - y_0)\delta(z - z_0)$ is a Delta Function that spikes at

(x_0, y_0, z_0), to the infinite Hyper-real number $\frac{1}{dxdydz}$.

3-Space allows avoiding a singular point of a different Delta Function, by wrapping around it an infinitesimal sphere centered at $\vec{r} = \vec{r}_0$, in order to pick a function value at $\vec{r} = \vec{r}_0$.

That Delta Function is the Spherical Delta function that is defined on the Hyper-real 3-Space.

The Spherical Delta Function spikes on the Infinitesimal Sphere $\left|\vec{r} - \vec{r}_0\right| = dr$, to the Hyper-real $\frac{1}{4\pi(dr)^2}$, and vanishes elsewhere, including the singular point $\vec{r} = \vec{r}_0$.

Unlike the Hyper-Real Linear Delta function [Dan4], the integration Surface does not contain the singularity at the Infinitesimal Ball's center. Instead, the Surface enwraps the singularity.

We use the Spherical Delta with the 2nd Green's identity to represent a symmetrically radial function $f(r)$ in the Hyper-real 3-Space.

1.

Hyper-real Line

Each real number α can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of infinitesimal hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the infinite hyper-reals.
4. The infinite hyper-reals are greater than any real number, yet strictly smaller than infinity.
5. The infinite hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a non-constant hyper-real.
7. The Hyper-reals are the totality of constant hyper-reals, a family of infinitesimals, a family of infinitesimals with

negative sign, a family of infinite hyper-reals, a family of infinite hyper-reals with negative sign, and non-constant hyper-reals.

8. The hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.
9. That line includes the real numbers separated by the non-constant hyper-reals. Each real number is the center of an interval of hyper-reals, that includes no other real number.
10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.
11. Zero is not an infinitesimal, because zero is not strictly greater than zero.
12. We do not add infinity to the hyper-real line.
13. The infinitesimals, the infinitesimals with negative signs, the infinite hyper-reals, and the infinite hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.
14. The hyper-real line is embedded in $\mathbb{R}^\infty$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the hyper-real onto the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal hyper-reals, or to the infinite hyper-reals, or to the non-constant hyper-reals.

16. No neighbourhood of a hyper-real is homeomorphic to an $\mathbb{R}^n$ ball. Therefore, the hyper-real line is not a manifold.

17. The hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

2.

Hyper-real Plane

The Hyper-real Plane is a cross product of two Hyper-real lines.

Each 2-vector of real numbers (α, β) can be represented by a Cauchy sequence of rational numbers, $(r_1, q_1),(r_2, q_2),(r_3, q_3)...$ so that $(r_n, q_n) \to (\alpha, \beta)$.

The constant sequence $(\alpha,\beta),(\alpha,\beta),(\alpha,\beta)...$ is a constant hyper-real 2-vector.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to $(0,0)$ sequences of 2-vectors $(\iota_1, o_1)(\iota_2, o_2),(\iota_3, o_3)...$ constitutes a family of infinitesimal hyper-real 2-vectors.
2. The infinitesimal 2-vectors are smaller than any real 2-vector, yet strictly greater than the zero 2-vector.
3. Their reciprocals $(\frac{1}{\iota_1}, \frac{1}{o_1}),(\frac{1}{\iota_2}, \frac{1}{o_2}),(\frac{1}{\iota_3}, \frac{1}{o_3}),...$ are the infinite hyper-real 2-vectors.
4. The infinite hyper-real 2-vectors are greater than any real 2-vector, yet strictly smaller than the infinity 2-vector.

5. The infinite hyper-real 2-vectors with negative signs are smaller than any real 2-vector, yet strictly greater than $(-\infty,-\infty)$.
6. The sum of a real 2-vector with an infinitesimal 2-vector is a non-constant hyper-real 2-vector.
7. The Hyper-real 2-vectors are the totality of
   a. constant hyper-real 2-vectors,
   b. a family of infinitesimal 2-vectors, with signs that may be $(+,+)$, $(+,-)$, $(-,+)$, or $(-,-)$,
   c. a family of infinite hyper-real 2-vectors with signs that may be $(+,+)$, $(+,-)$, $(-,+)$, or $(-,-)$, and
   d. non-constant hyper-real 2-vectors.
8. The hyper-real 2-vectors constitute the <u>Hyper-real Plane</u>.
9. That plane includes the real 2-vectors separated by the non-constant hyper-real 2-vectors. Each real 2-vector is the center of a disk of infinitesimal radius of hyper-real 2-vectors, that includes no other real 2-vector.
10. In particular, the zero 2-vector is separated from any real 2-vector by infinitesimal 2-vectors that lie in a disk of infinitesimal radius around the zero.
11. The Zero 2-vector is not an infinitesimal 2-vector, because zero is not strictly greater than zero.

12. We do not add the infinity 2-vector to the hyper-real Plane.

13. The infinitesimal 2-vectors, and the infinite hyper-real 2-vectors, are semi-groups with respect to addition. Neither set includes zero.

14. The hyper-real Plane is embedded in $\mathbb{R}^{\infty} \times \mathbb{R}^{\infty}$, and is not homeomorphic to the real Plane. There is no bi-continuous one-one mapping from the hyper-real Plane onto the real plane.

15. In particular, there are no points in the real Plane that can be assigned uniquely to the infinitesimal hyper-real 2-vectors, or to the infinite hyper-real 2-vectorss, or to the non-constant hyper-real 2-vectors.

16. No neighbourhood of a hyper-real 2-vector is homeomorphic to an $\mathbb{R}^{n} \times \mathbb{R}^{n}$ ball. Therefore, the hyper-real plane is not a manifold.

17. The hyper-real plane is not spanned by two elements, and it is not two-dimensional.

3.

Hyper-real Integral

In [Dan3], we defined the integral of a Hyper-real Function.

Let $f(x)$ be a hyper-real function on the interval $[a,b]$.

The interval may not be bounded.

$f(x)$ may take infinite hyper-real values, and need not be bounded.

At each

$$a \le x \le b\,,$$

there is a rectangle with base $[x - \frac{dx}{2}, x + \frac{dx}{2}]$, height $f(x)$, and area

$$f(x)dx\,.$$

We form the **Integration Sum** of all the areas for the x's that start at $x = a$, and end at $x = b$,

$$\sum_{x \in [a,b]} f(x)dx\,.$$

If for any infinitesimal dx, the Integration Sum has the same hyper-real value, then $f(x)$ is integrable over the interval $[a,b]$.

Then, we call the Integration Sum the integral of $f(x)$ from $x = a$, to $x = b$, and denote it by

$$\int_{x=a}^{x=b} f(x)dx \,.$$

If the hyper-real is infinite, then it is the integral over $[a,b]$,

If the hyper-real is finite,

$$\int_{x=a}^{x=b} f(x)dx = \text{ real part of the hyper-real}\,.\square$$

3.1 The countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers, $Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty.$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[a,b]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(x)dx$.

The Lower Integral is the Integration Sum where $f(x)$ is replaced by its lowest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

3.2 $$\sum_{x \in [a,b]} \left(\inf_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

The Upper Integral is the Integration Sum where $f(x)$ is replaced by its largest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

3.3 $$\sum_{x \in [a,b]} \left(\sup_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

If the integral is a finite hyper-real, we have

3.4 *A hyper-real function has a finite integral if and only if its upper integral and its lower integral are finite, and differ by an infinitesimal*.

4.

