Delta Function of a Complex Variable

H. Vic Dannon
vic0@comcast.net

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Abstract Ignorance of Infinitesimal Complex Calculus prevented the definition of the Complex Delta Function.

$\delta(\zeta - z)$ is a Plane Delta Function that spikes to $\dfrac{1}{2\pi i(\zeta - z)}$ on the Infinitesimal Disk $\left|\zeta - z\right| \le dr$, and vanishes outside it.

For $dr = \left\langle \dfrac{1}{n} \right\rangle$, $r = \left|z\right|$, and $\phi = \operatorname{Arg}(z)$

$$\delta(z) = \frac{1}{2\pi i} e^{-i\phi} \frac{1}{dr} e^{-\frac{r}{dr}}.$$

Delta has the Bessel Integral Representation

$$\delta(z) = \frac{1}{2\pi i} e^{-i\phi} r \int_{\Omega=0}^{\Omega=\infty} J_0(\Omega r)\Omega \, d\Omega$$

The Circulation of Delta along the Infinitesimal circle $\left|\zeta - z\right| = dr$

is
$$\oint_{\left|\zeta - z\right| = dr} \delta(\zeta - z) d\zeta = 1$$

And If $f(z)$ is Hyper-Complex Differentiable function at z

Then, $$\oint_{|\zeta - z| = dr} f(\zeta)\delta(\zeta - z)d\zeta = f(z).$$

For an analytic function on a simply connected domain, this Sifting through the function values by the Complex Delta Function is the Cauchy Integral Formula

Also the Residue of a Laurent Expansion of a singular Function follows from the sifting property of the Delta Function.

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References

Introduction

0.1 Dirac's Confusion about the Complex Delta Function

The confusion about the Complex Delta Function can be traced to Dirac. In [Dirac] at the bottom of page 778, he writes

"…,we must use the formula

$$\frac{d}{dz}\log z = \frac{1}{z} - i\pi\delta(z), \qquad (27)$$

in which the term

$$-i\pi\delta(z)$$

is required in order to make the integral of the right-hand side of (27) between the limits a, and $-a$ equal

$$\log(-1),$$

the integral of

$$\frac{1}{z}$$

between these limits being assumed to be zero."

Dirac made here four errors:

1st Error, $\underline{\log(-1) \neq -i\pi}$

Indeed, integrating the left-hand side of (27), for $a \neq 0$,

$$\int_{z=-a}^{z=a} \frac{d}{dz}\log z = \log z\Big|_{z=-a}^{z=a}$$

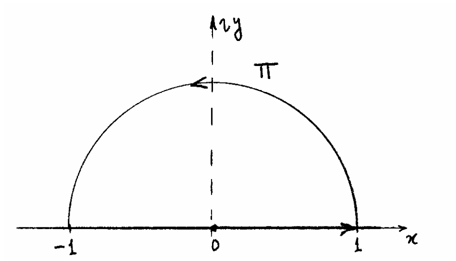
$$= \log a - \log(-a)$$

$$= \log \frac{a}{-a}$$

$$= \log(-1)$$

$$= \underbrace{\log\left|-1\right|}_{=0} + i\,\underbrace{\mathrm{Arg}(-1)}_{\pi}$$

$$= i\pi\,.$$

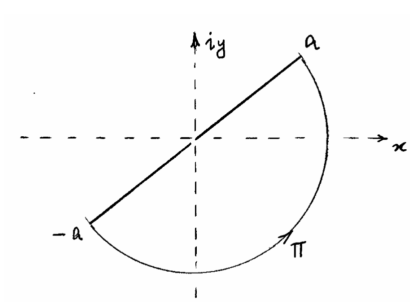


2nd Error $\underline{\displaystyle\int_{z=-a}^{z=a} \frac{1}{z}\,dz \neq 0}$

Indeed, by the Residue Theorem

$$\int_{z=-a}^{z=a} \frac{1}{z}\,dz = i\underbrace{\left[\mathrm{Arg}(a) - \mathrm{Arg}(-a)\right]}_{\pi}$$

$$= i\pi\,.$$



3rd Error $\underline{\dfrac{d}{dz}\log z \neq \dfrac{1}{z} - i\pi\delta(z)}$

Indeed, it is well known that without any exception

$$\frac{d}{dz}\log z = \frac{1}{z}.$$

4th Error $\underline{\displaystyle\int_{z=-a}^{z=a} \delta(z)dz \neq 1}$

Indeed, unlike the hyper-real Delta Function $\delta(x)$, which sifts through its singularity at $x = 0$, the Hyper-Complex Delta Function sifts along a line that encircles the singularity, and avoids it. In the proceeding, it will become clear that

$$\int_{z=-a}^{z=a} \delta(z)dz = \frac{1}{2\pi i} i \underbrace{[\mathrm{Arg}(a) - \mathrm{Arg}(-a)]}_{\pi}$$

$$= \frac{1}{2}.$$

Dirac's theory about the infinite distribution of electrons in the theory of the positron is flawed, because of Dirac's confusion about the Complex Delta Function.

0.2 The Hyper-Complex Delta Function

For z in the interior of γ, Cauchy Integral Formula gives an analytic $f(z)$ as the convolution of f with $\dfrac{1}{2\pi i}\dfrac{1}{\zeta}$.

$$f(z) = \oint_{\gamma} f(\zeta)\frac{1}{2\pi i}\frac{1}{\zeta - z}d\zeta$$

Thus, $\frac{1}{2\pi i}\frac{1}{\zeta - z}$ recovers the value of a complex function $f(\zeta)$ at the point z in the interior of a loop γ, by sifting through the values of $f(\zeta)$ on γ.

However, $\frac{1}{2\pi i}\frac{1}{\zeta - z}$ is the Hyper-Complex Delta Function only if the integration path is in the infinitesimal disk

$$\left|\zeta - z\right| \leq dr,$$

then, the sifting is performed by

$$\frac{1}{2\pi i}\frac{1}{\zeta - z}\chi_{\{|\zeta - z| \leq dr\}}(\zeta) \equiv \delta(\zeta - z).$$

We call $\delta(\zeta - z)$ the <u>Hyper-Complex Delta Function</u>.

In [Dan7], we showed that

$$\oint_{|\zeta - z| = dr} \frac{1}{2\pi i}\frac{1}{\zeta - z}d\zeta = 1.$$

Due to $\dfrac{1}{\zeta - z}$, the Complex Delta spikes to

$$\frac{1}{2\pi i}\frac{1}{d\zeta} = \frac{1}{2\pi i dr}e^{-i\,\mathrm{Arg}(\zeta - z)}$$

on the infinitesimal disk $\left|\zeta - z\right| \leq dr$.

The primitive of the Complex Delta on the disk is

$$\frac{1}{2\pi i}\,\mathrm{Log}(\zeta - z),$$

and the derivative of the Complex Delta on the disk is

$$\frac{1}{2\pi i}\frac{1}{(\zeta - z)^2}.$$

Since

$$\zeta - z = (dr)e^{-i\,\mathrm{Arg}(\zeta - z)},$$

is a hyper-complex infinitesimal, we need to recall the Hyper-Complex Plane that we introduced in [Dan7].

Since the Complex Delta Function is an extension of the Real Delta Function, we need to recall the Delta Function that we introduced in [Dan4].

1.

Hyper-real Line

Each real number α can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant Hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of infinitesimal Hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the infinite Hyper-reals.
4. The infinite Hyper-reals are greater than any real number, yet strictly smaller than infinity.
5. The infinite Hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a non-constant Hyper-real.

7. The Hyper-reals are the totality of constant Hyper-reals, a family of infinitesimals, a family of infinitesimals with negative sign, a family of infinite Hyper-reals, a family of infinite Hyper-reals with negative sign, and non-constant Hyper-reals.
8. The Hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.
9. That line includes the real numbers separated by the non-constant Hyper-reals. Each real number is the center of an interval of Hyper-reals, that includes no other real number.
10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.
11. Zero is not an infinitesimal, because zero is not strictly greater than zero.
12. We do not add infinity to the Hyper-real line.
13. The infinitesimals, the infinitesimals with negative signs, the infinite Hyper-reals, and the infinite Hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.

14. The Hyper-real line is embedded in $\mathbb{R}^{\infty}$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the Hyper-real onto the real line.
15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal Hyper-reals, or to the infinite Hyper-reals, or to the non-constant Hyper-reals.
16. No neighbourhood of a Hyper-real is homeomorphic to an $\mathbb{R}^n$ ball. Therefore, the Hyper-real line is not a manifold.
17. The Hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

2.

