

Sturm-Liouville Expansions of the Delta Function

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Abstract We expand the Delta Function in Series, and Integrals of Sturm-Liouville Eigen-functions.

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References

0.

Eigen-Functions Expansion of the Delta Function

Unlike Taylor's expansion of a function that requires its derivatives of any order, Fourier Integral or Fourier Series representation require no derivatives. Then, the function's projections on the orthogonal sequence of eigen-functions sum up to the function.

It is little known that the Fourier Series representation, and the Fourier Integral representation result from an expansion of the Delta Function:

For instance, the Fourier Integral representation,

$$\begin{aligned}
 f(x) &= \int_{\xi=-\infty}^{\xi=\infty} f(\xi)\delta(x - \xi)d\xi \\
 &= \int_{\xi=-\infty}^{\xi=\infty} f(\xi) \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} e^{i\omega(x-\xi)} d\omega d\xi \\
 &= \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} \underbrace{\int_{\xi=-\infty}^{\xi=\infty} f(\xi) e^{-i\omega\xi} d\xi}_{F(\omega)} e^{i\omega x} d\omega
 \end{aligned}$$

Here, the Fourier Transform $F(\omega)$ is the projection of $f(x)$ on $e^{i\omega x}$.

The eigen-function expansions of the Delta Function are at the root of the eigen-function expansion of any function.

The Delta Function can be defined only as a hyper-real function in infinitesimal Calculus.

We proceed with the definition of the Hyper-real line.

1.

Hyper-real Line

Each real number α can be represented by a Cauchy sequence of rational numbers, (r_1, r_2, r_3, \dots) so that $r_n \rightarrow \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, \dots)$ is a constant hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences (l_1, l_2, l_3, \dots) constitutes a family of infinitesimal hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{l_1}, \frac{1}{l_2}, \frac{1}{l_3}, \dots\right)$ are the infinite hyper-reals.
4. The infinite hyper-reals are greater than any real number, yet strictly smaller than infinity.
5. The infinite hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a non-constant hyper-real.
7. The Hyper-reals are the totality of constant hyper-reals, a family of infinitesimals, a family of infinitesimals with

- negative sign, a family of infinite hyper-reals, a family of infinite hyper-reals with negative sign, and non-constant hyper-reals.
8. The hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.
 9. That line includes the real numbers separated by the non-constant hyper-reals. Each real number is the center of an interval of hyper-reals, that includes no other real number.
 10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.
 11. Zero is not an infinitesimal, because zero is not strictly greater than zero.
 12. We do not add infinity to the hyper-real line.
 13. The infinitesimals, the infinitesimals with negative signs, the infinite hyper-reals, and the infinite hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.
 14. The hyper-real line is embedded in \mathbb{R}^∞ , and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the hyper-real onto the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal hyper-reals, or to the infinite hyper-reals, or to the non-constant hyper-reals.
16. No neighbourhood of a hyper-real is homeomorphic to an \mathbb{R}^n ball. Therefore, the hyper-real line is not a manifold.
17. The hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

2.

Hyper-real Function

2.1 Definition of a hyper-real function

$f(x)$ is a hyper-real function, iff it is from the hyper-reals into the hyper-reals.

This means that any number in the domain, or in the range of a hyper-real $f(x)$ is either one of the following

- real
- real + infinitesimal
- real – infinitesimal
- infinitesimal
- infinitesimal with negative sign
- infinite hyper-real
- infinite hyper-real with negative sign

Clearly,

2.2 *Every function from the reals into the reals is a hyper-real function.*

3.

Integral of a Hyper-real Function

In [Dan3], we defined the integral of a Hyper-real Function.

Let $f(x)$ be a hyper-real function on the interval $[a, b]$.

The interval may not be bounded.

$f(x)$ may take infinite hyper-real values, and need not be bounded.

At each

$$a \leq x \leq b,$$

there is a rectangle with base $[x - \frac{dx}{2}, x + \frac{dx}{2}]$, height $f(x)$, and area

$$f(x)dx.$$

We form the **Integration Sum** of all the areas for the x 's that start at $x = a$, and end at $x = b$,

$$\sum_{x \in [a, b]} f(x)dx.$$

If for any infinitesimal dx , the Integration Sum has the same hyper-real value, then $f(x)$ is integrable over the interval $[a, b]$.

Then, we call the Integration Sum the integral of $f(x)$ from $x = a$, to $x = b$, and denote it by

$$\int_{x=a}^{x=b} f(x)dx .$$

If the hyper-real is infinite, then it is the integral over $[a, b]$,

If the hyper-real is finite,

$$\int_{x=a}^{x=b} f(x)dx = \text{real part of the hyper-real. } \square$$

3.1 The countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers,

$Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and

we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = \dots = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = \dots \equiv \infty .$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[a, b]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(x)dx$.

The Lower Integral is the Integration Sum where $f(x)$ is replaced

by its lowest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

$$\mathbf{3.2} \quad \sum_{x \in [a, b]} \left(\inf_{x - \frac{dx}{2} \leq t \leq x + \frac{dx}{2}} f(t) \right) dx$$

The Upper Integral is the Integration Sum where $f(x)$ is replaced

by its largest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

$$\mathbf{3.3} \quad \sum_{x \in [a, b]} \left(\sup_{x - \frac{dx}{2} \leq t \leq x + \frac{dx}{2}} f(t) \right) dx$$

If the integral is a finite hyper-real, we have

3.4 *A hyper-real function has a finite integral if and only if its upper integral and its lower integral are finite, and differ by an infinitesimal.*

4.

Delta Function

In [Dan5], we have defined the Delta Function, and established its properties

1. The Delta Function is a hyper-real function defined from the

hyper-real line into the set of two hyper-reals $\left\{0, \frac{1}{dx}\right\}$. The

hyper-real 0 is the sequence $\langle 0, 0, 0, \dots \rangle$. The infinite hyper-

real $\frac{1}{dx}$ depends on our choice of dx .

2. We will usually choose the family of infinitesimals that is

spanned by the sequences $\left\langle \frac{1}{n} \right\rangle, \left\langle \frac{1}{n^2} \right\rangle, \left\langle \frac{1}{n^3} \right\rangle, \dots$. It is a

semigroup with respect to vector addition, and includes all

the scalar multiples of the generating sequences that are

non-zero. That is, the family includes infinitesimals with

negative sign. Therefore, $\frac{1}{dx}$ will mean the sequence $\langle n \rangle$.

Alternatively, we may choose the family spanned by the

sequences $\left\langle \frac{1}{2^n} \right\rangle, \left\langle \frac{1}{3^n} \right\rangle, \left\langle \frac{1}{4^n} \right\rangle, \dots$. Then, $\frac{1}{dx}$ will mean the

sequence $\langle 2^n \rangle$. Once we determined the basic infinitesimal dx , we will use it in the Infinite Riemann Sum that defines an Integral in Infinitesimal Calculus.

3. The Delta Function is strictly smaller than ∞

4. We define, $\delta(x) \equiv \frac{1}{dx} \mathcal{X}_{[-\frac{dx}{2}, \frac{dx}{2}]}(x)$,

$$\text{where } \mathcal{X}_{[-\frac{dx}{2}, \frac{dx}{2}]}(x) = \begin{cases} 1, & x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right] \\ 0, & \text{otherwise} \end{cases}$$

5. Hence,

$$\diamond \text{ for } x < 0, \delta(x) = 0$$

$$\diamond \text{ at } x = -\frac{dx}{2}, \delta(x) \text{ jumps from } 0 \text{ to } \frac{1}{dx},$$

$$\diamond \text{ for } x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right], \delta(x) = \frac{1}{dx}.$$

$$\diamond \text{ at } x = 0, \delta(0) = \frac{1}{dx}$$

$$\diamond \text{ at } x = \frac{dx}{2}, \delta(x) \text{ drops from } \frac{1}{dx} \text{ to } 0.$$

$$\diamond \text{ for } x > 0, \delta(x) = 0.$$

$$\diamond x\delta(x) = 0$$

6. If $dx = \langle \frac{1}{n} \rangle$, $\delta(x) = \left\langle \mathcal{X}_{[-\frac{1}{2}, \frac{1}{2}]}(x), 2\mathcal{X}_{[-\frac{1}{4}, \frac{1}{4}]}(x), 3\mathcal{X}_{[-\frac{1}{6}, \frac{1}{6}]}(x) \dots \right\rangle$

7. If $dx = \langle \frac{2}{n} \rangle$, $\delta(x) = \left\langle \frac{1}{2 \cosh^2 x}, \frac{2}{2 \cosh^2 2x}, \frac{3}{2 \cosh^2 3x}, \dots \right\rangle$

8. If $dx = \langle \frac{1}{n} \rangle$, $\delta(x) = \langle e^{-x}\chi_{[0,\infty)}, 2e^{-2x}\chi_{[0,\infty)}, 3e^{-3x}\chi_{[0,\infty)}, \dots \rangle$

9.
$$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1.$$

10.
$$\delta(\xi - x) = \frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} e^{-ik(\xi-x)}dk$$

5.