Delta Function

In [Dan4], we have defined the Delta Function, and established its properties

1. The Delta Function is a hyper-real function defined from the hyper-real line into the set of two hyper-reals $\left\{0, \frac{1}{dx}\right\}$. The hyper-real 0 is the sequence $\langle 0, 0, 0, ...\rangle$. The infinite hyper-real $\frac{1}{dx}$ depends on our choice of dx.
2. We will usually choose the family of infinitesimals that is spanned by the sequences $\left\langle \frac{1}{n}\right\rangle, \left\langle \frac{1}{n^2}\right\rangle, \left\langle \frac{1}{n^3}\right\rangle, ...$ It is a semigroup with respect to vector addition, and includes all the scalar multiples of the generating sequences that are non-zero. That is, the family includes infinitesimals with negative sign. Therefore, $\frac{1}{dx}$ will mean the sequence $\langle n\rangle$. Alternatively, we may choose the family spanned by the sequences $\left\langle \frac{1}{2^n}\right\rangle, \left\langle \frac{1}{3^n}\right\rangle, \left\langle \frac{1}{4^n}\right\rangle, ...$ Then, $\frac{1}{dx}$ will mean the

sequence $\left\langle 2^n \right\rangle$. Once we determined the basic infinitesimal dx, we will use it in the Infinite Riemann Sum that defines an Integral in Infinitesimal Calculus.

3. The Delta Function is strictly smaller than ∞

4. We define, $\delta(x) \equiv \frac{1}{dx}\chi_{[-\frac{dx}{2},\frac{dx}{2}]}(x)$,

   where $\chi_{[-\frac{dx}{2},\frac{dx}{2}]}(x) = \begin{cases} 1, x \in \left[-\frac{dx}{2},\frac{dx}{2}\right] \\ 0, \text{otherwise} \end{cases}$.

5. Hence,

   ❖ for $x < 0$, $\delta(x) = 0$

   ❖ at $x = -\frac{dx}{2}$, $\delta(x)$ jumps from 0 to $\frac{1}{dx}$,

   ❖ for $x \in \left[-\frac{dx}{2},\frac{dx}{2}\right]$, $\delta(x) = \frac{1}{dx}$.

   ❖ at $x = 0$, $\delta(0) = \frac{1}{dx}$

   ❖ at $x = \frac{dx}{2}$, $\delta(x)$ drops from $\frac{1}{dx}$ to 0.

   ❖ for $x > 0$, $\delta(x) = 0$.

6. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle \chi_{[-\frac{1}{2},\frac{1}{2}]}(x), 2\chi_{[-\frac{1}{4},\frac{1}{4}]}(x), 3\chi_{[-\frac{1}{6},\frac{1}{6}]}(x)... \right\rangle$

7. If $dx = \left\langle \frac{2}{n} \right\rangle$, $\delta(x) = \left\langle \frac{1}{2\cosh^2 x}, \frac{2}{2\cosh^2 2x}, \frac{3}{2\cosh^2 3x}, ... \right\rangle$

8. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle e^{-x}\chi_{[0,\infty)}, 2e^{-2x}\chi_{[0,\infty)}, 3e^{-3x}\chi_{[0,\infty)}, \ldots \right\rangle$

9. $$x\delta(x) = 0$$

10. $$\delta(x - x_0) \equiv \frac{1}{d(x - x_0)} \chi_{\left[x_0 - \frac{dx}{2}, x_0 + \frac{dx}{2}\right]}(x)$$

11. $$\delta^n(x) = \frac{1}{(dx)^n} \chi_{\left[-\frac{dx}{2}, \frac{dx}{2}\right]}(x), \qquad n = 2, 3, \ldots$$

12. $$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1.$$

13. $$\delta(x - \xi) = \frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} e^{ik(x-\xi)} dk$$

14. $$\delta(x, y) \equiv \delta(x)\delta(y) = \frac{1}{dx} \chi_{[-\frac{dx}{2}, \frac{dx}{2}]}(x) \frac{1}{dy} \chi_{[-\frac{dy}{2}, \frac{dy}{2}]}(y)$$

15. $$\delta(x, y, z) \equiv \delta(x)\delta(y)\delta(z)$$

$$= \frac{1}{dx} \chi_{[-\frac{dx}{2}, \frac{dx}{2}]}(x) \frac{1}{dy} \chi_{[-\frac{dy}{2}, \frac{dy}{2}]}(y) \frac{1}{dz} \chi_{[-\frac{dz}{2}, \frac{dz}{2}]}(z)$$

16. $$\delta(x, y, z, t) \equiv \delta(x)\delta(y)\delta(z)\delta(t)$$

$$= \frac{1}{dx} \chi_{[-\frac{dx}{2}, \frac{dx}{2}]}(x) \frac{1}{dy} \chi_{[-\frac{dy}{2}, \frac{dy}{2}]}(y) \frac{1}{dz} \chi_{[-\frac{dz}{2}, \frac{dz}{2}]}(z) \frac{1}{dt} \chi_{[-\frac{dt}{2}, \frac{dt}{2}]}(t)$$

5.

$\delta(\rho - \rho_0)\delta(\phi - \phi_0)$

5.1 $$\delta(\rho-\rho_0) = \frac{1}{d\rho}\chi_{[\rho_0 - \frac{d\rho}{2}, \rho_0 + \frac{d\rho}{2}]}(\rho), \;\; \rho \geq 0$$

5.2 $$\delta(\phi-\phi_0) = \frac{1}{d\phi}\chi_{[\phi_0 - \frac{d\phi}{2}, \phi_0 + \frac{d\phi}{2}]}(\phi), \;\; 0 \leq \phi \leq 2\pi$$

The product $\delta(\rho-\rho_0)\delta(\phi-\phi_0)$ defines a Delta Function that sifts along a line through its singularity at (ρ_0, ϕ_0).

5.3 $$\delta(\rho-\rho_0)\delta(\phi-\phi_0) = \frac{1}{d\rho}\chi_{[\rho_0 - \frac{d\rho}{2}, \rho_0 + \frac{d\rho}{2}]}(\rho)\frac{1}{d\phi}\chi_{[\phi_0 - \frac{d\phi}{2}, \phi_0 + \frac{d\phi}{2}]}(\phi)$$

Transforming between Polar and Cartesian Coordinates

$$\begin{matrix} x = \rho\cos\phi \\ y = \rho\sin\phi \end{matrix}, \qquad \begin{matrix} x_0 = \rho_0\cos\phi_0 \\ y_0 = \rho_0\sin\phi_0 \end{matrix}, \qquad \rho_0 > 0,$$

5.4 $$\delta(x-x_0)\delta(y-y_0) = \frac{1}{\rho_0}\delta(\rho-\rho_0)\delta(\phi-\phi_0),$$

Proof: $\delta(\rho-\rho_0)\delta(\phi-\phi_0)d\rho d\phi = \delta(x-x_0)\delta(y-y_0)dxdy$

$$\delta(\rho-\rho_0)\delta(\phi-\phi_0) = \delta(x-x_0)\delta(y-y_0)\left|\frac{\partial(x,y)}{\partial(\rho,\phi)}\right|$$

$$= \delta(x-x_0)\delta(y-y_0)\underbrace{\begin{vmatrix} \cos\phi & \sin\phi \\ -\rho\sin\phi & \rho\cos\phi \end{vmatrix}}_{\rho}$$

Both sides vanish unless $\rho = \rho_0 + \text{infinitesimal} \approx \rho_0$.

