The Fourier Integral, and Delta Function Exponential Representations

H. Vic Dannon
vic0user@gmail.com
May, 2025

Abstract From $\mathcal{F}\{\delta(x)\} = 1$, it follows that

$\delta(x)$ = ***inverse Fourier Transform of the unit function* 1**

$$\boxed{\delta(x) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x}d\omega}$$

This integral was used without identifying it as $\delta(x)$ by Laplace, Poisson, and Riemann. Then, the meaning of the integration over the exponentials is not clear.

By using the Hyper-real Delta Function, we show

$$\boxed{\left.\frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x}d\omega\right|_{x=0} = \delta(0) = \frac{1}{dx} = \text{infinite hyper-real like } N}$$

And

$$\boxed{\left.\frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x}d\omega\right|_{x\neq 0} = \delta(x)\big|_{x\neq 0} = 0}$$

Keywords: Infinitesimal, Infinite-Hyper-Real, Hyper-Real Function, Infinitesimal Calculus, Delta Function, Fourier Transform, Fourier Integral Theorem

2000 Mathematics Subject Classification 26E15; 26E20; 26A06; 97I40; 97I30; 44A10;

Contents

0. Introduction

By Fourier Integral Theorem

$$f(x) = \frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} \left(\int_{\xi=-\infty}^{\xi=\infty} f(\xi) e^{-ik\xi} d\xi \right) e^{ikx} dk$$

$$= \int_{\xi=-\infty}^{\xi=\infty} f(\xi) \left(\frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} e^{ik(x-\xi)} dk \right) d\xi$$

Thus, the integral $\frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} e^{-ik(\xi-x)} dk$ sifts through the values of the function $f(\xi)$, and picks its value at x.

Cauchy (1816), and Poisson (1815) derived the Fourier Integral Theorem by using the sifting property of the integral.

In the derivation of his Zeta Function, Riemann (1859) uses this sifting property repeatedly, without using a function notation for the integral $\frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} e^{-ik(\xi-x)} dk$. The derivations are in [Dan4].

However,

$$\xi = x \Rightarrow e^{-ik(\xi-x)} = 1,$$

and the integral $\frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} e^{-ik(\xi-x)} dk$ diverges.

Avoiding the singularity at $\xi = x$ does not recover the Fourier Integral Theorem, because without the singularity the Fourier integral equals zero.

Thus, the Fourier Integral Theorem cannot be written in the Calculus of Limits.

In Infinitesimal Calculus [Dan4], the singularity can be integrated over, and defines the Delta Function

$$\delta(x-\xi) = \frac{1}{2\pi}\int_{k=-\infty}^{k=\infty} e^{ik(x-\xi)}dk\,.$$

Then, the Fourier Integral theorem states the sifting property for the Delta Function

$$f(x) = \int_{\xi=-\infty}^{\xi=\infty} f(\xi)\delta(\xi - x)d\xi\,,$$

and for any hyper-real function $f(x)$, the Fourier Transform pairs converge, and the Fourier Integral Theorem holds.

In the Calculus of Limits, the Delta Function cannot be defined, and its sifting property does not apply.

That sifting property allows for a Hyper-real Fourier Integral Theorem for $f(x) \equiv 1$.

While $\int_{x=-\infty}^{x=\infty} \left|1\right| dx = \infty$, we have $1 = \frac{1}{2\pi}\int_{k=-\infty}^{k=\infty}\left(\int_{\xi=-\infty}^{\xi=\infty} e^{-ik\xi}d\xi\right)e^{ikx}dk\,.$

1.

Hyper-real Space

Each real number α can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of infinitesimal hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the infinite hyper-reals.
4. The infinite hyper-reals are greater than any real number, yet strictly smaller than infinity.
5. The infinite hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a non-constant hyper-real.
7. The Hyper-reals are the totality of the constant hyper-reals, the infinitesimals, the negative infinitesimals, the infinite

hyper-reals, the negative infinite hyper-reals, and the non-constant hyper-reals.

8. The hyper-reals constitute and aligned along the Hyper-real Space.

9. That space includes the real numbers separated by the non-constant hyper-reals. Each real number is the center of a cloud of hyper-reals, that includes no other real number.

10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.

