

# The Delta Function

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September, 2010

**Abstract** The Dirac Delta Function, the idealization of an impulse in Radar circuits, is a Hyper-Real function which definition and analysis require Infinitesimal Calculus, and Infinite Hyper-reals.

The controversy surrounding the Leibnitz Infinitesimals derailed the development of the Infinitesimal Calculus, and the Delta Function could not be defined and investigated properly.

For instance, it is labeled a “Generalized Function” although it generalizes no function.

Dirac’s intuitive definition by Delta’s sampling property

$$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1,$$

that avoids specifying  $\delta(0)$ , remains the main definition of the delta function, although the Delta Function is not Riemann integrable in the Calculus of Limits, and is not Lebesgue integrable in Measure Theory.

In fact, in the Calculus of Limits, only the Cauchy Principal Value

Integral of the Delta Function exists, and it equals zero.

Only in Infinitesimal Calculus, can the Delta Function be defined, differentiated, and integrated.

Infinitesimal Calculus allows us to resolve open problems such as

What is  $\delta(0)$ ?

How is  $x\delta(x)$  defined at  $x = 0$ ?

How is the Delta Function the derivative of a Step Function?

How do we integrate the Delta Function?

What is  $\delta(x^2)$ ?

What is  $\delta^2(x)$ ?

What is  $\delta(x^3)$ ?

What is  $\delta^3(x)$ ?

The Delta Function enables us to define the Fourier Transform with minimal requirements on the transformed function.

**Keywords:** Infinitesimal, Infinite-Hyper-Real, Hyper-Real, Cardinal, Infinity. Non-Archimedean, Non-Standard Analysis, Calculus, Limit, Continuity, Derivative, Integral,

**2000 Mathematics Subject Classification** 26E35; 26E30; 26E15; 26E20; 26A06; 26A12; 03E10; 03E55; 03E17; 03H15; 46S20; 97I40; 97I30.

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## References

# Introduction

## 0.1 Cauchy, Poisson, and Riemann

Cauchy (1816), and Poisson (1815) derived the Fourier Integral Theorem by using the sifting property of the Delta Function.

By Fourier Integral Theorem

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} \left( \int_{\xi=-\infty}^{\xi=\infty} f(\xi) e^{-ik\xi} d\xi \right) e^{ikx} dk \\ &= \int_{\xi=-\infty}^{\xi=\infty} f(\xi) \left( \frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} e^{-ik(\xi-x)} dk \right) d\xi \end{aligned}$$

Denoting

$$\frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} e^{-ik(\xi-x)} dk \equiv \delta(\xi - x),$$

the Delta Function is the Fourier Transform of the constant function 1,

And Fourier Integral theorem states the sifting property for the Delta Function

$$f(x) = \int_{\xi=-\infty}^{\xi=\infty} f(\xi) \delta(\xi - x) d\xi.$$

In the derivation of his Zeta Function, Riemann (1859) uses this sifting property repeatedly, without using a function notation for

the integral  $\frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} e^{-ik(\xi-x)} dk$  that represents the Delta function

$\delta(\xi - x)$ . The derivations are in [Dan4, p.84, p.90, p.97].

The derivations were not supplied by Riemann. Riemann's 1859 paper, as well as much of Riemann's published writings, outlines ideas, and states results without proof.

In particular, the representation of Delta that follows from the Fourier Integral Theorem does not hold in the Calculus of Limits.

Indeed,

$$\xi = x \Rightarrow e^{-ik(\xi-x)} = 1,$$

and the integral

$$\frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} e^{-ik(\xi-x)} dk$$

diverges.

Avoiding the singularity at  $\xi = x$  does not recover the Theorem, because without the singularity the integral equals zero.

Thus, the Fourier Integral Theorem cannot be written in the Calculus of Limits.

In other words, the indeterminate nature of singularities in the Calculus of Limits does not allow the Fourier Integral Theorem to hold.

## 0.2 Dirac

The Delta Function can be realized as a Radar transmission Pulse. A Radar Transmission has to be pulsed because a continuous wave train will not allow us to measure the time interval  $\tau$  between transmission and reception, and determine the range of the target by  $r = \frac{1}{2}c\tau$ .

Thus, a transmission lasts very short time. Then, the Radar system converts into a receiver for the reflected signal. This process of transmitting and receiving repeats thousands of time per second, in order to follow a moving target.

The Radar pulse envelops a carrier wave of very short wavelength. Radar carrier waves went down from centimeters to micrometers of light wavelength.

Since the illuminating power dissipation is proportional to  $\frac{1}{r^2}$ , the short electromagnetic wave-train has an electric field that seems nearly infinite, although the pulse power is finite.

Dirac (1930) was familiar with Radar Pulses when he defined the Delta Function in [Dirac, p.71] through the sifting property,

$$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1,$$

and

$$\delta(x) = 0, \text{ for all } x \neq 0.$$

Dirac definition left open the question of the nearly infinite amplitude at  $x = 0$ . That is,  $\delta(0)$  was left undefined.

Then, the sifting property does not hold.

Indeed, since  $\delta(0)$  is infinite, the integration of  $\delta(x)$  has to skip the point  $x = 0$ , and only the Cauchy Principal Value of the

integral  $\int_{x=-\infty}^{x=\infty} \delta(x)dx$  may exist. Then,

$$\lim_{n \rightarrow \infty} \left( \int_{x=-\infty}^{x=-\frac{1}{n}} \delta(x)dx + \int_{x=\frac{1}{n}}^{x=\infty} \delta(x)dx \right) = 0,$$

in contradiction to  $\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1$ .

### 0.3 Laurent Schwartz

Laurent Schwartz presents his Delta Distribution as follows

[Schwartz, p. 82]

$$\text{Let } Y = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

*Then, for any  $\varphi(x)$  infinitely differentiable, that vanishes at  $\infty$ , and at  $-\infty$ ,*

$$\begin{aligned}
\int_{x=-\infty}^{x=\infty} Y'(x)\varphi(x)dx &= - \int_{x=-\infty}^{x=\infty} Y(x)\varphi'(x)dx \\
&= - \int_{x=0}^{x=\infty} \varphi'(x)dx = -\varphi(x)\Big|_{x=0}^{x=\infty} \\
&= \varphi(0) \\
&= \int_{x=-\infty}^{x=\infty} \delta(x)\varphi(x)dx
\end{aligned}$$

*Thus,*

$$Y' = \delta.$$

Since  $Y(x)$  is not defined at  $x = 0$ ,  $Y'(0)$  is not defined, and the conclusion  $Y' = \delta$ , avoids  $\delta(0)$ .

That is, Schwartz' Definition is as incomplete as Dirac's.

Furthermore, the equality

$$\varphi(0) = \int_{x=-\infty}^{x=\infty} \delta(x)\varphi(x)dx$$

is the definition of the Delta Function by its sifting property that does not hold.

Since  $\delta(0)$  is not defined, the integration has to skip the point  $x = 0$ , and the integral is the Cauchy Principal Value Integral



$$\lim_{n \rightarrow \infty} \left( \int_{x=-\infty}^{x=-\frac{1}{n}} \delta(x)\varphi(x)dx + \int_{x=\frac{1}{n}}^{x=\infty} \delta(x)\varphi(x)dx \right) = 0$$

Like the Dirac Delta, the Schwartz Delta avoids  $\delta(0)$ , and postulates the sifting property.

## 0.4 Delta Sequence

Attempts to get back to the singular Delta Function, replaced the Delta Function by a Delta Sequence of functions that converge to the Delta Function. For instance,

$$\delta_n(x) = n \chi_{[-\frac{1}{2n}, \frac{1}{2n}]}(x) = \begin{cases} n, & x \in [-\frac{1}{2n}, \frac{1}{2n}] \\ 0, & \text{otherwise} \end{cases}.$$

Then, the delta Function is defined as the limit

$$\delta(x) = \lim_{n \rightarrow \infty} \delta_n(x).$$

The sequential approach is reviewed in [Mikusinski], and is used in Mathematical Physics texts.

However, the Delta Sequence contradicts Dirac's definition.