Therefore, we can replace ρ with ρ_0.□

5.5 $\begin{matrix} x = \rho\cos\phi \\ y = \rho\sin\phi \end{matrix}$, $\rho_0 = 0 \quad \Rightarrow \quad \delta(x)\delta(y) = \dfrac{1}{2\pi\rho}\delta(\rho)$

$$= \frac{1}{2\pi\rho d\rho}\chi_{[-\frac{d\rho}{2},\frac{d\rho}{2}]}(\rho)$$

Proof:

$$\delta(\rho)\delta(\phi)d\rho d\phi = \delta(x)\delta(y)dxdy$$

$$\delta(\rho)\delta(\phi) = \delta(x)\delta(y)\left|\frac{\partial(x,y)}{\partial(\rho,\phi)}\right|$$

$$= \delta(x)\delta(y)\underbrace{\begin{vmatrix} \cos\phi & \sin\phi \\ -\rho\sin\phi & \rho\cos\phi \end{vmatrix}}_{\rho}.$$

$$\delta(x)\delta(y)d\phi = \frac{1}{\rho}\delta(\rho)\delta(\phi)d\phi$$

Since $\rho_0 = 0$, ϕ may take any value in $[0, 2\pi]$, and we integrate over it. Then,

$$\delta(x)\delta(y)\underbrace{\int_{\phi=0}^{\phi=2\pi} d\phi}_{2\pi} = \frac{1}{\rho}\delta(\rho)\underbrace{\int_{\phi=0}^{\phi=2\pi} \delta(\phi)d\phi}_{1}.$$

Therefore,

$$\delta(x)\delta(y) = \frac{1}{2\pi\rho}\delta(\rho)$$

$$= \frac{1}{2\pi\rho d\rho}\chi_{[-\frac{d\rho}{2},\frac{d\rho}{2}]}(\rho). \square$$

6.

Circular Delta Function $\delta_{\text{Circle}}(\vec{\rho})$

The Circular Delta Function has its origins in Vector Calculus, where it was used implicitly to sift through the values of a function, and obtain its value at the singularity of $\log\left|\vec{\rho} - \vec{\rho}_0\right|$.

Then, using limits, there is no way to define that implicit function, because any definition of a Delta Function requires infinitesimals [Dan4].

In Infinitesimal Vector Calculus [Dan5], we used it implicitly, to represent a harmonic function on a planar simply connected domain.

6.1 Circular Delta Definition

We define the Circular Delta as the Hyper-real Function

$$\delta_{\text{Circle}}(\vec{\rho} - \vec{\rho}_0) = \frac{1}{2\pi d\rho} \chi_{\{|\vec{\rho} - \vec{\rho}_0| = d\rho\}},$$

from the Hyper-real line into the set $\left\{0, \frac{1}{2\pi d\rho}\right\}$.

The infinite Hyper-real value $\frac{1}{2\pi d\rho}$, appears only on the infinitesimal Circle $\left|\vec{\rho} - \vec{\rho}_0\right| = d\rho$.

Elsewhere, the circular Delta Function vanishes.

In particular, at the singularity at $\vec{\rho} = \vec{\rho}_0$, it vanishes,

6.2 $$\delta_{\text{Circle}}(0) = 0.$$

Proof: $$\delta_{\text{Circle}}(0) = \frac{1}{2\pi d\rho} \times 0 = 0. \square$$

7.

Circulation of Circular Delta

The Circular Hyper-real Delta Function is singular on the infinitesimal circle $\left|\vec{\rho} - \vec{\rho}_0\right| = d\rho$.

Its integration path encircles the singularity of $\dfrac{1}{\left|\vec{\rho} - \vec{\rho}_0\right|}$ at $\vec{\rho} = \vec{\rho}_0$.

The Circulation of Delta along the infinitesimal circle is 1.

7.1 Circulation of Circular Delta

$$\oint_{\left|\vec{\rho}-\vec{\rho}_0\right|=d\rho} \delta_{\text{Circle}}(\vec{\rho} - \vec{\rho}_0)dl = 1$$

Proof: Since $dl = d\rho d\alpha$,

$$\oint_{\left|\vec{\rho}-\vec{\rho}_0\right|=d\rho} \delta_{\text{Circle}}(\vec{\rho} - \vec{\rho}_0)dl = \oint_{\left|\vec{\rho}-\vec{\rho}_0\right|=d\rho} \frac{1}{2\pi d\rho} d\alpha d\rho$$

$$= \frac{1}{2\pi} \oint_{\left|\vec{\rho}-\vec{\rho}_0\right|=d\rho} d\alpha$$

$$= \frac{1}{2\pi} \int_{\alpha=0}^{\alpha=2\pi} d\alpha$$

$$= 1. \square$$

8.

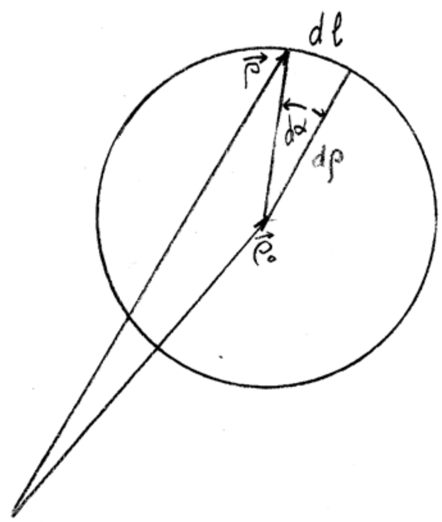
Sifting by $\delta_{\text{Circle}}(\vec{\rho})$

The sifting property of the circular delta function, serves to represent a Hyper-real radially-symmetric function $f(\rho)$ differentiable at $\vec{\rho} = \vec{\rho}_0$, on the infinitesimal circle where the circular delta spikes.

8.1 *If* $f(\rho)$ *is Hyper-real function Differentiable at* $\vec{\rho} = \vec{\rho}_0$

Then, $$\oint_{|\vec{\rho}-\vec{\rho}_0|=d\rho} f(\rho)\delta_{\text{Circle}}(\vec{\rho} - \vec{\rho}_0)dl = f(\rho_0)$$

Proof:



Since f is differentiable at $\vec{\rho}_0$, then, on the circle $|\vec{\rho} - \vec{\rho}_0| = d\rho$,

$$f(\rho) = f(\rho_0 + \left|\vec{\rho} - \vec{\rho}_0\right|)$$

$$= f(\rho_0) + f'(\rho_0)\left|\vec{\rho} - \vec{\rho}_0\right|,$$

Therefore,

$$\oint_{\left|\vec{\rho}-\vec{\rho}_0\right|=d\rho} f(\rho)\delta_{\text{Circle}}(\vec{\rho} - \vec{\rho}_0)dl =$$

$$= \oint_{\left|\vec{\rho}-\vec{\rho}_0\right|=d\rho} [f(\rho_0) + f'(\rho_0)\underbrace{\left|\vec{\rho} - \vec{\rho}_0\right|}_{d\rho}]\underbrace{\delta_{\text{Circle}}(\vec{\rho} - \vec{\rho}_0)}_{\frac{1}{2\pi d\rho}}\underbrace{dl}_{(d\rho)(d\alpha)}$$

$$= f(\rho_0)\underbrace{\oint_{\left|\vec{\rho}-\vec{\rho}_0\right|=d\rho} \delta_{\text{Circle}}(\vec{\rho} - \vec{\rho}_0)dl}_{1} + \frac{1}{2\pi}f'(\rho_0)d\rho\underbrace{\oint_{\left|\vec{\rho}-\vec{\rho}_0\right|=d\rho} d\alpha}_{2\pi}$$