11. Zero is not an infinitesimal, because zero is not strictly greater than zero.

12. We do not add infinity to the hyper-real space.

13. The infinitesimals, the infinitesimals with negative signs, the infinite hyper-reals, and the infinite hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.

14. Any hyper-real number belongs to $\mathbb{R}^\infty$. Thus, the hyper-real space is embedded in $\mathbb{R}^\infty$. But there is no bi-continuous one-one mapping from the hyper-real space onto the real line. That is, the hyper-real space is not homeomorphic to the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal hyper-reals, or to the infinite hyper-reals, or to the non-constant hyper-reals.
16. No neighbourhood of a hyper-real number is homeomorphic to an $\mathbb{R}^n$ ball. Therefore, the hyper-real space is not a manifold.
17. The hyper-real space is totally ordered like a line, but unlike a line, it is not spanned by one element, and it is not one-dimensional.

2.

Hyper-real Functions in $Span\{x^n\}\Big|_{n=0}^{n=\infty}$

A Function with no infinite hyper-real values is called a <u>constant hyper-real function</u>. Its Taylor Series expansion is infinite linear combination of the power functions

$$\{x^0, x^1, x^2, x^3, \ldots x^n, \ldots\}\Big|_{n=0}^{n=\infty}$$

In hyper-real space, the function is the infinite vector of the partial sums of its Taylor Series Expansion.

For instance, the hyper-real exponential function is

$$\begin{bmatrix} \frac{1}{0!}x^0 \\ \frac{1}{0!}x^0 + \frac{1}{1!}x^1 \\ \frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 \\ \frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \\ ------------------ \\ \frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots + \frac{1}{n!}x^n \\ ------------------ \\ \\ \end{bmatrix}$$

3.

Integral of a Hyper-real Function

In [Dan3], we defined the integral of a Hyper-real Function.

Let $f(x)$ be a hyper-real function on the interval $[a,b]$.

The interval may not be bounded.

$f(x)$ may take infinite hyper-real values, and need not be bounded.

At each

$$a \le x \le b\,,$$

there is a rectangle with base $[x - \frac{dx}{2}, x + \frac{dx}{2}]$, height $f(x)$, and area

$$f(x)dx\,.$$

We form the **Integration Sum** of all the areas for the x's that start at $x = a$, and end at $x = b$,

$$\sum_{x \in [a,b]} f(x)dx\,.$$

This is a hyper-real number.

If for any infinitesimal dx, the Integration Sum has the same hyper-real value, then $f(x)$ is integrable over the interval $[a,b]$.

Then, the Integration Sum is the integral of $f(x)$ from $x = a$, to $x = b$, denoted by

$$\int_{x=a}^{x=b} f(x)dx\,.$$

The integral may be equal to an infinite hyper-real.

If the hyper-real is finite,

$$\int_{x=a}^{x=b} f(x)dx = \text{ real part of the hyper-real}.\square$$

3.1 The Countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers, $Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty.$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[a,b]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(x)dx$.

The Lower Integral is the Integration Sum where $f(x)$ is replaced by its lowest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

3.2 $$\sum_{x \in [a,b]} \left(\inf_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

The Upper Integral is the Integration Sum where $f(x)$ is replaced by its largest value on each interval $[x-\frac{dx}{2}, x+\frac{dx}{2}]$

3.3
$$\sum_{x\in[a,b]}\left(\sup_{x-\frac{dx}{2}\le t\le x+\frac{dx}{2}} f(t)\right)dx$$

If the integral is a finite hyper-real, we have

3.4 *A hyper-real function has a finite integral if and only if its upper integral and its lower integral are finite, and differ by an infinitesimal.*

4.

The Hyper-Real Delta Function

In [Dan5], we have defined the Delta Function, and established its properties

1. The Delta Function is a hyper-real function defined from the hyper-real space into the set of two hyper-reals $\left\{0, \frac{1}{dx}\right\}$. The hyper-real 0 is the sequence $\left\langle 0, 0, 0, ... \right\rangle$. The infinite hyper-real $\frac{1}{dx}$ depends on our choice of dx.
2. We will usually choose the family of infinitesimals that is spanned by the sequences $\left\langle \frac{1}{n} \right\rangle, \left\langle \frac{1}{n^2} \right\rangle, \left\langle \frac{1}{n^3} \right\rangle, ...$ It is a semigroup with respect to vector addition, and includes all the scalar multiples of the generating sequences that are non-zero. That is, the family includes infinitesimals with negative sign. Therefore, $\frac{1}{dx}$ will mean the sequence $\left\langle n \right\rangle$. Alternatively, we may choose the family spanned by the sequences $\left\langle \frac{1}{2^n} \right\rangle, \left\langle \frac{1}{3^n} \right\rangle, \left\langle \frac{1}{4^n} \right\rangle, ...$ Then, $\frac{1}{dx}$ will mean the sequence $\left\langle 2^n \right\rangle$. Once we determined the basic infinitesimal

