Delta Function, the Fourier Transform, and Fourier Integral Theorem

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Abstract The Fourier Integral Theorem guarantees that the Fourier Transform and its Inverse are well defined operations, so that the inversed transform is the originally transformed function. It is believed to hold in the Calculus of Limits under some highly restrictive sufficient conditions. In fact,

The Theorem does not hold in the Calculus of Limits under any conditions,

because evaluating the Fourier Integral requires the integration of

$$\int_{k=-\infty}^{k=\infty} e^{ik(x-\xi)}dk,$$

that diverges at $x=\xi$.

Only in Infinitesimal Calculus, the integral is the Hyper-real Delta Function

$$\delta(x-\xi)=\frac{1}{2\pi}\int_{k=-\infty}^{k=\infty} e^{ik(x-\xi)}dk,$$

and the Fourier Integral Theorem states the sifting property for the Delta Function

$$f(x) = \int_{\xi=-\infty}^{\xi=\infty} f(\xi)\delta(\xi - x)d\xi .$$

In infinitesimal Calculus we can integrate over singularities, and the Fourier Integral Theorem holds

$$f(x) = \frac{1}{2\pi}\int_{k=-\infty}^{k=\infty}\left(\int_{\xi=-\infty}^{\xi=\infty} f(\xi)e^{-ik\xi}d\xi\right)e^{ikx}dk ,$$

where the Integrals are Hyper-real.

The highly restrictive conditions for the Fourier Integral Theorem, in the Calculus of Limits, are irrelevant to the simplest functions, such as constants, and useless for singular functions.

In particular, the singular $\delta(x)$ violates these conditions

- the Hyper-real Delta $\delta(x)$ is not defined in the Calculus of Limits, and is not Piecewise Continuous.
- $\delta'(x)$ is not defined, and is not Piecewise Continuous in any bounded interval.

But in Infinitesimal Calculus, $\delta(x)$ satisfies the Hyper-real Fourier integral Theorem

$$\delta(x) = \frac{1}{2\pi}\int_{k=-\infty}^{k=\infty}\left(\int_{\xi=-\infty}^{\xi=\infty} \delta(\xi)e^{-ik\xi}d\xi\right)e^{ikx}dk .$$

Also, the constant function $f(x) \equiv 1$ violates the sufficient conditions' requirement of absolute integrability, $\int_{x=-\infty}^{x=\infty} \left|1\right| dx = \infty$.

But in Infinitesimal Calculus, $f(x) \equiv 1$ satisfies the Hyper-real Fourier integral Theorem

$$1 = \frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} \left(\int_{\xi=-\infty}^{\xi=\infty} e^{-ik\xi} d\xi \right) e^{ikx} dk .$$

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Introduction

By Fourier Integral Theorem

$$f(x) = \frac{1}{2\pi}\int_{k=-\infty}^{k=\infty}\left(\int_{\xi=-\infty}^{\xi=\infty} f(\xi)e^{-ik\xi}d\xi\right)e^{ikx}dk$$

$$= \int_{\xi=-\infty}^{\xi=\infty} f(\xi)\left(\frac{1}{2\pi}\int_{k=-\infty}^{k=\infty} e^{ik(x-\xi)}dk\right)d\xi$$

Thus, the integral $\frac{1}{2\pi}\int_{k=-\infty}^{k=\infty} e^{-ik(\xi-x)}dk$ sifts through the values of the function $f(\xi)$, and picks its value at x.

Cauchy (1816), and Poisson (1815) derived the Fourier Integral Theorem by using the sifting property of the integral.

In the derivation of his Zeta Function, Riemann (1859) uses this sifting property repeatedly, without using a function notation for the integral $\frac{1}{2\pi}\int_{k=-\infty}^{k=\infty} e^{-ik(\xi-x)}dk$. The derivations are in [Dan4, p.84, p.90, p.97].

However,

$$\xi = x \Rightarrow e^{-ik(\xi-x)} = 1,$$

and the integral $\frac{1}{2\pi}\int_{k=-\infty}^{k=\infty} e^{-ik(\xi-x)}dk$ diverges.