$$= f(\rho_0) + f'(\rho_0)d\rho$$

$$= f(\rho_0) + \text{infinitesimal}$$

Therefore,

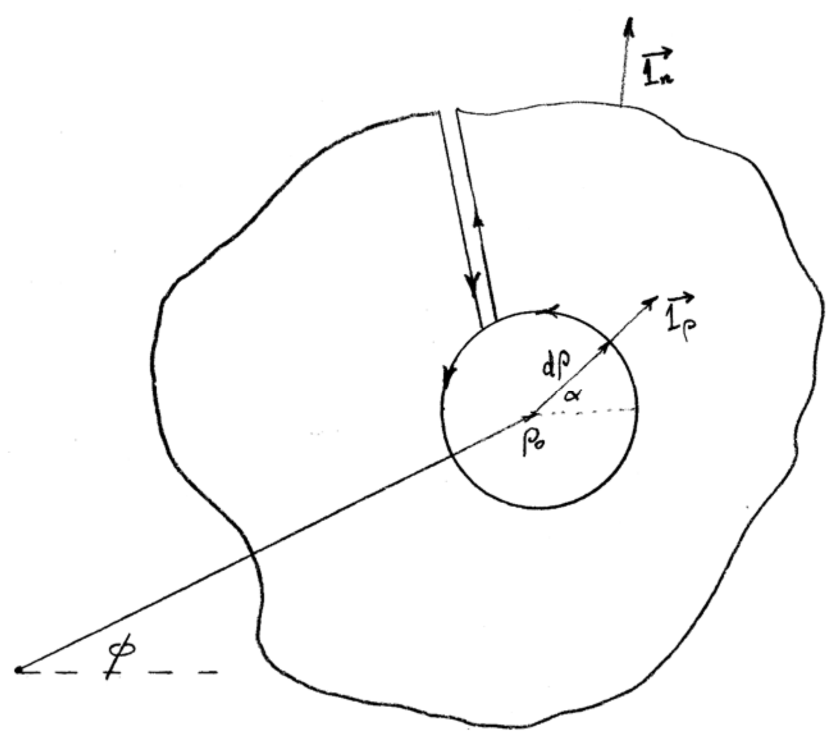
$$\oint_{\left|\vec{\rho}-\vec{\rho}_0\right|=d\rho} f(\rho)\delta_{\text{Circle}}(\vec{\rho} - \vec{\rho}_0)dl = f(\rho_0).\square$$

9.

Representing $f(\rho)$

Let $f(\rho)$ be Hyper-real differentiable radial function in a plane domain D, that contains a loop γ.

Use an integration path along γ, that includes an infinitesimal circle of radius $d\rho$ centered at ρ_0.



9.1 $$= \frac{1}{2\pi}\oint_{\gamma}\left\{ f(\rho)\frac{1}{\rho-\rho_0}\vec{1}_\rho \cdot \vec{1}_n - \log\left|\vec{\rho}-\vec{\rho}_0\right|\frac{\partial f}{\partial n}\right\} dl$$

$$-\frac{1}{2\pi}\oiint_{\substack{\text{Area between}\\ \gamma\text{ and disk}}}[f''(\rho)+\tfrac{1}{\rho}f'(\rho)]\log\left|\vec{\rho}-\vec{\rho}_0\right|\rho d\phi d\rho$$

Proof: In the 2nd Polar Green Identity [Dan5], put

$$g(\rho)=\log\left|\vec{\rho}-\vec{\rho}_0\right|.$$

Then,

$$\nabla\cdot\nabla g(\rho)=\nabla\cdot\nabla\log\left|\vec{\rho}-\vec{\rho}_0\right|$$

$$=\nabla\cdot(\partial_\rho\log\left|\vec{\rho}-\vec{\rho}_0\right|)\vec{1}_\rho$$

$$=\nabla\cdot\left(\frac{1}{\rho-\rho_0}\vec{1}_\rho\right)$$

$$=\frac{1}{\rho-\rho_0}\frac{\partial}{\partial(\rho-\rho_0)}\left((\rho-\rho_0)\frac{1}{\rho-\rho_0}\right)$$

$$=0.$$

$$\nabla\cdot\nabla f(\rho)=\nabla\cdot f'(\rho)$$

$$=\frac{1}{\rho}\partial_\rho\left(\rho f'(\rho)\right)$$

$$=f''(\rho)+\frac{1}{\rho}f'(\rho)$$

In the domain enclosed between γ and the infinitesimal circle, $\log\left|\vec{\rho}-\vec{\rho}_0\right|$ is at most an infinite hyper-real, and

$$\oiint_{\substack{\text{Area between}\\ \gamma\text{ and circle}}}[\log(\rho-\rho_0)\underbrace{\nabla^2 f}_{f''+\frac{1}{\rho}f'}-V\underbrace{\nabla^2\log\left|\vec{\rho}-\vec{\rho}_0\right|}_{0}]\rho d\phi d\rho=$$

$$= \oiint_{\substack{\text{Area between} \\ \gamma \text{ and circle}}} [f''(\rho) + \tfrac{1}{\rho} f'(\rho)] \log\left|\vec{\rho} - \vec{\rho}_0\right| \rho d\phi d\rho .$$

The line integrals between γ , and the circle are of opposite signs, and cancel each other.

Therefore, by the 2nd Polar Green Identity,

$$\oiint_{\substack{\text{Area between} \\ \gamma \text{ and circle}}} [f''(\rho) + \tfrac{1}{\rho} f'(\rho)] \log\left|\vec{\rho} - \vec{\rho}_0\right| \rho d\phi d\rho =$$

$$= \oint_{\left|\vec{\rho}-\vec{\rho}_0\right|=d\rho} [\log\left|\vec{\rho} - \vec{\rho}_0\right| \nabla f \cdot \vec{1}_n - f(\rho)(\nabla \log\left|\vec{\rho} - \vec{\rho}_0\right|) \cdot \vec{1}_n] dl +$$

$$+ \oint_{\gamma} [\log\left|\vec{\rho} - \vec{\rho}_0\right| \nabla f \cdot \vec{1}_n - f(\rho)(\nabla \log\left|\vec{\rho} - \vec{\rho}_0\right|) \cdot \vec{1}_n] dl$$

The first integral along the circle is infinitesimal because

$$\oint_{\text{circle}} \log \underbrace{\left|\vec{\rho} - \vec{\rho}_0\right|}_{d\rho} \nabla f \cdot \underbrace{\vec{1}_n}_{\vec{1}_\rho} \underbrace{dl}_{d\alpha d\rho} = \oint_{\text{circle}} \log(d\rho) f'(\rho) d\alpha d\rho .$$

By Bernoulli-L'Hospital rule, for an infinitesimal ε ,

$$\varepsilon \log \varepsilon = \frac{\log \varepsilon}{\frac{1}{\varepsilon}} = \frac{D_\varepsilon(\log \varepsilon)}{D_\varepsilon(\frac{1}{\varepsilon})} = \frac{\frac{1}{\varepsilon}}{-\frac{1}{\varepsilon^2}} = -\varepsilon .$$

That is,

$$(d\rho)\log(d\rho) = \text{infinitesimal} .$$

Hence,

$$\oint_{\text{circle}} f'(\rho)(d\rho)\log(d\rho) d\alpha = (\text{infinitesimal}) \times \int_{\alpha=0}^{\alpha=2\pi} d\alpha = \text{infinitesimal} .$$

The second integral along the circle is

$$-\oint_{|\vec{\rho}-\vec{\rho}_0|=d\rho} f(\rho)(\underbrace{\nabla \log|\vec{\rho}-\vec{\rho}_0|}_{\frac{1}{|\vec{\rho}-\vec{\rho}_0|}\vec{1}_\rho=\frac{1}{d\rho}\vec{1}_\rho})\cdot \underbrace{\vec{1}_n}_{\vec{1}_\rho}\, dl = -2\pi \oint_{|\vec{\rho}-\vec{\rho}_0|=d\rho} f(\rho)\frac{1}{2\pi d\rho}\, dl$$

$$= -2\pi \oint_{|\vec{\rho}-\vec{\rho}_0|=d\rho} f(\rho)\delta_{\text{Circle}}(\vec{\rho}-\vec{\rho}_0) dl$$

$$= -2\pi f(\rho_0)$$

Therefore,

$$f(\rho_0) = -\frac{1}{2\pi}\oint_\gamma \left\{\log(\rho-\rho_0)\underbrace{\nabla f\cdot \vec{1}_n}_{\frac{\partial f}{\partial n}} - f(\rho)(\underbrace{\nabla \log(\rho-\rho_0)}_{\frac{1}{\rho-\rho_0}\vec{1}_\rho})\cdot \vec{1}_n\right\} dl$$

$$-\frac{1}{2\pi}\oiint_{\substack{\text{Area between}\\ \gamma \text{ and circle}}} [f''(\rho)+\tfrac{1}{\rho}f'(\rho)]\log|\vec{\rho}-\vec{\rho}_0|\rho d\phi d\rho$$

$$= \frac{1}{2\pi}\oint_\gamma \left\{f(\rho)\frac{1}{\rho-\rho_0}\vec{1}_\rho\cdot\vec{1}_n - \log|\vec{\rho}-\vec{\rho}_0|\frac{\partial f}{\partial n}\right\}dl$$

$$-\frac{1}{2\pi}\oiint_{\substack{\text{Area between}\\ \gamma \text{ and circle}}} [f''(\rho)+\tfrac{1}{\rho}f'(\rho)]\log|\vec{\rho}-\vec{\rho}_0|\rho d\phi d\rho \,.\square$$

10.

