Delta Function and Hilbert Transform

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Abstract The Hilbert Integral Theorem guarantees that the Hilbert Transform and its Inverse are well defined operations, so that inversion yields the original function that generated the Transform. It is Fundamental to the theory of the Hilbert Transform, but has not been obtained until now. We establish it here for any hyper-real function in Infinitesimal Calculus, from the properties of the Delta Function.

Nor were Conditions for the Existence of the Hilbert Transform established. In general, it may seem that the existence of the Hilbert Transform requires an analytic function, but conditions have been given for functions that are absolutely integrable. That is, functions with $\int \left| g(\xi) \right|^p d\xi < \infty$. We use the Delta Function, in Infinitesimal Calculus to establish that any hyper-real function has a Hilbert Transform.

The Transform gives rise to two Delta Functions, known in optical coherence, $\delta_+(x)$, and $\delta_-(x)$. We derive new formulas for these functions.

Keywords: Infinitesimal, Infinite-Hyper-Real, Hyper-Real, Cardinal, Infinity. Non-Archimedean, Non-Standard Analysis, Calculus, Limit, Continuity, Derivative, Integral, Delta Function, Fourier Transform, Hilbert Transform, Filter,

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Introduction

0.1 Transfer Function

A Filter composed of a Resistance R, and an Inductance L in series, driven by a harmonic signal at frequency ω, has the impedance $R + i\omega L$, and the Transfer Function

$$\frac{1}{R + i\omega L} = \underbrace{\frac{R}{R^2 + \omega^2 L^2}}_{u(\omega,0)} + i\underbrace{\frac{-\omega L}{R^2 + \omega^2 L^2}}_{v(\omega,0)}.$$

The functions $u(\omega, 0)$, and $v(\omega, 0)$ can be extended to the analytic functions

$$u(z) = \frac{R}{R^2 + z^2 L^2},$$

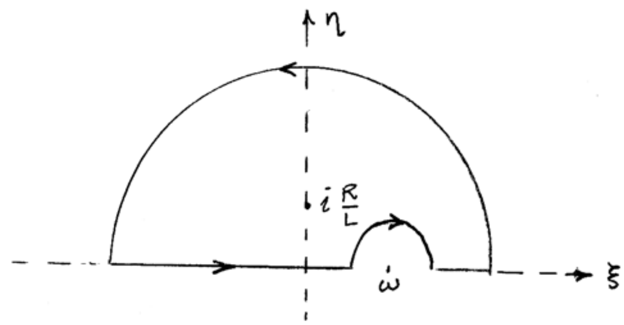
and

$$v(z) = \frac{-zL}{R^2 + z^2 L^2}.$$

It follows that

$$\boxed{\text{Cauchy Principal Value of} \int_{\xi=-\infty}^{\xi=\infty} \frac{u(\xi)}{\xi - \omega} d\xi = \pi v(\omega, 0)}.$$

To see that, note that on the contour γ



$$\oint_{\gamma} \frac{u(z)}{z-\omega} dz = \frac{R}{L^2} \oint_{\gamma} \frac{1}{(z - i\frac{R}{L})(z + i\frac{R}{L})(z-\omega)} dz$$

$$= \frac{R}{L^2} 2\pi i \mathrm{Res} \left\{ \frac{1}{(z - i\frac{R}{L})(z + i\frac{R}{L})(z-\omega)} \right\}_{z = i\frac{R}{L}}$$

$$= \frac{R}{L^2} 2\pi i \lim_{z \to i\frac{R}{L}} \left\{ \frac{1}{(z + i\frac{R}{L})(z-\omega)} \right\}$$

$$= \frac{R}{L^2} 2\pi i \frac{1}{2i\frac{R}{L}(i\frac{R}{L} - \omega)}$$

$$= \pi \frac{1}{iR - \omega L}$$

$$= \frac{-i\pi R - \pi\omega L}{R^2 + \omega^2 L^2}.$$

On the semicircle in the upper-half plane

$$\int \frac{u(z)}{z-\omega} dz \to 0, \text{ as } z \to \infty.$$

On the semi-circle that bypasses ω

$$\oint \frac{u(z)}{z-\omega}dz = \frac{R}{L^2}\oint \frac{1}{(z-i\frac{R}{L})(z+i\frac{R}{L})(z-\omega)}dz$$