Indeed, as  $n \rightarrow \infty$ ,

$$\delta(0) = \lim_{n \rightarrow \infty} \delta_n(0) = \lim_{n \rightarrow \infty} n = \infty.$$

Then, the integration of  $\delta(x)$  has to skip the point  $x = 0$ .

That is, only the Cauchy Principal Value of the integral

$\int_{x=-\infty}^{x=\infty} \delta(x)dx$  may exist. The Principal Value is

$$\lim_{n \rightarrow \infty} \left( \int_{x=-\infty}^{x=-\frac{1}{n}} \delta(x)dx + \int_{x=\frac{1}{n}}^{x=\infty} \delta(x)dx \right) = 0.$$

That is, the sifting  $\int_{x=-\infty}^{x=\infty} \delta_n(x)dx = 1$ , is not preserved for the limit

of the Delta sequence,  $\delta(x) = \lim_{n \rightarrow \infty} \delta_n(x)$ .  $\square$

## 0.5 The Hyper-real Delta Function

The above attempts failed because the Delta Function is a hyper-real function. A function from the hyper-reals into the hyper-reals.

By resolving the problem of the infinitesimals [Dan2], we obtained the Infinite Hyper-reals that are strictly smaller than  $\infty$ , and can serve to supply the value of the Delta Function at the singularity.

The attempts to get by with Calculus restricted to the real line, deprived Calculus of its full power. In Infinitesimal Calculus, [Dan3], we differentiate over a jump discontinuity of a step function, and obtain the Delta Function. We can integrate over a singularity, and obtain a finite value.

Here, we present the Delta Function, and the properties of the Delta Function in Infinitesimal Calculus.

In particular, we resolve open problems such as

What is  $\delta(0)$ ?

How is  $x\delta(x)$  defined at  $x = 0$ ?

How is the Delta Function the derivative of a Step Function?

How do we integrate the Delta Function?

What is  $\delta(x^2)$ ?

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**1.****Hyper-real Line**

Each real number  $\alpha$  can be represented by a Cauchy sequence of rational numbers,  $(r_1, r_2, r_3, \dots)$  so that  $r_n \rightarrow \alpha$ .

The constant sequence  $(\alpha, \alpha, \alpha, \dots)$  is a constant hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences  $(l_1, l_2, l_3, \dots)$  constitutes a family of infinitesimal hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals  $\left(\frac{1}{l_1}, \frac{1}{l_2}, \frac{1}{l_3}, \dots\right)$  are the infinite hyper-reals.
4. The infinite hyper-reals are greater than any real number, yet strictly smaller than infinity.
5. The infinite hyper-reals with negative signs are smaller than any real number, yet strictly greater than  $-\infty$ .
6. The sum of a real number with an infinitesimal is a non-constant hyper-real.
7. The Hyper-reals are the totality of constant hyper-reals, a family of infinitesimals, a family of infinitesimals with

- negative sign, a family of infinite hyper-reals, a family of infinite hyper-reals with negative sign, and non-constant hyper-reals.
8. The hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.
  9. That line includes the real numbers separated by the non-constant hyper-reals. Each real number is the center of an interval of hyper-reals, that includes no other real number.
  10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs,  $-dx$ .
  11. Zero is not an infinitesimal, because zero is not strictly greater than zero.
  12. We do not add infinity to the hyper-real line.
  13. The infinitesimals, the infinitesimals with negative signs, the infinite hyper-reals, and the infinite hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.
  14. The hyper-real line is embedded in  $\mathbb{R}^\infty$ , and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the hyper-real onto the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal hyper-reals, or to the infinite hyper-reals, or to the non-constant hyper-reals.
16. No neighbourhood of a hyper-real is homeomorphic to an  $\mathbb{R}^n$  ball. Therefore, the hyper-real line is not a manifold.
17. The hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

## 2.

# Delta Function Definition

## 2.1 Domain and Range

The Delta Function is a hyper-real function defined from the hyper-real line into the set of two hyper-reals

$$\left\{ 0, \frac{1}{dx} \right\}.$$

The hyper-real 0 is the sequence  $\langle 0, 0, 0, \dots \rangle$ .

The infinite hyper-real  $\frac{1}{dx}$  depends on our choice of  $dx$ . We will usually choose the family of infinitesimals that is spanned by the sequences  $\langle \frac{1}{n} \rangle, \langle \frac{1}{n^2} \rangle, \langle \frac{1}{n^3} \rangle, \dots$ . It is a semigroup with respect to vector addition, and includes all the scalar multiples of the generating sequences that are non-zero. That is, the family includes infinitesimals with negative sign.

Therefore,  $\frac{1}{dx}$  will mean the sequence  $\langle n \rangle$ .

Alternatively, we may choose the family spanned by the sequences  $\langle \frac{1}{2^n} \rangle, \langle \frac{1}{2^{2n}} \rangle, \langle \frac{1}{2^{3n}} \rangle, \dots$ . Then,  $\frac{1}{dx}$  will mean the sequence  $\langle 2^n \rangle$ .

Once we determined the basic infinitesimal  $dx$ , we will use it in the Infinite Riemann Sum that defines an Integral in

Infinitesimal Calculus.

## 2.2 The Delta Function is strictly smaller than $\infty$

*Proof:* Since  $dx > 0$ ,  $\frac{1}{dx} < \infty$ .

## 2.3 Definition of the Delta Function

We define,

$$\delta(x) \equiv \frac{1}{dx} \chi_{\left[-\frac{dx}{2}, \frac{dx}{2}\right]}(x),$$

where

$$\chi_{\left[-\frac{dx}{2}, \frac{dx}{2}\right]}(x) = \begin{cases} 1, & x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right] \\ 0, & \text{otherwise} \end{cases}.$$

This means that

for  $x < -\frac{1}{2}dx$ ,  $\delta(x) = 0$

at  $x = -\frac{1}{2}dx$ ,  $\delta(x)$  jumps from 0 to  $\frac{1}{dx}$ ,

for  $x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right]$ ,  $\delta(x) = \frac{1}{dx}$ .

at  $x = 0$ ,  $\delta(0) = \frac{1}{dx}$

at  $x = \frac{1}{2}dx$ ,  $\delta(x)$  drops from  $\frac{1}{dx}$  to 0.

for  $x > \frac{1}{2}dx$ ,  $\delta(x) = 0$ .



$$\mathbf{2.4} \quad \delta(x) = \frac{1}{dx} \begin{cases} 1, x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right] \\ 0, \text{otherwise} \end{cases} \text{ is the sequence } \left\langle \begin{cases} \frac{1}{i^n}, x \in \left[-\frac{i_n}{2}, \frac{i_n}{2}\right] \\ 0, \text{otherwise} \end{cases} \right\rangle$$

where  $dx = \langle i_n \rangle$ .

Namely, as a hyper-real function the value of Delta at the singularity is the infinite hyper-real  $\frac{1}{dx}$  which is a sequence, an infinite vector with countably many components.