$\delta(r - r_0)\delta(\theta - \theta_0)\delta(\phi - \phi_0)$

10.1 $\delta(r - r_0) = \dfrac{1}{dr}\chi_{[r_0 - \frac{dr}{2}, r_0 + \frac{dr}{2}]}(r),\ \ r \geq 0$

10.2 $\delta(\theta - \theta_0) = \dfrac{1}{d\theta}\chi_{[\theta_0 - \frac{d\theta}{2}, \theta_0 + \frac{d\theta}{2}]}(\theta),\ \ 0 \leq \theta \leq \pi$

10.3 $\delta(\phi - \phi_0) = \dfrac{1}{d\phi}\chi_{[\phi_0 - \frac{d\phi}{2}, \phi_0 + \frac{d\phi}{2}]}(\phi),\ \ 0 \leq \phi \leq 2\pi$

The product $\delta(r - r_0)\delta(\theta - \theta_0)\delta(\phi - \phi_0)$ defines a Delta Function that sifts along a line through its singularity at (r_0, θ_0, ϕ_0).

10.4 $\delta(r - r_0, \theta - \theta_0, \phi - \phi_0) \equiv \delta(r - r_0)\delta(\theta - \theta_0)\delta(\phi - \phi_0)$

$$= \frac{1}{dr}\chi_{[r_0 - \frac{dr}{2}, r_0 + \frac{dr}{2}]}(r)\frac{1}{d\theta}\chi_{[\theta_0 - \frac{d\theta}{2}, \theta_0 + \frac{d\theta}{2}]}(\theta)\frac{1}{d\phi}\chi_{[\phi_0 - \frac{d\phi}{2}, \phi_0 + \frac{d\phi}{2}]}(\phi)$$

Transforming between Spherical and Cartesian Coordinates

$$\begin{aligned} x &= r\sin\theta\cos\phi & x_0 &= r_0\sin\theta_0\cos\phi_0 \\ y &= r\sin\theta\sin\phi\,, & y_0 &= r_0\sin\theta_0\sin\phi_0\,, \quad r_0 > 0 \\ z &= r\cos\theta & z_0 &= r_0\cos\theta_0 \end{aligned}$$

10.5 $\delta(x-x_0)\delta(y-y_0)\delta(z-z_0) = \dfrac{1}{r_0^2\sin\theta_0}\delta(r-r_0)\delta(\theta-\theta_0)\delta(\phi-\phi_0)$

Proof:

$$\delta(r-r_0)\delta(\theta-\theta_0)\delta(\phi-\phi_0)drd\theta d\phi = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)dxdydz$$

$$\delta(r-r_0)\delta(\theta-\theta_0)\delta(\phi-\phi_0) = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\underbrace{\left|\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\right|}_{r^2\sin\theta}$$

Both sides vanish unless

$$r = r_0 + \text{infinitesimal} \approx r_0,$$

and

$$\theta = \theta_0 + \text{infinitesimal} \approx \theta_0.$$

Therefore, we can replace r with r_0, and θ with θ_0.

$$\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)r_0^2\sin\theta_0 = \delta(r-r_0)\delta(\theta-\theta_0)\delta(\phi-\phi_0).\Box$$

10.6 $\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi\,, \\ z &= r\cos\theta \end{aligned}$ $r_0 = 0 \Rightarrow \delta(x)\delta(y)\delta(z) = \dfrac{1}{4\pi r^2}\delta(r)$

$$= \frac{1}{4\pi r^2 dr}\chi_{[-\frac{dr}{2},\frac{dr}{2}]}(r)$$

Proof:

$$\delta(r)\delta(\theta)\delta(\phi)drd\theta d\phi = \delta(x)\delta(y)\delta(z)dxdydz$$

$$\delta(r)\delta(\theta)\delta(\phi) = \delta(x)\delta(y)\delta(z)\underbrace{\left|\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\right|}_{r^2\sin\theta}$$

Since $r_0 = 0$, ϕ may take any value in $[0,2\pi]$, θ may take any value in $[0,\pi]$, and we integrate over them. Then,

$$r^2\sin\theta d\theta \underbrace{\int_{\phi=0}^{\phi=2\pi} d\phi}_{2\pi}\, \delta(x)\delta(y)\delta(z) = \delta(r)\delta(\theta)d\theta \underbrace{\int_{\phi=0}^{\phi=2\pi} \delta(\phi)d\phi}_{1}$$

$$2\pi r^2\delta(x)\delta(y)\delta(z)\underbrace{\int_{\theta=0}^{\theta=\pi} \sin\theta d\theta}_{-\cos\theta\big|_{\theta=0}^{\theta=\pi}=2} = \delta(r)\underbrace{\int_{\theta=0}^{\theta=\pi} \delta(\theta)d\theta}_{1}.$$

$$4\pi r^2\delta(x)\delta(y)\delta(z) = \delta(r).\square$$

11.

Spherical Delta Function $\delta_{\text{Sphere}}(\vec{r})$

The Spherical Delta Function has its origins in Vector Calculus, where it was used implicitly to sift through the values of a function, and obtain its value at the singularity of $\frac{1}{\left|\vec{r}-\vec{r}_0\right|}$.

Then, using limits, there is no way to define that implicit function, because the definition of the Delta Function requires infinitesimals [Dan4].

In Infinitesimal Vector Calculus [Dan5], we used it implicitly, to represent a harmonic function on a 3-space simply connected domain.

11.1 Spherical Delta Definition

We define the Spherical Delta as the Hyper-real Function

$$\delta_{\text{Sphere}}(\vec{r}-\vec{r}_0)=\frac{1}{4\pi(dr)^2}\chi_{\{\left|\vec{r}-\vec{r}_0\right|=dr\}}(\vec{r}),$$

from the Hyper-real line into the set $\left\{0,\frac{1}{4\pi(dr)^2}\right\}$.

The infinite Hyper-real value $\frac{1}{4\pi(dr)^2}$, appears only on the infinitesimal Sphere $\left|\vec{r}-\vec{r}_0\right|=dr$.

Elsewhere, the Spherical Delta Function vanishes.

In particular, at the singularity at $\vec{r} = \vec{r}_0$, it vanishes.

11.2 $$\delta_{\text{Sphere}}(0) = 0\,.$$

Proof: $$\delta_{\text{Sphere}}(0) = \frac{1}{4\pi(dr)^2} \times 0 = 0\,.$$

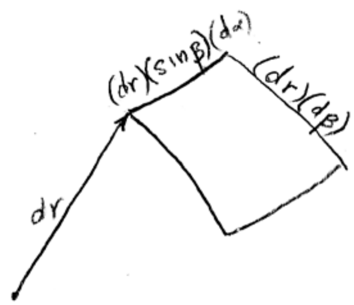
12.