Hyper-real Integral

In [Dan3], we defined the integral of a Hyper-real Function.

Let $f(x)$ be a Hyper-real function on the interval $[a,b]$.

The interval may not be bounded.

$f(x)$ may take infinite Hyper-real values, and need not be bounded.

At each

$$a \le x \le b,$$

there is a rectangle with base $[x - \frac{dx}{2}, x + \frac{dx}{2}]$, height $f(x)$, and area

$$f(x)dx.$$

We form the **Integration Sum** of all the areas for the x's that start at $x = a$, and end at $x = b$,

$$\sum_{x \in [a,b]} f(x)dx.$$

If for any infinitesimal dx, the Integration Sum has the same Hyper-real value, then $f(x)$ is integrable over the interval $[a,b]$.

Then, we call the Integration Sum the integral of $f(x)$ from $x = a$, to $x = b$, and denote it by

$$\int_{x=a}^{x=b} f(x)dx\,.$$

If the Hyper-real is infinite, then it is the integral over $[a,b]$,

If the Hyper-real is finite,

$$\int_{x=a}^{x=b} f(x)dx = \text{ real part of the hyper-real}\,.\square$$

2.1 The countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers, $Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty.$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[a,b]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(x)dx$.

The Lower Integral is the Integration Sum where $f(x)$ is replaced by its lowest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

2.2
$$\sum_{x \in [a,b]} \left(\inf_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

The Upper Integral is the Integration Sum where $f(x)$ is replaced by its largest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

2.3
$$\sum_{x \in [a,b]} \left(\sup_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

If the integral is a finite Hyper-real, we have

2.4 *A Hyper-real function has a finite integral if and only if its upper integral and its lower integral are finite, and differ by an infinitesimal.*

3.

Delta Function

In [Dan5], we defined the Delta Function, and established its properties

1. The Delta Function is a Hyper-real function defined from the Hyper-real line into the set of two Hyper-reals $\left\{0, \frac{1}{dx}\right\}$. The Hyper-real 0 is the sequence $\langle 0,0,0,...\rangle$. The infinite Hyper-real $\frac{1}{dx}$ depends on our choice of dx.
2. We will usually choose the family of infinitesimals that is spanned by the sequences $\left\langle \frac{1}{n}\right\rangle, \left\langle \frac{1}{n^2}\right\rangle, \left\langle \frac{1}{n^3}\right\rangle,\ldots$ It is a semigroup with respect to vector addition, and includes all the scalar multiples of the generating sequences that are non-zero. That is, the family includes infinitesimals with negative sign. Therefore, $\frac{1}{dx}$ will mean the sequence $\langle n\rangle$. Alternatively, we may choose the family spanned by the

sequences $\left\langle \frac{1}{2^n} \right\rangle, \left\langle \frac{1}{3^n} \right\rangle, \left\langle \frac{1}{4^n} \right\rangle, \ldots$ Then, $\frac{1}{dx}$ will mean the sequence $\left\langle 2^n \right\rangle$. Once we determined the basic infinitesimal dx, we will use it in the Infinite Riemann Sum that defines an Integral in Infinitesimal Calculus.

3. The Delta Function is strictly smaller than ∞

4. We define, $\delta(x) \equiv \frac{1}{dx} \chi_{\left[-\frac{dx}{2}, \frac{dx}{2}\right]}(x)$,

   where $\chi_{\left[-\frac{dx}{2}, \frac{dx}{2}\right]}(x) = \begin{cases} 1, x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right] \\ 0, \text{otherwise} \end{cases}$.

5. Hence,

   - for $x < 0$, $\delta(x) = 0$
   - at $x = -\frac{dx}{2}$, $\delta(x)$ jumps from 0 to $\frac{1}{dx}$,
   - for $x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right]$, $\delta(x) = \frac{1}{dx}$.
   - at $x = 0$, $\delta(0) = \frac{1}{dx}$
   - at $x = \frac{dx}{2}$, $\delta(x)$ drops from $\frac{1}{dx}$ to 0.
   - for $x > 0$, $\delta(x) = 0$.
   - $x\delta(x) = 0$

6. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle \chi_{[-\frac{1}{2},\frac{1}{2}]}(x), 2\chi_{[-\frac{1}{4},\frac{1}{4}]}(x), 3\chi_{[-\frac{1}{6},\frac{1}{6}]}(x) ... \right\rangle$

7. If $dx = \left\langle \frac{2}{n} \right\rangle$, $\delta(x) = \left\langle \dfrac{1}{2\cosh^2 x}, \dfrac{2}{2\cosh^2 2x}, \dfrac{3}{2\cosh^2 3x}, ... \right\rangle$

8. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle e^{-x}\chi_{[0,\infty)}, 2e^{-2x}\chi_{[0,\infty)}, 3e^{-3x}\chi_{[0,\infty)}, ... \right\rangle$

9. $$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1.$$

4.

The Fourier Transform

In [Dan6], we defined the Fourier Transform and established its properties

1. $\mathcal{F}\left\{\delta(x)\right\} = 1$

2. $\delta(x)$ = *the inverse Fourier Transform of the unit function* 1

$$= \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x} d\omega$$

$$= \int_{\nu=-\infty}^{\nu=\infty} e^{2\pi i x} d\nu\,, \quad \omega = 2\pi\nu$$

3. $\left.\frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x} d\omega\right|_{x=0} = \frac{1}{dx} =$ an infinite Hyper-real

$$\left.\int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x} d\omega\right|_{x\neq 0} = 0$$

4. Fourier Integral Theorem

$$f(x) = \frac{1}{2\pi}\int_{k=-\infty}^{k=\infty}\left(\int_{\xi=-\infty}^{\xi=\infty} f(\xi)e^{-ik\xi}d\xi\right)e^{ikx}dk$$

does not hold in the Calculus of Limits, under any conditions.

5. Fourier Integral Theorem in Infinitesimal Calculus

If $f(x)$ *is a Hyper-real function,*

Then,

- *the Fourier Integral Theorem holds.*
- $\int\limits_{x=-\infty}^{x=\infty} f(x)e^{-i\alpha x}dx$ ***converges to*** $F(\alpha)$
- $\frac{1}{2\pi}\int\limits_{\alpha=-\infty}^{\alpha=\infty} F(\alpha)e^{-i\alpha x}d\alpha$ ***converges to*** $f(x)$

6. 2-Dimesional Fourier Transform

$$\mathcal{F}\left\{f(x,y)\right\} = \int\limits_{y=-\infty}^{y=\infty}\int\limits_{x=-\infty}^{x=\infty} f(x,y)e^{-i\omega_x x - i\omega_y y}dxdy$$

$$= \int\limits_{y=-\infty}^{y=\infty}\int\limits_{x=-\infty}^{x=\infty} f(x,y)e^{-2\pi i(\nu_x x + \nu_y y)}dxdy\,, \quad \begin{aligned}\omega_x &= 2\pi\nu_x \\ \omega_y &= 2\pi\nu_y\end{aligned}$$

8. 2-Dimesional Inverse Fourier Transform

$$\mathcal{F}^{-1}\left\{F(\omega_x,\omega_y)\right\} = \frac{1}{(2\pi)^2}\int\limits_{\omega_y=-\infty}^{\omega_y=\infty}\int\limits_{\omega_x=-\infty}^{\omega_x=\infty} F(\omega_x,\omega_y)e^{i(\omega_x x + \omega_y y)}d\omega_x d\omega_y$$

$$= \int_{\nu_y=-\infty}^{\nu_y=\infty} \int_{\nu_x=-\infty}^{\nu_x=\infty} F(2\pi\nu_x, 2\pi\nu_y) e^{2\pi i(\nu_x x+\nu_y y)} d\nu_x d\nu_y \,, \quad \begin{aligned} \omega_x &= 2\pi\nu_x \\ \omega_y &= 2\pi\nu_y \end{aligned}$$