dx, we will use it in the Infinite Riemann Sum that defines an Integral in Infinitesimal Calculus.

3. The Delta Function is strictly smaller than ∞

4. $\delta(x) \equiv \frac{1}{dx}\chi_{\left[-\frac{dx}{2},\frac{dx}{2}\right]}(x)$ where $\chi_{\left[-\frac{dx}{2},\frac{dx}{2}\right]}(x) = \begin{cases} 1, x \in \left[-\frac{dx}{2},\frac{dx}{2}\right] \\ 0, \text{otherwise} \end{cases}$.

   - for $x < 0$, $\delta(x) = 0$
   - at $x = -\frac{dx}{2}$, $\delta(x)$ jumps from 0 to $\frac{1}{dx}$,
   - for $x \in \left[-\frac{dx}{2},\frac{dx}{2}\right]$, $\delta(x) = \frac{1}{dx}$.
   - at $x = 0$, $\delta(0) = \frac{1}{dx} =< 1,2,3,...n,... >$
   - at $x = \frac{dx}{2}$, $\delta(x)$ drops from $\frac{1}{dx}$ to 0.
   - for $x > 0$, $\delta(x) = 0$.
   - $x\delta(x) = 0$

5. $dx = \left\langle \frac{1}{n} \right\rangle \Rightarrow \delta(x) = \left\langle 1,2,3...,n,... \right\rangle \chi_{[-\frac{1}{2}dx,\frac{1}{2}dx]}(x)$

6. $dx = \left\langle \frac{2}{n} \right\rangle \Rightarrow \delta(x) = \frac{1}{2}\left\langle \frac{1}{\cosh^2 x}, \frac{2}{\cosh^2 2x}, \frac{3}{\cosh^2 3x}, ... \right\rangle \chi_{[-\frac{1}{2}dx,\frac{1}{2}dx]}(x)$

7. $dx = \left\langle \frac{1}{n} \right\rangle \Rightarrow \delta(x) = \left\langle e^{-x}, 2e^{-2x}, 3e^{-3x}, ... \right\rangle \chi_{\left[-\frac{dx}{2},\frac{dx}{2}\right]}(x)$

8. $$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1.$$

9. $\delta(x-\xi)=\frac{1}{2\pi}\int_{k=-\infty}^{k=\infty}e^{ik(x-\xi)}dk$

5.

$\delta(x)$ and the Fourier Transform

5.1 $\boxed{\mathcal{F}\left\{\delta(x)\right\} = 1}$

Proof: For any dx, the Integration Sum for the function

$$\delta(x)e^{-i\omega x} = \frac{1}{dx}e^{-i\omega x}\chi[-\tfrac{dx}{2},\tfrac{dx}{2}]$$

has only one hyper-real term

$$\frac{1}{dx}e^{-i\omega x}\chi[-\tfrac{dx}{2},\tfrac{dx}{2}]dx = e^{-i\omega x}\chi[-\tfrac{dx}{2},\tfrac{dx}{2}].$$

Therefore, the Fourier Transform

$$\mathcal{F}\left\{\delta(x)\right\} = \int_{x=-\infty}^{x=\infty}\delta(x)e^{-i\omega x}dx$$

exists.

Since $e^{-i\omega x}\chi[-\tfrac{dx}{2},\tfrac{dx}{2}] = 1$, it is a finite hyper-real.

Therefore, the Fourier Transform equals to the constant part of this hyper-real.