### 3.

## Delta Function Plots

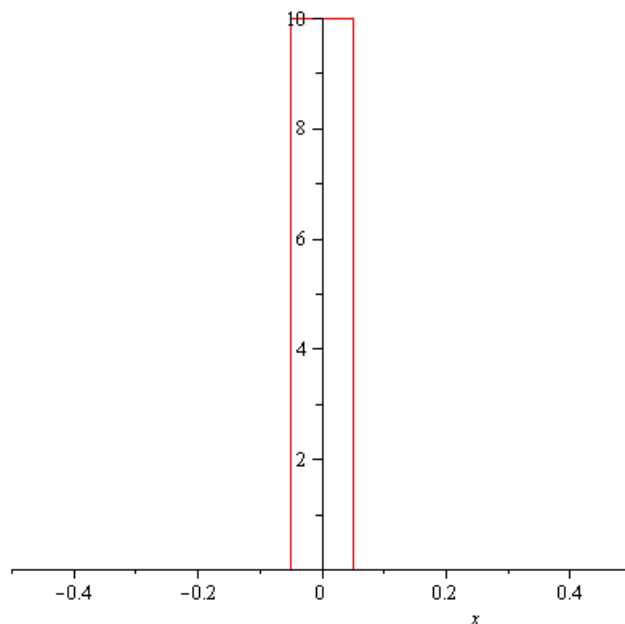
### 3.1 Delta Plot for $dx = \langle \frac{1}{n} \rangle$

If  $i_n = \frac{1}{n}$ , Delta is the infinite Hyper-Real number,

$$\delta(x) = \left\langle \chi_{[-1,1]}(x), 2\chi_{[-\frac{1}{2},\frac{1}{2}]}(x), 3\chi_{[-\frac{1}{3},\frac{1}{3}]}(x), \dots \right\rangle$$

We plot in Maple the 10<sup>th</sup> component with

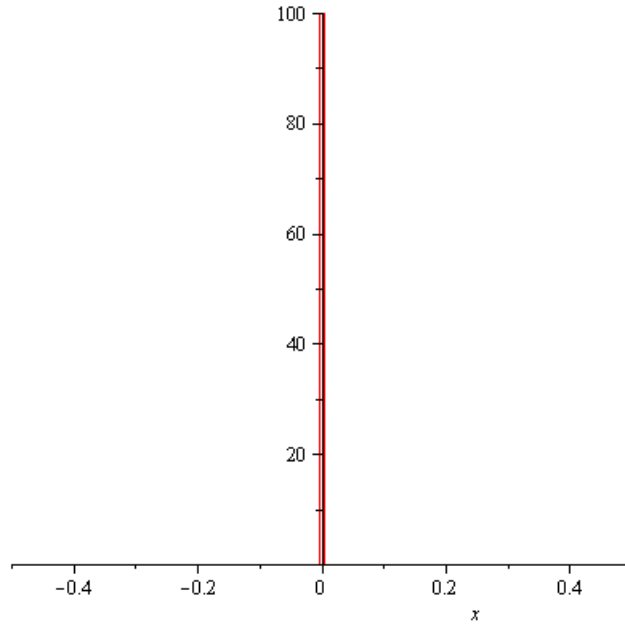
$$\text{plot} \left( \left\{ \begin{array}{l} 10 \text{ } |x| \leq \frac{1}{20} \\ 0 \text{ } |x| > \frac{1}{20} \end{array} \right. , x = -0.5 .. 0.5 \right)$$



Similarly, we use

$$\text{plot} \left( \left\{ \begin{array}{l} 100 \quad |x| \leq \frac{1}{200} \\ 0 \quad |x| > \frac{1}{200} \end{array} \right. , x = -0.5 \dots 0.5 \right)$$

to plot an the 100<sup>th</sup> component of Delta



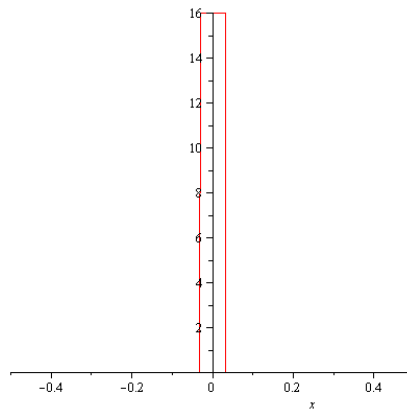
### 3.2 Delta with $dx = \left\langle \frac{1}{2^n} \right\rangle$ is

$$\delta(x) = \left\langle 2\chi_{[-\frac{1}{4}, \frac{1}{4}]}(x), 4\chi_{[-\frac{1}{8}, \frac{1}{8}]}(x), 8\chi_{[-\frac{1}{16}, \frac{1}{16}]}(x), \dots \right\rangle$$

We use

$$\text{plot} \left( \left\{ \begin{array}{l} 16 \quad |x| \leq \frac{1}{32} \\ 0 \quad |x| > \frac{1}{32} \end{array} \right. , x = -0.5 \dots 0.5 \right)$$

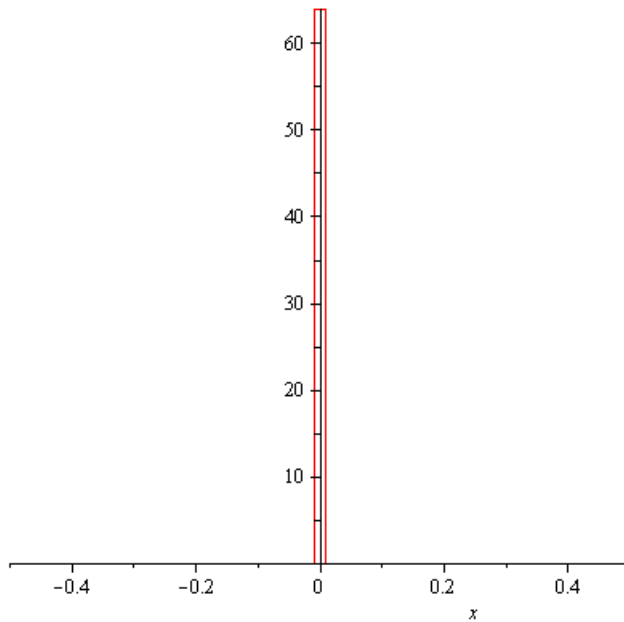
to plot the 4<sup>th</sup> component of Delta



Similarly, we use

$$plot \left( \begin{cases} 64 & |x| \leq \frac{1}{128} \\ 0 & |x| > \frac{1}{128} \end{cases}, x = -0.5 .. 0.5 \right)$$

to plot the 6<sup>th</sup> component of Delta,



## 4.

# Delta Function Properties

**4.1**  $x\delta(x) = 0$

*Proof:*  $x \neq 0 \Rightarrow \delta(x) = 0 \Rightarrow x\delta(x) = 0.$

$$x = 0 \Rightarrow x\delta(x) = 0\delta(0) = \frac{0}{dx} = 0, \text{ since } dx > 0.$$

**4.2** 
$$\delta(x - x_0) \equiv \frac{1}{d(x - x_0)} \mathcal{X}_{[x_0 - \frac{dx}{2}, x_0 + \frac{dx}{2}]}(x)$$

$$= \frac{1}{dx} \mathcal{X}_{[x_0 - \frac{dx}{2}, x_0 + \frac{dx}{2}]}(x)$$

**4.3** 
$$\delta^n(x) = \frac{1}{(dx)^n} \mathcal{X}_{[-\frac{dx}{2}, \frac{dx}{2}]}(x), \quad n = 2, 3, \dots$$

That is,  $\delta^n(0)$  spikes to  $\frac{1}{(dx)^n}$ , which is greater than  $\frac{1}{dx}$ .

**4.4** 
$$\delta(x, y) \equiv \delta(x)\delta(y) = \frac{1}{dx} \mathcal{X}_{[-\frac{dx}{2}, \frac{dx}{2}]}(x) \frac{1}{dy} \mathcal{X}_{[-\frac{dy}{2}, \frac{dy}{2}]}(y)$$

## 5.

**Delta Sequence**  $\delta_n(x) = n \frac{1}{2 \cosh^2 nx}$

Depending on the choice of the infinitesimal  $dx = \langle i_n \rangle$ , there are many Delta Sequences that lead to the Delta Function,  $\delta(x)$ .