$$= \frac{R}{L^2}(-\pi)i\mathrm{Res}\left\{\frac{1}{(z-i\frac{R}{L})(z+i\frac{R}{L})(z-\omega)}\right\}_{z=\omega}$$

$$= -\frac{R}{L^2}\pi i \lim_{z\to\omega}\left\{\frac{1}{(z-i\frac{R}{L})(z+i\frac{R}{L})}\right\}$$

$$= -\frac{R}{L^2}\pi i \frac{1}{(\omega-i\frac{R}{L})(\omega+i\frac{R}{L})}$$

$$= \frac{-i\pi R}{R^2+\omega^2 L^2}.$$

Therefore,

$$\text{Principal Value of} \int_{\xi=-\infty}^{\xi=\infty} \frac{u(\xi)}{\xi-\omega}d\xi = \frac{-i\pi R-\pi\omega L}{R^2+\omega^2L^2} - \frac{-i\pi R}{R^2+\omega^2L^2}$$

$$= \pi\frac{-\omega L}{R^2+\omega^2L^2}$$

That is,

$$v(\omega,0) = \frac{1}{\pi}\left\{\text{princial value of} \int_{\xi=-\infty}^{\xi=\infty}\frac{u(\xi,0)}{\xi-\omega}d\xi\right\}$$

$$= \frac{1}{\pi}\left\{\int_{\xi=-\infty}^{\xi=\omega-o}\frac{u(\xi,0)}{\xi-\omega}d\xi + \int_{\xi=\omega+o}^{\xi=\infty}\frac{u(\xi,0)}{\xi-\omega}d\xi\right\},$$

where o is an infinitesimal.

Similarly, it can be shown that

$$u(\omega,0) = -\frac{1}{\pi}\left\{\text{princial value of} \int_{\xi=-\infty}^{\xi=\infty} \frac{v(\xi,0)}{\xi-\omega}d\xi\right\}$$

$$= -\frac{1}{\pi}\left\{\int_{\xi=-\infty}^{\xi=\omega-o} \frac{v(\xi,0)}{\xi-\omega}d\xi + \int_{\xi=\omega+o}^{\xi=\infty} \frac{v(\xi,0)}{\xi-\omega}d\xi\right\},$$

where o is an infinitesimal.

0.2 Hilbert Transform

$v(\omega,0)$ is the Hilbert Transform of $u(\omega,0)$

$$v(\omega,0) = \mathcal{H}u\,,$$

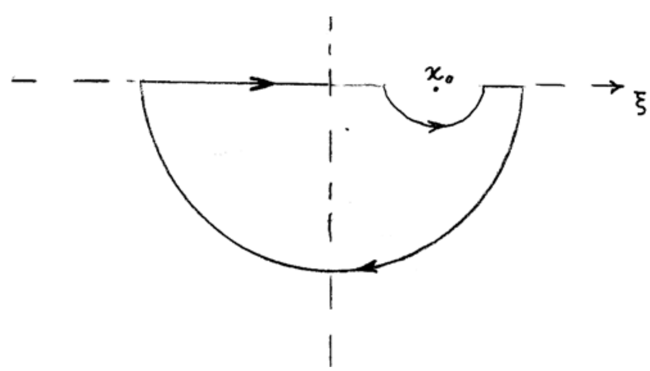
and $u(\omega,0)$ is the Inverse Hilbert Transform of $v(\omega,0)$

$$u(\omega,0) = \mathcal{H}^{-1}v\,.$$

In general, let

$$f(z) = u(x,y) + iv(x,y)$$

be analytic in $y < 0$, and consider the contour



By Cauchy Theorem,

$$\oint_{\gamma} \frac{f(z)}{z-x_0}\,dz = 0\,.$$

On the semi-circle that bypasses x_0,

$$\oint \frac{f(z)}{z-x_0}\,dz = i\pi \mathrm{Res}\left\{\frac{f(z)}{z-x_0}\right\}_{z=x_0}$$

$$= i\pi \lim_{z\to x_0} f(z) = i\pi f(x_0)$$

If on the semi-circle in the lower half plane

$$\int \frac{f(z)}{z-x_0}dz \to 0\text{, as } z\to\infty.$$

then

$$\text{princial value of} \int_{\xi=-\infty}^{\xi=\infty} \frac{f(\xi)}{\xi-x_0}d\xi + i\pi f(x_0) = 0\,.$$