9. 2-Dimesional Fourier Integral Theorem

$$f(x,y) = \frac{1}{(2\pi)^2} \int_{\omega_y=-\infty}^{\omega_y=\infty} \int_{\omega_x=-\infty}^{\omega_x=\infty} \left(\int_{\eta=-\infty}^{\eta=\infty} \int_{\xi=-\infty}^{\xi=\infty} f(\xi,\eta) e^{-i\omega_x\xi - i\omega_y\eta} d\xi d\eta \right) e^{i(\omega_x x+\omega_y y)} d\omega_x d\omega_y$$

$$= \int_{\eta=-\infty}^{\eta=\infty} \int_{\xi=-\infty}^{\xi=\infty} f(\xi,\eta) \left(\frac{1}{2\pi} \int_{\omega_x=-\infty}^{\omega_x=\infty} e^{i\omega_x(x-\xi)} d\omega_x \right) d\xi \left(\frac{1}{2\pi} \int_{\omega_y=-\infty}^{\omega_y=\infty} e^{i\omega_y(y-\eta)} d\omega_y \right) d\eta$$

$$= \int_{\eta=-\infty}^{\eta=\infty} \int_{\xi=-\infty}^{\xi=\infty} f(\xi,\eta) \left(\int_{\nu_x=-\infty}^{\nu_x=\infty} e^{2\pi i\nu_x(x-\xi)} d\nu_x \right) d\xi \left(\int_{\nu_y=-\infty}^{\nu_y=\infty} e^{2\pi i\nu_y(y-\eta)} d\nu_y \right) d\eta \,, \quad \begin{aligned} \omega_x &= 2\pi\nu_x \\ \omega_y &= 2\pi\nu_y \end{aligned}$$

10. 2-Dimesional Delta Function

$$\delta(x,y) = \left(\frac{1}{2\pi} \int_{\omega_x=-\infty}^{\omega_x=\infty} e^{i\omega_x x} d\omega_x \right) \left(\frac{1}{2\pi} \int_{\omega_y=-\infty}^{\omega_y=\infty} e^{i\omega_y y} d\omega_y \right)$$

$$= \left(\int_{\nu_x=-\infty}^{\nu_x=\infty} e^{2\pi i\nu_x x} d\nu_x \right) \left(\int_{\nu_y=-\infty}^{\nu_y=\infty} e^{2\pi i\nu_y y} d\nu_y \right), \quad \begin{aligned} \omega_x &= 2\pi\nu_x \\ \omega_y &= 2\pi\nu_y \end{aligned}$$

5.

Hyper-Complex Plane

Each complex number $\alpha + i\beta$ can be represented by a Cauchy sequence of rational complex numbers, $\langle r_1 + is_1, r_2 + is_2, r_3 + is_3 ... \rangle$ so that $r_n + is_n \to \alpha + i\beta$.

The constant sequence $(\alpha + i\beta, \alpha + i\beta, \alpha + i\beta, ...)$ is a Constant Hyper-Complex Number.

In [Dan2] we established that,

1. Any set of sequences $(\iota_1 + io_1, \iota_2 + io_2, \iota_3 + io_3, ...)$, where $(\iota_1, \iota_2, \iota_3, ...)$ belongs to one family of infinitesimal hyper reals, and $(o_1, o_2, o_3, ...)$ belongs to another family of infinitesimal hyper-reals, constitutes a family of infinitesimal hyper-complex numbers.
2. Each hyper-complex infinitesimal has a polar representation $dz = (dr)e^{i\phi} = o_* e^{i\phi}$, where $dr = o_*$ is an infinitesimal, and $\phi = \arg(dz)$.
3. The infinitesimal hyper-complex numbers are smaller in length, than any complex number, yet strictly greater than

zero.

4. Their reciprocals $\left(\frac{1}{\iota_1 + io_1}, \frac{1}{\iota_2 + io_2}, \frac{1}{\iota_3 + io_3}, ...\right)$ are the infinite hyper-complex numbers.
5. The infinite hyper-complex numbers are greater in length than any complex number, yet strictly smaller than infinity.
6. The sum of a complex number with an infinitesimal hyper-complex is a non-constant hyper-complex.
7. The Hyper-Complex Numbers are the totality of constant hyper-complex numbers, a family of hyper-complex infinitesimals, a family of infinite hyper-complex, and non-constant hyper-complex.
8. The Hyper-Complex Plane is the direct product of a Hyper-Real Line by an imaginary Hyper-Real Line.
9. In Cartesian Coordinates, the Hyper-Real Line serves as an x coordinate line, and the imaginary as an iy coordinate line.
10. In Polar Coordinates, the Hyper-Real Line serves as a Range r line, and the imaginary as an $i\theta$ coordinate. Radial symmetry leads to Polar Coordinates.
11. The Hyper-Complex Plane includes the complex numbers separated by the non-constant hyper-complex

numbers. Each complex number is the center of a disk of hyper-complex numbers, that includes no other complex number.

12. In particular, zero is separated from any complex number by a disk of complex infinitesimals.

13. Zero is not a complex infinitesimal, because the length of zero is not strictly greater than zero.

14. We do not add infinity to the hyper-complex plane.

15. The hyper-complex plane is embedded in $\mathbb{C}^{\infty}$, and is not homeomorphic to the Complex Plane $\mathbb{C}$. There is no bi-continuous one-one mapping from the hyper-complex Plane onto the Complex Plane.

16. In particular, there are no points in the Complex Plane that can be assigned uniquely to the hyper-complex infinitesimals, or to the infinite hyper-complex numbers, or to the non-constant hyper-complex numbers.

17. No neighbourhood of a hyper-complex number is homeomorphic to a $\mathbb{C}^n$ ball. Therefore, the Hyper-Complex Plane is not a manifold.

18. The Hyper-Complex Plane is not spanned by two elements, and is not two-dimensional.

6.

Hyper-Complex Path Integral

Following the definition of the Hyper-real Integral in [Dan3], the Hyper-Complex Integral of $f(z)$ over a path $z(t)$, $t \in [\alpha, \beta]$, in its domain, is the sum of the areas $f(z)z'(t)dt = f(z)dz(t)$ of the rectangles with base $z'(t)dt = dz$, and height $f(z)$.

6.1 Hyper-Complex Path Integral Definition

Let $f(z)$ be hyper-complex function, defined on a domain in the Hyper-Complex Plane. The domain may not be bounded.

$f(z)$ may take infinite hyper-complex values, and need not be bounded.

Let $z(t)$, $t \in [\alpha, \beta]$, be a path, $\gamma(a, b)$, so that $dz = z'(t)dt$, and $z'(t)$ is continuous.

For each t, there is a hyper-complex rectangle with base $[z(t) - \frac{dz}{2}, z(t) + \frac{dz}{2}]$, height $f(z)$, and area $f(z(t))dz(t)$.

We form the **Integration Sum** of all the areas that start at $z(\alpha) = a$, and end at $z(\beta) = b$,

$$\sum_{t\in[\alpha,\beta]} f(z(t))dz(t).$$

If for any infinitesimal $dz = z'(t)dt$, the Integration Sum equals the same hyper-complex number, then $f(z)$ is Hyper-Complex Integrable over the path $\gamma(a,b)$.

Then, we call the Integration Sum the Hyper-Complex Integral of $f(z)$ over the $\gamma(a,b)$, and denote it by $\int\limits_{\gamma(a,b)} f(z)dz$.

If the hyper-complex number is an infinite hyper-complex, then it equals $\int\limits_{\gamma(a,b)} f(z)dz$.

If the hyper-complex number is finite, then its constant part equals $\int\limits_{\gamma(a,b)} f(z)dz$.□

The Integration Sum may take infinite hyper-complex values, such as $\frac{1}{dz}$, but may not equal to ∞.

The Hyper-Complex Integral of the function $f(z) = \frac{1}{|z|}$ over a path that goes through $z = 0$ diverges.

6.2 The Countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers, $Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty.$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[\alpha, \beta]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(z)dz$.

6.3 Continuous $f(z)$ is Path-Integrable

Hyper-Complex $f(z)$ *Continuous on* D *is Path-Integrable on* D

Proof:

Let $z(t)$, $t \in [\alpha, \beta]$, be a path, $\gamma(a, b)$, so that $dz = z'(t)dt$, and $z'(t)$ is continuous. Then,

$$f(z(t))z'(t) = \big(u(x(t), y(t)) + iv(x(t), y(t))\big)\big(x'(t) + iy'(t)\big)$$

$$= \underbrace{\Big[u(x(t),y(t))x'(t) - v(x(t),y(t))y'(t)\Big]}_{U(t)} +$$

$$+i\underbrace{\Big[u(x(t),y(t))y'(t) + v(x(t),y(t))x'(t)\Big]}_{V(t)}$$

$$= U(t) + iV(t),$$

where $U(t)$, and $V(t)$ are Hyper-Real Continuous on $[\alpha, \beta]$.