Surface Circulation of Spherical Delta

12.1 Surface Circulation of the Spherical Delta

$$\oint\limits_{|\vec{r}-\vec{r}_0|=dr} \delta_{\text{Sphere}}(\vec{r}-\vec{r}_0)dS = 1$$

Proof:



Since $dS = [(dr)(\sin\beta)(d\alpha)][(dr)(d\beta)]$,

$$\oiint\limits_{|\vec{r}-\vec{r}_0|=dr} \delta_{\text{Sphere}}(\vec{r}-\vec{r}_0)dS = \oiint\limits_{|\vec{r}-\vec{r}_0|=dr} \frac{1}{4\pi(dr)^2}\chi_{\{|\vec{r}-\vec{r}_0|=dr\}}(dr)^2(\sin\beta)d\beta d\alpha$$

$$= \frac{1}{4\pi}\oiint\limits_{|\vec{r}-\vec{r}_0|=dr}(\sin\beta)d\beta d\alpha$$

$$= \frac{1}{4\pi} \underbrace{\int_{\beta=0}^{\beta=\pi} \sin\beta d\beta}_{2} \underbrace{\int_{\alpha=0}^{\alpha=2\pi} d\alpha}_{2\pi}$$

$$= 1. \square$$

13.

Sifting by $\delta_{\text{Sphere}}(\vec{r} - \vec{r}_0)$

The sifting property of the Spherical delta function, serves to represent a Hyper-real radially-symmetric function $f(r)$ differentiable at $\vec{r} = \vec{r}_0$, on the infinitesimal Sphere where the Spherical delta spikes.

13.1 *If* $f(r)$ *is Hyper-real function Differentiable at* $\vec{r} = \vec{r}_0$

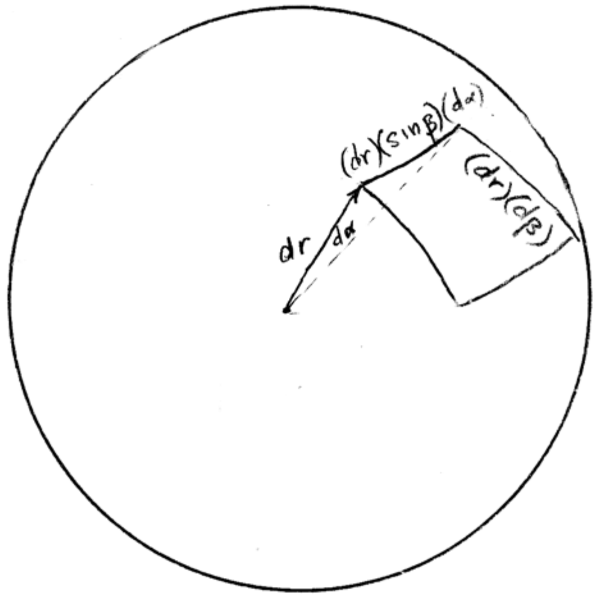
Then, $$\oiint_{|\vec{r}-\vec{r}_0|=dr} f(r)\delta_{\text{Sphere}}(\vec{r} - \vec{r}_0)dS = f(r_0)$$

Proof:

Since f is differentiable at $\vec{r}_0$, then, on the Sphere $|\vec{r} - \vec{r}_0| = dr$,

$$f(r) = f(r_0 + [r - r_0])$$

$$= f(r_0) + f'(r_0)|\vec{r} - \vec{r}_0|,$$



Therefore,

$$\oiint_{|\vec{r}-\vec{r}_0|=dr} f(r)\delta_{\text{Sphere}}(\vec{r}-\vec{r}_0)dS =$$

$$= \oiint_{|\vec{r}-\vec{r}_0|=dr} [f(r_0) + f'(r_0)\underbrace{|\vec{r}-\vec{r}_0|}_{dr}]\underbrace{\delta_{\text{Sphere}}(\vec{r}-\vec{r}_0)}_{\frac{1}{4\pi(dr)^2}}\underbrace{dS}_{(dr)^2\sin\beta d\beta d\alpha}$$

$$= f(r_0)\underbrace{\oiint_{|\vec{r}-\vec{r}_0|=dr}\delta_{\text{Sphere}}(\vec{r}-\vec{r}_0)dS}_{1} + \frac{1}{4\pi}f'(r_0)dr\underbrace{\oiint_{|\vec{r}-\vec{r}_0|=dr}\sin\beta d\beta d\alpha}_{4\pi}$$

$$= f(r_0) + f'(r_0)dr$$

$$= f(r_0) + \text{infinitesimal}$$

Therefore,

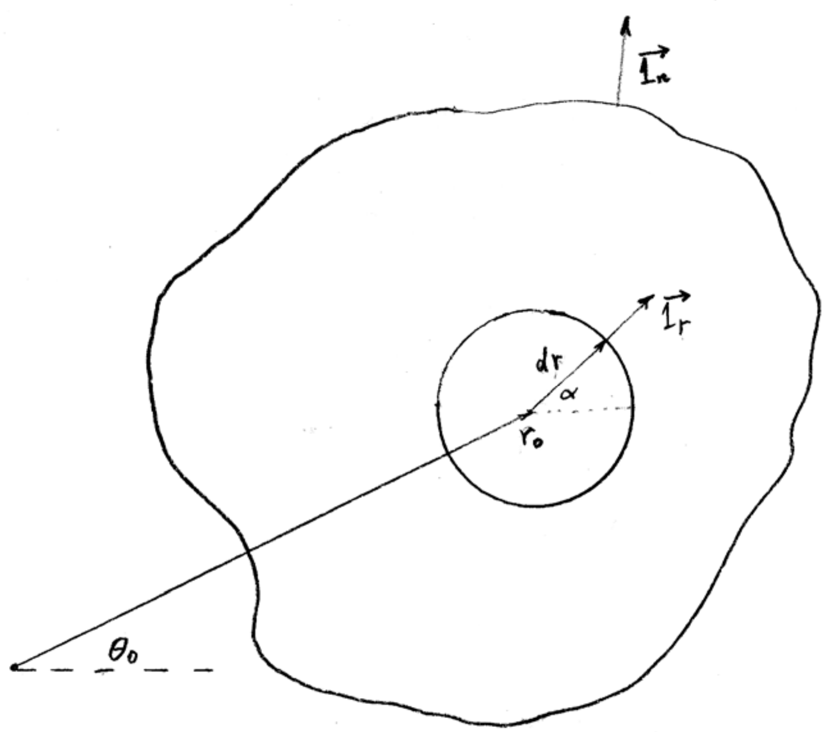
$$\oiint_{|\vec{r}-\vec{r}_0|=dr} f(r)\delta_{\text{Sphere}}(\vec{r}-\vec{r}_0)dS = f(r_0).\square$$

14.

Representing $f(r)$

Let $f(r)$ be Hyper-real differentiable Harmonic function in a volume D, bounded by the closed surface $S = \partial D$.

Integrate over a surface that includes $S = \partial D$, and an infinitesimal sphere of radius dr centered at r_0.