**5.1** Each  $\delta_n(x) = n \frac{1}{2 \cosh^2 nx}$

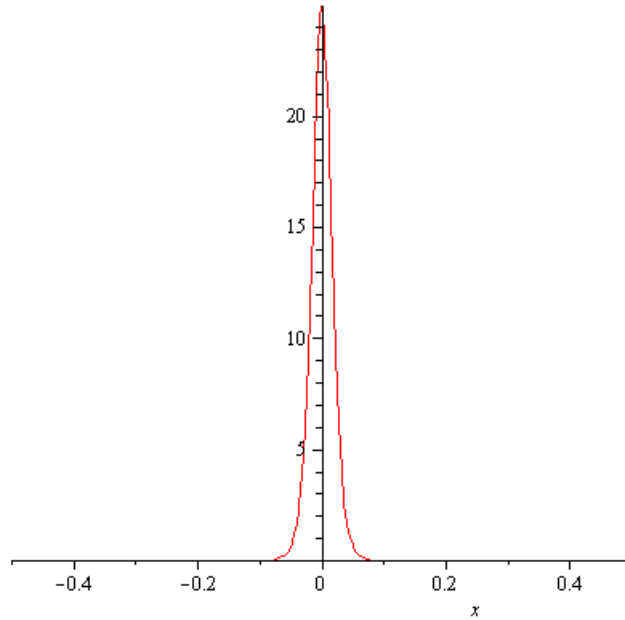
- has the sifting property  $\int_{x=-\infty}^{x=\infty} \delta_n(x) dx = 1$
- is continuous
- peaks at  $x = 0$  to  $\delta_n(0) = \frac{n}{2}$

*Proof:*  $\int_{x=-\infty}^{x=\infty} n \frac{1}{2 \cosh^2 nx} dx = n \frac{\tanh nx}{2n} \Big|_{x=-\infty}^{x=\infty} = \frac{1}{2}(1 - (-1)) = 1. \square$

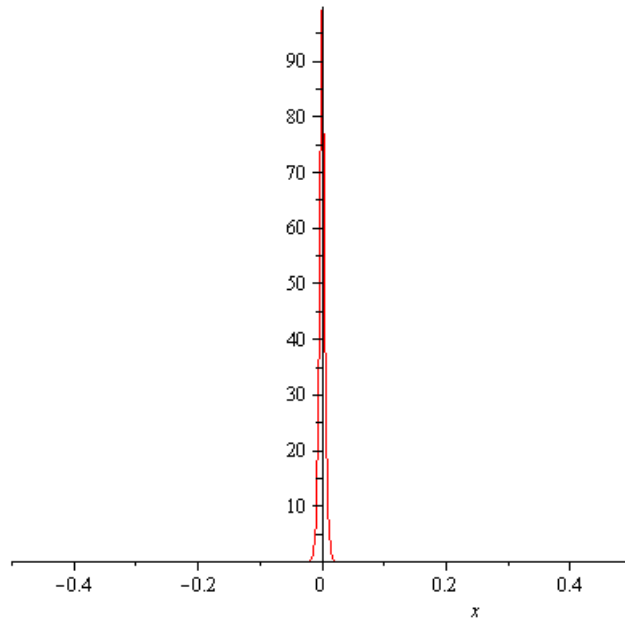
The sequence represents the hyper-real Delta Function

**5.2** If  $i_n = \frac{2}{n}$ ,  $\delta(x) = \left\langle \frac{1}{2 \cosh^2 x}, \frac{2}{2 \cosh^2 2x}, \frac{3}{2 \cosh^2 3x}, \dots \right\rangle$

$plot\left(\frac{50}{2 \cosh^2(50x)}, x=-0.5..0.5\right)$  plots in Maple, the 50<sup>th</sup> component,



$plot\left(\frac{200}{2 \cosh^2(200x)}, x=-0.5..0.5\right)$  plots in Maple the 200<sup>th</sup> component,



## 6.

### Delta Sequence $\delta_n(x) = ne^{-nx}\chi_{[0,\infty)}(x)$

**6.1** Each  $\delta_n(x) = ne^{-nx}\chi_{[0,\infty)}$

- has the sifting property  $\int_{x=-\infty}^{x=\infty} \delta_n(x)dx = 1$
- is continuous hyper-real function
- peaks at  $x = 0$  to  $\delta_n(0) = n$

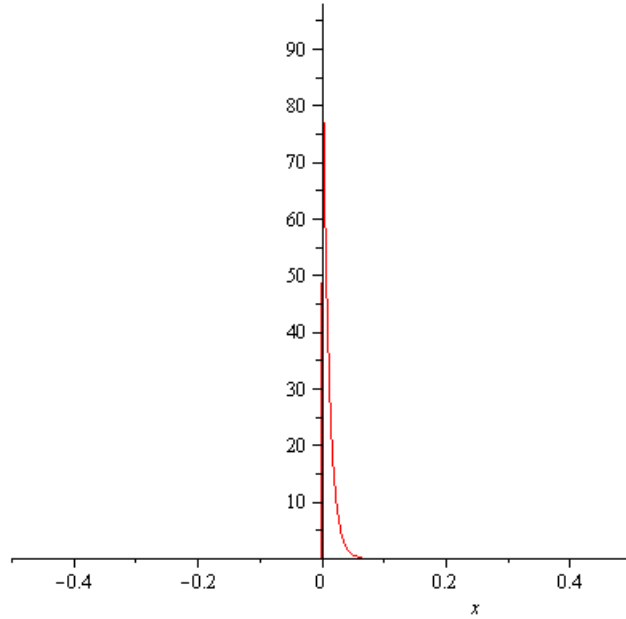
*Proof:* 
$$\int_{x=-\infty}^{x=\infty} ne^{-nx}\chi_{[0,\infty)}(x)dx = \int_{x=0}^{x=\infty} ne^{-nx}dx = n \frac{e^{-nx}}{-n} \Big|_{x=0}^{x=\infty} = 1. \square$$

The sequence represents the hyper-real Delta Function

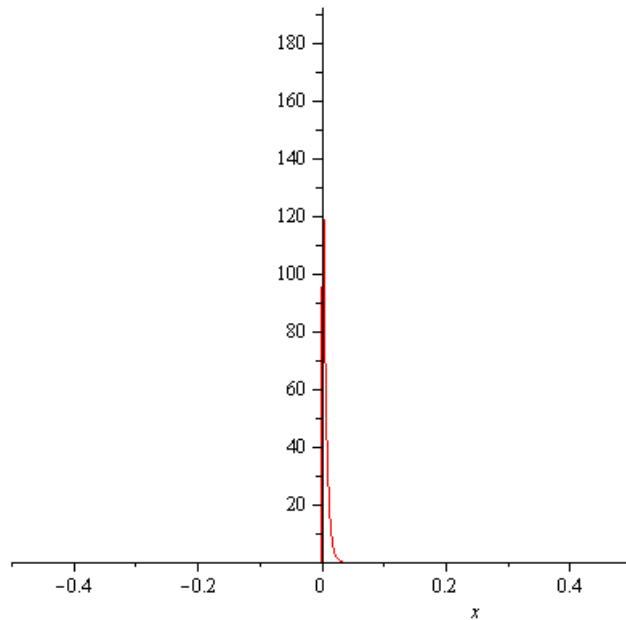
**6.2** If  $i_n = \frac{1}{n}$ ,  $\delta(x) = \langle e^{-x}\chi_{[0,\infty)}, 2e^{-2x}\chi_{[0,\infty)}, 3e^{-3x}\chi_{[0,\infty)}, \dots \rangle$



$plot\left(\left\{\begin{array}{l} 0 \quad x < 0 \\ 100e^{-100x} \quad x \geq 0 \end{array}\right\}, x = -0.5 \dots 0.5\right)$  plots in Maple the 100<sup>th</sup> component,



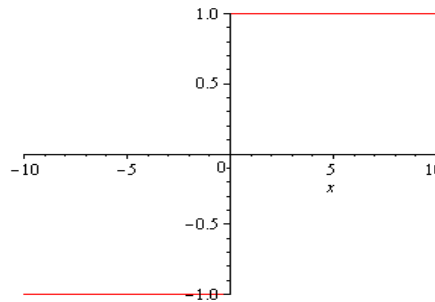
$plot\left(\left\{\begin{array}{l} 0 \quad x < 0 \\ 200e^{-200x} \quad x \geq 0 \end{array}\right\}, x = -0.5 \dots 0.5\right)$  plots in Maple the 200<sup>th</sup> component,