Therefore, by [Dan3, 12.4], $U(t)$, and $V(t)$ are integrable on $[\alpha, \beta]$.

Hence, $f(z(t))z'(t)$ is integrable on $[\alpha, \beta]$.

Since

$$\int_{t=\alpha}^{t=\beta} f(z(t))z'(t)dt = \int_{\gamma(a,b)} f(z)dz,$$

$f(z)$ is Path-Integrable on $\gamma(a,b)$.$\square$

7.

Hyper-Complex Delta $\delta(z)$

7.1 Domain and Range of $\delta(z)$

The Hyper-Complex Delta Function is defined from the Hyper-Complex plane into the set of two hyper-complex numbers,

$$\left\{0, \frac{1}{2\pi i dz}\right\}.$$

The hyper-complex 0 is the sequence $\langle 0, 0, 0, ... \rangle$.

The infinite hyper-complex

$$\frac{1}{2\pi i}\frac{1}{dz} = \frac{1}{2\pi i}\frac{1}{dr}e^{-i\phi}$$

depends on

$$\operatorname{Arg} z = \phi \,,$$

and on our choice of the infinitesimal dr.

We will usually choose the family of infinitesimals that is spanned by the sequences $\left\langle \frac{1}{n} \right\rangle, \left\langle \frac{1}{n^2} \right\rangle, \left\langle \frac{1}{n^3} \right\rangle, \ldots$ It is a semigroup with respect to vector addition, and includes all the scalar multiples of the generating sequences that are non-zero. That is, the family includes infinitesimals with negative sign.

Therefore, $\frac{1}{dr}$ will mean the sequence $\langle n \rangle$.

Alternatively, we may choose the family spanned by the sequences $\left\langle \frac{1}{2^n} \right\rangle, \left\langle \frac{1}{2^{2n}} \right\rangle, \left\langle \frac{1}{2^{3n}} \right\rangle$,... Then, $\frac{1}{dr}$ will mean the sequence $\langle 2^n \rangle$.

7.2 $\delta(z)$ *is an infinite hyper-complex function on the infinitesimal hyper-complex disk* $|z| \leq dr$, *In particular,*

$$\boxed{\delta(z) < \infty}$$

Proof: Since $dr > 0$, we have $\frac{1}{dr} < \infty$, and $\frac{1}{2\pi i dr} e^{-i\phi} < \infty$.

7.3 Definition of $\delta(z - z_0)$

For any dz,

$$\delta(z - z_0) \equiv \frac{1}{2\pi i dz} \chi_{\{|z - z_0| \leq |dz|\}}(z),$$

$$= \frac{1}{2\pi i(dx + idy)} \begin{cases} 1, & |z - z_0| \leq \sqrt{(dx)^2 + (dy)^2} \\ 0, & \text{otherwise} \end{cases}$$

Thus,

- on the disk, $|z - z_0| \leq dr$, $\delta(z - z_0) = \frac{1}{2\pi i} \frac{1}{dz}$.
- off the disk, for $|z - z_0| > dr$, $\delta(z - z_0) = 0$.

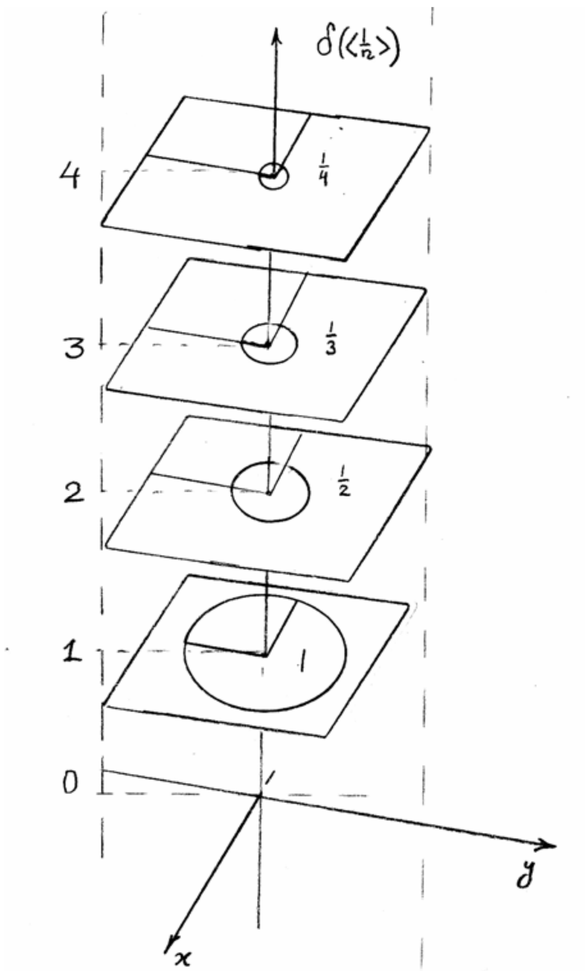
8.

$\delta(z)$ Plots

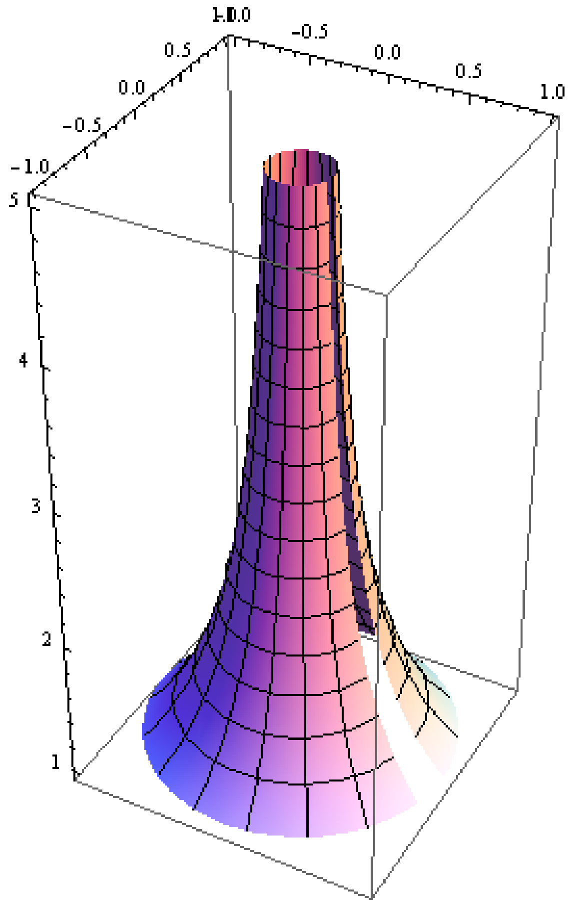
8.1 Delta Plot for

$$\delta(z) = \frac{1}{2\pi i} e^{-i\phi} \left\langle \chi_{\{|z| \le 1\}}(z), 2\chi_{\{|z| \le \frac{1}{2}\}}(z), 3\chi_{\{|z| \le \frac{1}{3}\}}(z), ... \right\rangle,$$

is a sequence of shrinking disks with increasing heights



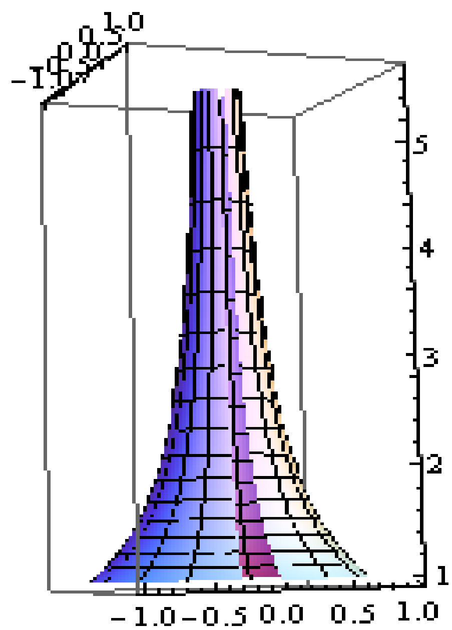
The disks are in the body generated by revolving $\frac{1}{|z|}=\frac{1}{r}$, over the Complex plane



8.2 Delta Plot for

$$\delta(z)=\frac{1}{2\pi i}e^{-i\phi}\left\langle \chi_{\left\{|z|\le\frac{1}{2^0}\right\}}(z), 2\chi_{\left\{|z|\le\frac{1}{2^1}\right\}}(z), 4\chi_{\left\{|z|\le\frac{1}{2^2}\right\}}(z), 8\chi_{\left\{|z|\le\frac{1}{2^3}\right\}}(z)...\right\rangle$$

is a sequence of shrinking disks with increasing heights that lie in the body of revolution of $\dfrac{1}{\left|2^z\right|} = \dfrac{1}{2^x}$.