Since the constant hyper-real in $[-\tfrac{dx}{2},\tfrac{dx}{2}]$ is zero, the constant

hyper-real part of $e^{-i\omega x}\chi[-\tfrac{dx}{2},\tfrac{dx}{2}]$ is

$$e^{-i\omega 0} = 1.$$

That is

$$\mathcal{F}\left\{\delta(x)\right\} = 1.\square$$

Consequently,

5.2 $\delta(x)$ = ***inverse Fourier Transform of the unit function*** **1**

$$\boxed{\delta(x) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x} d\omega}$$

$\omega = 2\pi\nu \Rightarrow$

$$\boxed{\delta(x) = \int_{\nu=-\infty}^{\nu=\infty} e^{2\pi i\nu x} d\nu}$$

Thus,

5.3 $$\boxed{\delta(0) = \frac{1}{2\pi}\int_{\omega=-N}^{\omega=N} e^{i\omega\cdot 0} d\omega = \frac{1}{2\pi}\int_{\omega=-N}^{\omega=N} d\omega = \frac{1}{2\pi}\ 2N = N}$$

And since $x \neq 0 \Rightarrow \frac{1}{2\pi}\int_{\omega=-N}^{\omega=N} e^{i\omega x} d\omega = \delta(x) = 0$,

5.4 $$\boxed{\frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x} d\omega = 0}$$

6.

Delta in $span\left\{ \int_{\omega=-2\pi n}^{\omega=2\pi n} e^{i\omega x} d\omega \right\}_{n=1}^{n=\infty}$

From

$$\delta(x) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x} d\omega \,,$$

$\delta(x)$ is the infinite vector with components $\int_{\omega=-2\pi n}^{\omega=2\pi n} e^{i\omega x} d\omega$,

$\delta(x)$ is in $span\left\{ \int_{\omega=-2\pi n}^{\omega=2\pi n} e^{i\omega x} d\omega \right\}_{n=1}^{n=\infty}$,

$$\delta(x) = \frac{1}{2\pi}\begin{bmatrix} \int_{\omega=-2\pi}^{\omega=2\pi} e^{i\omega x} d\omega \\ \int_{\omega=-4\pi}^{\omega=4\pi} e^{i\omega x} d\omega \\ \int_{\omega=-6\pi}^{\omega=6\pi} e^{i\omega x} d\omega \\ \int_{\omega=-8\pi}^{\omega=8\pi} e^{i\omega x} d\omega \\ \int_{\omega=-10\pi}^{\omega=10\pi} e^{i\omega x} d\omega \\ -\,-\,-\,-\,- \\ -\,-\,-\,-\,- \\ \int_{\omega=-2\pi n}^{\omega=2\pi n} e^{i\omega x} d\omega \\ -\,-\,-\,-\,- \\ -\,-\,-\,-\,- \end{bmatrix} = \begin{cases} < n >, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

7.

Delta in $span\left\{\int_{\omega=0}^{\omega=n}\cos(\omega x)d\omega\right\}_{n=1}^{n=\infty}$

7.1 $\delta(x) = \frac{1}{\pi}\int_{\omega=0}^{\omega=\infty}\cos(\omega x)d\omega$

$$= 2\int_{\nu=0}^{\nu=\infty}\cos(2\pi\nu x)d\nu$$

Proof: $\delta(x) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty}e^{i\omega x}d\omega$

$$= \frac{1}{2\pi}\left\{\int_{\omega=-\infty}^{\omega=0}e^{i\omega x}d\omega + \int_{\omega=0}^{\omega=\infty}e^{i\omega x}d\omega\right\}$$

$$= \frac{1}{2\pi}\left\{\int_{\omega=0}^{\omega=\infty}e^{-i\omega x}d\omega + \int_{\omega=0}^{\omega=\infty}e^{i\omega x}d\omega\right\}$$

$$= \frac{1}{\pi}\int_{\omega=0}^{\omega=\infty}\cos(\omega x)d\omega . \square$$

Therefore,

7.2 $\delta(x)$ is in $span\left\{\int_{\omega=0}^{\omega=n}\cos(\omega x)d\omega\right\}_{n=1}^{n=\infty}$

$$\delta(x) = \frac{1}{\pi}\begin{bmatrix} \int\limits_{\omega=0}^{\omega=2\pi} \cos(\omega x)d\omega \\ \int\limits_{\omega=0}^{\omega=4\pi} \cos(\omega x)d\omega \\ \int\limits_{\omega=0}^{\omega=6\pi} \cos(\omega x)d\omega \\ \int\limits_{\omega=0}^{\omega=8\pi} \cos(\omega x)d\omega \\ \int\limits_{\omega=0}^{\omega=10\pi} \cos(\omega x)d\omega \\ ------ \\ ------ \\ \int\limits_{\omega=0}^{\omega=2\pi n} \cos(\omega x)d\omega \\ ------ \\ ------ \end{bmatrix} = \begin{cases} < n >, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

8.

