Delta Function and the Poisson Summation Formula

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Abstract In the Calculus of Limits, the Poisson Summation Formula holds under unnecessary conditions.

Using Delta function, and Periodic Delta Functions, we present here an Infinitesimal Calculus proof, that is free of unnecessary conditions.

Then, we apply the formula to a particular series.

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Contents

Calculus of Limits Limitations on the Poisson Summation Formula

In the Calculus of Limits, the Poisson Summation Formula is claimed to hold under restricting conditions: For instance, the version presented in [Apostol, p.332]:

<u>Theorem 11.24</u>

If $f(x) \geq 0$, for $-\infty < x < \infty$

$\int_{x=-\infty}^{x=\infty} f(x)dx$ exists as an improper Riemann Integral

f increases on $-\infty < x \leq 0$

f decreases on $0 \leq x < \infty$

Then
$$\sum_{m=-\infty}^{m=\infty} \frac{f(m+) + f(m-)}{2},$$

and

$$\sum_{m=-\infty}^{m=\infty} \int_{x=-\infty}^{x=\infty} f(x)e^{-2\pi imx}dx$$

are absolutely convergent to the same limit.

In fact, it is likely that even under these restrictions, the formula does not hold.

However, since no scientist may ever care about these restrictions, we will skip an examination that will render them meaningless, and establish the Formula in Infinitesimal Calculus, free of restricting conditions

For a Hyper-real Function in Infinitesimal Calculus, we obtain under no limitations

$$\sum_{m=-\infty}^{m=\infty} f(m) = \sum_{m=-\infty}^{m=\infty} F(2\pi m).$$

1.

Hyper-real Line

The minimal domain and range, needed for the definition and analysis of a hyper-real function, is the hyper-real line.

Each real number α can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of infinitesimal hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the infinite hyper-reals.
4. The infinite hyper-reals are greater than any real number, yet strictly smaller than infinity.
5. The infinite hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a non-constant hyper-real.

7. The Hyper-reals are the totality of constant hyper-reals, a family of infinitesimals, a family of infinitesimals with negative sign, a family of infinite hyper-reals, a family of infinite hyper-reals with negative sign, and non-constant hyper-reals.
8. The hyper-reals are totally ordered, and aligned along a line: the <u>Hyper-real Line</u>.
9. That line includes the real numbers separated by the non-constant hyper-reals. Each real number is the center of an interval of hyper-reals, that includes no other real number.
10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.
11. Zero is not an infinitesimal, because zero is not strictly greater than zero.
12. We do not add infinity to the hyper-real line.
13. The infinitesimals, the infinitesimals with negative signs, the infinite hyper-reals, and the infinite hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.
14. The hyper-real line is embedded in $\mathbb{R}^{\infty}$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the hyper-real onto the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal hyper-reals, or to the infinite hyper-reals, or to the non-constant hyper-reals.

16. No neighbourhood of a hyper-real is homeomorphic to an $\mathbb{R}^n$ ball. Therefore, the hyper-real line is not a manifold.

17. The hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

2.

Integral of a Hyper-real Function

In [Dan3], we defined the integral of a Hyper-real Function.

Let $f(x)$ be a hyper-real function on the interval $[a,b]$.

The interval may not be bounded.

$f(x)$ may take infinite hyper-real values, and need not be bounded.

At each

$$a \le x \le b \,,$$

there is a rectangle with base $[x - \frac{dx}{2}, x + \frac{dx}{2}]$, height $f(x)$, and area

$$f(x)dx \,.$$

We form the **Integration Sum** of all the areas for the x's that start at $x = a$, and end at $x = b$,

$$\sum_{x \in [a,b]} f(x)dx \,.$$

If for any infinitesimal dx, the Integration Sum has the same hyper-real value, then $f(x)$ is integrable over the interval $[a,b]$.