9.

$\delta(z)$ Properties

9.1 $$\boxed{\delta(z-z_0)=\frac{1}{2\pi i}\frac{1}{dr}e^{-i(\phi-\phi_0)}\chi_{\{|z-z_0|\leq dr\}}(z)},$$

$$\phi=\arg z\,,\ \phi_0=\arg z_0$$

9.2 $$\boxed{\delta(0)=\frac{1}{2\pi i dr}}$$

9.3 $$\boxed{(\delta(z))^n=\frac{1}{(2\pi i)^n}\frac{1}{(dr)^n}e^{-in\phi}\chi_{\{|z|\leq dr\}}(z)},\quad n=2,3,\ldots$$

10.

$\delta(f(z))$

10.1 $\boxed{\delta(az) = \frac{1}{a}\delta(z)}$

Proof: $\delta(az) = \frac{1}{2\pi i}\frac{1}{d(az)}\chi_{\{|az| \le dr\}}(z)$

$$= \frac{1}{a}\frac{1}{2\pi i}\frac{1}{dz}\chi_{\left\{|z| \le \frac{1}{|a|}dr\right\}}(z)$$

$$= \frac{1}{a}\frac{1}{2\pi i}\frac{1}{dz}\chi_{\{|z| \le |dz|\}}(z)$$

$$= \frac{1}{a}\frac{1}{2\pi i}\delta(z).\square$$

10.2 $z_1 =$ **only zero of** $f(z)$, $f'(z_1) \neq 0 \Rightarrow$

$$\Rightarrow \boxed{\delta(f(z)) = \frac{1}{f'(z_1)}\delta(z - z_1)}$$

Proof: $\delta(f(z)) = \delta\left(f(z) - f(z_1)\right)$

For $z - z_1 =$ infinitesimal,

$$= \delta\left(f'(z_1)(z - z_1) \right)$$

$$= \frac{1}{f'(z_1)} \delta(z - z_1), \text{ by 10.1.}\square$$

10.3 z_1, z_2 **are the only zeros of** $f(z); f'(z_1), f'(z_2) \neq 0 \ \Rightarrow$

$$\Rightarrow \quad \boxed{\delta(f(z)) = \frac{1}{f'(z_1)} \delta(z - z_1) + \frac{1}{f'(z_2)} \delta(z - z_2)}$$

Proof: $\delta(f(z)) = \delta\left(f(z) - f(z_1) \right) + \delta\left(f(z) - f(z_2) \right)$

For $z - z_1 = \text{infinitesimal}$, $z - z_2 = \text{infinitesimal}$,

$$\delta(f(z)) = \delta\left(f'(z_1)(z - z_1) \right) + \delta\left(f'(z_2)(z - z_2) \right)$$

$$= \frac{1}{f'(z_1)} \delta(z - z_1) + \frac{1}{f'(z_2)} \delta(z - z_2).\square$$

10.4 $$\boxed{\delta(z^2 - a^2) = \frac{1}{2a} \delta(z - a) + \frac{1}{2a} \delta(z + a)}$$

10.5 $$\boxed{\delta\left((z - a)(z - b) \right) = \frac{1}{a - b} \delta(z - a) + \frac{1}{b - a} \delta(z - b)}$$

10.6 $z_1,...z_n$ **are the only zeros of** $f(z)$; $f'(z_1),..,f'(z_n) \neq 0 \Rightarrow$

$$\boxed{\delta(f(z)) = \frac{1}{f'(z_1)}\delta(z - z_1) + ... + \frac{1}{f'(z_n)}\delta(z - z_n)}$$

10.7

$$\boxed{\delta(\sin z) = .. + \delta(z + 2\pi) - \delta(z + \pi) + +\delta(z) - \delta(z - \pi) + \delta(z - 2\pi) + ..}$$

Proof: **The zeros of** $\sin z$ **are** $... - 2\pi, -\pi, 0, \pi, 2\pi, ...$

and ...,$\cos(-\pi) = -1$, $\cos(0) = 1$, $\cos(\pi) = -1$,... $\square$

11.

Polar Representation of $\delta(z)$

11.1 $$\boxed{\delta(z) = \frac{1}{2\pi i} e^{-i\,\mathrm{Arg}(z)} \delta(|z|)}$$

<u>*Proof*</u>:

$$\delta(z) = \frac{1}{2\pi i dz} \chi_{\{|\zeta| \le |dz|\}}(z), \qquad \textbf{where } \chi_{\{|\zeta| \le |dz|\}}(z) = \begin{cases} 0, |\zeta| > |dz| \\ 1, |\zeta| \le |dz| \end{cases}$$

$$= \frac{dx - idy}{2\pi i[(dx)^2 + (dy)^2]} \chi_{\left\{|\zeta| \le \sqrt{(dx)^2 + (dy)^2}\right\}}(z)$$

$$= \frac{\dfrac{dx}{\sqrt{(dx)^2 + (dy)^2}} - i \dfrac{dy}{\sqrt{(dx)^2 + (dy)^2}}}{2\pi i \sqrt{(dx)^2 + (dy)^2}} \chi_{\left\{|\zeta| \le \sqrt{(dx)^2 + (dy)^2}\right\}}(z)$$

$$= \frac{\cos\phi - i\sin\phi}{2\pi i} \underbrace{\frac{1}{d\rho} \chi_{\{|\zeta| \le d\rho\}}(z)}_{\delta(\rho)}, \quad \textbf{where } \begin{matrix} d\rho = \sqrt{(dx)^2 + (dy)^2} \\ \phi = \mathrm{Arg}(z) \end{matrix}$$

$$= \frac{1}{2\pi i} e^{-i\,\mathrm{Arg}(z)} \delta(\rho), \quad \rho = |z| = |x + iy| = \sqrt{x^2 + y^2}.$$

12.

$\delta(\rho)$

12.1 Each component of $\left\langle \frac{1}{e^{\rho}}, \frac{2}{e^{2\rho}}, \frac{3}{e^{3\rho}}, ... \right\rangle$

- has the sifting property: $\int\limits_{\rho=0}^{\rho=\infty} ne^{-n\rho}dr = n\frac{e^{-n\rho}}{-n}\Bigg|_{\rho=0}^{\rho=\infty} = 1.$
- is continuous Hyper-real function
- peaks at $\rho = 0$ to n

Therefore,

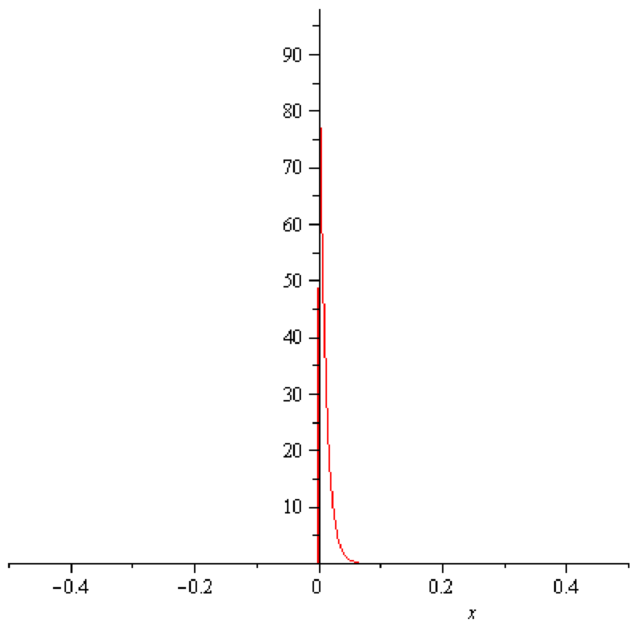
12.2 $\left\langle e^{-\rho}, 2e^{-2\rho}, 3e^{-3\rho}, ... \right\rangle$ represents

the Hyper-Real Delta Function $\delta(\rho)$

12.3 *For* $d\rho = \left\langle \frac{1}{n} \right\rangle$, $\boxed{\delta(\rho) = \frac{1}{d\rho} e^{-\frac{\rho}{d\rho}}}$

12.4 $plot\left(\begin{cases} 0 & x < 0 \\ 100e^{-100x} & x \geq 0 \end{cases}, x = -0.5\,..0.5\right)$ plots in Maple