Avoiding the singularity at $\xi = x$ does not recover the Fourier Integral Theorem, because without the singularity the Fourier integral equals zero.

Thus, the Fourier Integral Theorem cannot be written in the Calculus of Limits.

In Infinitesimal Calculus [Dan4], the singularity can be integrated over, and defines the Delta Function

$$\delta(x-\xi) = \frac{1}{2\pi}\int_{k=-\infty}^{k=\infty} e^{ik(x-\xi)}dk\,.$$

Then, the Fourier Integral theorem states the sifting property for the Delta Function

$$f(x) = \int_{\xi=-\infty}^{\xi=\infty} f(\xi)\delta(\xi - x)d\xi\,,$$

and for any hyper-real function $f(x)$, the Fourier Transform pairs converge, and the Fourier Integral Theorem holds.

In the Calculus of Limits, the Delta Function cannot be defined, and its sifting property does not apply.

That sifting property allows for a Hyper-real Fourier Integral Theorem for $f(x) \equiv 1$.

While $\int\limits_{x=-\infty}^{x=\infty} \left|1\right| dx = \infty$, we have $1 = \frac{1}{2\pi}\int\limits_{k=-\infty}^{k=\infty}\left(\int\limits_{\xi=-\infty}^{\xi=\infty} e^{-ik\xi}d\xi\right)e^{ikx}dk\,.$

1.

Hyper-real Line

Each real number α can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of infinitesimal hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the infinite hyper-reals.
4. The infinite hyper-reals are greater than any real number, yet strictly smaller than infinity.
5. The infinite hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a non-constant hyper-real.
7. The Hyper-reals are the totality of constant hyper-reals, a family of infinitesimals, a family of infinitesimals with

negative sign, a family of infinite hyper-reals, a family of infinite hyper-reals with negative sign, and non-constant hyper-reals.

8. The hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.
9. That line includes the real numbers separated by the non-constant hyper-reals. Each real number is the center of an interval of hyper-reals, that includes no other real number.
10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.
11. Zero is not an infinitesimal, because zero is not strictly greater than zero.
12. We do not add infinity to the hyper-real line.
13. The infinitesimals, the infinitesimals with negative signs, the infinite hyper-reals, and the infinite hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.
14. The hyper-real line is embedded in $\mathbb{R}^{\infty}$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the hyper-real onto the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal hyper-reals, or to the infinite hyper-reals, or to the non-constant hyper-reals.

16. No neighbourhood of a hyper-real is homeomorphic to an $\mathbb{R}^n$ ball. Therefore, the hyper-real line is not a manifold.

17. The hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

2.

Integral of a Hyper-real Function

In [Dan3], we defined the integral of a Hyper-real Function.

Let $f(x)$ be a hyper-real function on the interval $[a,b]$.

The interval may not be bounded.

$f(x)$ may take infinite hyper-real values, and need not be bounded.

At each

$$a \leq x \leq b\,,$$

there is a rectangle with base $[x - \frac{dx}{2}, x + \frac{dx}{2}]$, height $f(x)$, and area

$$f(x)dx\,.$$

We form the **Integration Sum** of all the areas for the x's that start at $x = a$, and end at $x = b$,

$$\sum_{x \in [a,b]} f(x)dx\,.$$

If for any infinitesimal dx, the Integration Sum has the same hyper-real value, then $f(x)$ is integrable over the interval $[a,b]$.

Then, we call the Integration Sum the integral of $f(x)$ from $x = a$, to $x = b$, and denote it by

$$\int_{x=a}^{x=b} f(x)dx \,.$$

If the hyper-real is infinite, then it is the integral over $[a,b]$,

If the hyper-real is finite,

$$\int_{x=a}^{x=b} f(x)dx = \text{ real part of the hyper-real} \,.\square$$

2.1 The countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers, $Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty \,.$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[a,b]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(x)dx$.