Delta Sequence $\delta_n(x) = \frac{n}{\pi}\left(\frac{\sin(nx)}{nx}\right)^2$

We show that the Hyper-real Delta Function is represented by the Delta Sequence

$$\delta_n(x) = \frac{n}{\pi}\left(\frac{\sin(nx)}{nx}\right)^2.$$

The nth component of the Hyper-real Delta is $\frac{n}{\pi}\left(\frac{\sin(nx)}{nx}\right)^2$. That is,

$$\delta(x) = \left\langle \frac{n}{\pi}\left(\frac{\sin(nx)}{nx}\right)^2 \right\rangle = \begin{cases} < n >, & x = 0 \\ 0, & x \neq 0 \end{cases}.$$

8.1 Each $\delta_n(x) = \frac{n}{\pi}\left(\frac{\sin(nx)}{nx}\right)^2$

- has the sifting property $\int_{x=-\infty}^{x=\infty} \delta_n(x)dx = 1$,
- is a continuous Hyper-real function,
- peaks at $x = 0$ to $\delta_n(0) = \frac{1}{\pi}n$.

Proof:

$$\int_{x=-\infty}^{x=\infty} \delta_n(x)dx = \frac{n}{\pi}\int_{x=-\infty}^{x=\infty} \frac{\sin^2(nx)}{n^2x^2}dx = \frac{1}{\pi n}2\underbrace{\int_{x=0}^{x=\infty}\frac{\sin^2(nx)}{x^2}dx}_{\frac{1}{2}\pi n,\text{by [Spiegel2]}} = 1.\square$$

To see that $\delta_n(0) = \frac{1}{\pi}n$, we use the infinitesimal $\left\langle \frac{1}{n^2} \right\rangle$. Then,

$$\delta_n(\tfrac{1}{n^2}) = \frac{n}{\pi}\left(\frac{\sin(\frac{1}{n})}{\frac{1}{n}}\right)^2 .$$

Since for any infinitesimal o, $\dfrac{\sin(o)}{o} = 1$, we have

$$\delta_n(\tfrac{1}{n^2}) = \frac{n}{\pi},$$

and $\delta_n(0) = \frac{1}{\pi}n.\square$

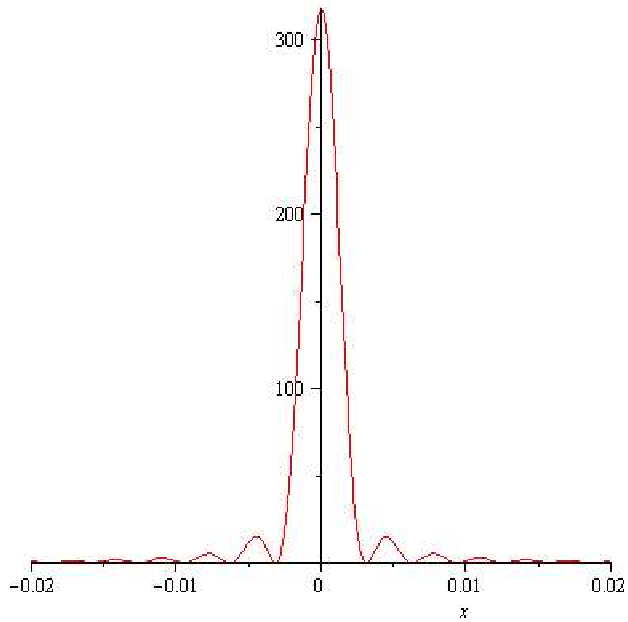
Therefore,

8.2 *The sequence represents the Hyper-real Delta*

$$\delta(x) = \frac{1}{\pi}\left\langle \frac{\sin^2(x)}{x^2}, \frac{\sin^2(2x)}{2x^2}, \frac{\sin^2(3x)}{3x^2}, \dots \right\rangle.$$