That is,

$$\text{p.v.}\int_{\xi=-\infty}^{\xi=\infty} \frac{u(\xi,0)+iv(\xi,0)}{\xi-x_0}d\xi + i\pi\big(u(x_0,0)+iv(x_0,0)\big) = 0$$

Therefore,

$$v(x_0,0) = \frac{1}{\pi}\,p.v.\int_{\xi=-\infty}^{\xi=\infty} \frac{u(\xi,0)}{\xi-x_0}\,d\xi = \mathcal{H}u$$

$$u(x_0,0) = -\frac{1}{\pi}\,p.v.\int_{\xi=-\infty}^{\xi=\infty} \frac{v(\xi,0)}{\xi-x_0}\,d\xi = \mathcal{H}^{-1}v\,.$$

0.3 Existence of the Hilbert Transform

In general, it may seem that the existence of the Hilbert Transform requires an analytic function, but conditions have been given for functions that are absolutely integrable. That is, functions with $\int \left| g(\xi) \right|^p d\xi < \infty$. We use the Delta Function, in Infinitesimal Calculus to establish that any hyper-real function has a Hilbert Transform.

0.4 The Hilbert Integral Theorem

The Hilbert Integral Theorem guarantees that the Hilbert Transform and its Inverse are well defined operations, so that inversion yields the original function that generated the Transform. We shall establish it for any hyper-real function in Infinitesimal Calculus, from the properties of the Delta Function.

0.5 Delta functions and the Hilbert Transform

The Hilbert transform gives rise to two Delta Functions, known in optical coherence, $\delta_{+}(x)$, and $\delta_{-}(x)$. We derive new formulas for these Delta Functions.

1.

Hyper-real Line

Each real number α can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of infinitesimal hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the infinite hyper-reals.
4. The infinite hyper-reals are greater than any real number, yet strictly smaller than infinity.
5. The infinite hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a non-constant hyper-real.
7. The Hyper-reals are the totality of constant hyper-reals, a family of infinitesimals, a family of infinitesimals with

negative sign, a family of infinite hyper-reals, a family of infinite hyper-reals with negative sign, and non-constant hyper-reals.

8. The hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.

9. That line includes the real numbers separated by the non-constant hyper-reals. Each real number is the center of an interval of hyper-reals, that includes no other real number.

10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.

11. Zero is not an infinitesimal, because zero is not strictly greater than zero.

12. We do not add infinity to the hyper-real line.

13. The infinitesimals, the infinitesimals with negative signs, the infinite hyper-reals, and the infinite hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.

14. The hyper-real line is embedded in $\mathbb{R}^\infty$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the hyper-real onto the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal hyper-reals, or to the infinite hyper-reals, or to the non-constant hyper-reals.

16. No neighbourhood of a hyper-real is homeomorphic to an $\mathbb{R}^n$ ball. Therefore, the hyper-real line is not a manifold.

17. The hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

2.

Integral of a Hyper-real Function

In [Dan3], we defined the integral of a Hyper-real Function.

Let $f(x)$ be a hyper-real function on the interval $[a,b]$.

The interval may not be bounded.

$f(x)$ may take infinite hyper-real values, and need not be bounded.

At each

$$a \leq x \leq b,$$

there is a rectangle with base $[x - \frac{dx}{2}, x + \frac{dx}{2}]$, height $f(x)$, and area

$$f(x)dx.$$

We form the **Integration Sum** of all the areas for the x's that start at $x = a$, and end at $x = b$,

$$\sum_{x \in [a,b]} f(x)dx.$$

If for any infinitesimal dx, the Integration Sum has the same hyper-real value, then $f(x)$ is integrable over the interval $[a,b]$.

Then, we call the Integration Sum the integral of $f(x)$ from $x = a$, to $x = b$, and denote it by

$$\int_{x=a}^{x=b} f(x)dx\,.$$

If the hyper-real is infinite, then it is the integral over $[a,b]$,

If the hyper-real is finite,

$$\int_{x=a}^{x=b} f(x)dx = \text{ real part of the hyper-real}\,.\square$$

2.1 The countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers, $Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty\,.$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[a,b]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(x)dx$.