The Lower Integral is the Integration Sum where $f(x)$ is replaced by its lowest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

2.2
$$\sum_{x \in [a,b]} \left(\inf_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

The Upper Integral is the Integration Sum where $f(x)$ is replaced by its largest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

2.3
$$\sum_{x \in [a,b]} \left(\sup_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

If the integral is a finite hyper-real, we have

2.4 *A hyper-real function has a finite integral if and only if its upper integral and its lower integral are finite, and differ by an infinitesimal.*

3.

Delta Function

In [Dan5], we have defined the Delta Function, and established its properties

1. The Delta Function is a hyper-real function defined from the hyper-real line into the set of two hyper-reals $\left\{0, \dfrac{1}{dx}\right\}$. The hyper-real 0 is the sequence $\langle 0, 0, 0, ... \rangle$. The infinite hyper-real $\dfrac{1}{dx}$ depends on our choice of dx.
2. We will usually choose the family of infinitesimals that is spanned by the sequences $\left\langle \dfrac{1}{n} \right\rangle, \left\langle \dfrac{1}{n^2} \right\rangle, \left\langle \dfrac{1}{n^3} \right\rangle, \ldots$ It is a semigroup with respect to vector addition, and includes all the scalar multiples of the generating sequences that are non-zero. That is, the family includes infinitesimals with negative sign. Therefore, $\dfrac{1}{dx}$ will mean the sequence $\langle n \rangle$. Alternatively, we may choose the family spanned by the sequences $\left\langle \dfrac{1}{2^n} \right\rangle, \left\langle \dfrac{1}{3^n} \right\rangle, \left\langle \dfrac{1}{4^n} \right\rangle, \ldots$ Then, $\dfrac{1}{dx}$ will mean the

sequence $\langle 2^n \rangle$. Once we determined the basic infinitesimal dx, we will use it in the Infinite Riemann Sum that defines an Integral in Infinitesimal Calculus.

3. The Delta Function is strictly smaller than ∞

4. We define, $\delta(x) \equiv \frac{1}{dx}\chi_{[-\frac{dx}{2},\frac{dx}{2}]}(x)$,

$$\text{where} \quad \chi_{[-\frac{dx}{2},\frac{dx}{2}]}(x) = \begin{cases} 1, x \in \left[-\frac{dx}{2},\frac{dx}{2}\right] \\ 0, \text{otherwise} \end{cases}.$$

5. Hence,

❖ for $x < 0$, $\delta(x) = 0$

❖ at $x = -\frac{dx}{2}$, $\delta(x)$ jumps from 0 to $\frac{1}{dx}$,

❖ for $x \in \left[-\frac{dx}{2},\frac{dx}{2}\right]$, $\delta(x) = \frac{1}{dx}$.

❖ at $x = 0$, $\delta(0) = \frac{1}{dx}$

❖ at $x = \frac{dx}{2}$, $\delta(x)$ drops from $\frac{1}{dx}$ to 0.

❖ for $x > 0$, $\delta(x) = 0$.

❖ $x\delta(x) = 0$

6. If $dx = \langle \frac{1}{n} \rangle$, $\delta(x) = \left\langle \chi_{[-\frac{1}{2},\frac{1}{2}]}(x), 2\chi_{[-\frac{1}{4},\frac{1}{4}]}(x), 3\chi_{[-\frac{1}{6},\frac{1}{6}]}(x)... \right\rangle$

7. If $dx = \langle \frac{2}{n} \rangle$, $\delta(x) = \left\langle \frac{1}{2\cosh^2 x}, \frac{2}{2\cosh^2 2x}, \frac{3}{2\cosh^2 3x}, ... \right\rangle$

8. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle e^{-x}\chi_{[0,\infty)}, 2e^{-2x}\chi_{[0,\infty)}, 3e^{-3x}\chi_{[0,\infty)}, \ldots \right\rangle$

9. $$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1.$$

10. $$\delta(x-\xi) = \frac{1}{2\pi}\int_{k=-\infty}^{k=\infty} e^{ik(x-\xi)}dk$$

4.