14.1 $$= \frac{1}{4\pi} \oiint_S \left\{ \frac{1}{\left|\vec{r} - \vec{r}_0\right|} \frac{\partial V}{\partial n} + V(r,\theta,\phi) \frac{1}{\left|\vec{r} - \vec{r}_0\right|^2} \vec{1}_r \cdot \vec{1}_n \right\} r^2 \sin\theta d\theta d\phi$$

$$-\underset{\substack{\text{Volume between}\\ \text{S and sphere}}}{\oiiint} \frac{1}{\left|\vec{r}-\vec{r}_0\right|}[f''(r)+\tfrac{2}{r}f'(r)]r^2\sin\theta drd\theta d\phi$$

Proof: In the 2nd Spherical Green Identity [Dan5], put

$$g(r)=\frac{1}{\left|\vec{r}-\vec{r}_0\right|}.$$

Then,

$$\nabla\cdot\nabla g(r)=\nabla\cdot\nabla\frac{1}{\left|\vec{r}-\vec{r}_0\right|}$$

$$=\nabla\cdot(\partial_r\frac{1}{\left|\vec{r}-\vec{r}_0\right|}\vec{1}_r)$$

$$=\nabla\cdot\left(\frac{-1}{\left|\vec{r}-\vec{r}_0\right|^2}\vec{1}_r\right)$$

$$=\frac{1}{\left|\vec{r}-\vec{r}_0\right|^2}\frac{\partial}{\partial(r-r_0)}\left(\left|\vec{r}-\vec{r}_0\right|^2\frac{-1}{\left|\vec{r}-\vec{r}_0\right|^2}\right)$$

$$=0.$$

$$\nabla\cdot\nabla f(r)=\nabla\cdot\nabla f(r)$$

$$=\nabla\cdot f'(r)\vec{1}_r$$

$$=\tfrac{1}{r^2}\partial_r[r^2f'(r)]$$

$$=f''(r)+\tfrac{2}{r}f'(r)$$

In the domain enclosed between S and the infinitesimal sphere,

$$\iiint\limits_{\substack{\text{Volume between}\\ \text{S and sphere}}} \left\{ \frac{1}{\left|\vec{r}-\vec{r}_0\right|} \underbrace{\nabla^2 V}_{f''(r)+\frac{2}{r}f'(r)} - V\,\underbrace{\nabla^2 \frac{1}{\left|\vec{r}-\vec{r}_0\right|}}_{0} \right\} r^2 \sin\theta dr d\theta d\phi =$$

$$= \iiint\limits_{\substack{\text{Volume between}\\ \text{S and sphere}}} \frac{1}{\left|\vec{r}-\vec{r}_0\right|} [f''(r) + \tfrac{2}{r} f'(r)] r^2 \sin\theta dr d\theta d\phi$$

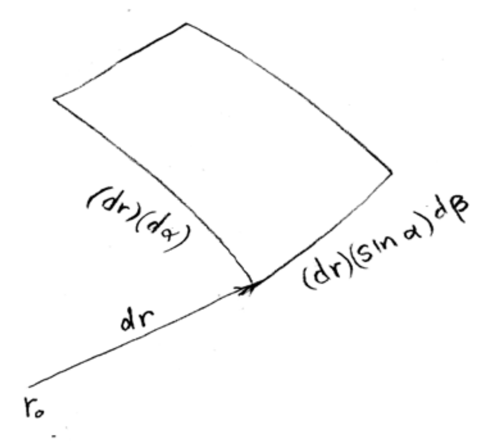
By the 2nd Spherical Green Identity, this equals to the integrals over S, and the Sphere. That is,

$$\iiint\limits_{\substack{\text{Volume between}\\ \text{S and sphere}}} \frac{1}{\left|\vec{r}-\vec{r}_0\right|} [f''(r) + \tfrac{2}{r} f'(r)] r^2 \sin\theta dr d\theta d\phi =$$

$$= \oiint\limits_{\left|\vec{r}-\vec{r}_0\right|=dr} \left\{ \frac{1}{\left|\vec{r}-\vec{r}_0\right|} f'(r) \cdot \vec{1}_n - f(r)(\nabla \frac{1}{\left|\vec{r}-\vec{r}_0\right|}) \cdot \vec{1}_n \right\} dS$$

$$+ \oiint\limits_{S} \left\{ \frac{1}{\left|\vec{r}-\vec{r}_0\right|} f'(r) \cdot \vec{1}_n - f(r)(\nabla \frac{1}{\left|\vec{r}-\vec{r}_0\right|}) \cdot \vec{1}_n \right\} dS$$

The first integral over the sphere is infinitesimal. Indeed,



$$\oiint_{|\vec{r}-\vec{r}_0|=dr} \frac{1}{|\vec{r}-\vec{r}_0|} \underbrace{\nabla f}_{f'(r)\vec{1}_r} \cdot \underbrace{\vec{1}_n}_{\vec{1}_r} \underbrace{dS}_{(dr)^2(d\alpha)(\sin\alpha)(d\beta)} = \oiint_{|\vec{r}-\vec{r}_0|=dr} \underbrace{f'(r)dr}_{\text{infinitesimal}} (\sin\alpha)d\alpha d\beta\,.$$

$$= (\text{infinitesimal}) \times \underbrace{\int_{\alpha=0}^{\alpha=\pi} (\sin\alpha)d\alpha}_{2} \underbrace{\int_{\beta=0}^{\beta=2\pi} d\beta}_{2\pi}$$

The second integral over the sphere surface is

$$-\oiint_{|\vec{r}-\vec{r}_0|=dr} f(r)(\underbrace{\nabla\frac{1}{|\vec{r}-\vec{r}_0|}}_{\frac{-1}{|\vec{r}-\vec{r}_0|^2}\vec{1}_r}) \cdot \underbrace{\vec{1}_n}_{\vec{1}_r} \underbrace{dS}_{(dr)^2(\sin\alpha)(d\alpha)(d\beta)} =$$

$$= 4\pi \oiint_{|\vec{r}-\vec{r}_0|=dr} f(r) \underbrace{\frac{1}{|\vec{r}-\vec{r}_0|^2}}_{\delta_{\text{Sphere}}(\vec{r}-\vec{r}_0)} dS = 4\pi f(r_0).$$

Therefore,

$$f(r_0) = \frac{1}{4\pi}\oiint_S \left\{ \frac{1}{|\vec{r}-\vec{r}_0|} f'(r)\cdot\vec{1}_n - f(r)(\nabla\frac{1}{|\vec{r}-\vec{r}_0|})\cdot\vec{1}_n \right\} r^2 \sin\theta d\theta d\phi$$

$$-\iiint\limits_{\substack{\text{Volume between}\\ \text{S and sphere}}} \frac{1}{\left|\vec{r}-\vec{r}_0\right|}[f''(r)+\tfrac{2}{r}f'(r)]r^2\sin\theta drd\theta d\phi$$

$$=\frac{1}{4\pi}\oiint\limits_{S}\left\{\frac{1}{\left|\vec{r}-\vec{r}_0\right|}\frac{\partial V}{\partial n}+V(r,\theta,\phi)\frac{1}{\left|\vec{r}-\vec{r}_0\right|^2}\vec{1}_r\cdot\vec{1}_n\right\}r^2\sin\theta d\theta d\phi$$

$$-\iiint\limits_{\substack{\text{Volume between}\\ \text{S and sphere}}} \frac{1}{\left|\vec{r}-\vec{r}_0\right|}[f''(r)+\tfrac{2}{r}f'(r)]r^2\sin\theta drd\theta d\phi.\square$$

15.