The Lower Integral is the Integration Sum where $f(x)$ is replaced by its lowest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

2.2
$$\sum_{x \in [a,b]} \left(\inf_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

The Upper Integral is the Integration Sum where $f(x)$ is replaced by its largest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

2.3
$$\sum_{x \in [a,b]} \left(\sup_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

If the integral is a finite hyper-real, we have

2.4 *A hyper-real function has a finite integral if and only if its upper integral and its lower integral are finite, and differ by an infinitesimal.*

3.

Delta Function

In [Dan5], we have defined the Delta Function, and established its properties

1. The Delta Function is a hyper-real function defined from the hyper-real line into the set of two hyper-reals $\left\{0, \frac{1}{dx}\right\}$. The hyper-real 0 is the sequence $\left\langle 0, 0, 0, ... \right\rangle$. The infinite hyper-real $\frac{1}{dx}$ depends on our choice of dx.
2. We will usually choose the family of infinitesimals that is spanned by the sequences $\left\langle \frac{1}{n} \right\rangle, \left\langle \frac{1}{n^2} \right\rangle, \left\langle \frac{1}{n^3} \right\rangle, \ldots$ It is a semigroup with respect to vector addition, and includes all the scalar multiples of the generating sequences that are non-zero. That is, the family includes infinitesimals with negative sign. Therefore, $\frac{1}{dx}$ will mean the sequence $\left\langle n \right\rangle$. Alternatively, we may choose the family spanned by the sequences $\left\langle \frac{1}{2^n} \right\rangle, \left\langle \frac{1}{3^n} \right\rangle, \left\langle \frac{1}{4^n} \right\rangle, \ldots$ Then, $\frac{1}{dx}$ will mean the

sequence $\langle 2^n \rangle$. Once we determined the basic infinitesimal dx, we will use it in the Infinite Riemann Sum that defines an Integral in Infinitesimal Calculus.

3. The Delta Function is strictly smaller than ∞

4. We define, $\delta(x) \equiv \frac{1}{dx}\chi_{[-\frac{dx}{2},\frac{dx}{2}]}(x)$,

$$\text{where } \chi_{[-\frac{dx}{2},\frac{dx}{2}]}(x) = \begin{cases} 1, x \in \left[-\frac{dx}{2},\frac{dx}{2}\right] \\ 0, \text{otherwise} \end{cases}.$$

5. Hence,

❖ for $x < 0$, $\delta(x) = 0$

❖ at $x = -\frac{dx}{2}$, $\delta(x)$ jumps from 0 to $\frac{1}{dx}$,

❖ for $x \in \left[-\frac{dx}{2},\frac{dx}{2}\right]$, $\delta(x) = \frac{1}{dx}$.

❖ at $x = 0$, $\delta(0) = \frac{1}{dx}$

❖ at $x = \frac{dx}{2}$, $\delta(x)$ drops from $\frac{1}{dx}$ to 0.

❖ for $x > 0$, $\delta(x) = 0$.

❖ $x\delta(x) = 0$

6. If $dx = \langle \frac{1}{n} \rangle$, $\delta(x) = \left\langle \chi_{[-\frac{1}{2},\frac{1}{2}]}(x), 2\chi_{[-\frac{1}{4},\frac{1}{4}]}(x), 3\chi_{[-\frac{1}{6},\frac{1}{6}]}(x) ... \right\rangle$

7. If $dx = \langle \frac{2}{n} \rangle$, $\delta(x) = \left\langle \frac{1}{2\cosh^2 x}, \frac{2}{2\cosh^2 2x}, \frac{3}{2\cosh^2 3x}, ... \right\rangle$

8. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle e^{-x}\chi_{[0,\infty)}, 2e^{-2x}\chi_{[0,\infty)}, 3e^{-3x}\chi_{[0,\infty)}, \ldots \right\rangle$

9. $$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1.$$

10. $$\delta(x-\xi) = \frac{1}{2\pi}\int_{k=-\infty}^{k=\infty} e^{ik(x-\xi)}dk$$

4.