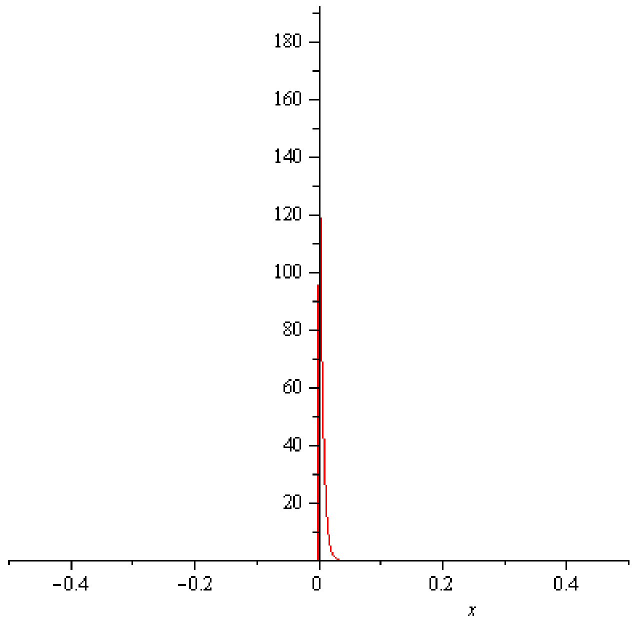
the 100^{th} component of $\delta(r)$,



12.5

$plot\left(\begin{cases} 0 & x < 0 \\ 200e^{-200x} & x \geq 0 \end{cases}, x = -0.5\,..0.5\right)$ plots in Maple

the 200^{th} component of $\delta(r)$


180
160
140
120
100
80
60
40
20
-0.4
-0.2
0
0.2
0.4
x

13.

$\delta(\rho)$ and $\delta(x)\delta(y)$

13.1 $$\delta(\rho) = \frac{1}{d\rho}\chi_{[-\frac{d\rho}{2},\frac{d\rho}{2}]}(\rho), \ \ \rho \geq 0$$

13.2 $$\delta(\phi) = \frac{1}{d\phi}\chi_{[-\frac{d\phi}{2},\frac{d\phi}{2}]}(\phi), \ \ 0 \leq \phi \leq 2\pi$$

Transforming from Polar to Cartesian Coordinates, $\begin{aligned} x &= \rho\cos\phi \\ iy &= i\rho\sin\phi \end{aligned}$

13.3 $$\delta(\rho)\delta(\phi) = \rho\delta(x)\delta(y)$$

<u>*Proof*</u>: $$\delta(\rho)\underbrace{\delta(i\phi)}_{\frac{1}{i}\delta(\phi)} = \delta(x)\underbrace{\delta(iy)}_{\frac{1}{i}\delta(y)}\left|\frac{\partial(x,iy)}{\partial(\rho,i\phi)}\right|$$

$$\delta(\rho)\delta(\phi) = \delta(x)\delta(y)\underbrace{\begin{vmatrix} \cos\phi & i\sin\phi \\ i\rho\sin\phi & \rho\cos\phi \end{vmatrix}}_{\rho}.\square$$

Integrating over ϕ,

13.4 $$\boxed{\delta(\rho) = 2\pi\rho\delta(x)\delta(y)}$$

Proof: $$\delta(\rho)\underbrace{\int_{\phi=0}^{\phi=2\pi}\delta(\phi)d\phi}_{1} = \rho\delta(x)\delta(y)\underbrace{\int_{\phi=0}^{\phi=2\pi}d\phi}_{2\pi}.\square$$

14.

Cartesian Representation of $\delta(z)$

14.1 $$\boxed{\delta(z) = \frac{1}{i} e^{-i\,\mathrm{Arg}(z)} \rho \left(\frac{1}{2\pi} \int_{\omega_x = -\infty}^{\omega_x = \infty} e^{i\omega_x x} d\omega_x \right) \left(\frac{1}{2\pi} \int_{\omega_y = -\infty}^{\omega_y = \infty} e^{i\omega_y y} d\omega_y \right)}$$

Proof: By 12.1,

$$\delta(z) = \frac{1}{2\pi i} e^{-i\,\mathrm{Arg}(z)} \delta(\rho)$$

By 13.4,

$$= \frac{1}{2\pi i} e^{-i\,\mathrm{Arg}(z)} 2\pi \rho \delta(x) \delta(y)$$

$$= \frac{1}{i} e^{-i\,\mathrm{Arg}(z)} \rho \delta(x) \delta(y)$$

By 4.10,

$$= \frac{1}{i} e^{-i\,\mathrm{Arg}(z)} \rho \left(\int_{\nu_x = -\infty}^{\nu_x = \infty} e^{2\pi i \nu_x x} d\nu_x \right) \left(\int_{\nu_y = -\infty}^{\nu_y = \infty} e^{2\pi i \nu_y y} d\nu_y \right),$$

Denoting $\begin{aligned} \omega_x &= 2\pi \nu_x \\ \omega_y &= 2\pi \nu_y \end{aligned}$,

$$= \frac{1}{i} e^{-i\,\mathrm{Arg}(z)} \rho \left(\frac{1}{2\pi} \int_{\omega_x = -\infty}^{\omega_x = \infty} e^{i\omega_x x} d\omega_x \right) \left(\frac{1}{2\pi} \int_{\omega_y = -\infty}^{\omega_y = \infty} e^{i\omega_y y} d\omega_y \right) . \square$$

15.

Bessel Integral Representation of $\delta(z)$

15.1 $$\delta(x)\delta(y) = 2\pi \int_{\upsilon=0}^{\upsilon=\infty} J_0(2\pi\upsilon\rho)\upsilon d\upsilon, \quad \upsilon = \left|\nu_x + i\nu_y\right|$$

$$= \frac{1}{2\pi}\int_{\Omega=0}^{\Omega=\infty} J_0(\Omega\rho)\Omega\, d\Omega, \quad \Omega = \left|\omega_x + i\omega_y\right|$$

$=$ Inverse 2D-Bessel-Fourier Transform of 1.

$$\omega = 2\pi\nu, \quad \omega = \omega_x + i\omega_y, \quad \nu = \nu_x + i\nu_y$$

<u>*Proof*</u>: By 4.2,

$$\delta(x)\delta(y) = \left(\int_{\nu_x=-\infty}^{\nu_x=\infty} e^{2\pi i\nu_x x} d\nu_x\right)\left(\int_{\nu_y=-\infty}^{\nu_y=\infty} e^{2\pi i\nu_y y} d\nu_y\right).$$

Substitute

$$\begin{aligned} x &= \rho\cos\phi & \nu_x &= \upsilon\cos\beta \\ y &= \rho\sin\phi & \nu_y &= \upsilon\sin\beta \\ \rho &= \left|x + iy\right| & \upsilon &= \left|\nu_x + i\nu_y\right| \end{aligned}$$

Then,

$$\nu_x x + \nu_y y = \upsilon\rho(\cos\beta\cos\theta + \sin\beta\sin\theta) = \upsilon\rho\cos(\beta - \theta).$$

Integrating with respect to υ, and β,

$$= \int_{\upsilon=0}^{\upsilon=\infty} \left(\int_{\beta=0}^{\beta=2\pi} e^{2\pi i\upsilon\rho\cos(\beta-\theta)} d\beta \right) \upsilon d\upsilon$$

Denoting $\alpha = \beta - \phi$,

$$\int_{\beta=0}^{\beta=2\pi} e^{2\pi i\upsilon\rho\cos(\beta-\phi)} d\beta = \int_{\alpha=-\phi}^{\alpha=2\pi-\phi} e^{2\pi i\upsilon\rho\cos\alpha} d\alpha .$$

Since $e^{2\pi i\upsilon\rho\cos\alpha}$ is periodic with period 2π,

$$= \int_{\alpha=0}^{\alpha=2\pi} e^{2\pi i\upsilon\rho\cos\alpha} d\alpha ,$$

$$= \int_{\alpha=0}^{\alpha=2\pi} \left(1 + \frac{2\pi i\upsilon\rho\cos\alpha}{1} + \frac{(2\pi i\upsilon\rho\cos\alpha)^2}{2!} + \frac{(2\pi i\upsilon\rho\cos\alpha)^3}{3!} + ... \right) d\alpha .$$