## 7.

# Primitive of Delta Function

**7.1**  $\delta(x)$  is the derivative of  $g(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$



*Proof:* At the jump over  $[-dx, 0]$ , from  $-1$  to  $0$ , for any  $dx$ ,

$$\frac{g(0) - g(0 - dx)}{dx} = \frac{0 - (-1)}{dx} = \frac{1}{dx}$$

Therefore, the left derivative at  $x = 0$  is

$$g'(0-) = \frac{1}{dx}.$$

At the jump over  $[0, dx]$ , from  $0$  to  $1$ , for any  $dx$ ,

$$\frac{g(0 + dx) - g(0)}{dx} = \frac{1 - 0}{dx} = \frac{1}{dx}$$

Therefore, the right derivative at  $x = 0$  is

$$g'(0+) = \frac{1}{dx}.$$

Since the right and left derivatives are equal, the derivative at  $x = 0$  is

$$g'(0) = \frac{1}{dx} = \delta(0). \square$$

Since at  $x \neq 0$ ,  $g'(x) = 0$ , we have,

$$g'(x) = \frac{1}{dx} \chi\left[-\frac{dx}{2}, \frac{dx}{2}\right]. \square$$

**7.2**  $\delta(x)$  is the Principal Value derivative of  $h(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$

*Proof:* For any  $dx$ ,

$$\frac{h(0) - h(-dx)}{dx} = \frac{0}{dx} = 0 \quad \Rightarrow h'(0-) = 0.$$

$$\frac{h(dx) - h(0)}{dx} = \frac{1 - 0}{dx} = \frac{1}{dx} \quad \Rightarrow h'(0+) = \frac{1}{dx}.$$

Therefore,  $h(x)$  has no derivative at  $x = 0$ .  $\square$

But since

$$\frac{h\left(\frac{dx}{2}\right) - h\left(-\frac{dx}{2}\right)}{dx} = \frac{1 - 0}{dx} = \frac{1}{dx},$$

The principal value derivative of  $h(x)$  at  $x = 0$  is

$$\text{p.v.}h'(0) = \frac{1}{dx}. \square$$

Since at  $x \neq 0$ ,  $\text{p.v.}h'(x) = 0$ , we have,  $\text{p.v.}h'(x) = \frac{1}{dx} \chi\left[-\frac{dx}{2}, \frac{dx}{2}\right]. \square$

**8.**

$$\delta(f(x))$$

$$\mathbf{8.1} \quad \delta(ax) = \frac{1}{|a|} \delta(x)$$

*Proof:*  $\delta(ax)adx = \delta(ax)d(ax) = 1 = \delta(x)dx.$

We divide both sides by  $adx$ , and put  $|a|$ , because the Delta's on both sides are positive.  $\square$

**8.2** *If  $\xi_1$  is the only zero of  $f(x)$ , and  $f'(\xi_1) \neq 0$ ,*

*Then,* 
$$\delta(f(x)) = \frac{1}{|f'(\xi_1)|} \delta(x - \xi_1)$$

*Proof:* 
$$\delta(f(x)) = \delta(f(x) - f(\xi_1))$$

For  $x - \xi_1 = \text{infinitesimal}$ ,

$$= \delta(f'(\xi_1)(x - \xi_1))$$

By 8.1,

$$= \frac{1}{|f'(\xi_1)|} \delta(x - \xi_1). \square$$

**8.3** If  $\xi_1, \xi_2$  are the only zeros of  $f(x)$ , and  $f'(\xi_1), f'(\xi_2) \neq 0$

$$\text{Then, } \delta(f(x)) = \frac{1}{|f'(\xi_1)|} \delta(x - \xi_1) + \frac{1}{|f'(\xi_2)|} \delta(x - \xi_2)$$

$$\text{Proof: } \delta(f(x)) = \delta(f(x) - f(\xi_1)) + \delta(f(x) - f(\xi_2))$$

$$\text{If } x - \xi_1 = \text{infinitesimal, } f(x) - f(\xi_1) = f'(\xi_1)(x - \xi_1)$$

$$\text{If } x - \xi_2 = \text{infinitesimal, } f(x) - f(\xi_2) = f'(\xi_2)(x - \xi_2)$$

Either way,

$$\begin{aligned} \delta(f(x)) &= \delta(f'(\xi_1)(x - \xi_1)) + \delta(f'(\xi_2)(x - \xi_2)) \\ &= \frac{1}{|f'(\xi_1)|} \delta(x - \xi_1) + \frac{1}{|f'(\xi_2)|} \delta(x - \xi_2). \square \end{aligned}$$

$$\mathbf{8.4} \quad \delta(x^2 - a^2) = \frac{1}{2|a|} \delta(x - a) + \frac{1}{2|a|} \delta(x + a)$$

$$\mathbf{8.5} \quad \delta((x - a)(x - b)) = \frac{1}{|a - b|} \delta(x - a) + \frac{1}{|b - a|} \delta(x - b)$$

**8.6** If  $\xi_1, \dots, \xi_n$  are the only zeros of  $f(x)$ , and  $f'(\xi_1), \dots, f'(\xi_n) \neq 0$

$$\text{Then, } \delta(f(x)) = \frac{1}{|f'(\xi_1)|} \delta(x - \xi_1) + \dots + \frac{1}{|f'(\xi_n)|} \delta(x - \xi_n)$$

**8.7** If  $\xi_1, \xi_2, \dots$  are zeros of  $f(x)$ , and  $f'(\xi_1), f'(\xi_2), \dots \neq 0$

$$\text{Then, } \delta(f(x)) = \frac{1}{|f'(\xi_1)|} \delta(x - \xi_1) + \frac{1}{|f'(\xi_n)|} \delta(x - \xi_n) + \dots$$

**8.8**  $\delta(\sin x) = \dots + \delta(x + 2\pi) + \delta(x + \pi) + \delta(x) + \delta(x - \pi) + \delta(x - 2\pi) + \dots$

*Proof:* The zeros of  $\sin x$  are  $\dots - 2\pi, -\pi, 0, \pi, 2\pi, \dots$

$$\text{and } |\cos(n\pi)| = 1. \square$$

## 9.

$$\delta(x^n)$$

$$\mathbf{9.1} \quad \delta(x^2) = \frac{1}{2x dx} \mathcal{X}_{[-x dx, x dx]}(x), \quad x > 0$$

*Proof:*

$$\begin{aligned} \delta(x^2) &= \frac{1}{d(x^2)} \mathcal{X}_{\left[-\frac{d(x^2)}{2}, \frac{d(x^2)}{2}\right]}(x) \\ &= \frac{1}{2x dx} \mathcal{X}_{\left[-\frac{d(x^2)}{2}, \frac{d(x^2)}{2}\right]}(x), \end{aligned}$$

where to ensure  $d(x^2) > 0$ , we must have

$$x > 0.$$

The amplitude and domain of the  $\delta(x^2)$  spike depend on  $x$ .

For instance,

$$\mathbf{9.2} \quad \frac{1}{2x dx} \mathcal{X}_{[-x dx, x dx]}(x) \Big|_{x=\frac{1}{2}} = \delta(x)$$

$$\mathbf{9.3} \quad \frac{1}{2x dx} \mathcal{X}_{[-x dx, x dx]}(x) \Big|_{x=\frac{dx}{2}} = \frac{1}{(dx)^2} \mathcal{X}_{\left[-\frac{(dx)^2}{2}, \frac{(dx)^2}{2}\right]}(x) \leq \delta^2(x)$$

$$\begin{aligned}
 \mathbf{9.4} \quad \delta(x^n) &= \frac{1}{d(x^n)} \mathcal{X}_{\left[-\frac{d(x^n)}{2}, \frac{d(x^n)}{2}\right]}(x) \\
 &= \frac{1}{nx^{n-1}dx} \mathcal{X}_{\left[-\frac{n}{2}x^{n-1}dx, \frac{n}{2}x^{n-1}dx\right]}(x), \quad x > 0
 \end{aligned}$$

The amplitude and domain of the  $\delta(x^n)$  spike depend on  $x$ .