$\delta(x)$ and the Hilbert Transform

4.1 $\mathcal{H}\{\delta(\cdot)\} = -\dfrac{1}{\pi x}$

Proof: For any infinitesimal $d\xi$, the Integration Sum for the function

$$\frac{1}{\xi - x}\delta(\xi) = \frac{1}{\xi - x}\frac{1}{d\xi}\chi_{[-\frac{d\xi}{2},\frac{d\xi}{2}]}(\xi)$$

has only the unique hyper-real term

$$\frac{1}{\xi - x}\frac{1}{d\xi}\chi_{[-\frac{d\xi}{2},\frac{d\xi}{2}]}(\xi)d\xi = \frac{1}{\xi - x}\chi_{[-\frac{d\xi}{2},\frac{d\xi}{2}]}(\xi).$$

Therefore, the Hilbert Transform

$$\mathcal{H}\{\delta(\cdot)\} = \frac{1}{\pi}\, p.v. \int_{\xi=-\infty}^{\xi=\infty} \frac{1}{\xi - x}\delta(\xi)d\xi$$

exists, and equals the constant hyper-real part of $\frac{1}{\pi}\frac{1}{\xi - x}\chi_{[-\frac{d\xi}{2},\frac{d\xi}{2}]}(\xi)$,

$$\frac{1}{\pi}\frac{1}{\xi - x}\chi_{[-\frac{d\xi}{2},\frac{d\xi}{2}]}(\xi)\bigg|_{\xi=0} = -\frac{1}{\pi x}.$$

That is

$$\mathcal{H}\{\delta(\cdot)\} = -\frac{1}{\pi x}.\square$$

Consequently,

4.2 $\delta(x)$ $=$ *the inverse Hilbert Transform of the function* $-\dfrac{1}{\pi x}$

$$= \frac{1}{\pi^2}\, p.v. \int_{t=-\infty}^{t=\infty} \frac{1}{t(t-x)}\, dt\,.$$

Proof:
$$\delta(x) = -\frac{1}{\pi}\, p.v. \int_{t=-\infty}^{t=\infty} -\frac{1}{\pi t}\frac{1}{t-x}\, dt$$
$$= \frac{1}{\pi^2}\, p.v. \int_{t=-\infty}^{t=\infty} \frac{1}{t(t-x)}\, dt\,.\square$$

4.3
$$\left.\frac{1}{\pi^2}\, p.v. \int_{t=-\infty}^{t=\infty} \frac{1}{t(t-x)}\, dt\right|_{x=0} = \frac{1}{dx} = \text{an infinite hyper-real}$$
$$\left.\frac{1}{\pi^2}\, p.v. \int_{t=-\infty}^{t=\infty} \frac{1}{t(t-x)}\, dt\right|_{x\neq 0} = 0$$

Proof: $\delta(0) = \dfrac{1}{dx}$.

$$\delta(x)\big|_{x\neq 0} = 0\,.\square$$

4.4
$$\delta(\xi - x) = \frac{1}{\pi^2}\, p.v. \int_{\tau=-\infty}^{\tau=\infty} \frac{1}{(\tau-\xi)(\tau-x)}\, d\tau\,.$$

$$= \frac{1}{\pi^2} \int_{\tau=-\infty}^{\tau=\xi-o} \frac{1}{(\tau-\xi)(\tau-x)} d\tau + \int_{\tau=\xi+o}^{\tau=x-o} \frac{1}{(\tau-\xi)(\tau-x)} d\tau + \int_{\tau=x+o}^{\tau=\infty} \frac{1}{(\tau-\xi)(\tau-x)} d\tau$$

for any infinitesimal o.

Proof: $\delta(\xi - x) = \frac{1}{\pi^2}\, p.v. \int_{t=-\infty}^{t=\infty} \frac{1}{t(t-[\xi-x])} dt$

The change of variable, $\tau = t + x$, gives

$$= \frac{1}{\pi^2}\, p.v. \int_{\tau=-\infty}^{\tau=\infty} \frac{1}{(\tau-x)(\tau-\xi)} d\tau$$

$$= \frac{1}{\pi^2}\, p.v. \int_{\tau=-\infty}^{\tau=\infty} \frac{1}{\xi-x} \left\{ \frac{1}{\tau-\xi} - \frac{1}{\tau-x} \right\} d\tau\,.\square$$

4.5 $\delta(\xi - x) = \frac{1}{\pi^2}\, p.v. \int_{\tau=-\infty}^{\tau=\infty} \frac{1}{\xi-x} \left\{ \frac{1}{\tau-\xi} - \frac{1}{\tau-x} \right\} d\tau$

Proof: Apply 4.4.$\square$

5.