$\delta(x)$ and the Fourier Transform

4.1 $\mathcal{F}\left\{\delta(x)\right\} = 1$

Proof: For any infinitesimal dx, the Integration Sum for the function

$$\delta(x)e^{-i\omega x} = \frac{1}{dx}e^{-i\omega x}\chi[-\tfrac{dx}{2},\tfrac{dx}{2}]$$

has only the unique hyper-real term

$$\frac{1}{dx}e^{-i\omega x}\chi[-\tfrac{dx}{2},\tfrac{dx}{2}]dx = e^{-i\omega x}\chi[-\tfrac{dx}{2},\tfrac{dx}{2}].$$

Therefore, the Fourier Transform

$$\mathcal{F}\left\{\delta(x)\right\} = \int\limits_{x=-\infty}^{x=\infty}\delta(x)e^{-i\omega x}dx$$

exists.

Since $e^{-i\omega x}\chi[-\frac{dx}{2},\frac{dx}{2}]$ is bounded by 1, it is a finite hyper-real.

Therefore, the Fourier Transform equals to the constant part of this hyper-real.

Since the constant hyper-real in $[-\frac{dx}{2},\frac{dx}{2}]$ is zero, the constant

hyper-real part of $e^{-i\omega x}\chi[-\frac{dx}{2},\frac{dx}{2}]$ is

$$e^{-i\omega 0} = 1.$$

That is

$$\mathcal{F}\left\{\delta(x)\right\} = 1.\square$$

Consequently,

4.2 $\delta(x) =$ *the inverse Fourier Transform of the unit function* 1

$$= \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x}d\omega$$

$$= \int_{\nu=-\infty}^{\nu=\infty} e^{2\pi i x}d\nu, \qquad \omega = 2\pi\nu$$

Thus,

4.3 $$\frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x}d\omega\Bigg|_{x=0} = \frac{1}{dx} = \text{ an infinite hyper-real}$$

$$\int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x}d\omega\Bigg|_{x\neq 0} = 0$$

Proof: $\delta(0) = \dfrac{1}{dx}$.

$$\delta(x)\Big|_{x\neq 0} = 0.\square$$

5.

Fourier Integral Theorem

The Fourier Integral Theorem is the Fundamental Theorem of the Fourier Transform Theory.

It guarantees that the Fourier Transform and its Inverse are well defined operations, so that inversion yields the originally transformed function.

It is well known to hold in the Calculus of Limits under given conditions.

In fact, it does not hold in the Calculus of Limits under any conditions.

That failure is due to the inadequacy of the Calculus of Limits for dealing with singularities

5.1 Fourier Integral Theorem does not hold in the Calculus of Limits

Proof: By the Fourier Integral Theorem

$$f(x) = \frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} \left(\int_{\xi=-\infty}^{\xi=\infty} f(\xi) e^{-ik\xi} d\xi \right) e^{ikx} dk$$

$$= \int_{\xi=-\infty}^{\xi=\infty} f(\xi)\left(\frac{1}{2\pi}\int_{k=-\infty}^{k=\infty} e^{ik(x-\xi)}dk\right)d\xi \,.$$

However,

$$x = \xi \Rightarrow e^{ik(x-\xi)} = 1\,,$$

and the integral

$$\frac{1}{2\pi}\int_{k=-\infty}^{k=\infty} e^{-ik(\xi-x)}dk\,,$$

diverges.

That is, the Fourier Integral Theorem cannot be written in the Calculus of Limits.

Avoiding the singularity at $\xi = x$ does not recover the Theorem, because without the singularity the integral equals zero.

Furthermore,

5.2 Calculus of Limits Conditions are not sufficient for the Fourier Integral Theorem

Proof: The Calculus of Limits Conditions are

1. Piecewise Continuity of $f(x)$, and $f'(x)$ in any bounded interval.

2. convergence of $\int_{x=-\infty}^{x=\infty} |f(x)|\,dx$

3. At a discontinuity point, $f(x)$ is replaced by $\frac{1}{2}\big(f(x+0)+f(x-0)\big)$.

It is clear from 5.1 that even an infinitely differentiable $f(x)$ will not resolve the singularity of the Delta Function.