For instance,

$$\mathbf{9.5} \quad \frac{1}{nx^{n-1}dx} \mathcal{X}_{\left[-\frac{n}{2}x^{n-1}dx, \frac{n}{2}x^{n-1}dx\right]}(x) \Big|_{x=\left(\frac{1}{n}\right)^{\frac{1}{n-1}}} = \delta(x)$$



**10.**

$$\delta(x^n - (dx)^n)$$

While  $x^2 - (\frac{dx}{2})^2$  is infinitesimally close to  $x^2$ ,  $\delta(x^2 - (\frac{dx}{2})^2)$  is different from  $\delta(x^2)$ .

$$\mathbf{10.1} \quad \delta\left(x^2 - \left(\frac{dx}{2}\right)^2\right) = \frac{1}{dx} \delta\left(x - \frac{dx}{2}\right) + \frac{1}{dx} \delta\left(x + \frac{dx}{2}\right)$$

*Proof:* By 8.4, since  $dx > 0$ .  $\square$

10.1 has two positive spikes. For instance,

$$\begin{aligned} \mathbf{10.2} \quad \left[ \frac{1}{dx} \delta\left(x - \frac{dx}{2}\right) + \frac{1}{dx} \delta\left(x + \frac{dx}{2}\right) \right]_{x=0} &= \frac{1}{dx} \delta\left(-\frac{dx}{2}\right) + \frac{1}{dx} \delta\left(\frac{dx}{2}\right) \\ &= \frac{1}{(dx)^2} \mathcal{X}_{[-dx, 0]}(x) + \frac{1}{(dx)^2} \mathcal{X}_{[0, dx]}(x). \end{aligned}$$

Similarly,  $\delta(x^3 - (dx)^3)$  has three positive spikes.

$$\mathbf{10.3} \quad \delta(x^3 - (dx)^3) = \frac{1}{3(dx)^2} \left( \delta(x - dx) + \delta\left(x - e^{i\frac{2\pi}{3}} dx\right) + \delta\left(x - e^{2i\frac{2\pi}{3}} dx\right) \right).$$

*Proof:*  $x^3 - (dx)^3$  has the three zeros

$$x_1 = dx, \quad x_2 = e^{i\frac{2\pi}{3}} dx, \quad \text{and} \quad x_3 = e^{2i\frac{2\pi}{3}} dx.$$

Since  $f'(x) = 3x^2$ , and since  $|x_2|^2 = |x_3|^2 = (dx)^2$ , by 6.6 we obtain

$$\delta(x^3 - (dx)^3) = \frac{1}{3(dx)^2} \left( \delta(x - dx) + \delta\left(x - e^{i\frac{2\pi}{3}} dx\right) + \delta\left(x - e^{2i\frac{2\pi}{3}} dx\right) \right). \square$$

$\delta(x^n - (dx)^n)$  has  $n$  positive spikes.

$$\mathbf{10.4} \delta(x^n - (dx)^n) = \frac{1}{n(dx)^{n-1}} \left( \delta(x - dx) + \delta\left(x - e^{\frac{2\pi}{n}} dx\right) + \dots + \delta\left(x - e^{(n-1)\frac{2\pi}{n}} dx\right) \right)$$

*Proof:*  $x^n - (dx)^n$  has the  $n$  zeros

$$x_1 = dx, \quad x_2 = e^{i\frac{2\pi}{n}} dx, \dots, x_n = e^{(n-1)i\frac{2\pi}{n}} dx.$$

Since  $f'(x) = nx^{n-1}$ , and since  $|x_1|^{n-1} = \dots = |x_n|^{n-1} = (dx)^{n-1}$ ,

by 6.6 we obtain

$$\delta(x^n - (dx)^n) = \frac{1}{n(dx)^{n-1}} \left( \delta(x - dx) + \delta\left(x - e^{\frac{2\pi}{n}} dx\right) + \dots + \delta\left(x - e^{(n-1)\frac{2\pi}{n}} dx\right) \right). \square$$

# 11.

## Integral of $\delta(x)$

### 11.1 Integral of a Hyper-real Function

Let  $f(x)$  be a hyper-real function on the interval  $[a, b]$ .

$f(x)$  may take infinite hyper-real values, and need not be bounded.

At each

$$a \leq x \leq b,$$

there is a rectangle with base  $[x - \frac{dx}{2}, x + \frac{dx}{2}]$ , height  $f(x)$ , and area

$$f(x)dx.$$

We form the **Integration Sum** of all the areas for the  $x$ 's that start at  $x = a$ , and end at  $x = b$ ,

$$\sum_{x \in [a, b]} f(x)dx.$$

If for any infinitesimal  $dx$ , the Integration Sum has the same hyper-real value, then  $f(x)$  is integrable over the interval  $[a, b]$ .

Then, we call the Integration Sum the integral of  $f(x)$  from  $x = a$ , to  $x = b$ , and denote it by

$$\int_{x=a}^{x=b} f(x)dx .$$

If the hyper-real is infinite, then it is the integral over  $[a, b]$ ,

If the hyper-real is finite,

$$\int_{x=a}^{x=b} f(x)dx = \text{real part of the hyper-real. } \square$$

## 11.2 The countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers,

$Card\mathbb{N}$ , equals the number of Real Numbers,  $Card\mathbb{R} = 2^{Card\mathbb{N}}$ , and

we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = \dots = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = \dots \equiv \infty .$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval  $[a, b]$ , and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many  $f(x)dx$ .

The Lower Integral is the Integration Sum where  $f(x)$  is replaced

by its lowest value on each interval  $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

$$\mathbf{11.3} \quad \sum_{x \in [a, b]} \left( \inf_{x - \frac{dx}{2} \leq t \leq x + \frac{dx}{2}} f(t) \right) dx$$

The Upper Integral is the Integration Sum where  $f(x)$  is replaced by its largest value on each interval  $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

$$\mathbf{11.4} \quad \sum_{x \in [a, b]} \left( \sup_{x - \frac{dx}{2} \leq t \leq x + \frac{dx}{2}} f(t) \right) dx$$

If the integral is a finite hyper-real, we have

**11.5** *A hyper-real function has a finite integral if and only if its upper integral and its lower integral are finite, and differ by an infinitesimal.*

$$\mathbf{11.6} \quad \int_{x=-\infty}^{x=\infty} \delta(x) dx = 1.$$

*Proof:* The only term in the integration Sum is  $\frac{1}{dx} dx = 1$ .

Both the upper integral, and the lower integral are equal to

$$\frac{1}{dx} dx = 1. \square$$

## 12.

# The Principal Value Derivative of Delta: the Dipole Function

We have seen in 2.3 that

at  $x = -\frac{dx}{2}$ ,  $\delta(x)$  jumps from 0 to  $\frac{1}{dx}$ ,

for  $x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right]$ ,  $\delta(x) = \frac{1}{dx}$ . In particular,  $\delta(0) = \frac{1}{dx}$

at  $x = \frac{dx}{2}$ ,  $\delta(x)$  drops from  $\frac{1}{dx}$  to 0.

Here, we show

- ❖ in 12.1, that  $\delta(x)$  has no derivative at  $x = -\frac{1}{2}dx$ , but the Principal Value Derivative over the jump at  $x = -\frac{1}{2}dx$ , is a Positive Impulse Function.
- ❖ in 12.2, that  $\delta(x)$  has no derivative at  $x = \frac{1}{2}dx$ , but the Principal Value Derivative over the jump at  $x = \frac{1}{2}dx$ , is a Negative Impulse Function.

We sum up 12.1, and 12.2. in 12.3.

Namely,  $\delta(x)$  has no derivative at  $x = 0$ , but the Principal Value Derivative over the two jumps is a Dipole Function.

That Dipole function is a positive Impulse Function followed by a negative Impulse function.

Both the positive, and the negative impulses have jumps far greater than the jump of the generating delta function.

### 12.1 The Principal Value Derivative of Delta at $x = -\frac{1}{2} dx$

$\delta(x)$  has no derivative at  $x = -\frac{dx}{2}$ .