Convergent Series

In [Dan8], we defined convergence of infinite series in Infinitesimal Calculus

5.1 Sequence Convergence to a finite hyper-real a

$$a_n \rightarrow a \text{ iff } \langle a_n \rangle - \langle a \rangle = \text{infinitesimal.}$$

5.2 Sequence Convergence to an infinite hyper-real A

$$a_n \rightarrow A \text{ iff } \langle a_n \rangle \text{ represents the infinite hyper-real } A.$$

5.3 Series Convergence to a finite hyper-real s

$$a_1 + a_2 + \dots \rightarrow s \text{ iff } \langle a_1 + \dots + a_n \rangle - \langle s \rangle = \text{infinitesimal.}$$

5.4 Series Convergence to an Infinite Hyper-real S

$$a_1 + a_2 + \dots \rightarrow S \text{ iff}$$

$$\langle a_1 + \dots + a_n \rangle \text{ represents the infinite hyper-real } S.$$

6.

Hyper-real Sturm-Liouville Problem

The Hyper-real Sturm-Liouville equation is the second order Hyper-real linear differential equation for the Hyper-real function $y(x)$,

$$-y''(x) + q(x) = \lambda y(x),$$

on an interval that may be bounded, or may be the whole Hyper-real line.

The hyper-real function $q(x)$ is (assumed in the literature to be) continuous on the interval, and bounded at its endpoints.

The choice of the number λ (which may be real or complex), allows the equation with boundary conditions, at the interval endpoints to become an eigen-value problem:

λ_n is an eigen-value, and $\psi_n(x)$ is the corresponding Hyper-real eigen-function iff

$$-\psi_n''(x) + q(x)\psi_n(x) = \lambda_n \psi_n(x).$$

The eigen-functions are orthogonal, over the interval.

$$\int_{x=a}^{x=b} \psi_n(x)\psi_m(x)dx = 0, \quad \text{for } n \neq m.$$

7.

Delta Expansion in Non-Normalized Eigen-functions

As described in [Titchmarsh, Chapter I], given numbers

$$\alpha, \text{ and } \beta,$$

the Sturm-Liouville Problem on the interval with endpoints a , and b has solutions

$$\phi_\alpha(x, \lambda), \text{ with } \phi_\alpha(a, \lambda) = \sin \alpha, \text{ and } \phi_\alpha'(a, \lambda) = -\cos \alpha,$$

$$\chi_\beta(x, \lambda), \text{ with } \chi_\beta(b, \lambda) = \sin \beta, \text{ and } \chi_\beta'(b, \lambda) = -\cos \beta,,$$

which are entire functions of λ .

Then,

$$\begin{aligned} \frac{d}{dx} \underbrace{\begin{vmatrix} \phi_\alpha(x, \lambda) & \chi_\beta(x, \lambda) \\ \phi_\alpha'(x, \lambda) & \chi_\beta'(x, \lambda) \end{vmatrix}}_{W[\phi_\alpha, \chi_\beta]} &= \underbrace{\begin{vmatrix} \phi_\alpha'(x, \lambda) & \chi_\beta'(x, \lambda) \\ \phi_\alpha''(x, \lambda) & \chi_\beta''(x, \lambda) \end{vmatrix}}_0 + \begin{vmatrix} \phi_\alpha(x, \lambda) & \chi_\beta(x, \lambda) \\ \phi_\alpha''(x, \lambda) & \chi_\beta''(x, \lambda) \end{vmatrix} \\ &= \phi_\alpha \chi_\beta'' - \chi_\beta \phi_\alpha'' \\ &= \phi_\alpha (q - \lambda) \chi_\beta - \chi_\beta (q - \lambda) \phi_\alpha \\ &= 0. \end{aligned}$$

Hence, the Wronskian $W[\phi_\alpha, \chi_\beta]$ is a function of λ alone:

$$W[\phi_\alpha, \chi_\beta] = \omega(\lambda).$$

Now, if the only zeros of $\omega(\lambda)$ are the simple zeros

$$\lambda_0, \lambda_1, \lambda_2, \lambda_3, \dots,$$

Then, for each $n = 0, 1, 2, 3, \dots$

$$0 = \omega(\lambda_n) = \begin{vmatrix} \phi_\alpha(x, \lambda_n) & \chi_\beta(x, \lambda_n) \\ \phi_\alpha'(x, \lambda_n) & \chi_\beta'(x, \lambda_n) \end{vmatrix}.$$

That is, for each $n = 0, 1, 2, 3, \dots$ there is a number k_n so that

$$\chi_\beta(x, \lambda_n) = k_n \phi_\alpha(x, \lambda_n).$$

Titchmarsh applied the Residue Theorem to obtain the coefficients in the Sturm-Liouville expansion of $f(x)$.

Following Titchmarsh, we conclude that the Hyper-real Sturm-Liouville expansion of a Hyper-real function $f(x)$ in the Hyper-real eigen-functions $\phi_\alpha(x, \lambda_n)$ is

$$\begin{aligned} f(x) &= \sum_{n=0}^{n=\infty} \left(\frac{k_n}{\omega'(\lambda_n)} \int_{\xi=a}^{\xi=b} f(\xi) \phi_\alpha(\xi, \lambda_n) d\xi \right) \phi_\alpha(x, \lambda_n) \\ &= \sum_{n=0}^{n=\infty} \left(\frac{k_n}{\omega'(\lambda_n)} \sum_{\xi=a}^{\xi=b} f(\xi) \phi_\alpha(\xi, \lambda_n) d\xi \right) \phi_\alpha(x, \lambda_n) \end{aligned}$$

Exchanging summation order

$$\begin{aligned} &= \sum_{\xi=a}^{\xi=b} \sum_{n=0}^{n=\infty} \frac{k_n}{\omega'(\lambda_n)} \phi_\alpha(x, \lambda_n) \phi_\alpha(\xi, \lambda_n) f(\xi) d\xi \\ &= \int_{\xi=a}^{\xi=b} \underbrace{\sum_{n=0}^{n=\infty} \frac{k_n}{\omega'(\lambda_n)} \phi_\alpha(x, \lambda_n) \phi_\alpha(\xi, \lambda_n)}_{\delta(x-\xi)} f(\xi) d\xi \end{aligned}$$

Therefore,

7.1 *The Hyper-real Delta Function Expansion in non-normalized eigen-functions is*

$$\delta(x - \xi) = \sum_{n=0}^{n=\infty} \frac{k_n}{\omega'(\lambda_n)} \phi_\alpha(x, \lambda_n) \phi_\alpha(\xi, \lambda_n)$$

That is,

7.2 *The Hyper-real Delta Function is the infinite sequence*

$$\left\langle \frac{k_0}{\omega'(\lambda_0)} \phi_\alpha(x, \lambda_0) \phi_\alpha(\xi, \lambda_0) + \dots + \frac{k_n}{\omega'(\lambda_n)} \phi_\alpha(x, \lambda_n) \phi_\alpha(\xi, \lambda_n) \right\rangle.$$

8.

Fourier-Sine Expansion of Delta

Associated with $y''(x) + \lambda y(x) = 0$

& $\alpha = \beta = 0$

Two independent solutions are

$$\cos \sqrt{\lambda}x, \text{ and } \sin \sqrt{\lambda}x.$$

For $\alpha = 0$,

$$\phi_0(x, \lambda) = -\frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda}(x - a)$$

satisfies the Boundary conditions

$$\phi_0(a, \lambda) = \sin 0 = 0,$$

and

$$\phi_0'(a, \lambda) = -\cos 0 = -1.$$

For $\beta = 0$,

$$\chi_0(x, \lambda) = \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda}(b - x)$$

satisfies the Boundary conditions

$$\chi_0(b, \lambda) = \sin 0 = 0,$$

and

$$\chi_0'(b, \lambda) = -\cos 0 = -1,$$

Therefore,

$$\begin{aligned}\omega(\lambda) &= \left| \begin{array}{cc} -\frac{\sin \sqrt{\lambda}(x-a)}{\sqrt{\lambda}} & \frac{\sin \sqrt{\lambda}(b-x)}{\sqrt{\lambda}} \\ -\cos \sqrt{\lambda}(x-a) & -\cos \sqrt{\lambda}(b-x) \end{array} \right| \\ &= \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda}(b-a).\end{aligned}$$