In particular, the Calculus of Limits Conditions are not sufficient for the Fourier Integral Theorem.

In Infinitesimal Calculus, the Fourier Integral Theorem holds for any Hyper-Real function

5.3 Fourier Integral Theorem for hyper-real $f(x)$

If $f(x)$ is hyper-real function,

Then, the Fourier Integral Theorem holds.

$$f(x)=\frac{1}{2\pi}\int_{k=-\infty}^{k=\infty}\left(\int_{\xi=-\infty}^{\xi=\infty} f(\xi)e^{-ik\xi}d\xi\right)e^{ikx}dk$$

Proof: In Infinitesimal Calculus, the Integration Sum

$$\int_{\xi=-\infty}^{\xi=\infty} f(\xi)\delta(x-\xi)d\xi$$

yields $f(x)$. That is,

$$f(x)=\int_{\xi=-\infty}^{\xi=\infty} f(\xi)\delta(x-\xi)d\xi\,.$$

Since Delta is the Inverse Fourier Transform of the function 1,

$\delta(x-\xi)$ equals the Integration Sum

$$\frac{1}{2\pi}\int_{k=-\infty}^{k=\infty} e^{ik(x-\xi)}dk\,,$$

which vanishes at any $x \neq \xi$, and equals $\frac{1}{dx}$ at $x = \xi$.

Substituting in the Integration Sum for $f(x)$,

$$f(x) = \int_{\xi=-\infty}^{\xi=\infty} f(\xi)\left(\frac{1}{2\pi}\int_{k=-\infty}^{k=\infty} e^{ik(x-\xi)}dk\right)d\xi\,.$$

By changing the Summation order,

$$f(x) = \frac{1}{2\pi}\int_{k=-\infty}^{k=\infty}\left(\int_{\xi=-\infty}^{\xi=\infty} f(\xi)e^{-ik\xi}d\xi\right)e^{ikx}dk\,.\square$$

Then, the Fourier transform of $f(x)$

$$\int_{x=-\infty}^{x=\infty} f(x)e^{-i\alpha x}dx\,,$$

converges to a hyper-real function $F(\alpha)$, some of its values may be infinite hyper-reals, like the Delta Function.

And the Inverse Fourier Transform of $F(\alpha)$

$$\frac{1}{2\pi}\int_{\alpha=-\infty}^{\alpha=\infty} F(\alpha)e^{-i\alpha x}d\alpha$$

converges to the hyper-real function $f(x)$.

5.4 *If* $f(x)$ *is hyper-real function,*

Then,

- *the hyper-real integral* $\int_{x=-\infty}^{x=\infty} f(x)e^{-i\alpha x}dx$ *converges to* $F(\alpha)$
- *the hyper-real integral* $\frac{1}{2\pi}\int_{\alpha=-\infty}^{\alpha=\infty} F(\alpha)e^{i\alpha x}d\alpha$ *converges to* $f(x)$

The value of the Hyper-real Fourier Integral Theorem is demonstrated by the following examples:

5.5 *The Hyper-real Delta* $\delta(x)$ *is not defined in the Calculus of Limits, and is not Piecewise Continuous.* $\delta'(x)$ *is not defined, and is not Piecewise Continuous in any bounded interval.*

But $\delta(x)$ *satisfies the Hyper-real Fourier integral Theorem*

$$\delta(x) = \frac{1}{2\pi}\int_{k=-\infty}^{k=\infty}\left(\int_{\xi=-\infty}^{\xi=\infty} \delta(\xi)e^{-ik\xi}d\xi\right)e^{ikx}dk$$

Proof: By the sifting property of the Hyper-real Delta

$$\int_{\xi=-\infty}^{\xi=\infty} \delta(\xi)e^{-ik\xi}d\xi = 1\,.$$

Substituting this representation of 1 into $\delta(x) = \dfrac{1}{2\pi}\displaystyle\int_{k=-\infty}^{k=\infty} e^{ikx}dk\,,$

$$\delta(x) = \frac{1}{2\pi}\int_{k=-\infty}^{k=\infty}\left(\int_{\xi=-\infty}^{\xi=\infty} \delta(\xi)e^{-ik\xi}d\xi\right)e^{ikx}dk\,.\square$$