*The Principal Value Derivative of  $\delta(x)$  at  $x = -\frac{1}{2} dx$ , is the Positive Impulse function  $\frac{1}{(dx)^2} \chi[-dx, 0]$ .*

*Proof:* The left derivative of  $\delta(x)$  at  $x = -\frac{1}{2} dx$  is

$$\frac{\delta(-\frac{dx}{2}) - \delta(-dx)}{-\frac{dx}{2} + dx} = \frac{\frac{1}{dx} - 0}{\frac{dx}{2}} = \frac{2}{(dx)^2}$$

The right derivative of  $\delta(x)$  at  $x = -\frac{1}{2} dx$  is

$$\frac{\delta(0) - \delta(-\frac{dx}{2})}{0 - (-\frac{dx}{2})} = \frac{\frac{1}{dx} - \frac{1}{dx}}{\frac{dx}{2}} = 0.$$

Since the left and right derivatives are unequal,  $\delta(x)$  has no derivative at  $x = -\frac{1}{2} dx$ .  $\square$

The Principal Value Derivative at  $x = -\frac{1}{2}dx$  is

$$\frac{\delta(0) - \delta(-dx)}{0 - (-dx)} = \frac{\frac{1}{dx} - 0}{dx} = \frac{1}{(dx)^2}.$$

It is the Positive Impulse Function  $\frac{1}{(dx)^2} \chi[-dx, 0]. \square$

## 12.2 The Principal Value Derivative of Delta at $x = \frac{dx}{2}$

$\delta(x)$  has no derivative at  $x = \frac{1}{2}dx$ .

The Principal Value Derivative of  $\delta(x)$  at  $x = \frac{1}{2}dx$ , is the Negative

Impulse Function  $-\frac{1}{(dx)^2} \chi[0, dx]$ .

*Proof:* The Left Derivative of  $\delta(x)$  at  $x = \frac{1}{2}dx$  is

$$\frac{\delta(\frac{dx}{2}) - \delta(0)}{\frac{dx}{2} - 0} = \frac{\frac{1}{dx} - \frac{1}{dx}}{\frac{dx}{2}} = \frac{0}{\frac{dx}{2}} = 0.$$

The Right Derivative of  $\delta(x)$  at  $x = \frac{1}{2}dx$  is

$$\frac{\delta(dx) - \delta(\frac{dx}{2})}{dx - \frac{dx}{2}} = \frac{0 - \frac{1}{dx}}{\frac{dx}{2}} = -\frac{2}{(dx)^2}.$$

Since the Left and Right Derivatives are unequal,  $\delta(x)$  has no derivative at  $x = \frac{1}{2}dx. \square$



The Principal Value Derivative at  $x = \frac{1}{2} dx$  is

$$\frac{\delta(dx) - \delta(0)}{dx} = \frac{0 - \frac{1}{dx}}{dx} = -\frac{1}{(dx)^2}.$$

It is the Negative Impulse Function  $-\frac{1}{(dx)^2} \chi[0, dx]$ .  $\square$

### 12.3 The Principal Value Derivative of Delta at $x = 0$

$\delta(x)$  has no derivative at  $x = 0$ .

The Principal Value Derivative of  $\delta(x)$ , p.v.D $\delta(x)$

is the Dipole Function

$$Dipole(x) = \frac{1}{(dx)^2} \chi[-dx, 0] - \frac{1}{(dx)^2} \chi[0, dx].$$

*Proof:* The Left Derivative of  $\delta(x)$  at  $x = 0$  is

$$\frac{\delta(0) - \delta(-dx)}{dx} = \frac{\frac{1}{dx} - 0}{dx} = \frac{1}{(dx)^2}$$

The Right Derivative of  $\delta(x)$  at  $x = 0$  is

$$\frac{\delta(dx) - \delta(0)}{dx} = \frac{0 - \frac{1}{dx}}{dx} = -\frac{1}{(dx)^2}.$$

Since the left and right derivatives are unequal,  $\delta(x)$  has no derivative at  $x = 0$ .  $\square$

The Principal Value Derivative of  $\delta(x)$  is

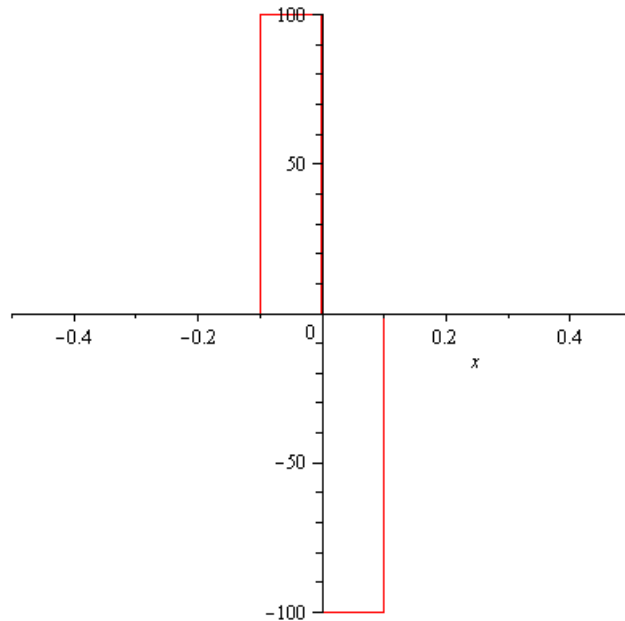
$$\frac{\delta(x + \frac{dx}{2}) - \delta(x - \frac{dx}{2})}{dx} = \frac{1}{dx} \delta(x + \frac{dx}{2}) - \frac{1}{dx} \delta(x - \frac{dx}{2}).$$

It is the Dipole Function  $\frac{1}{(dx)^2} \chi[-dx, 0] - \frac{1}{(dx)^2} \chi[0, dx]. \square$

If  $dx = \langle \frac{1}{n} \rangle$ , this is the sequence

$$Dipole(x) = \langle n^2 \chi[-\frac{1}{n}, 0] \rangle - \langle n^2 \chi[0, \frac{1}{n}] \rangle.$$

Then, a Maple plot of the 10<sup>th</sup> component of  $Dipole(x)$  is



# 13.

## The 2<sup>nd</sup> Principal Value Derivative of Delta: the 4-Pole Function

The 2<sup>nd</sup> Principal Value Derivative of  $\delta(x)$  is the 4-pole Function

$$\begin{aligned} \mathbf{13.1} \quad (\text{p.v.D})^2\delta(x) &= \frac{1}{(dx)^2}(\delta(x + dx) - 2\delta(x) + \delta(x - dx)) \\ &= \frac{1}{(dx)^3} \left\{ \chi\left[-\frac{3dx}{2}, -\frac{dx}{2}\right] - 2\chi\left[-\frac{dx}{2}, \frac{dx}{2}\right] + \chi\left[\frac{dx}{2}, \frac{3dx}{2}\right] \right\}. \end{aligned}$$

$$\begin{aligned} \textit{Proof:} \quad (\text{p.v.D})^2\delta(x) &= \frac{\textit{Dipole}(x + \frac{dx}{2}) - \textit{Dipole}(x - \frac{dx}{2})}{dx}, \\ &= \frac{1}{(dx)^2}(\delta(x + dx) - 2\delta(x) + \delta(x - dx)) \\ &= \frac{1}{(dx)^3} \left\{ \chi\left[-\frac{3dx}{2}, -\frac{dx}{2}\right] - 2\chi\left[-\frac{dx}{2}, \frac{dx}{2}\right] + \chi\left[\frac{dx}{2}, \frac{3dx}{2}\right] \right\}. \square \end{aligned}$$

The 4-pole Function has four Impulse Functions

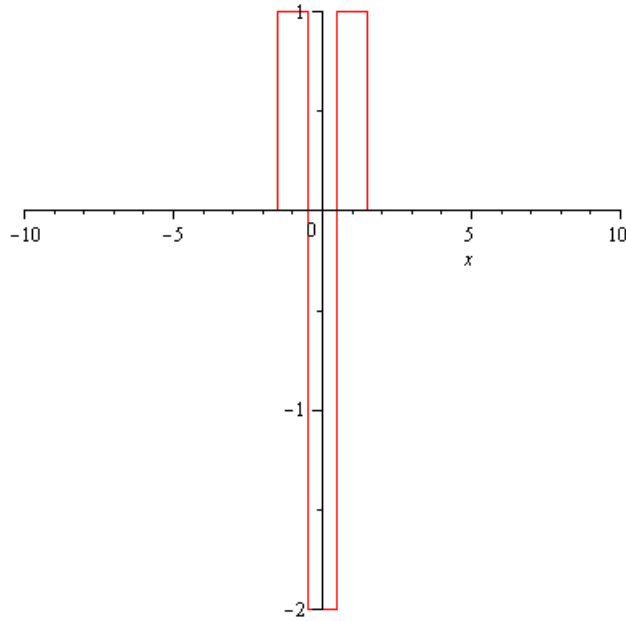
- a Positive Impulse  $\frac{1}{(dx)^2}\delta(x + dx)$  centered at  $x = -dx$ ,

- two Negative Impulses  $-2 \frac{1}{(dx)^2} \delta(x)$  centered at  $x = 0$ ,
- a Positive Impulse  $\frac{1}{(dx)^2} \delta(x - dx)$  centered at  $x = dx$ .