Then, we call the Integration Sum the integral of $f(x)$ from $x = a$, to $x = b$, and denote it by

$$\int_{x=a}^{x=b} f(x)dx \,.$$

If the hyper-real is infinite, then it is the integral over $[a,b]$, If the hyper-real is finite,

$$\int_{x=a}^{x=b} f(x)dx = \text{ real part of the hyper-real}.\square$$

2.1 The countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers,

$Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty.$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[a,b]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(x)dx$.

The Lower Integral is the Integration Sum where $f(x)$ is replaced by its lowest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

2.2
$$\sum_{x\in[a,b]} \left(\inf_{x-\frac{dx}{2} \le t \le x+\frac{dx}{2}} f(t) \right) dx$$

The Upper Integral is the Integration Sum where $f(x)$ is replaced by its largest value on each interval $[x-\frac{dx}{2}, x+\frac{dx}{2}]$

2.3 $$\sum_{x\in[a,b]}\left(\sup_{x-\frac{dx}{2}\le t\le x+\frac{dx}{2}} f(t)\right)dx$$

If the integral is a finite hyper-real, we have

2.4 *A hyper-real function has a finite integral if and only if its upper integral and its lower integral are finite, and differ by an infinitesimal.*

3.

Delta Function

In [Dan5], we have defined the Delta Function, and established its properties

1. The Delta Function is a hyper-real function defined from the hyper-real line into the set of two hyper-reals $\left\{0, \frac{1}{dx}\right\}$. The hyper-real 0 is the sequence $\left\langle 0,0,0,...\right\rangle$. The infinite hyper-real $\frac{1}{dx}$ depends on our choice of dx.
2. We will usually choose the family of infinitesimals that is spanned by the sequences $\left\langle \frac{1}{n}\right\rangle, \left\langle \frac{1}{n^2}\right\rangle, \left\langle \frac{1}{n^3}\right\rangle, \ldots$ It is a semigroup with respect to vector addition, and includes all the scalar multiples of the generating sequences that are non-zero. That is, the family includes infinitesimals with negative sign. Therefore, $\frac{1}{dx}$ will mean the sequence $\left\langle n\right\rangle$. Alternatively, we may choose the family spanned by the sequences $\left\langle \frac{1}{2^n}\right\rangle, \left\langle \frac{1}{3^n}\right\rangle, \left\langle \frac{1}{4^n}\right\rangle, \ldots$ Then, $\frac{1}{dx}$ will mean the sequence $\left\langle 2^n\right\rangle$. Once we determined the basic infinitesimal

3. The Delta Function is strictly smaller than ∞

4. We define, $\delta(x) \equiv \frac{1}{dx} \chi_{[-\frac{dx}{2},\frac{dx}{2}]}(x)$,

$$\text{where} \quad \chi_{[-\frac{dx}{2},\frac{dx}{2}]}(x) = \begin{cases} 1, x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right] \\ 0, \text{otherwise} \end{cases}.$$

5. Hence,

❖ for $x < 0$, $\delta(x) = 0$

❖ at $x = -\frac{dx}{2}$, $\delta(x)$ jumps from 0 to $\frac{1}{dx}$,

❖ for $x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right]$, $\delta(x) = \frac{1}{dx}$.

❖ at $x = 0$, $\delta(0) = \frac{1}{dx}$

❖ at $x = \frac{dx}{2}$, $\delta(x)$ drops from $\frac{1}{dx}$ to 0.

❖ for $x > 0$, $\delta(x) = 0$.