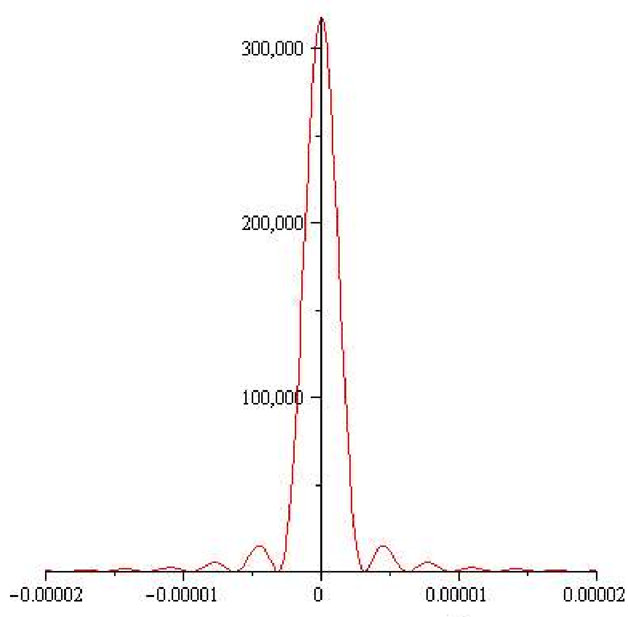
8.3 $plot\left(\frac{1}{1000\,\pi}\frac{\sin^2(1000\,x)}{x^2},\ x = -\frac{2}{100}..\frac{2}{100}\right)$

plots in Maple the 1000th component, that peaks at $\dfrac{1000}{\pi} \approx 318$.



8.4 $$plot\left(\frac{1}{10^6\pi}\frac{\sin^2(10^6 x)}{x^2},\ x=-\frac{2}{10^5}..\frac{2}{10^5}\right)$$

plots in Maple the 10^6 component, that peaks at $\frac{10^6}{\pi} \approx 318,310$.



To show the relation between the infinitesimal dx, and this Hyper-real $\delta(x)$, we note

8.5 *<u>If</u>* dx *is given by* $i_n = \frac{1}{n}$,

<u>Then</u> This Hyper-real $\delta(x)$

H peaks to $\frac{1}{\pi dx}$.

H may be written symbolically by $\delta(x) = \frac{1}{\pi dx}\left(\frac{\sin(\frac{x}{dx})}{\frac{x}{dx}}\right)^2$

References

[Dan1] Dannon, H. Vic, "*Well-Ordering of the Reals, Equality of all Infinities, and the Continuum Hypothesis*" in Gauge Institute Journal Vol.6, No. 2, May 2010;

[Dan2] Dannon, H. Vic, "*Infinitesimals*" in Gauge Institute Journal Vol.6, No. 4, November 2010;

[Dan3] Dannon, H. Vic, "*Infinitesimal Calculus*" in Gauge Institute Journal Vol.7, No. 4, November 2011;

[Dan4] Dannon, H. Vic, "*Riemann's Zeta Function: the Riemann Hypothesis Origin, the Factorization Error, and the Count of the Primes*", in Gauge Institute Journal of Math and Physics, Vol.5, No. 4, November 2009.

A 2011 revision, published by the Gauge Institute, follows the development of the Infinitesimal Calculus [Dan3], that is necessary to establish Riemann results.

[Dan5] Dannon, H. Vic, "*The Delta Function*" in Gauge Institute Journal Vol.8, No. 1, February 2012;

[Riemann] Riemann, Bernhard, "*On the Representation of a Function by a Trigonometric Series*".

(1) In "*Collected Papers, Bernhard Riemann*", translated from the 1892 edition by Roger Baker, Charles Christenson, and Henry Orde, Paper XII, Part 5, Conditions for the existence of a definite integral, pages 231-232, Part 6, Special Cases, pages 232-234. Kendrick press, 2004

(2) In "*God Created the Integers*" Edited by Stephen Hawking, Part 5, and Part 6, pages 836-840, Running Press, 2005.

[Spiegel1] Spiegel, Murray, "*Fourier Analysis with applications to Boundary value Problems*" Schaum's Outline Series, McGraw-Hill, 1974.

p.80 has the sufficient conditions for the Fourier integral Theorem

[Spiegel2] Spiegel, Murray, "*Mathematical Handbook of Formulas and Tables*" Schaum's Outline Series, McGraw-Hill, 1968. p. 96, #15.36