If  $dx = \langle \frac{1}{n} \rangle$ , this is the sequence

$$4pole(x) = \left\langle n^3 \chi\left[-\frac{3}{2n}, -\frac{1}{2n}\right] \right\rangle - \left\langle 2n^3 \chi\left[-\frac{1}{2n}, \frac{1}{2n}\right] \right\rangle + \left\langle n^3 \chi\left[\frac{1}{2n}, \frac{3}{2n}\right] \right\rangle.$$

Then, a Maple plot of a component of  $4pole(x)$  is



*The x axis units are  $\frac{1}{n}$ . The y axis units are  $n^3$ .*

# 14.

## Higher Principal Value Derivatives of Delta

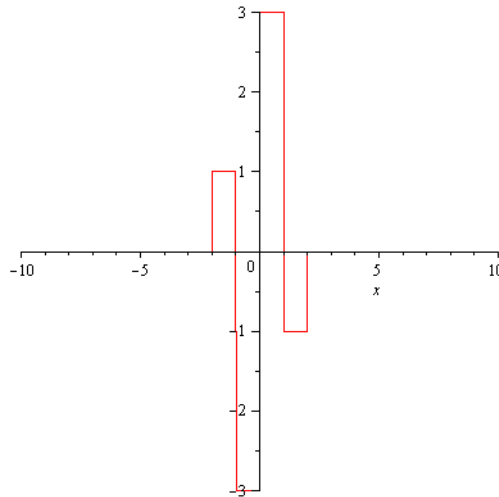
**14.1** *The 3<sup>rd</sup> Principal Value Derivative of  $\delta(x)$ ,  $(\text{p.v.D})^3\delta(x)$  is the 8-pole Function*

$$\delta_{pole}(x) = \frac{1}{(dx)^4} (\chi[-2dx, -dx] - 3\chi[-dx, 0] + 3\chi[0, dx] - \chi[dx, 2dx]).$$

If  $dx = \langle \frac{1}{n} \rangle$ , this is the sequence

$$\delta_{pole}(x) = \langle n^4 \chi[-\frac{2}{n}, -\frac{1}{n}] \rangle - \langle 3n^4 \chi[-\frac{1}{n}, 0] \rangle + \langle 3n^4 \chi[0, \frac{1}{n}] \rangle - \langle n^4 \chi[\frac{1}{n}, \frac{2}{n}] \rangle.$$

Then, a Maple plot of a component of  $\delta_{pole}(x)$  is



*The x axis units are  $\frac{1}{n}$ . The y axis units are  $n^4$ .*

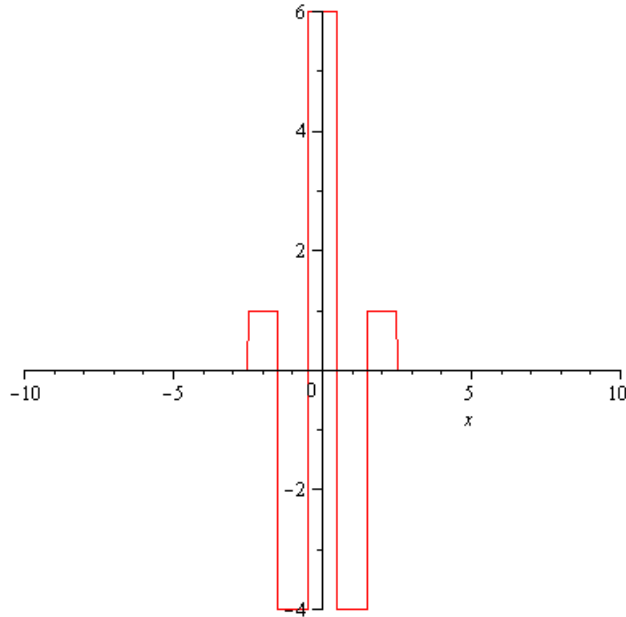
**14.2** *The 4<sup>th</sup> Principal Value Derivative of  $\delta(x)$ ,  $(p.v.D)^4\delta(x)$  is the 16-pole Function*

$$16pole(x) = \frac{1}{(dx)^5} \left( \chi\left[-\frac{5dx}{2}, -\frac{3dx}{2}\right] - 4\chi\left[-\frac{3dx}{2}, -\frac{dx}{2}\right] + \right. \\ \left. + 6\chi\left[-\frac{dx}{2}, \frac{dx}{2}\right] - 4\chi\left[\frac{dx}{2}, \frac{3dx}{2}\right] + \chi\left[\frac{3dx}{2}, \frac{5dx}{2}\right] \right).$$

If  $dx = \left\langle \frac{1}{n} \right\rangle$ , this is the sequence

$$16pole(x) = \left( n^5\chi\left[-\frac{5}{2n}, -\frac{3}{2n}\right] - 4n^5\chi\left[-\frac{3}{2n}, -\frac{1}{2n}\right] + \right. \\ \left. + 6n^5\chi\left[-\frac{1}{2n}, \frac{1}{2n}\right] - 4n^5\chi\left[\frac{1}{2n}, \frac{3}{2n}\right] + n^5\chi\left[\frac{3}{2n}, \frac{5}{2n}\right] \right).$$

Then, a Maple plot of a component of  $16pole(x)$  is



*The x axis units are  $\frac{1}{n}$ . The y axis units are  $n^5$ .*

Using the Binomial coefficients,

**14.3** *The  $k^{\text{th}}$  Principal Value Derivative of  $\delta(x)$ , (p.v.D) $^k \delta(x)$  is*

$$\begin{aligned} 2^k \text{pole}(x) &= \frac{2^{k-1} \text{pole}(x + \frac{dx}{2}) - 2^{k-1} \text{pole}(x - \frac{dx}{2})}{dx} \\ &= \frac{1}{(dx)^{k+1}} \left( \chi\left[-\frac{(k+1)dx}{2}, -\frac{(k-1)dx}{2}\right] - \binom{k}{1} \chi\left[-\frac{(k-1)dx}{2}, -\frac{(k-3)dx}{2}\right] + \right. \\ &\quad \left. + \binom{k}{2} \chi\left[-\frac{(k-1)dx}{2}, -\frac{(k-3)dx}{2}\right] + \dots + (-1)^k \chi\left[\frac{(k-1)dx}{2}, \frac{(k+1)dx}{2}\right] \right). \end{aligned}$$

If  $dx = \left\langle \frac{1}{n} \right\rangle$ , this is the sequence

$$\begin{aligned} &\left\langle n^{k+1} \chi\left[-\frac{(k+1)}{2n}, -\frac{(k-1)}{2n}\right] \right\rangle - \left\langle n^{k+1} \binom{n}{1} \chi\left[-\frac{(n-1)}{2}, -\frac{(n-3)}{2}\right] \right\rangle + \\ &\quad + \left\langle n^{k+1} \binom{k}{2} \chi\left[-\frac{k-1}{2n}, -\frac{k-3}{2n}\right] \right\rangle + \dots + \left\langle (-1)^k n^{k+1} \chi\left[\frac{k-1}{2n}, \frac{k+1}{2n}\right] \right\rangle \end{aligned}$$

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