❖ $x\delta(x) = 0$

6. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle \chi_{[-\frac{1}{2},\frac{1}{2}]}(x), 2\chi_{[-\frac{1}{4},\frac{1}{4}]}(x), 3\chi_{[-\frac{1}{6},\frac{1}{6}]}(x) ... \right\rangle$

7. If $dx = \left\langle \frac{2}{n} \right\rangle$, $\delta(x) = \left\langle \frac{1}{2\cosh^2 x}, \frac{2}{2\cosh^2 2x}, \frac{3}{2\cosh^2 3x}, ... \right\rangle$

8. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle e^{-x}\chi_{[0,\infty)}, 2e^{-2x}\chi_{[0,\infty)}, 3e^{-3x}\chi_{[0,\infty)}, ... \right\rangle$

9. $$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1.$$

10. $$\delta(\xi - x) = \frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} e^{-ik(\xi - x)} dk$$

4.

Periodic Delta Function $\delta_{Periodic}(\xi - x)$

In [Dan5], we defined the Periodic Delta Function, and related it to the Dirichlet Kernel in Infinitesimal Calculus

1) Periodic Delta Function Definition

$$\delta_{Periodic}(\xi - x) = ... + \delta(\xi - x + 2) + \delta(\xi - x) + \delta(\xi - x - 2) + ...$$

is a periodic hyper-real Delta function, with period $T = 2$.

2) Fourier Transform of $\delta_{Periodic}(x)$

$$\mathcal{F}\left\{\delta_{Periodic}(x)\right\} = ... + e^{-i4\pi\nu} + 1 + e^{i4\pi\nu} + ...$$

3) Fourier Integral Theorem for $\delta_{Periodic}(x)$

$$\mathcal{F}^{-1}\mathcal{F}\left\{\delta_{Periodic}(x)\right\} = \delta_{Periodic}(x)$$

4) Dirichlet Sequence Definition

The Fourier Series partial sums

$$\mathcal{S}_n\left\{f(x)\right\} = \int_{\xi=-1}^{\xi=1} f(\xi)\underbrace{\left\{\tfrac{1}{2}e^{-in\pi(\xi-x)} + ... + \tfrac{1}{2}e^{-i\pi(\xi-x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi-x)} + ... + \tfrac{1}{2}e^{in\pi(\xi-x)}\right\}}_{\text{Dirichlet Sequence}} d\xi \,.$$

give rise to the Dirichlet Sequence

$$D_n(\xi - x) = \tfrac{1}{2}e^{-in\pi(\xi-x)} + ... + \tfrac{1}{2}e^{-i\pi(\xi-x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi-x)} + ... + \tfrac{1}{2}e^{in\pi(\xi-x)}$$

$$= \tfrac{1}{2} + \cos\pi(\xi - x) + \cos 2\pi(\xi - x) + ... + \cos n\pi(\xi - x)$$

$$= \frac{\sin(n + \frac{1}{2})\pi(\xi - x)}{2\sin\frac{1}{2}\pi(\xi - x)}, \quad n = 0,1,2,..$$

5) Dirichlet Sequence is a Periodic Delta Sequence

Each $D_n(x) = \dfrac{\sin(n + \frac{1}{2})\pi x}{2\sin\frac{1}{2}\pi x}$, $n = 0,1,2,3,...$

- *has the sifting property on each interval*,

$$.. \int_{x=-5}^{x=-3} D_n(x)dx = 1; \quad \int_{x=-3}^{x=-1} D_n(x)dx = 1; \quad \int_{x=-1}^{x=1} D_n(x)dx = 1..$$

- *is a continuous function*
- *peaks on each of these intervals to* $\lim\limits_{x \to 2m} D_n(x) = n + \frac{1}{2}$.

6) *Dirichlet Sequence Represents* $\delta_{Periodic}(\xi - x)$

$$\delta_{Periodic}(\xi - x) = \left\langle \frac{\sin\frac{2n+1}{2}\pi(\xi - x)}{2\sin\frac{1}{2}\pi(\xi - x)} \right\rangle$$

$$= \left\langle \tfrac{1}{2} + \cos\pi(\xi - x) + \cos 2\pi(\xi - x) + ... + \cos n\pi(\xi - x) \right\rangle$$

$$= \left\langle \tfrac{1}{2} e^{-in\pi(\xi - x)} + ... + \tfrac{1}{2} e^{-i\pi(\xi - x)} + \tfrac{1}{2} + \tfrac{1}{2} e^{i\pi(\xi - x)} + ... + \tfrac{1}{2} e^{in\pi(\xi - x)} \right\rangle.$$