The integrals of the odd powers vanish, and we have

$$= 2\pi - \frac{(2\pi\upsilon\rho)^2}{2^2} 2\pi + \frac{(2\pi\upsilon\rho)^4}{2^2 \cdot 4^2} 2\pi - \frac{(2\pi\upsilon\rho)^6}{2^2 \cdot 4^2 \cdot 6^2} 2\pi + ...$$

$$= 2\pi J_0(2\pi\upsilon\rho) .$$

Therefore,

$$\delta(x)\delta(y) = 2\pi \int_{\upsilon=0}^{\upsilon=\infty} J_0(2\pi\upsilon\rho)\upsilon d\upsilon , \qquad \upsilon = \left| \nu_x + i\nu_y \right|$$

Put $\omega = 2\pi\nu$, $\omega = \omega_x + i\omega_y$, $\nu = \nu_x + i\nu_y$

$$= \frac{1}{2\pi} \int_{\Omega=0}^{\Omega=\infty} J_0(\Omega\rho)\Omega d\Omega , \qquad \Omega = \left| \omega_x + i\omega_y \right| . \square$$

Thus,

15.2 $$\boxed{\delta(\rho) = \rho \int_{\Omega=0}^{\Omega=\infty} J_0(\Omega\rho)\Omega\, d\Omega}$$

Proof: By 13.4,

$$\delta(\rho) = 2\pi\rho\delta(x)\delta(y)$$

By 15.1,

$$= 2\pi\rho \frac{1}{2\pi} \int_{\Omega=0}^{\Omega=\infty} J_0(\Omega\rho)\Omega\, d\Omega\,.\square$$

Hence,

15.3 $$\boxed{\delta(z) = \frac{1}{2\pi i} e^{-i\,\mathrm{Arg}(z)} \rho \int_{\Omega=0}^{\Omega=\infty} J_0(\Omega\rho)\Omega\, d\Omega}$$

Proof: By 11.1,

$$\delta(z) = \frac{1}{2\pi i} e^{-i\,\mathrm{Arg}(z)} \delta(\rho)$$

By 15.2,

$$= \frac{1}{2\pi i} e^{-i\,\mathrm{Arg}(z)} \rho \int_{\Omega=0}^{\Omega=\infty} J_0(\Omega\rho)\Omega\, d\Omega\,.\square$$

16.

Primitive and Derivatives of $\delta(z)$

The Hyper-Complex Log Function is the primitive of $\delta(z)$, on the Hyper-Complex Plane

16.1 $$\boxed{\delta(\zeta - z) = \frac{d}{dz}\frac{1}{2\pi i}\big(\mathrm{Log}(\zeta - z)\big)\chi_{\{|\zeta - z| \le dr\}}(\zeta)}$$

16.2 $$\boxed{\frac{d}{dz}\delta(\zeta - z) = \frac{1}{2\pi i}\frac{1}{(\zeta - z)^2}\chi_{\{|\zeta - z| \le dr\}}(z)}$$

That is,

- on the disk $\left|\zeta - z\right| \le dr$, $\dfrac{d}{dz}\delta(\zeta - z) = \dfrac{1}{2\pi i}\dfrac{1}{(dr)^2}e^{-2i\theta}$.
- off the disk, in $\left|\zeta - z\right| > dr$, $\dfrac{d}{dz}\delta(\zeta - z) = 0$.

16.3 $$\boxed{\frac{d^k}{dz^k}\delta(\zeta - z) = \frac{1}{2\pi i}\frac{k!}{(\zeta - z)^{k+1}}\chi_{\{|z| \le dr\}}(z)}$$

That is,

- on the disk, $\left|\zeta - z\right| \le dr$, $\frac{d^k}{dz^k}\delta(\zeta - z) = \frac{k!}{2\pi i}\frac{1}{(dr)^{k+1}}e^{-i(k+1)\theta}$,
- off the disk, on $\left|\zeta - z\right| > dr$, $\frac{d^k}{dz^k}\delta(\zeta - z) = 0$.

16.4 $\delta(z)$ *is differentiable, and integrable to any order,*

But it has no Taylor Series, it is Not analytic,

Hence, its integral along a closed path does not vanish:

$$\oint_{\gamma} \delta(z)dz \ne 0$$

17.

Circulation of $\delta(z)$

The Hyper-complex Delta Function is Not Analytic.

It is singular on the infinitesimal disk $\left|\zeta - z\right| \le dr$.

Integrating along a path that encircles its singularity at $\zeta = z$, the Circulation of Delta along the infinitesimal circle is 1.

17.1 $$\boxed{\oint_{\left|\zeta-z\right|=dr} \delta(\zeta - z)d\zeta = 1}.$$

<u>*Proof*</u>: Put

$$\zeta - z = (dr)e^{i\alpha}$$

$$d\zeta = i(dr)e^{i\alpha}d\alpha .$$

Then,

$$\frac{1}{2\pi i}\oint_{\left|\zeta-z\right|=dr} \frac{1}{\zeta - z}d\zeta = \frac{1}{2\pi i}\int_{\alpha=0}^{\alpha=2\pi} \frac{1}{(dr)e^{i\alpha}} i(dr)e^{i\alpha}d\alpha ,$$

Since $(dr)e^{i\alpha} \neq 0$, for any infinitesimal dr, and any α,

$$= \frac{1}{2\pi i} i \int_{\alpha=0}^{\alpha=2\pi} d\alpha = 1 .\square$$

18.

Sifting by $\delta(\zeta - z)$ and $\frac{d}{dz}\delta(\zeta - z)$

18.1 Sifting by $\delta(\zeta - z)$

If $f(z)$ is Hyper-Complex Differentiable function at z

Then, $$\boxed{\oint_{|\zeta - z| = dr} f(\zeta)\delta(\zeta - z)d\zeta = f(z)}$$

Proof:

Since $f(\zeta)$ is differentiable at z, on the circle $\zeta - z = (dr)e^{i\alpha}$,

$$\zeta = z + (dr)e^{i\alpha},$$

$$d\zeta = i(dr)e^{i\alpha}d\alpha$$

$$f(z + (dr)e^{i\alpha}) = f(z) + f'(z)(dr)e^{i\alpha},$$

$$\oint_{|\zeta - z| = dr} f(\zeta)\delta(\zeta - z)d\zeta =$$

$$= f(z)\underbrace{\oint_{|\zeta - z| = dr} \delta(\zeta - z)d\zeta}_{1} + f'(z)dr \oint_{|\zeta - z| = dr} e^{i\alpha}\frac{1}{2\pi i}\underbrace{\frac{1}{\zeta - z}}_{\frac{1}{dr}e^{-i\alpha}}\underbrace{d\zeta}_{i(dr)e^{i\alpha}d\alpha}$$

$$= f(z) + \frac{1}{2\pi i}f'(z)(dr)\int_{\alpha=0}^{\alpha=2\pi} e^{i\alpha}\frac{1}{(dr)e^{i\alpha}}i(dr)e^{i\alpha}d\alpha$$

$$= f(z) + \frac{1}{2\pi i} f'(z)(dr) \underbrace{\int_{\alpha=0}^{\alpha=2\pi} e^{i\alpha} d\alpha}_{0}$$

$$= f(z). \square$$

18.2 $$\boxed{\frac{d}{dz} f(z) = \oint_{|\zeta - z| = dr} f(\zeta) \frac{d}{dz} \delta(\zeta - z) d\zeta}$$

Proof: $$\frac{d}{dz} f(z) = \frac{1}{2\pi i} \oint_{|\zeta - z| = dr} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$$

$$= \oint_{|\zeta - z| = dr} f(\zeta) \frac{d}{dz} \delta(\zeta - z) d\zeta . \square$$

18.3 $$\boxed{\frac{d^k}{dz^k} f(z) = \oint_{|\zeta - z| = dr} f(\zeta) \frac{d^k}{dz^k} \delta(\zeta - z) d\zeta}$$

Proof: $$\frac{d}{dz} f(z) = \frac{k!}{2\pi i} \oint_{|\zeta - z| = dr} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta$$

$$= \oint_{|\zeta - z| = dr} f(\zeta) \frac{d^k}{dz^k} \delta(\zeta - z) d\zeta . \square$$

19.