Hence, the zeros of $\omega(\lambda)$ are

$$\lambda_n = \left(\frac{n\pi}{b-a} \right)^2, \quad n = 1, 2, 3, \dots$$

$$\begin{aligned}\omega'(\lambda) &= \frac{d}{d\lambda} \left\{ \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda}(b-a) \right\} \\ &= \frac{1}{\lambda} \left\{ \left[\cos \sqrt{\lambda}(b-a) \right] \frac{b-a}{2\sqrt{\lambda}} \sqrt{\lambda} - \frac{1}{2\sqrt{\lambda}} \sin \sqrt{\lambda}(b-a) \right\}.\end{aligned}$$

$$\begin{aligned}\omega'(\lambda_n) &= \frac{1}{2\lambda_n} (b-a) \cos \underbrace{\sqrt{\lambda_n}(b-a)}_{n\pi} \\ &= \frac{1}{2\lambda_n} (b-a) (-1)^n.\end{aligned}$$

$$\begin{aligned}k_n &= \frac{\chi_0(x, \lambda_n)}{\phi_0(x, \lambda_n)} \\ &= -\frac{\sin \sqrt{\lambda_n}(b-x)}{\sin \sqrt{\lambda_n}(x-a)} \\ &= -\frac{\sin \frac{b-x}{b-a} n\pi}{\sin \frac{x-a}{b-a} n\pi} \\ &= -\frac{\sin [n\pi - \frac{x-a}{b-a} n\pi]}{\sin \frac{x-a}{b-a} n\pi}\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sin \frac{x-a}{b-a} n\pi} \left\{ \underbrace{-\sin(n\pi) \cos \frac{x-a}{b-a} n\pi}_0 + \underbrace{\cos(n\pi) \sin \frac{x-a}{b-a} n\pi}_{(-1)^n} \right\} \\
&= (-1)^n.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\delta(x - \xi) &= \sum_{n=1}^{n=\infty} \frac{k_n}{\omega'(\lambda_n)} \phi_\alpha(\xi, \lambda_n) \phi_\alpha(x, \lambda_n) \\
&= \sum_{n=1}^{n=\infty} \frac{(-1)^n}{b-a} \frac{\sin \sqrt{\lambda_n} (x-a)}{\sqrt{\lambda_n}} \frac{\sin \sqrt{\lambda_n} (\xi-a)}{\sqrt{\lambda_n}} \\
&= \sum_{n=1}^{n=\infty} \frac{2}{b-a} \sin \left(n\pi \frac{x-a}{b-a} \right) \sin \left(n\pi \frac{\xi-a}{b-a} \right)
\end{aligned}$$

8.1 The Fourier Sine expansion of Delta Function in $[a, b]$

$$\boxed{\delta(x - \xi) = \frac{2}{b-a} \sum_{n=1}^{n=\infty} \sin \left(n\pi \frac{x-a}{b-a} \right) \sin \left(n\pi \frac{\xi-a}{b-a} \right)}$$

Namely,

8.2 The Hyper-real Delta in $[a, b]$ is the infinite sequence

$$\frac{2}{b-a} \left\langle \sin \left(\pi \frac{x-a}{b-a} \right) \sin \left(\pi \frac{\xi-a}{b-a} \right) + \dots + \sin \left(n\pi \frac{x-a}{b-a} \right) \sin \left(n\pi \frac{\xi-a}{b-a} \right) \right\rangle_{n=1}^{n=\infty}$$

9.

Fourier-Cosine Expansion of Delta Associated with

$$y''(x) + \lambda y(x) = 0 \quad \& \quad \alpha = \beta = \frac{1}{2} \pi$$

Similarly to the former expansion, for $\alpha = \beta = \frac{1}{2} \pi$, we obtain

9.1 *Fourier Cosine expansion of the Delta Function in $[a, b]$*

$$\delta(x - \xi) = \frac{1}{b - a} \left\{ 1 + 2 \sum_{n=1}^{n=\infty} \cos \left(n\pi \frac{x - a}{b - a} \right) \cos \left(n\pi \frac{\xi - a}{b - a} \right) \right\}$$

Namely,

9.2 *The Hyper-real Delta in $[a, b]$ is the infinite sequence*

$$\frac{1}{b - a} \left\langle 1 + 2 \cos \left(\pi \frac{x - a}{b - a} \right) \cos \left(\pi \frac{\xi - a}{b - a} \right) + \dots + 2 \cos \left(n\pi \frac{x - a}{b - a} \right) \cos \left(n\pi \frac{\xi - a}{b - a} \right) \right\rangle_{n=1}^{n=\infty}$$

10.

Fourier-Sine Expansion of Delta

Associated with $y''(x) + \lambda y(x) = 0$ & $\alpha = 0$

For $\alpha = 0$,

$$\phi_0(x, \lambda) = -\frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}}$$

satisfies the Boundary conditions

$$\phi_0(a, \lambda) = \sin 0 = 0,$$

and

$$\phi_0'(a, \lambda) = -\cos 0 = -1.$$

The solution

$$\chi_\beta(x, \lambda) = \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda}(b-x) \cos \beta + \cos[\sqrt{\lambda}(b-x)] \sin \beta$$

satisfies the Boundary conditions

$$\chi_\beta(b, \lambda) = \sin \beta,$$

and

$$\chi_\beta'(b, \lambda) = -\cos \beta,$$

Therefore, at $x = b$,

$$\omega(\lambda) = \begin{vmatrix} -\frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda}b & \sin \beta \\ -\cos \sqrt{\lambda}b & -\cos \beta \end{vmatrix}$$

Hence, the zeros of $\omega(\lambda)$ are the roots of

$$\frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda} b \cos \beta = -\cos \sqrt{\lambda} b \sin \beta.$$

That is,

$$\tan \sqrt{\lambda_n} b = -\sqrt{\lambda_n} \tan \beta.$$

[Titchmarsh, p. 17] obtains

$$\omega'(\lambda_n) = \frac{1}{2\lambda_n} b \cos \beta \cos(b\sqrt{\lambda_n}) \{1 + \lambda_n \tan^2 \beta + \frac{1}{b} \tan \beta\}$$

$$k_n = \cos \beta \cos(b\sqrt{\lambda_n}) \{1 + \lambda_n \tan^2 \beta\}$$

Therefore,

$$\begin{aligned} \delta(x - \xi) &= \sum_{n=0}^{n=\infty} \frac{k_n}{\omega'(\lambda_n)} \phi_\alpha(\xi, \lambda_n) \phi_\alpha(x, \lambda_n) \\ &= \frac{2}{b} \sum_{n=0}^{n=\infty} \frac{1 + \lambda_n \tan^2 \beta}{1 + \lambda_n \tan^2 \beta + \frac{1}{b} \tan \beta} \sin(x\sqrt{\lambda_n}) \sin(\xi\sqrt{\lambda_n}) \end{aligned}$$

10.1 The Fourier Sine expansion of Delta in $[a, b]$

$$\boxed{\delta(x - \xi) = \frac{2}{b} \sum_{n=0}^{n=\infty} \frac{1 + \lambda_n \tan^2 \beta}{1 + \lambda_n \tan^2 \beta + \frac{1}{b} \tan \beta} \sin(x\sqrt{\lambda_n}) \sin(\xi\sqrt{\lambda_n})}$$

Namely,

10.2 The Hyper-real Delta in $[a, b]$ is the infinite sequence

$$\frac{2}{b} \left\langle \sum_{j=0}^{j=n} \frac{1 + \lambda_j \tan^2 \beta}{1 + \lambda_j \tan^2 \beta + \frac{1}{b} \tan \beta} \sin(x\sqrt{\lambda_j}) \sin(\xi\sqrt{\lambda_j}) \right\rangle_{n=0}^{n=\infty}$$

11.**Fourier-Bessel Expansion of Delta
Associated with**

$$u''(x) + \left(\lambda - \frac{\nu^2 - \frac{1}{4}}{x^2}\right)u(x) = 0 \quad \& \quad \alpha = \beta = 0$$

Put

$$u(x) = \sqrt{x}y(x),$$

and obtain Bessel's equation

$$y''(x) + \frac{1}{x}y'(x) + \left(\lambda - \frac{\nu^2}{x^2}\right)y(x) = 0.$$

Two independent solutions to Bessel's equation are

$$J_\nu(x\sqrt{\lambda}), \text{ and } Y_\nu(x\sqrt{\lambda}).$$

Two independent solutions to $u''(x) + \left(\lambda - \frac{\nu^2 - \frac{1}{4}}{x^2}\right)u(x) = 0$ are

$$\sqrt{x}J_\nu(x\sqrt{\lambda}), \text{ and } \sqrt{x}Y_\nu(x\sqrt{\lambda}).$$

For $\alpha = 0$,

$$\phi_0(x, \lambda) = \frac{1}{2}\pi\sqrt{ax}\{J_\nu(x\sqrt{\lambda})Y_\nu(a\sqrt{\lambda}) - Y_\nu(x\sqrt{\lambda})J_\nu(a\sqrt{\lambda})\}$$

satisfies the Boundary conditions

$$\phi_0(a, \lambda) = \sin 0 = 0,$$

and

$$\phi_0'(a, \lambda) = -\cos 0 = -1.$$

For $\beta = 0$,

$$\chi_0(x, \lambda) = \frac{1}{2} \pi \sqrt{bx} \{J_\nu(x\sqrt{\lambda})Y_\nu(b\sqrt{\lambda}) - Y_\nu(x\sqrt{\lambda})J_\nu(b\sqrt{\lambda})\}$$