7) Hyper-real Dirichlet Kernel

$$D_{irichlet}(\xi - x) = \begin{cases} \left\langle \frac{1}{2} + n \right\rangle, & \xi - x = 2m \\ 0, & \xi - x \neq 2m \end{cases}$$

8) ***Let*** $N = \frac{1}{dx}$ ***be an infinite Hyper-real. Then***

$$D_{irichlet}(\xi - x) = \frac{\sin(N + \frac{1}{2})\pi(\xi - x)}{2\sin\frac{1}{2}\pi(\xi - x)}$$

$$= \tfrac{1}{2} + \cos\pi(\xi - x) + \cos 2\pi(\xi - x) + ... + \cos N\pi(\xi - x)$$

$$= \tfrac{1}{2}e^{-iN\pi(\xi - x)} + .. + \tfrac{1}{2}e^{-i\pi(\xi - x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi - x)} + .. + \tfrac{1}{2}e^{iN\pi(\xi - x)}$$

$$= \delta(\xi - x + 2N) + ... + \delta(\xi - x + 2) + \delta(\xi - x) + \delta(\xi - x - 2) + ... + \delta(\xi - x - 2N)$$

$$= \delta_{Periodic}(\xi - x)$$

9) $... + \delta(\xi - x + 2) + \delta(\xi - x) + \delta(\xi - x - 2) + ... =$

$$= .. + \tfrac{1}{2}e^{-i2\pi(\xi - x)} + \tfrac{1}{2}e^{-i\pi(\xi - x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi - x)} + \tfrac{1}{2}e^{i2\pi(\xi - x)} + ..$$

$$= \tfrac{1}{2} + \cos\pi(\xi - x) + \cos 2\pi(\xi - x) + \cos 3\pi(\xi - x) + ...$$

10) $... + \delta(\theta - \phi + 2\pi) + \delta(\theta - \phi) + \delta(\theta - \phi - 2\pi) + ... =$

$$= .. + \tfrac{1}{2}e^{-i2(\theta - \phi)} + \tfrac{1}{2}e^{-i(\theta - \phi)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i(\theta - \phi)} + \tfrac{1}{2}e^{i2(\theta - \phi)} + ..$$

$$= \tfrac{1}{2} + \cos(\theta - \phi) + \cos 2(\theta - \phi) + \cos 3(\theta - \phi) + ...$$

5.

The Poisson Summation Formula in Infinitesimal Calculus

By [Dan4], in Infinitesimal Calculus, a Hyper-real function has a Fourier Transform

$$\mathcal{F}\{f\}(\omega) = F(\omega),$$

without the Calculus of Limits conditions that in fact do not guarantee the Fourier Transform for any function.

For such hyper-real function in Infinitesimal Calculus we need no Calculus of Limits conditions to prove the Poisson Summation Formula:

5.1 Poisson Summation Formula

For a Hyper-real function $f(x)$ *in Infinitesimal Calculus*

$$... + f(-2) + f(-1) + f(0) + f(1) + f(2) + ... =$$

$$= ... + F(-4\pi) + F(-2\pi) + F(0) + F(2\pi) + F(4\pi) + ...$$

<u>*Proof*</u>:

To apply the Fourier Transform to the $f(n)$ Summation, we define

$$S(x) = ... + f(x-2) + f(x-1) + f(x) + f(x+1) + f(x+2) + ...$$

$$= ... + \int\limits_{u=-\infty}^{u=\infty} f(u)\delta(x+2-u])du + \int\limits_{u=-\infty}^{u=\infty} f(u)\delta(x+1-u)du +$$

$$+\int_{u=-\infty}^{u=\infty} f(u)\delta(x-u)du +$$

$$+\int_{u=-\infty}^{u=\infty} f(u)\delta(x-1-u)du + \int_{u=-\infty}^{u=\infty} f(u)\delta(x-2-u)du + ...$$

$$= \int_{u=-\infty}^{u=\infty} f(u)\{..+\delta(x+2-u)+\delta(x+1-u)+\delta(x-u)+$$

$$+\delta(x-1-u)+\delta(x-2-u)+..\}du\,.$$

$$= f(x) * \{\underbrace{...+\delta(x+2)+\delta(x+1)+\delta(x)+\delta(x-1)+\delta(x-2)+...}_{\text{Periodic Delta}}\}$$

Thus, $S(x)$ is the convolution of f with a Periodic Delta Function, which was defined, and discussed extensively in [Dan5]

By the Convolution Theorem, $S(x)$ Fourier Transforms into the product of the Transforms of f, and the Periodic Delta

$$\mathcal{F}\{S(x)\}\Big|_{\omega} = \mathcal{F}\{f(x)\}\Big|_{\omega}\mathcal{F}\{...+\delta(x+2)+\delta(x+1)+$$

$$+\delta(x)+\delta(x-1)+\delta(x-2)+..\}\Big|_{\omega}$$

$$= F(\omega)\{..+\mathcal{F}\delta(x+2)+\mathcal{F}\delta(x+1)+$$

$$+\mathcal{F}\delta(x)+\mathcal{F}\delta(x-1)+\mathcal{F}\delta(x-2)+..\}\Big|_{\omega}$$

Since $\mathcal{F}\delta(x+2)\Big|_{\omega} = \int_{x=-\infty}^{x=\infty}\delta(x+2)e^{-i\omega x}dx = e^{2i\omega}$,

$$\mathcal{F}\delta(x+1)\Big|_{\omega} = \int_{x=-\infty}^{x=\infty}\delta(x+1)e^{-i\omega x}dx = e^{i\omega},$$

$$\mathcal{F}\delta(x)\Big|_{\omega} = \int_{x=-\infty}^{x=\infty}\delta(u-x)e^{-i\omega x}du = e^{0} = 1,$$

$$\mathcal{F}\delta(x-1)\Big|_{\omega} = \int\limits_{x=-\infty}^{x=\infty} \delta(x-1)e^{-i\omega x}dx = e^{-i\omega},$$

$$\mathcal{F}\delta(x-2)\Big|_{\omega} = \int\limits_{x=-\infty}^{x=\infty} \delta(x-2)e^{-i\omega x}du = e^{2i\omega}$$

$$\mathcal{F}\{S(u)\}\Big|_{\omega} = F(\omega)\{... + e^{2i\omega} + e^{i\omega} + 1 + +e^{-i\omega} + e^{-2i\omega} + ...\}$$

By [Dan5, 8.4, and 8.5], the Hyper-real Dirichlet Kernel

$$\underbrace{.. + e^{2i\omega} + e^{i\omega} + 1 + +e^{-i\omega} + e^{-2i\omega} + ...}_{\text{Dirichlet Kernel}}$$

equals the Periodic Delta

$$... + 2\delta\left(\frac{\omega}{\pi} + 4\right) + 2\delta\left(\frac{\omega}{\pi} + 2\right) + 2\delta\left(\frac{\omega}{\pi}\right) + 2\delta\left(\frac{\omega}{\pi} - 2\right) + 2\delta\left(\frac{\omega}{\pi} - 4\right) + ...$$

$$= 2\pi\{.. + \delta\big(\omega + 4\pi\big) + \delta\big(\omega + 2\pi\big) + \delta\big(\omega\big) + \delta\big(\omega - 2\pi\big) + \delta\big(\omega - 4\pi\big) + ..\}$$