Hilbert Integral Theorem

The Fundamental Theorem of the Hilbert Transform Theory is the Hilbert Integral Theorem.

It guarantees that the Hilbert Transform and its Inverse are well defined operations, that when inverted, yield the original function.

The use of the Delta Function, and Infinitesimal Calculus Integration, are necessary to establish the Hilbert Integral Theorem.

Consequently, it cannot be established in the Calculus of Limits which is inadequate for dealing with singularities, and it does not hold in the Calculus of Limits under any conditions.

5.1 Hilbert Integral Theorem

If $u(x)$ is hyper-real function,

Then, the Hilbert Integral Theorem holds.

$$\boxed{u(x) = -\frac{1}{\pi}\, p.v. \int_{\tau=-\infty}^{\tau=\infty} \frac{1}{\tau - x}\left(\frac{1}{\pi}\, p.v. \int_{\xi=-\infty}^{\xi=\infty} \frac{u(\xi)}{\xi - \tau}\, d\xi\right) d\tau}$$

Proof:

In Infinitesimal Calculus, the Integration Sum

$$\int_{\xi=-\infty}^{\xi=\infty} u(\xi)\delta(\xi - x)d\xi$$

yields $u(x)$. That is,

$$u(x) = \int_{\xi=-\infty}^{\xi=\infty} u(\xi)\delta(\xi - x)d\xi .$$

By 4.4, $\delta(\xi - x)$ equals the Integration Sum

$$\delta(\xi - x) = \frac{1}{\pi^2} p.v.\int_{\tau=-\infty}^{\tau=\infty} \frac{1}{(\tau - \xi)(\tau - x)} d\tau ,$$

where the principal value means that τ has to skip both ξ, and x.

Substituting in the Integration Sum for $u(x)$,

$$u(x) = \frac{1}{\pi}\int_{\xi=-\infty}^{\xi=\infty} u(\xi)\left(\frac{1}{\pi} p.v.\int_{\tau=-\infty}^{\tau=\infty} \frac{1}{(\tau - \xi)(\tau - x)} d\tau\right)d\xi$$

The terms in this Integration Sum are zero whenever $\xi \neq x$. The only nonzero term appears when $\xi = x$.

Changing the Summation order, ξ needs to skip τ,and the principal value is necessary in the ξ integration as well. Therefore,

$$u(x) = \frac{1}{\pi} p.v.\int_{\tau=-\infty}^{\tau=\infty} \frac{1}{\tau - x}\left(\frac{1}{\pi} p.v.\int_{\xi=-\infty}^{\xi=\infty} \frac{u(\xi)}{\tau - \xi} d\xi\right)d\tau$$

$$u(x) = -\frac{1}{\pi} p.v. \int_{\tau=-\infty}^{\tau=\infty} \frac{1}{\tau - x} \left(\frac{1}{\pi} p.v. \int_{\xi=-\infty}^{\xi=\infty} \frac{u(\xi)}{\xi - \tau} d\xi \right) d\tau . \square$$

Then, the Hilbert transform of $u(\xi)$,

$$\frac{1}{\pi} p.v. \int_{\xi=-\infty}^{\xi=\infty} \frac{u(\xi)}{\xi - \tau} d\xi ,$$

converges to a hyper-real function $H(\tau)$, some of its values may be infinite hyper-reals, like the Delta Function.

And the Inverse Hilbert Transform of $H(\tau)$

$$-\frac{1}{\pi} p.v. \int_{\tau=-\infty}^{\tau=\infty} \frac{H(\tau)}{\tau - x} d\tau$$

converges to the hyper-real function $u(x)$.

5.2 *If* $u(x)$ *is hyper-real function,*

Then,

- *the hyper-real integral* $\frac{1}{\pi} p.v. \int_{\xi=-\infty}^{\xi=\infty} \frac{u(\xi)}{\xi - \tau} d\xi$ *converges to* $H(\tau)$
- *the hyper-real integral* $-\frac{1}{\pi} p.v. \int_{\tau=-\infty}^{\tau=\infty} \frac{H(\tau)}{\tau - x} d\tau$ *converges to* $u(x)$

6.