satisfies the Boundary conditions

$$\chi_0(b, \lambda) = \sin 0 = 0,$$

and

$$\chi_0'(b, \lambda) = -\cos 0 = -1,$$

[Titchmarsh, p.18] obtains

$$\begin{aligned} \omega(\lambda) &= \begin{vmatrix} \phi_0(x, \lambda) & \chi_0(x, \lambda) \\ \partial_x \phi_0(x, \lambda) & \partial_x \chi_0(x, \lambda) \end{vmatrix} \\ &= \frac{1}{2} \pi \sqrt{ab} \{J_\nu(a\sqrt{\lambda})Y_\nu(b\sqrt{\lambda}) - Y_\nu(a\sqrt{\lambda})J_\nu(b\sqrt{\lambda})\}. \\ \omega'(\lambda) &= - \left\{ \frac{a}{2\sqrt{\lambda}} \frac{J_\nu'(a\sqrt{\lambda})}{J_\nu(b\sqrt{\lambda})} + \frac{b}{2\sqrt{\lambda}} \frac{J_\nu'(b\sqrt{\lambda})}{J_\nu(b\sqrt{\lambda})} \right\} \omega(\lambda) - \frac{\sqrt{ab}}{2\lambda} \left[\frac{J_\nu(b\sqrt{\lambda})}{J_\nu(a\sqrt{\lambda})} - \frac{J_\nu(a\sqrt{\lambda})}{J_\nu(b\sqrt{\lambda})} \right] \\ \omega'(\lambda_n) &= - \frac{\sqrt{ab}}{2\lambda_n} \left[\frac{J_\nu^2(b\sqrt{\lambda_n}) - J_\nu^2(a\sqrt{\lambda_n})}{J_\nu(a\sqrt{\lambda_n})J_\nu(b\sqrt{\lambda_n})} \right] \\ k_n &= \sqrt{\frac{b}{a} \frac{J_\nu(b\sqrt{\lambda_n})}{J_\nu(a\sqrt{\lambda_n})}}, \end{aligned}$$

where λ_n are the zeros of $\omega(\lambda)$.

Therefore,

$$\delta(x - \xi) = \sum_{n=0}^{n=\infty} \frac{k_n}{\omega'(\lambda_n)} \phi_\alpha(\xi, \lambda_n) \phi_\alpha(x, \lambda_n)$$

$$\begin{aligned}
&= \frac{\pi^2}{2} \sqrt{x\xi} \sum_{n=0}^{n=\infty} \frac{\lambda_n J_\nu^2(b\sqrt{\lambda_n})}{J_\nu^2(a\sqrt{\lambda_n}) - J_\nu^2(b\sqrt{\lambda_n})} \times \\
&\quad \times \left\{ J_\nu(x\sqrt{\lambda_n}) Y_\nu(a\sqrt{\lambda_n}) - Y_\nu(x\sqrt{\lambda_n}) J_\nu(a\sqrt{\lambda_n}) \right\} \times \\
&\quad \times \left\{ J_\nu(\xi\sqrt{\lambda_n}) Y_\nu(a\sqrt{\lambda_n}) - Y_\nu(\xi\sqrt{\lambda_n}) J_\nu(a\sqrt{\lambda_n}) \right\}
\end{aligned}$$

11.1 The Fourier Bessel expansion of Delta in $[a, b]$

$$\begin{aligned}
\delta(x - \xi) &= \frac{\pi^2}{2} \sqrt{x\xi} \sum_{n=0}^{n=\infty} \frac{\lambda_n J_\nu^2(b\sqrt{\lambda_n})}{J_\nu^2(a\sqrt{\lambda_n}) - J_\nu^2(b\sqrt{\lambda_n})} \times \\
&\quad \times \left\{ J_\nu(x\sqrt{\lambda_n}) Y_\nu(a\sqrt{\lambda_n}) - Y_\nu(x\sqrt{\lambda_n}) J_\nu(a\sqrt{\lambda_n}) \right\} \times \\
&\quad \times \left\{ J_\nu(\xi\sqrt{\lambda_n}) Y_\nu(a\sqrt{\lambda_n}) - Y_\nu(\xi\sqrt{\lambda_n}) J_\nu(a\sqrt{\lambda_n}) \right\}
\end{aligned}$$

Namely,

11.2 The Hyper-real Delta in $[a, b]$ is the infinite sequence

$$\begin{aligned}
&\frac{\pi^2}{2} \sqrt{x\xi} \left\langle \sum_{j=0}^{j=n} \frac{\lambda_j J_\nu^2(b\sqrt{\lambda_j})}{J_\nu^2(a\sqrt{\lambda_j}) - J_\nu^2(b\sqrt{\lambda_j})} \times \right. \\
&\quad \times \left\{ J_\nu(x\sqrt{\lambda_j}) Y_\nu(a\sqrt{\lambda_j}) - Y_\nu(x\sqrt{\lambda_j}) J_\nu(a\sqrt{\lambda_j}) \right\} \times \\
&\quad \left. \times \left\{ J_\nu(\xi\sqrt{\lambda_j}) Y_\nu(a\sqrt{\lambda_j}) - Y_\nu(\xi\sqrt{\lambda_j}) J_\nu(a\sqrt{\lambda_j}) \right\} \right\rangle_{n=0}^{n=\infty}
\end{aligned}$$

12.

Delta Expansion in Orthonormal Eigen-functions

The Hyper-real eigen-functions of a Hyper-real Sturm-Liouville problem over the interval with endpoints a , and b ,

$$\psi_0(x), \psi_1(x), \psi_2(x), \dots$$

can be normalized so that

$$\int_{x=a}^{x=b} \psi_n(x)\psi_m(x)dx = \delta_{nm}.$$

Then, a hyper-real function $f(x)$ may be expanded in them by

$$\begin{aligned} f(x) &= \sum_{n=0}^{n=\infty} \left(\int_{\xi=a}^{\xi=b} f(\xi)\psi_n(\xi)d\xi \right) \psi_n(x). \\ &= \sum_{n=0}^{n=\infty} \sum_{\xi=a}^{\xi=b} f(\xi)\psi_n(\xi)d\xi \psi_n(x) \end{aligned}$$

Exchanging summation order,

$$\begin{aligned} &= \sum_{\xi=a}^{\xi=b} \left\{ \sum_{n=0}^{n=\infty} \psi_n(x)\psi_n(\xi) \right\} f(\xi)d\xi \\ &= \int_{\xi=a}^{\xi=b} \underbrace{\sum_{n=0}^{n=\infty} \psi_n(x)\psi_n(\xi)}_{\delta(x-\xi)} f(\xi)d\xi. \end{aligned}$$

Therefore,

12.1 *The Hyper-real Delta Function expansion in orthonormal Sturm-Liouville eigen-functions is*

$$\delta(x - \xi) = \sum_{n=0}^{n=\infty} \psi_n(x)\psi_n(\xi).$$

That is,

12.2 *The Hyper-real Delta Function is the infinite sequence*

$$\langle \psi_0(x)\psi_0(\xi) + \psi_1(x)\psi_1(\xi) + \dots + \psi_n(x)\psi_n(\xi) \rangle.$$

13.

Fourier-Hermit Expansion of Delta Associated with

$$u''(x) + (\lambda - x^2)u(x) = 0, \text{ for any } x \text{ real}$$

Put

$$u(x) = e^{-\frac{1}{2}x^2} y(x),$$

and obtain Hermit's equation

$$y''(x) + 2xy'(x) + (\lambda - 1)y(x) = 0,$$

The eigen values are

$$\lambda_n = 2n + 1, \quad n = 0, 1, 2, 3, \dots$$

and the corresponding eigen functions are Hermit Polynomials of degree n ,

$$H_n(x).$$

Therefore,

$$e^{-\frac{1}{2}x^2} H_n(x) \text{ solve } u''(x) + (\lambda - x^2)u(x) = 0.$$

[Titchmarsh, p. 75] shows that the Normalized eigen functions are

$$\psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-\frac{1}{2}x^2} H_n(x).$$

Therefore,

$$\begin{aligned}\delta(x - \xi) &= \sum_{n=0}^{n=\infty} \psi_n(x)\psi_n(\xi) \\ &= \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}\xi^2} \sum_{n=0}^{n=\infty} \frac{1}{2^n n!} H_n(x)H_n(\xi)\end{aligned}$$

13.1 *The Fourier-Hermit expansion of Delta in real x , and ξ*

$$\boxed{\delta(x - \xi) = \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}\xi^2} \sum_{n=0}^{n=\infty} \frac{1}{2^n n!} H_n(x)H_n(\xi)}$$

Namely,

13.2 *The Hyper-real Delta in real x , and ξ is the infinite sequence*

$$\frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}\xi^2} \left\langle H_0(x)H_0(\xi) + \frac{1}{2^1} H_1(x)H_1(\xi) \dots + \frac{1}{2^n n!} H_n(x)H_n(\xi) \right\rangle_{n=0}^{n=\infty}$$

14.