Substituting the Periodic Delta into $\mathcal{F}\{S(u)\}\Big|_{\omega}$,

$$\mathcal{F}\{S(x)\}\Big|_{\omega} = F(\omega)2\pi\{... + \delta\big(\omega + 4\pi\big) + \delta\big(\omega + 2\pi\big) +$$

$$+\delta\big(\omega\big) + \delta\big(\omega - 2\pi\big) + \delta\big(\omega - 4\pi\big) + ...\}$$

$$= \{.. + F(-4\pi)2\pi\delta\big(\omega + 4\pi\big) + F(-2\pi)2\pi\delta\big(\omega + 2\pi\big) +$$

$$+F(0)2\pi\delta(\omega) + F(2\pi)2\pi\delta(\omega - 2\pi) + F(4\pi)2\pi\delta(\omega - 4\pi) + ...\}.$$

Since

$$\mathcal{F}^{-1}\{2\pi\delta\big(\omega + 4\pi\big)\} = \int\limits_{\omega=-\infty}^{\omega=\infty} \delta\big(\omega + 4\pi\big)e^{i\omega x}d\omega = e^{i(-4\pi)x}$$

$$\mathcal{F}^{-1}\{2\pi\delta\big(\omega + 2\pi\big)\} = \int\limits_{\omega=-\infty}^{\omega=\infty} \delta\big(\omega + 2\pi\big)e^{i\omega x}d\omega = e^{i(-2\pi)x}$$

$$\mathcal{F}^{-1}\{2\pi\delta\big(\omega\big)\} = \int_{\omega=-\infty}^{\omega=\infty} \delta\big(\omega\big)e^{i\omega x}d\omega = e^{0} = 1$$

$$\mathcal{F}^{-1}\{2\pi\delta\big(\omega - 2\pi\big)\} = \int_{\omega=-\infty}^{\omega=\infty} \delta\big(\omega - 2\pi\big)e^{i\omega x}d\omega = e^{i2\pi x}$$

$$\mathcal{F}^{-1}\{2\pi\delta\big(\omega - 4\pi\big)\} = \int_{\omega=-\infty}^{\omega=\infty} \delta\big(\omega - 4\pi\big)e^{i\omega x}d\omega = e^{i4\pi x}$$

$$S(x) = .. + F(-4\pi)e^{i(-4\pi)x} + F(-2\pi)e^{i(-2\pi)x} +$$

$$+F(0) + F(2\pi)e^{i2\pi x} + F(4\pi)e^{i4\pi x} + ...$$

$$S(0) = ... + F(-4\pi) + F(-2\pi) + F(0) + F(2\pi) + F(4\pi) + ...$$

Therefore,

$$... + f(-2) + f(-1) + f(0) + f(1) + f(2) + ... = S(0) =$$

$$= ... + F(-4\pi) + F(-2\pi) + F(0) + F(2\pi) + F(4\pi) + ... \square$$

The Calculus of Limits proofs under their bizarre conditions, mask the fact that

5.2 *the Formula does not indicate which terms in the* $f(n)$ ***summation contribute to whichever terms in the*** $F(n)$ ***summation***

Many statements of the formula attempt to hint at 1.2 by using an index n for the f summation, and an index m for the F summation.

This does not clarify the fact that the whole f summation equals the whole F summation.

6.

An Inquiry

Professor Thomas Radil inquired about applying the Formula to the summation of a series:

Rapidly convergent form of a series

I have encountered a series as follows,

$$S = \sum_{m=-\infty}^{m=\infty} \frac{\cos(\beta\sqrt{(md)^2 + v^2})}{[(md)^2 + v^2]} \tag{1}$$

where $v > 0$ and real number and d is spacing. Both v and d have units of meters. β is wave number having units of 1/meter. Let $\tau = md$, we have Eq. (1) as,

$$S = \sum_{m=-\infty}^{m=\infty} \frac{\cos(\beta\sqrt{\tau^2 + v^2})}{[\tau^2 + v^2]} = \sum_{m=-\infty}^{m=\infty} f(\tau) \tag{2}$$