$\delta(\vec{r})$ in 4 Spherical Coordinates

15.1 $\delta(r - r_0) = \frac{1}{dr}\chi_{[r_0 - \frac{1}{2}dr, r_0 + \frac{1}{2}dr]}(r), \;\; r \geq 0$

15.2 $\delta(\theta - \theta_0) = \frac{1}{d\theta}\chi_{[\theta_0 - \frac{1}{2}d\theta, \theta_0 + \frac{1}{2}d\theta]}(\theta), \;\; 0 \leq \theta \leq \pi$

15.3 $\delta(\phi - \phi_0) = \frac{1}{d\phi}\chi_{[\phi_0 - \frac{1}{2}d\phi, \phi_0 + \frac{1}{2}d\phi]}(\phi), \;\; 0 \leq \phi \leq \pi$

15.4 $\delta(\alpha - \alpha_0) = \frac{1}{d\alpha}\chi_{[\alpha_0 - \frac{1}{2}d\alpha, \alpha_0 + \frac{1}{2}d\alpha]}(\alpha), \;\; 0 \leq \alpha \leq 2\pi$

15.5 $\delta(r - r_0, \theta - \theta_0, \phi - \phi_0, \alpha - \alpha_0) \equiv \delta(r - r_0)\delta(\theta - \theta_0)\delta(\phi - \phi_0)\delta(\alpha - \alpha_0) =$

$$= \frac{1}{dr}\chi_{[r_0 - \frac{dr}{2}, r_0 + \frac{dr}{2}]}(r)\frac{1}{d\theta}\chi_{[\theta_0 - \frac{d\theta}{2}, \theta_0 + \frac{d\theta}{2}]}(\theta)\frac{1}{d\phi}\chi_{[\phi_0 - \frac{d\phi}{2}, \phi_0 + \frac{d\phi}{2}]}(\phi)\frac{1}{d\alpha}\chi_{[\alpha_0 - \frac{d\alpha}{2}, \alpha_0 + \frac{d\alpha}{2}]}(\alpha)$$

15.6 $\begin{aligned} x &= r\sin\theta\sin\phi\sin\alpha \\ y &= r\sin\theta\sin\phi\cos\alpha \\ z &= r\sin\theta\cos\phi \\ t &= r\cos\theta \end{aligned}$, $\begin{aligned} x_0 &= r_0\sin\theta_0\sin\phi_0\sin\alpha_0 \\ y_0 &= r_0\sin\theta_0\sin\phi_0\cos\alpha_0 \\ z_0 &= r_0\sin\theta_0\cos\phi_0 \\ t_0 &= r_0\cos\theta_0 \end{aligned}$, $r_0 > 0 \Rightarrow$

$$\delta(x - x_0)\delta(y - y_0)\delta(z - z_0)\delta(t - t_0) =$$

$$= \frac{1}{r_0^3\sin^2\theta_0\sin\phi_0}\delta(r - r_0)\delta(\theta - \theta_0)\delta(\phi - \phi_0)\delta(\alpha - \alpha_0)$$

Proof:

$$\delta(r - r_0)\delta(\theta - \theta_0)\delta(\phi - \phi_0)\delta(\alpha - \alpha_0)drd\theta d\phi d\alpha =$$

$$= \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)\delta(t - t_0)dxdydzdt$$

$$\delta(r - r_0)\delta(\theta - \theta_0)\delta(\phi - \phi_0)\delta(\alpha - \alpha_0) =$$

$$= \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)\delta(t - t_0)\underbrace{\left|\frac{\partial(x,y,z,t)}{\partial(r,\theta,\phi,\alpha)}\right|}_{r^3\sin^2\theta\sin\phi}$$

Both sides vanish unless

$$r = r_0 + \text{infinitesimal} \approx r_0,$$

$$\theta = \theta_0 + \text{infinitesimal} \approx \theta_0,$$

and

$$\phi = \phi_0 + \text{infinitesimal} \approx \phi_0.$$

Therefore, we can replace r with r_0, θ with θ_0, and ϕ with ϕ_0

$$r_0^3\sin^2\theta_0\sin\phi_0\delta(x - x_0)\delta(y - y_0)\delta(z - z_0)\delta(t - t_0) =$$

$$= \delta(r - r_0)\delta(\theta - \theta_0)\delta(\phi - \phi_0)\delta(\alpha - \alpha_0).\square$$

15.7
$$\begin{aligned} x &= r\sin\theta\sin\phi\sin\alpha \\ y &= r\sin\theta\sin\phi\cos\alpha \\ z &= r\sin\theta\cos\phi \\ t &= r\cos\theta \end{aligned}, \quad r_0 > 0 \ \Rightarrow\ \delta(x)\delta(y)\delta(z)\delta(t) = \frac{1}{2\pi^2 r^3}\delta(r)$$

Proof:

$$\delta(r)\delta(\theta)\delta(\phi)\delta(\alpha)drd\theta d\phi d\alpha = \delta(x)\delta(y)\delta(z)\delta(t)dxdydzdt$$

$$\delta(r)\delta(\theta)\delta(\phi)\delta(\alpha) = \delta(x)\delta(y)\delta(z)\delta(t)\underbrace{\left|\frac{\partial(x,y,z,t)}{\partial(r,\theta,\phi,\alpha)}\right|}_{r^3\sin^2\theta\sin\phi}$$

Since $r_0 = 0$, α **may take any value in** $[0,2\pi]$, ϕ **may take any value in** $[0,\pi]$, **and** θ **may take any value in** $[0,\pi]$, **and we integrate over these intervals**

$$r^3\sin^2\theta\sin\phi d\theta d\phi \underbrace{\int_{\alpha=0}^{\alpha=2\pi} d\alpha}_{2\pi}\,\delta(x)\delta(y)\delta(z)\delta(t) = \delta(r)\delta(\theta)\delta(\phi)d\theta d\phi \underbrace{\int_{\alpha=0}^{\alpha=2\pi}\delta(\alpha)d\alpha}_{1}$$

$$2\pi r^3\sin^2\theta d\theta \underbrace{\int_{\phi=0}^{\phi=\pi}\sin\phi d\phi}_{2}\,\delta(x)\delta(y)\delta(z)\delta(t) = \delta(r)\delta(\theta)d\theta \underbrace{\int_{\phi=0}^{\phi=\pi}\delta(\phi)d\phi}_{1}$$

$$4\pi r^3 \underbrace{\int_{\theta=0}^{\theta=\pi}\sin^2\theta d\theta}_{\frac{1}{2}\pi}\,\delta(x)\delta(y)\delta(z)\delta(t) = \delta(r)\underbrace{\int_{\theta=0}^{\theta=\pi}\delta(\theta)d\theta}_{1}$$

$$2\pi^2 r^3\delta(x)\delta(y)\delta(z)\delta(t) = \delta(r).\square$$

16.

3-Spherical Delta $\delta_{\text{3-Sphere}}(\vec{r} - \vec{r}_0)$

The surface area of a 3-sphere of radius r is

$$2\pi^2 r^3$$

The surface area of a unit 3-sphere is $2\pi^2$.

16.1 3-Spherical Delta Function Definition

We define the Spherical Delta as the Hyper-real Function

$$\delta_{\text{3-Sphere}}(\vec{r} - \vec{r}_0) = \frac{1}{2\pi^2 (dr)^3} \chi_{\{|\vec{r}-\vec{r}_0|=dr\}}(\vec{r}),$$

from the Hyper-real line into the set $\left\{0, \frac{1}{2\pi^2 (dr)^3}\right\}$.

The infinite Hyper-real value $\frac{1}{2\pi^2 (dr)^3}$, appears only on the infinitesimal Sphere $\left|\vec{r} - \vec{r}_0\right| = dr$.

Elsewhere, the Spherical Delta Function vanishes.

In particular, at the singularity at $\vec{r} = \vec{r}_0$, it vanishes.

16.2 $$\delta_{\text{3-Sphere}}(0) = 0.$$

16.3 Sifting of $\delta_{\text{3-Sphere}}$

$$\oint_{|\vec{r}-\vec{r}_0|=dr} \delta_{\text{3-Sphere}}(\vec{r}-\vec{r}_0)dS^{(3)} = 1$$

The sifting property of the 3-Sphere delta function, serves to represent a Hyper-real radially-symmetric function $f(r)$ differentiable at $\vec{r} = \vec{r}_0$, on the infinitesimal Sphere where the Spherical delta spikes.

16.4 *If* $f(r)$ *is Hyper-real function Differentiable at* $\vec{r} = \vec{r}_0$

Then, $$\oiiint_{|\vec{r}-\vec{r}_0|=dr} f(r)\delta_{\text{3-Sphere}}(\vec{r}-\vec{r}_0)dS = f(r_0)$$

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