Fourier-Legendre Expansion of Delta Associated with

$$u''(\theta) + \left[\lambda + \frac{1}{4} \frac{1}{\cos^2 \theta}\right]u(\theta) = 0, \quad -\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$$

Legendre's equation with $m = 0$ is

$$(1 - x^2)y''(x) + 2xy'(x) + (\lambda - \frac{1}{4})y(x) = 0.$$

The eigen values are

$$\lambda_n = (n + \frac{1}{2})^2, \quad n = 0, 1, 2, 3, \dots$$

and the corresponding eigen functions are Legendre Polynomials of degree n ,

$$P_n(x).$$

Put

$$x = \sin \theta,$$

and obtain

$$y''(\theta) - y'(\theta) \tan \theta + (\lambda - \frac{1}{4})y(\theta) = 0.$$

Put

$$y(\theta) = \frac{1}{\sqrt{\cos \theta}} u(\theta),$$

and obtain

$$u''(\theta) + \left[\lambda + \frac{1}{4}(1 + \tan^2 \theta)\right]u(\theta) = 0.$$

Substituting $\frac{1}{\cos^2 \theta} = 1 + \tan^2 \theta$,

$$u''(\theta) + \left[\lambda + \frac{1}{4} \frac{1}{\cos^2 \theta}\right]u(\theta) = 0,$$

Therefore,

$$\sqrt{\cos \theta} P_n(\sin \theta) \text{ solve } u''(\theta) + \left(\lambda + \frac{1}{4 \cos^2 \theta}\right)u(\theta) = 0.$$

[Titchmarsh, p. 79] shows that

14.1 *The Normalized eigen-functions are*

$$\psi_n(\theta) = \sqrt{n + \frac{1}{2}} \sqrt{\cos \theta} P_n(\sin \theta).$$

Therefore,

$$\begin{aligned} \delta(\theta - \varphi) &= \sum_{n=0}^{n=\infty} \psi_n(\theta) \psi_n(\varphi) \\ &= \sqrt{\cos \theta} \sqrt{\cos \varphi} \sum_{n=0}^{n=\infty} \left(n + \frac{1}{2}\right) P_n(\sin \theta) P_n(\sin \varphi) \end{aligned}$$

14.2 *The Fourier-Legendre expansion of Delta in*

$$-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$$

$$\boxed{\delta(\theta - \varphi) = \sqrt{\cos \theta} \sqrt{\cos \varphi} \sum_{n=0}^{n=\infty} \left(n + \frac{1}{2}\right) P_n(\sin \theta) P_n(\sin \varphi)}$$

Namely,

14.3 *The Hyper-real Delta in $-\frac{1}{2}\pi < \theta, \varphi < \frac{1}{2}\pi$ is the infinite sequence*

$$\sqrt{\cos \theta} \sqrt{\cos \varphi} \left\langle \frac{1}{2} P_0(\sin \theta) P_0(\sin \varphi) + \dots \left(n + \frac{1}{2}\right) P_n(\sin \theta) P_n(\sin \varphi) \right\rangle_{n=0}^{n=\infty}$$

15.

Fourier-Legendre Expansion of Delta Associated with

$$u''(\theta) + \left[\lambda + \left(\frac{1}{4} - m^2\right) \frac{1}{\cos^2 \theta}\right] u(\theta) = 0, \quad -\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$$

Legendre's equation with for $m = 0, 1, 2, 3, \dots$, is

$$(1 - x^2)y''(x) + 2xy'(x) + \left[\lambda - \frac{1}{4} - \frac{1}{1-x^2} m^2\right] y(x) = 0.$$

The eigen values are

$$\lambda_n^m = \left(n - m + \frac{1}{2}\right)^2, \quad n = 2m, 2m + 1, 2m + 2, \dots$$

and the corresponding eigen functions are Legendre Functions

$$P_n^m(x).$$

Put

$$x = \sin \theta,$$

and obtain

$$y''(\theta) - y'(\theta) \tan \theta + \left[\lambda - \frac{1}{4} - m^2 \frac{1}{\cos^2 \theta}\right] y(\theta) = 0.$$

Put

$$y(\theta) = \frac{1}{\sqrt{\cos \theta}} u(\theta),$$

and obtain

$$u''(\theta) + \left[\lambda + \frac{1}{4}(1 + \tan^2 \theta) - \frac{1}{4} \frac{1}{\cos^2 \theta} m^2\right] u(\theta) = 0.$$

Substituting $\frac{1}{\cos^2 \theta} = 1 + \tan^2 \theta$,

$$u''(\theta) + \left[\lambda + \left(\frac{1}{4} - m^2\right) \frac{1}{\cos^2 \theta}\right] u(\theta) = 0,$$

Therefore,

$$\sqrt{\cos \theta} P_n^m(\sin \theta) \text{ solve } u''(\theta) + \left[\lambda + \left(\frac{1}{4} - m^2\right) \frac{1}{\cos^2 \theta}\right] u(\theta) = 0.$$

[Titchmarsh, p. 80] shows that

15.1 *The Normalized eigen-functions are*

$$\psi_n(\theta) = \sqrt{\frac{(n-2m)!}{n!}} \sqrt{n-m+\frac{1}{2}} \sqrt{\cos \theta} P_n^m(\sin \theta), \quad n = 2m, 2m+1, \dots$$

Therefore,

$$\begin{aligned} \delta(\theta - \varphi) &= \sum_{n=2m}^{n=\infty} \psi_n(\theta) \psi_n(\varphi) \\ &= \sqrt{\cos \theta} \sqrt{\cos \varphi} \sum_{n=2m}^{n=\infty} \frac{(n-2m)!}{n!} \left(n-m+\frac{1}{2}\right) P_n^m(\sin \theta) P_n^m(\sin \varphi) \end{aligned}$$

15.2 *The Fourier-Legendre expansion of Delta in $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$,*

for $m = 0, 1, 2, 3, \dots$, is

$$\delta(\theta - \varphi) = \sqrt{\cos \theta} \sqrt{\cos \varphi} \sum_{n=2m}^{n=\infty} \frac{(n-2m)!}{n!} \left(n-m+\frac{1}{2}\right) P_n^m(\sin \theta) P_n^m(\sin \varphi)$$

Namely,

15.3 *The Hyper-real Delta in $-\frac{1}{2}\pi < \theta, \varphi < \frac{1}{2}\pi$ is the sequence*

$$\sqrt{\cos \theta} \sqrt{\cos \varphi} \left\langle \sum_{k=2m}^{k=n} \frac{(k-2m)!}{k!} \left(k-m+\frac{1}{2}\right) P_k^m(\sin \theta) P_k^m(\sin \varphi) \right\rangle_{n=m}^{n=\infty}$$

16.

Fourier-Bessel Expansion of Delta Associated with

$$y''(x) + \left[\lambda - \frac{1}{x^2} \left(\nu^2 - \frac{1}{4}\right)\right]y(x) = 0, \quad 0 < x < b$$

Two independent solutions are

$$\sqrt{x}J_\nu(x\sqrt{\lambda}), \quad \text{and} \quad \sqrt{x}Y_\nu(x\sqrt{\lambda})$$

For $\boxed{\nu \geq 1}$

By [Titchmarsh, p. 82], the eigen-values

$$\lambda_n \text{ are the zeros of } J_\nu(b\sqrt{\lambda}), \quad n = 1, 2, 3, \dots$$

and the normalized eigen-functions are

$$\frac{\sqrt{2}}{bJ_\nu'(b\sqrt{\lambda_n})} J_\nu(x\sqrt{\lambda_n}).$$

Therefore,

16.1 For $\nu \geq 1$

the Fourier-Bessel expansion of Delta in $0 < x < b$ is

$$\boxed{\delta(x - \xi) = \frac{2}{b^2} \sqrt{x\xi} \sum_{n=1}^{n=\infty} \frac{1}{J_\nu'(b\sqrt{\lambda_n})} J_\nu(x\sqrt{\lambda_n}) J_\nu(\xi\sqrt{\lambda_n}), \quad \nu \geq 1}$$

Namely,

16.2 For $\nu \geq 1$

the Hyper-real Delta in $0 < x, \xi < b$ is the infinite sequence

$$\frac{2}{b^2} \sqrt{x\xi} \left\langle \sum_{k=1}^{k=n} \frac{1}{J_\nu'(b\sqrt{\lambda_k})} J_\nu(x\sqrt{\lambda_k}) J_\nu(\xi\sqrt{\lambda_k}) \right\rangle_{n=1}^{n=\infty}$$

For $\boxed{0 < \nu < 1, \nu \neq \frac{1}{2}}$

By [Titchmarsh, p. 83], the eigen-values

$$\lambda_n \text{ are the zeros of } cJ_\nu(b\sqrt{\lambda}) - \lambda^\nu J_{-\nu}(b\sqrt{\lambda}),$$

where $c = \text{const.}$

$$r_n = -\text{Res} \left\{ \sqrt{\lambda} \frac{c\sqrt{\lambda^{-\nu}} J_\nu'(b\sqrt{\lambda}) - \sqrt{\lambda^\nu} J_{-\nu}'(b\sqrt{\lambda})}{c\sqrt{\lambda^{-\nu}} J_\nu(b\sqrt{\lambda}) - \sqrt{\lambda^\nu} J_{-\nu}(b\sqrt{\lambda})} + \frac{1}{2b} \right\}_{\lambda=\lambda_n}$$

and the normalized eigen-functions are

$$\sqrt{|r_n|} \frac{\pi\sqrt{b}}{2\sin\nu\pi} \sqrt{x} J_\nu(b\sqrt{\lambda_n}) \left\{ c\lambda_n^{-\nu} J_\nu(x\sqrt{\lambda_n}) - J_{-\nu}(x\sqrt{\lambda_n}) \right\}.$$