Applying Poisson Summation Formula (PSF) to the above Eq., we can write it as,

$$S = \frac{1}{d} \sum_{m=-\infty}^{m=\infty} F(w)\bigg|_{w=\frac{2\pi m}{d}} \tag{3}$$

where $F(w)$ is Fourier Transform of $f(\tau)$ and given as,

$$\begin{aligned} F(w) &= \int_{-\infty}^{+\infty} e^{iw\tau} \frac{\cos(\beta\sqrt{\tau^2 + v^2})}{[\tau^2 + v^2]} d\tau \\ &= 2\int_{0}^{+\infty} \cos(w\tau) \frac{\cos(\beta\sqrt{\tau^2 + v^2})}{[\tau^2 + v^2]} d\tau \end{aligned} \tag{4}$$

Using Ref. [1], page 482, formula 3.876-5, we can write $F(w)$ as follows,

$$F(w) = \frac{\pi}{v} e^{-vw}, \qquad 0 < \beta < w \tag{5}$$

It should be noted that $0 < \beta < w$ is only satisfied if we choose $w = 2\pi|m|/d$ and exclude the $m = 0$ term where $|\cdot|$ is absolute value. Using this information in Eq. (3), we obtain the summation in the following form,

$$\begin{aligned} S &= \frac{1}{d} \sum_{m=-\infty, m\neq 0}^{m=\infty} \frac{\pi}{v} e^{-v2\pi|m|/d} \\ &= \frac{2\pi}{vd} \sum_{m=1}^{m=\infty} e^{-v2\pi|m|/d} \end{aligned} \tag{6}$$

The original summation or series given in Eq. (1) has $m=$ 0 term included, but after applying PSF, the obtained summation given by Eq. (6) does not

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have $m = 0$ term.

Q.1 - Is it true that a summation given by Eq. (1) when transformed using PSF (Eq. 6) must not have $m = 0$ term?

Q.2 - Should we need to write Poisson transformed Eq. (6) as follows to include m=0 term as below,

$$\begin{aligned} S &= \frac{2\pi}{vd}\sum_{m=1}^{m=\infty} e^{-v2\pi|m|/d} + (m = 0 \text{ term of Eq.}(1)) \\ &= \frac{2\pi}{vd}\sum_{m=1}^{m=\infty} e^{-v2\pi|m|/d} + \frac{\cos(\beta|v|)}{v^2} \end{aligned} \quad (7)$$

Q.3 - Whether Eq. (6) or Eq. (7) is actual rapidly convergent form representation of original Eq. (1).

References

[1] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products,* 7th edition, Academic Press, Burlington, MA, USA, 2007.

7.

Applying Poisson Summation to 6.

$$F\left(\tfrac{1}{d}2\pi m\right) = \frac{\pi}{v}e^{-\frac{v}{d}2\pi m}$$

$$\frac{1}{d}F(0) = \frac{1}{d}\lim_{m\to 0}\frac{\pi}{v}e^{-\frac{v}{d}2\pi m} = \frac{\pi}{vd}$$

The rest of the series transforms into what you have

$$\frac{2\pi}{vd}e^{-\frac{2\pi v}{d}}\underbrace{\left(1 + e^{-2\frac{2\pi v}{d}} + e^{-3\frac{2\pi v}{d}} + e^{-4\frac{2\pi v}{d}} + ...\right)}_{\frac{1}{1-e^{-\frac{2\pi v}{d}}}} = \frac{2\pi}{vd}\frac{1}{e^{\frac{2\pi v}{d}} - 1}$$

Therefore, by the Poisson Summation Formula,

$$S = \frac{2\pi}{vd}\left(\frac{1}{2} + \frac{1}{e^{\frac{2\pi v}{d}} - 1}\right).\square$$

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