$\delta_+(t)$, $\delta\_(t)$, and the Hilbert transform

6.1 Definition of $\delta_+(x)$, and $\delta\_(x)$

$$\boxed{\delta\_(x) \equiv \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=0} e^{i\omega t}d\omega}$$

$$\boxed{\delta_+(x) \equiv \frac{1}{2\pi}\int_{\omega=0}^{\omega=\infty} e^{i\omega t}d\omega}$$

6.2 $$\boxed{\delta(x) = \delta\_(x) + \delta_+(x)}$$

Proof: 3.10.□

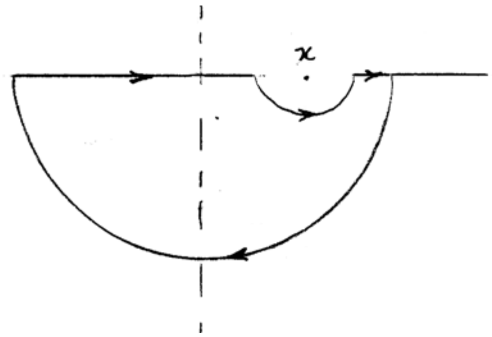
6.3 For $\omega > 0$, $$\boxed{\mathcal{H}\left\{e^{-i\omega x}\right\} = -ie^{-i\omega x}}$$

That is, for $\omega > 0$, $\mathcal{H}$ delays the phase of $e^{-i\omega x}$ by $-\frac{\pi}{2}$.

Proof:

$$\mathcal{H}\left\{e^{-i\omega x}\right\} = \frac{1}{\pi}\, p.v. \int_{\xi=-\infty}^{\xi=\infty} \frac{e^{-i\omega\xi}}{\xi - x}d\xi$$

In $y < 0$, consider the contour γ



By Cauchy Theorem,

$$\oint_{\gamma} \frac{e^{-i\omega z}}{z-x} dz = 0 .$$

On the semi-circle that bypasses x,

$$\oint \frac{e^{-i\omega z}}{z-x} dz = i\pi \mathrm{Res}\left\{\frac{e^{-i\omega z}}{z-x}\right\}_{z=x}$$

$$= i\pi \lim_{z\to x} e^{-i\omega z}$$

$$= i\pi e^{-i\omega x} .$$

By Jordan's Lemma, on the semi-circle in the lower half plane

$$\int \frac{e^{-i\omega z}}{z-x} dz \to 0, \text{ as } z \to \infty .$$

Therefore,

$$\text{princial value of} \int_{\xi=-\infty}^{\xi=\infty} \frac{e^{-i\omega \xi}}{\xi - x} d\xi + i\pi e^{-i\omega x} = 0 .$$

That is,

$$\frac{1}{\pi} p.v. \int_{\xi=-\infty}^{\xi=\infty} \frac{e^{-i\omega \xi}}{\xi - x} d\xi = -ie^{-i\omega x} . \square$$

6.4 $$\boxed{\mathcal{H}\left\{\delta\_(x)\right\} = -i\delta\_(x)}$$

That is, $\mathcal{H}$ *rotates* $\delta\_(x)$ *by* $-\frac{\pi}{2}$

Proof:

$$\mathcal{H}\left\{\delta\_(x)\right\} = \mathcal{H}\left\{\frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=0} e^{i\omega t}d\omega\right\}$$

$$= \mathcal{H}\left\{\frac{1}{2\pi}\int_{\omega=0}^{\omega=\infty} e^{-i\omega t}d\omega\right\}$$

$$= \frac{1}{2\pi}\int_{\omega=0}^{\omega=\infty} \mathcal{H}\left\{e^{-i\omega t}\right\}d\omega$$

$$= \frac{1}{2\pi}\int_{\omega=0}^{\omega=\infty} -ie^{-i\omega t}d\omega$$

$$= -i\frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=0} e^{i\omega t}d\omega$$