Therefore,

16.3 If $0 < \nu < 1, \nu \neq \frac{1}{2}$, then

the Fourier-Bessel expansion of Delta in $0 < x < b$ is

$$\delta(x - \xi) = \frac{\pi^2 b}{4 \sin^2 \nu \pi} \sqrt{x \xi} \sum_{n=1}^{n=\infty} |r_n| J_\nu^2(b\sqrt{\lambda_n}) \left\{ c \lambda_n^{-\nu} J_\nu(x\sqrt{\lambda_n}) - J_{-\nu}(x\sqrt{\lambda_n}) \right\} \times \\ \times \left\{ c \lambda_n^{-\nu} J_\nu(\xi\sqrt{\lambda_n}) - J_{-\nu}(\xi\sqrt{\lambda_n}) \right\}$$

Namely,

16.4 If $0 < \nu < 1$, $\nu \neq \frac{1}{2}$, then

the Hyper-real Delta in $0 < x, \xi < b$ is the infinite sequence

$$\frac{\pi^2 b}{4 \sin^2 \nu \pi} \sqrt{x \xi} \left\langle \sum_{k=1}^{k=n} |r_k| J_\nu^2(b\sqrt{\lambda_k}) \left\{ c \lambda_k^{-\nu} J_\nu(x\sqrt{\lambda_k}) - J_{-\nu}(x\sqrt{\lambda_k}) \right\} \times \right. \\ \left. \times \left\{ c \lambda_k^{-\nu} J_\nu(\xi\sqrt{\lambda_k}) - J_{-\nu}(\xi\sqrt{\lambda_k}) \right\} \right\rangle_{n=1}^{n=\infty}$$

16.5 If $0 < \nu < 1$, $\nu \neq \frac{1}{2}$, then

For $c = \text{infinite hyper-real}$, the expansion is in J_ν :

$$\delta(x - \xi) = \frac{\pi^2 b}{4 \sin^2 \nu \pi} c^2 \sqrt{x \xi} \sum_{n=1}^{n=\infty} |r_n| J_\nu^2(b\sqrt{\lambda_n}) \lambda_n^{-2\nu} J_\nu(x\sqrt{\lambda_n}) J_\nu(\xi\sqrt{\lambda_n})$$

For $c = \text{infinitesimal}$, the expansion is in $J_{-\nu}$:

$$\delta(x - \xi) = \frac{\pi^2 b}{4 \sin^2 \nu \pi} \sqrt{x \xi} \sum_{n=1}^{n=\infty} |r_n| J_\nu^2(b\sqrt{\lambda_n}) J_{-\nu}(x\sqrt{\lambda_n}) J_{-\nu}(\xi\sqrt{\lambda_n})$$

17.

Fourier-Bessel Expansion of Delta

Associated with $y''(x) + [\lambda - x]y(x) = 0,$

$0 < x < \infty,$ **with** $\alpha = 0$

By [Titchmarsh, p. 91], the eigen-values

$$\lambda_n \text{ are the zeros of } J_{\frac{1}{3}}\left(\frac{2}{3}\lambda_n^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3}\lambda_n^{\frac{3}{2}}\right), \quad n = 1, 2, 3, \dots$$

and the normalized eigen-functions are

$$\frac{-1}{\lambda_n \left[J_{\frac{2}{3}}\left(\frac{2}{3}\lambda_n^{\frac{3}{2}}\right) - J_{-\frac{2}{3}}\left(\frac{2}{3}\lambda_n^{\frac{3}{2}}\right) \right]} \sqrt{\lambda_n - x} \left\{ J_{\frac{1}{3}}\left(\frac{2}{3}[\lambda_n - x]^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3}[\lambda_n - x]^{\frac{3}{2}}\right) \right\}$$

Therefore,

17.1 *the Fourier-Bessel expansion of Delta in $0 < x < \infty$ is*

$$\delta(x - \xi) = \sum_{n=1}^{n=\infty} \frac{1}{\lambda_n^2 \left[J_{\frac{2}{3}}\left(\frac{2}{3}\lambda_n^{\frac{3}{2}}\right) - J_{-\frac{2}{3}}\left(\frac{2}{3}\lambda_n^{\frac{3}{2}}\right) \right]^2} \sqrt{\lambda_n - x} \sqrt{\lambda_n - \xi} \times$$

$$\times \left\{ J_{\frac{1}{3}}\left(\frac{2}{3}[\lambda_n - x]^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3}[\lambda_n - x]^{\frac{3}{2}}\right) \right\} \times$$

$$\times \left\{ J_{\frac{1}{3}}\left(\frac{2}{3}[\lambda_n - \xi]^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3}[\lambda_n - \xi]^{\frac{3}{2}}\right) \right\}$$

Namely,

17.2 *the Hyper-real Delta in $0 < x, \xi < \infty$ is the infinite sequence*

$$\left\langle \sum_{k=1}^{k=n} \frac{1}{\lambda_n^2 \left[J_{\frac{2}{3}}\left(\frac{2}{3}\lambda_k^{\frac{3}{2}}\right) - J_{-\frac{2}{3}}\left(\frac{2}{3}\lambda_k^{\frac{3}{2}}\right) \right]^2} \sqrt{\lambda_k - x} \sqrt{\lambda_k - \xi} \times \right.$$

$$\times \left\{ J_{\frac{1}{3}}\left(\frac{2}{3}[\lambda_k - x]^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3}[\lambda_k - x]^{\frac{3}{2}}\right) \right\} \times$$

$$\left. \times \left\{ J_{\frac{1}{3}}\left(\frac{2}{3}[\lambda_k - \xi]^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3}[\lambda_k - \xi]^{\frac{3}{2}}\right) \right\} \right\rangle_{n=1}^{n=\infty}$$

18.

Fourier-Bessel Expansion of Delta

Associated with $y''(x) + [\mu - x]y(x) = 0,$

$0 < x < \infty,$ **with** $\alpha = \frac{1}{2} \pi$

By [Titchmarsh, p. 92], the eigen-values

μ_n are the zeros of $J_{\frac{2}{3}}(\frac{2}{3}\mu^{\frac{3}{2}}) - J_{-\frac{2}{3}}(\frac{2}{3}\mu^{\frac{3}{2}}), n = 1, 2, 3, \dots$

and the normalized eigen-functions are

$$\frac{-1}{\mu_n \left[J_{\frac{1}{3}}(\frac{2}{3}\mu_n^{\frac{3}{2}}) + J_{-\frac{1}{3}}(\frac{2}{3}\mu_n^{\frac{3}{2}}) \right]} \sqrt{\mu_n - x} \left\{ J_{\frac{1}{3}}(\frac{2}{3}[\mu_n - x]^{\frac{3}{2}}) + J_{-\frac{1}{3}}(\frac{2}{3}[\mu_n - x]^{\frac{3}{2}}) \right\}$$

Therefore,

18.1 *the Fourier-Bessel expansion of Delta in $0 < x < \infty$ is*

$$\delta(x - \xi) = \sum_{n=1}^{n=\infty} \frac{1}{\mu_n^2 \left[J_{\frac{1}{3}}(\frac{2}{3}\mu_n^{\frac{3}{2}}) + J_{-\frac{1}{3}}(\frac{2}{3}\mu_n^{\frac{3}{2}}) \right]^2} \sqrt{\mu_n - x} \sqrt{\mu_n - \xi} \times$$

$$\times \left\{ J_{\frac{1}{3}}(\frac{2}{3}[\mu_n - x]^{\frac{3}{2}}) + J_{-\frac{1}{3}}(\frac{2}{3}[\mu_n - x]^{\frac{3}{2}}) \right\} \times$$

$$\times \left\{ J_{\frac{1}{3}}(\frac{2}{3}[\mu_n - \xi]^{\frac{3}{2}}) + J_{-\frac{1}{3}}(\frac{2}{3}[\mu_n - \xi]^{\frac{3}{2}}) \right\}$$

Namely,

18.2 *the Hyper-real Delta in $0 < x, \xi < \infty$ is the infinite sequence*

$$\left\langle \sum_{k=1}^{k=n} \frac{1}{\mu_n^2 \left[J_{\frac{1}{3}}\left(\frac{2}{3} \mu_k^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3} \mu_k^{\frac{3}{2}}\right) \right]^2} \sqrt{\mu_k - x} \sqrt{\mu_k - \xi} \times \right.$$

$$\times \left\{ J_{\frac{1}{3}}\left(\frac{2}{3} [\mu_k - x]^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3} [\mu_k - x]^{\frac{3}{2}}\right) \right\} \times$$

$$\left. \times \left\{ J_{\frac{1}{3}}\left(\frac{2}{3} [\mu_k - \xi]^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3} [\mu_k - \xi]^{\frac{3}{2}}\right) \right\} \right\rangle_{n=1}^{n=\infty}$$

19.

Delta Expansion in an Integral over a Continuum of Eigen-functions

If the eigen-values are only infinitesimally separated from each other, the series summation of eigen-functions over discrete eigen values is replaced by Hyper-real integration over the continuum of eigen-values. Then, the expansion is represented by an integral .

Titchmarsh applied the Residue Theorem to obtain the projections of a function $f(x)$ on the eigen-functions of Sturm-Liouville problems, with continuous spectrum of eigen-values.