$\delta(\zeta - z)$ and the Cauchy Integral Formula

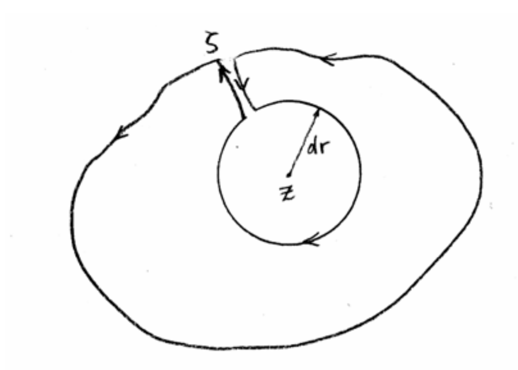
19.1 The Cauchy Integral Formula is Sifting by Delta

If $f(z)$ *is Hyper-Complex Differentiable function on a Hyper-Complex Simply-Connected Domain* D.

Then, for any loop γ, *and any point* z *in its interior*

$$\boxed{f(z) = \oint\limits_{|z-\zeta|=dr} f(\zeta)\delta(\zeta - z)d\zeta = \frac{1}{2\pi i}\oint\limits_{\gamma} \frac{f(\zeta)}{\zeta - z}d\zeta}$$

<u>*Proof*</u>: The Hyper-Complex function $\dfrac{f(\zeta)}{\zeta - z}$ is Differentiable on the Hyper-Complex Simply-Connected domain D, and on a path that includes γ and an infinitesimal circle about z.



Then, the integrals on the lines between γ and the circle have opposite signs and cancel each other.

The integral over the circle has a negative sign because its direction is clockwise, and by Cauchy Integral Theorem,

$$\oint_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta - \oint_{|\zeta - z| = dr} \frac{f(\zeta)}{\zeta - z} d\zeta = 0 .$$

Therefore,

$$\oint_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = \oint_{|\zeta - z| = dr} \frac{f(\zeta)}{\zeta - z} d\zeta$$

$$= 2\pi i \underbrace{\oint_{|\zeta - z| = dr} f(\zeta) \underbrace{\frac{1}{2\pi i} \frac{1}{\zeta - z}}_{\delta(\zeta - z)} d\zeta}_{f(z)} . \square$$

20.

$\delta(\zeta - z)$ and the Residue of a Laurent Expansion

20.1 Laurent Expansion of a Singular $f(z)$

If $f(z)$ *is Hyper-Complex Differentiable function on a Hyper-Complex disk* $0 < |z - z_0| < r$

Then, $f(z) = ... + a_{-3}(z - z_0)^{-3} + a_{-2}(z - z_0)^{-2} + a_{-1}(z - z_0)^{-1} +$

$$+a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ...$$

where for any loop γ*, and for any point* $z \neq z_0$ *in its interior*

$$a_{-k} = \frac{1}{2\pi i}\oint_{\gamma} f(\zeta)(\zeta - z_0)^{k-1} d\zeta, \quad k = 1, 2, ... ,$$

$$a_{-3} = \frac{1}{2\pi i}\oint_{\gamma} f(\zeta)(\zeta - z_0)^2 d\zeta$$

$$a_{-2} = \frac{1}{2\pi i}\oint_{\gamma} f(\zeta)(\zeta - z_0) d\zeta,$$

$$a_{-1} = \frac{1}{2\pi i}\oint_{\gamma} f(\zeta) d\zeta,$$

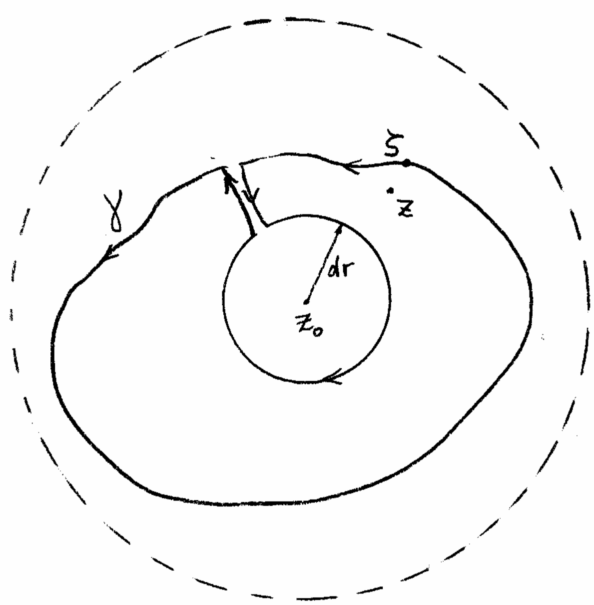
$$a_0 = \frac{1}{2\pi i}\oint_{\gamma} \frac{f(\zeta)}{\zeta - z_0} d\zeta,$$

$$a_1 = \frac{1}{2\pi i}\oint_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^2} d\zeta,$$

$$a_2 = \frac{1}{2\pi i}\oint_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^3} d\zeta$$

$$a_k = \frac{1}{2\pi i}\oint_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta, \quad k = 0,1,2,...$$

Proof: The Hyper-Complex Differentiable $f(z)$ satisfies Cauchy Integral Formula in the Hyper-Complex domain D, bounded by a path that includes γ and an infinitesimal circle about z_0



Then, the integrals on the lines between γ and the circle have opposite signs and cancel each other.

The integral over the circle has a negative sign because its direction is clockwise, and by Cauchy Integral Formula,

$$f(z) = \frac{1}{2\pi i}\left(\oint_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta - \oint_{|\zeta - z| = dr} \frac{f(\zeta)}{\zeta - z} d\zeta \right).$$

For ζ along γ,

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0}\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \frac{1}{\zeta - z_0}\left(1 + \frac{z - z_0}{\zeta - z_0} + \left(\frac{z - z_0}{\zeta - z_0}\right)^2 + \ldots \right).$$

Then,

$$\oint_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = \underbrace{\oint_{\gamma} \frac{f(\zeta)}{\zeta - z_0} d\zeta}_{2\pi i a_0(z_0)} + \underbrace{\oint_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^2} d\zeta}_{2\pi i a_1(z_0)}(z - z_0) + \underbrace{\oint_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^3} d\zeta}_{2\pi i a_2(z_0)}(z - z_0)^2 + \ldots$$

For ζ along the circle $|\zeta - z| = dr$,

$$-\frac{1}{\zeta - z} = \frac{1}{z - z_0}\frac{1}{1 - \frac{\zeta - z_0}{z - z_0}} = \frac{1}{z - z_0}\left(1 + \frac{\zeta - z_0}{z - z_0} + \left(\frac{\zeta - z_0}{z - z_0}\right)^2 + \ldots \right).$$

Then,

$$\oint \frac{-f(\zeta)}{\zeta - z} d\zeta = \underbrace{\oint f(\zeta) d\zeta}_{2\pi i a_{-1}(z_0)} \frac{1}{z - z_0} + \underbrace{\oint \frac{f(\zeta)}{\zeta - z_0} d\zeta}_{2\pi i a_{-2}(z_0)} \frac{1}{(z - z_0)^2} + \underbrace{\oint \frac{f(\zeta)}{(\zeta - z_0)^2} d\zeta}_{2\pi i a_{-3}(z_0)} \frac{1}{(z - z_0)^3} + \ldots$$

By the Cauchy Integral Theorem the integrals of a_{-1}, a_{-2}, a_{-3},... can be taken along γ. $\square$

20.2 $\delta(\zeta - z)$ and the Residue of a Laurent Expansion

If $f(z)$ *is Hyper-Complex function on a Hyper-Complex disk* $0 < |z - z_0| < r$ *so that*

$$f(z) = ... + a_{-3}(z - z_0)^{-3} + a_{-2}(z - z_0)^{-2} + a_{-1}(z - z_0)^{-1} +$$

$$+a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ...$$

Then, for any loop γ, *around* z_0

$$\oint_\gamma (z - z_0)^k dz = \begin{cases} 0, & k \neq -1 \\ 2\pi i \oint\limits_{|z - z_0| = dr} \delta(z - z_0) dz = 2\pi i, & k = -1 \end{cases}$$

Hence,

$$\frac{1}{2\pi i} \oint_\gamma f(\zeta) d\zeta = a_{-1},$$

<u>Proof</u>:

For any integer $k \neq -1$,

$$\oint_\gamma (z - z_0)^k dz = \frac{1}{k+1}(z - z_0)^{k+1}\Big|_{z=\alpha}^{z=\alpha} = 0$$

For $k = -1$,

$$\oint_\gamma (z - z_0)^{-1} dz = 2\pi i \oint\limits_{|z - z_0| = dr} \underbrace{\frac{1}{2\pi i}\frac{1}{z - z_0}}_{\delta(z - z_0)} dz = 2\pi i . \square$$

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