$$= -i\delta\_(x).\square$$

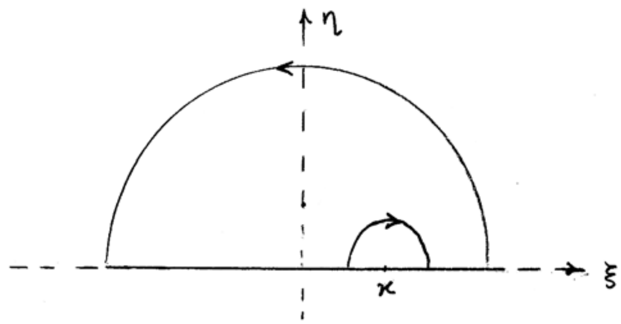
6.5 **For** $\omega > 0$, $$\mathcal{H}\left\{e^{i\omega x}\right\} = ie^{i\omega x}$$

That is, for $\omega > 0$, $\mathcal{H}$ *advances the phase of* $e^{i\omega x}$ *by* $\frac{\pi}{2}$.

Proof:

$$\mathcal{H}\left\{e^{i\omega x}\right\} = \frac{1}{\pi} p.v. \int_{\xi=-\infty}^{\xi=\infty} \frac{e^{i\omega\xi}}{\xi - x} d\xi$$

In $y > 0$, consider the contour γ



By Cauchy Theorem,

$$\oint_{\gamma} \frac{e^{i\omega z}}{z - x} dz = 0.$$

On the semi-circle that bypasses x,

$$\oint \frac{e^{i\omega z}}{z - x} dz = -i\pi \text{Res}\left\{\frac{e^{i\omega z}}{z - x}\right\}_{z=x}$$

$$= -i\pi \lim_{z \to x} e^{i\omega z}$$

$$= -i\pi e^{i\omega x}.$$

By Jordan's Lemma, on the semi-circle in the upper half plane

$$\int \frac{e^{i\omega z}}{z - x} dz \to 0, \text{ as } z \to \infty.$$

Therefore,

$$\text{princial value of} \int_{\xi=-\infty}^{\xi=\infty} \frac{e^{i\omega\xi}}{\xi - x} d\xi - i\pi e^{i\omega x} = 0 .$$

That is,

$$\frac{1}{\pi} p.v. \int_{\xi=-\infty}^{\xi=\infty} \frac{e^{i\omega\xi}}{\xi - x} d\xi = ie^{i\omega x} . \square$$

6.6 $$\boxed{\mathcal{H}\left\{\delta_+(x)\right\} = i\delta_+(x)}$$

That is, $\mathcal{H}$ ***rotates*** $\delta_+(x)$ ***by*** $\frac{\pi}{2}$

Proof:

$$\mathcal{H}\left\{\delta_+(x)\right\} = \mathcal{H}\left\{\frac{1}{2\pi}\int_{\omega=0}^{\omega=\infty} e^{i\omega t} d\omega\right\}$$

$$= \frac{1}{2\pi}\int_{\omega=0}^{\omega=\infty} \mathcal{H}\left\{e^{i\omega t}\right\} d\omega$$

$$= \frac{1}{2\pi}\int_{\omega=0}^{\omega=\infty} ie^{i\omega t} d\omega$$

$$= i\frac{1}{2\pi}\int_{\omega=0}^{\omega=\infty} e^{i\omega t} d\omega$$

$$= i\delta_+(x) . \square$$

6.7 $$\mathcal{H}\left\{\delta(x)\right\} = i\left(\delta_+(x) - \delta_-(x)\right)$$

Proof: 6.4, and 6.6.$\square$

6.8 $$\boxed{\delta_+(x) - \delta_-(x) = i\frac{1}{\pi x}}$$

Proof: By 6.7,

$$\delta_+(x) - \delta_-(x) = -i\mathcal{H}\left\{\delta(x)\right\}$$

$$= -i\frac{1}{\pi}\,p.v\int_{\xi=-\infty}^{\xi=\infty}\frac{\delta(\xi)}{\xi - x}d\xi$$

$$= -i\frac{1}{\pi}\left(\frac{1}{-x}\right)$$

$$= i\frac{1}{\pi x}.\square$$

6.9 $$\boxed{\delta_+(x) = \frac{1}{2}\delta(x) + i\frac{1}{\pi x}}$$

$$\boxed{\delta_-(x) = \frac{1}{2}\delta(x) - i\frac{1}{\pi x}}$$

Proof: 6.2, and 6.8.□

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