Following Titchmarsh, we expand the Delta Function in Integrals of Sturm-Liouville eigen-functions.

20.

Fourier-Cosine Integral of Delta Associated with $y''(x) + \lambda y(x) = 0,$

$$0 < x < \infty$$

The eigen-values

$$\lambda \equiv \omega^2,$$

are the interval of hyper-real positive numbers $(0, \infty)$.

By [Titchmarsh, p. 72], the Hyper-real function $f(x)$ is given for any hyper-real x by

$$\begin{aligned} f(x) &= \frac{1}{\pi} \int_{\xi=-\infty}^{\xi=\infty} \int_{\lambda=0}^{\lambda=\infty} \cos(x\sqrt{\lambda}) \cos(\xi\sqrt{\lambda}) \frac{1}{2\sqrt{\lambda}} d\lambda f(\xi) d\xi + \\ &\quad + \frac{1}{\pi} \int_{\xi=-\infty}^{\xi=\infty} \int_{\lambda=0}^{\lambda=\infty} \sin(x\sqrt{\lambda}) \sin(\xi\sqrt{\lambda}) \frac{1}{2\sqrt{\lambda}} d\lambda f(\xi) d\xi \\ &= \frac{1}{\pi} \int_{\xi=-\infty}^{\xi=\infty} \left\{ \int_{\lambda=0}^{\lambda=\infty} \underbrace{\{\cos(x\sqrt{\lambda}) \cos(\xi\sqrt{\lambda}) + \sin(x\sqrt{\lambda}) \sin(\xi\sqrt{\lambda})\}}_{\cos[(x-\xi)\sqrt{\lambda}]} \frac{1}{2\sqrt{\lambda}} d\lambda \right\} f(\xi) d\xi \\ &= \int_{\xi=-\infty}^{\xi=\infty} \underbrace{\frac{1}{\pi} \int_{\omega=0}^{\omega=\infty} \cos \omega(x - \xi) d\omega}_{\delta(x-\xi)} f(\xi) d\xi \end{aligned}$$

Therefore,

20.1 *The Fourier-Cosine Hyper-real Integral of Delta in*

$-\infty < x, \xi < \infty$ is

$$\delta(x - \xi) = \frac{1}{\pi} \int_{\omega=0}^{\omega=\infty} \cos \omega(x - \xi) d\omega$$

21.

Weber Formula for Fourier-Bessel Integral of Delta Associated with

$$y''(x) + \left[\lambda - \frac{1}{x^2} \left(\nu^2 - \frac{1}{4}\right)\right]y(x) = 0, \quad a < x < \infty$$

$$a > 0$$

The eigen-values

$$-\lambda \equiv s^2,$$

are the interval of hyper-real negative numbers $(-\infty, 0)$.

By [Titchmarsh, p. 87], the Hyper-real function $f(x)$ is given for hyper-real $a < x < \infty$ by

$$f(x) = \sqrt{x} \int_{\xi=a}^{\xi=\infty} \int_{s=0}^{s=\infty} \frac{1}{J_\nu^2(as) + Y_\nu^2(as)} \left\{ J_\nu(xs)Y_\nu(as) - J_\nu(as)Y_\nu(xs) \right\} \times \\ \times \left\{ J_\nu(\xi s)Y_\nu(as) - J_\nu(as)Y_\nu(\xi s) \right\} s ds \sqrt{\xi} f(\xi) d\xi$$

Therefore,

21.1 *The Fourier-Bessel Hyper-real Integral of Delta in*

$$a < x, \xi < \infty \text{ is}$$

$$\delta(x - \xi) = \sqrt{x\xi} \int_{s=0}^{s=\infty} \frac{1}{J_\nu^2(as) + Y_\nu^2(as)} \{ J_\nu(xs)Y_\nu(as) - J_\nu(as)Y_\nu(xs) \} \times$$

$$\times \{ J_\nu(\xi s)Y_\nu(as) - J_\nu(as)Y_\nu(\xi s) \} s ds$$

22.

Hankel Formula for Fourier-Bessel Integral of Delta Associated with

$$y''(x) + \left[\lambda - \frac{1}{x^2} \left(\nu^2 - \frac{1}{4}\right)\right]y(x) = 0, \quad 0 < x < \infty$$

$$\boxed{\nu > 1}$$

By [Titchmarsh, p. 88], the Hyper-real function $f(x)$ is given for any positive hyper-real x by

$$f(x) = \sqrt{x} \int_{\xi=0}^{\xi=\infty} \int_{s=0}^{s=\infty} J_{\nu}(xs)J_{\nu}(\xi s)sds\sqrt{\xi}f(\xi)d\xi$$

Therefore,

22.1 For $\underline{\nu > 1}$, the *Fourier-Bessel Hyper-real Integral of Delta* in hyper-real positive x , and ξ is

$$\boxed{\delta(x - \xi) = \sqrt{x\xi} \int_{s=0}^{s=\infty} J_{\nu}(xs)J_{\nu}(\xi s)sds}$$

$$\boxed{0 < \nu < 1, \text{ and } c < 0}$$

By [Titchmarsh, p. 89], the Hyper-real function $f(x)$ is given for any positive hyper-real x , and a constant $c < 0$ by

$$f(x) = \sqrt{x} \int_{\xi=0}^{\xi=\infty} \int_{s=0}^{s=\infty} \frac{1}{c^2 - 2cs^{2\nu} \cos \nu\pi + s^{4\nu}} \{cJ_\nu(xs) - s^{2\nu}J_{-\nu}(xs)\} \times \\ \times \{cJ_\nu(\xi s) - s^{2\nu}J_{-\nu}(\xi s)\} s ds \sqrt{\xi} f(\xi) d\xi$$

Therefore,

22.2 For $0 < \nu < 1$, and $c < 0$,

the Fourier-Bessel Hyper-real Integral of Delta in

hyper-real positive x , and ξ is

$$\boxed{\delta(x - \xi) = \sqrt{x\xi} \int_{s=0}^{s=\infty} \frac{1}{c^2 - 2cs^{2\nu} \cos \nu\pi + s^{4\nu}} \{cJ_\nu(xs) - s^{2\nu}J_{-\nu}(xs)\} \times \\ \times \{cJ_\nu(\xi s) - s^{2\nu}J_{-\nu}(\xi s)\} s ds}$$

$$\boxed{0 < \nu < 1, \text{ and } c > 0}$$

By [Titchmarsh, p. 90], the Hyper-real function $f(x)$ is given for any positive hyper-real x , and a constant $c > 0$ by

$$f(x) = \sqrt{x} \int_{\xi=0}^{\xi=\infty} \int_{s=0}^{s=\infty} \frac{1}{[c + \frac{2}{\pi} \log s]^2 + 1} \{cJ_0(xs) - Y_0(xs) + \frac{2}{\pi} J_0(xs) \log s\} \times$$

$$\begin{aligned} & \times \left\{ cJ_0(\xi s) - Y_0(\xi s) + \frac{2}{\pi} J_0(\xi s) \log s \right\} s ds \sqrt{\xi} f(\xi) d\xi \\ & + 2e^{-\pi c} \sqrt{x} K_0(xe^{-\frac{1}{2}\pi c}) \int_{\xi=0}^{\xi=\infty} \sqrt{\xi} K_0(\xi e^{-\frac{1}{2}\pi c}) f(\xi) d\xi \end{aligned}$$

Therefore,

22.3 For $0 < \nu < 1$, and $c > 0$,

the Fourier-Bessel Hyper-real Integral of Delta in hyper-real positive x , and ξ is

$$\begin{aligned} \delta(x - \xi) = & \sqrt{x\xi} \int_{s=0}^{s=\infty} \frac{1}{[c + \frac{2}{\pi} \log s]^2 + 1} \left\{ cJ_0(xs) - Y_0(xs) + \frac{2}{\pi} J_0(xs) \log s \right\} \times \\ & \times \left\{ cJ_0(\xi s) - Y_0(\xi s) + \frac{2}{\pi} J_0(\xi s) \log s \right\} s ds \\ & + 2e^{-\pi c} \sqrt{x\xi} K_0(xe^{-\frac{1}{2}\pi c}) K_0(\xi e^{-\frac{1}{2}\pi c}) \end{aligned}$$

23.

Fourier-Bessel Integral of Delta

Associated with $y''(x) + [\lambda + x]y(x) = 0,$

$$0 \leq x < \infty$$

The eigen-values

$$\lambda \equiv -\mu,$$

are the interval of hyper-real negative numbers $(-\infty, 0)$.

By [Titchmarsh, p. 93],

$$\phi(x, \mu) = \frac{2\pi}{3\sqrt{3}} \sqrt{\mu} \sqrt{|x - \mu|} \left\{ \begin{array}{l} \left| \begin{array}{ll} J_{\frac{1}{3}}\left(\frac{2}{3}|x - \mu|^{\frac{3}{2}}\right) & -I_{\frac{1}{3}}\left(\frac{2}{3}\mu^{\frac{3}{2}}\right) \\ J_{-\frac{1}{3}}\left(\frac{2}{3}|x - \mu|^{\frac{3}{2}}\right) & I_{-\frac{1}{3}}\left(\frac{2}{3}\mu^{\frac{3}{2}}\right) \end{array} \right|, & \mu \leq x \\ \left| \begin{array}{ll} I_{\frac{1}{3}}\left(\frac{2}{3}|x - \mu|^{\frac{3}{2}}\right) & I_{\frac{1}{3}}\left(\frac{2}{3}\mu^{\frac{3}{2}}\right) \\ I_{-\frac{1}{3}}\left(\frac{2}{3}|x - \mu|^{\frac{3}{2}}\right) & I_{-\frac{1}{3}}\left(\frac{2}{3}\mu^{\frac{3}{2}}\right) \end{array} \right|, & \mu \geq x \end{array} \right.$$

and the Hyper-real function $f(x)$ is given for $0 \leq x < \infty$ by

$$f(x) = \int_{\xi=0}^{\xi=\infty} \left\{ \frac{1}{\pi} \int_{\mu=0}^{\mu=\infty} \phi(x, \mu) \phi(\xi, \mu) d\mu \right\} f(\xi) d\xi$$

Therefore,

$$\delta(x - \xi) = \frac{1}{\pi} \int_{\mu=0}^{\mu=x} \phi(x, \mu) \phi(\xi, \mu) d\mu + \frac{1}{\pi} \int_{\mu=x}^{\mu=\infty} \phi(x, \mu) \phi(\xi, \mu) d\mu$$

$$\begin{aligned}
\delta(x - \xi) &= \frac{4\pi}{27} \int_{\mu=0}^{\mu=x} \mu \sqrt{|x - \mu|} \left| \begin{array}{cc} J_{\frac{1}{3}}(\frac{2}{3}|x - \mu|^{\frac{3}{2}}) & -I_{\frac{1}{3}}(\frac{2}{3}\mu^{\frac{3}{2}}) \\ J_{-\frac{1}{3}}(\frac{2}{3}|x - \mu|^{\frac{3}{2}}) & I_{-\frac{1}{3}}(\frac{2}{3}\mu^{\frac{3}{2}}) \end{array} \right| \times \\
&\quad \times \sqrt{|\xi - \mu|} \left\{ \begin{array}{l} \left| \begin{array}{cc} J_{\frac{1}{3}}(\frac{2}{3}|\xi - \mu|^{\frac{3}{2}}) & -I_{\frac{1}{3}}(\frac{2}{3}\mu^{\frac{3}{2}}) \\ J_{-\frac{1}{3}}(\frac{2}{3}|\xi - \mu|^{\frac{3}{2}}) & I_{-\frac{1}{3}}(\frac{2}{3}\mu^{\frac{3}{2}}) \end{array} \right|, & \mu \leq \xi \\ & d\mu \\ \left| \begin{array}{cc} I_{\frac{1}{3}}(\frac{2}{3}|\xi - \mu|^{\frac{3}{2}}) & I_{\frac{1}{3}}(\frac{2}{3}\mu^{\frac{3}{2}}) \\ I_{-\frac{1}{3}}(\frac{2}{3}|\xi - \mu|^{\frac{3}{2}}) & I_{-\frac{1}{3}}(\frac{2}{3}\mu^{\frac{3}{2}}) \end{array} \right|, & \mu \geq \xi \end{array} \right. \\
&+ \frac{4\pi}{27} \int_{\mu=x}^{\mu=\infty} \mu \sqrt{|x - \mu|} \left| \begin{array}{cc} I_{\frac{1}{3}}(\frac{2}{3}|x - \mu|^{\frac{3}{2}}) & I_{\frac{1}{3}}(\frac{2}{3}\mu^{\frac{3}{2}}) \\ I_{-\frac{1}{3}}(\frac{2}{3}|x - \mu|^{\frac{3}{2}}) & I_{-\frac{1}{3}}(\frac{2}{3}\mu^{\frac{3}{2}}) \end{array} \right| \times \\
&\quad \times \sqrt{|\xi - \mu|} \left\{ \begin{array}{l} \left| \begin{array}{cc} J_{\frac{1}{3}}(\frac{2}{3}|\xi - \mu|^{\frac{3}{2}}) & -I_{\frac{1}{3}}(\frac{2}{3}\mu^{\frac{3}{2}}) \\ J_{-\frac{1}{3}}(\frac{2}{3}|\xi - \mu|^{\frac{3}{2}}) & I_{-\frac{1}{3}}(\frac{2}{3}\mu^{\frac{3}{2}}) \end{array} \right|, & \mu \leq \xi \\ & d\mu \\ \left| \begin{array}{cc} I_{\frac{1}{3}}(\frac{2}{3}|\xi - \mu|^{\frac{3}{2}}) & I_{\frac{1}{3}}(\frac{2}{3}\mu^{\frac{3}{2}}) \\ I_{-\frac{1}{3}}(\frac{2}{3}|\xi - \mu|^{\frac{3}{2}}) & I_{-\frac{1}{3}}(\frac{2}{3}\mu^{\frac{3}{2}}) \end{array} \right|, & \mu \geq \xi \end{array} \right.
\end{aligned}$$

24.

Fourier-Bessel Integral of Delta Associated with

$$y''(x) + [\lambda + e^{2x}]y(x) = 0, \quad -\infty < x < \infty$$

By [Titchmarsh, p. 95], the Hyper-real function $f(x)$ is given for hyper-real $-\infty < x < \infty$ by

$$f(x) = \int_{\xi=-\infty}^{\xi=\infty} \left\{ 4 \sum_{n=1}^{n=\infty} n J_{2n}(e^x) J_{2n}(e^\xi) + \int_{\lambda=0}^{\lambda=\infty} \frac{1}{4 \sinh(\pi\sqrt{\lambda})} \times \right. \\ \left. \times \{J_{i\sqrt{\lambda}}(e^x) + J_{-i\sqrt{\lambda}}(e^x)\} \{J_{i\sqrt{\lambda}}(e^\xi) + J_{-i\sqrt{\lambda}}(e^\xi)\} d\lambda \right\} f(\xi) d\xi.$$

Therefore,

24.1 *The Fourier-Bessel Hyper-real Integral of Delta in x and ξ is*

$$\delta(x - \xi) = 4 \sum_{n=1}^{n=\infty} n J_{2n}(e^x) J_{2n}(e^\xi) + \int_{\lambda=0}^{\lambda=\infty} \frac{1}{4 \sinh(\pi\sqrt{\lambda})} \times \\ \times \{J_{i\sqrt{\lambda}}(e^x) + J_{-i\sqrt{\lambda}}(e^x)\} \{J_{i\sqrt{\lambda}}(e^\xi) + J_{-i\sqrt{\lambda}}(e^\xi)\} d\lambda$$

Alternatively, $f(x)$ is given for hyper-real $-\infty < x < \infty$ by

$$f(x) = \int_{\xi=-\infty}^{\xi=\infty} \left\{ \sum_{n=1}^{n=\infty} (4n+2) J_{2n+1}(e^x) J_{2n+1}(e^\xi) - \int_{\lambda=0}^{\lambda=\infty} \frac{1}{4 \sinh(\pi\sqrt{\lambda})} \times \right. \\ \left. \times \{J_{i\sqrt{\lambda}}(e^x) - J_{-i\sqrt{\lambda}}(e^x)\} \{J_{i\sqrt{\lambda}}(e^\xi) - J_{-i\sqrt{\lambda}}(e^\xi)\} d\lambda \right\} f(\xi) d\xi.$$

Therefore,

24.2 *The Fourier-Bessel Hyper-real Integral of Delta in x and ξ is*

$$\delta(x - \xi) = \sum_{n=1}^{n=\infty} (4n+2) J_{2n+1}(e^x) J_{2n+1}(e^\xi) - \int_{\lambda=0}^{\lambda=\infty} \frac{1}{4 \sinh(\pi\sqrt{\lambda})} \times \\ \times \{J_{i\sqrt{\lambda}}(e^x) - J_{-i\sqrt{\lambda}}(e^x)\} \{J_{i\sqrt{\lambda}}(e^\xi) - J_{-i\sqrt{\lambda}}(e^\xi)\} d\lambda$$

25.**Fourier-Bessel Integral of Delta
Associated with**

$$y''(x) + [\lambda - e^{2x}]y(x) = 0, \quad -\infty < x < \infty$$

By [Titchmarsh, p. 96], the Hyper-real function $f(x)$ is given for hyper-real $-\infty < x < \infty$ by

$$f(x) = \int_{\xi=-\infty}^{\xi=\infty} \frac{1}{\pi^2} \int_{\lambda=0}^{\lambda=\infty} K_{i\sqrt{\lambda}}(e^x) K_{i\sqrt{\lambda}}(e^\xi) \sinh(\pi\sqrt{\lambda}) d\lambda f(\xi) d\xi$$

Therefore,

25.1 *The Fourier-Bessel Hyper-real Integral of Delta in x and ξ is*

$$\delta(x - \xi) = \frac{1}{\pi^2} \int_{\lambda=0}^{\lambda=\infty} K_{i\sqrt{\lambda}}(e^x) K_{i\sqrt{\lambda}}(e^\xi) \sinh(\pi\sqrt{\lambda}) d\lambda$$

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