5.6 *The Hyper-real* $f(x) \equiv 1$*, does not satisfy* $\displaystyle\int_{x=-\infty}^{x=\infty} \left|f(x)\right|dx < \infty\,.$

But satisfies the Hyper-real Fourier integral Theorem

$$1 = \frac{1}{2\pi}\int_{k=-\infty}^{k=\infty}\left(\int_{\xi=-\infty}^{\xi=\infty} e^{-ik\xi}d\xi\right)e^{ikx}dk$$

Proof:

$$\frac{1}{2\pi}\int_{k=-\infty}^{k=\infty}\left(\int_{\xi=-\infty}^{\xi=\infty} e^{-ik\xi}d\xi\right)e^{ikx}dk = \int_{\xi=-\infty}^{\xi=\infty} \underbrace{\frac{1}{2\pi}\int_{k=-\infty}^{k=\infty} e^{ik(x-\xi)}dk}_{\delta(x-\xi)}d\xi = 1\,.\square$$

6.

Cosine representation of $\delta(x)$ and the Fourier Integral Transform

6.1 Cosine representation of Delta

$$\delta(x) = \frac{1}{\pi}\int_{\omega=0}^{\omega=\infty} \cos(\omega x)d\omega$$

$$= 2\int_{\nu=0}^{\nu=\infty} \cos(2\pi\nu x)d\nu$$

Proof: $\delta(x) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x}d\omega$

$$= \frac{1}{2\pi}\left\{\int_{\omega=-\infty}^{\omega=0} e^{i\omega x}d\omega + \int_{\omega=0}^{\omega=\infty} e^{i\omega x}d\omega\right\}$$

$$= \frac{1}{2\pi}\left\{\int_{\omega=0}^{\omega=\infty} e^{-i\omega x}d\omega + \int_{\omega=0}^{\omega=\infty} e^{i\omega x}d\omega\right\}$$

$$= \frac{1}{\pi}\int_{\omega=0}^{\omega=\infty} \cos(\omega x)d\omega$$

6.2 Cosine representation of the Fourier Integral Theorem

$$f(x) = \frac{1}{\pi} \int_{\omega=0}^{\omega=\infty} \int_{\xi=-\infty}^{\xi=\infty} f(\xi) \cos \omega(x-\xi) d\xi d\omega$$

$$= 2 \int_{\nu=0}^{\nu=\infty} \int_{\xi=-\infty}^{\xi=\infty} f(\xi) \cos 2\pi\nu(x-\xi) d\xi d\nu \,, \qquad \omega = 2\pi\nu$$

Proof: $f(x) = \int_{\xi=-\infty}^{\xi=\infty} f(\xi)\delta(x-\xi)d\xi$

$$= \int_{\xi=-\infty}^{\xi=\infty} f(\xi) \left\{ \frac{1}{\pi} \int_{\omega=0}^{\omega=\infty} \cos \omega(x-\xi) d\omega \right\} d\xi$$

$$= \frac{1}{\pi} \int_{\omega=0}^{\omega=\infty} \int_{\xi=-\infty}^{\xi=\infty} f(\xi) \cos \omega(x-\xi) d\xi d\omega \,.\square$$

7.

$\delta(x,y)$ and the Fourier Transform

7.1 2-Dimesional Delta Function

$$\delta\left(x,y\right)=\left(\frac{1}{2\pi}\int_{\omega_x=-\infty}^{\omega_x=\infty}e^{i\omega_x x}d\omega_x\right)\left(\frac{1}{2\pi}\int_{\omega_y=-\infty}^{\omega_y=\infty}e^{i\omega_y y}d\omega_y\right)$$

$$=\left(\int_{\nu_x=-\infty}^{\nu_x=\infty}e^{2\pi i\nu_x x}d\nu_x\right)\left(\int_{\nu_y=-\infty}^{\nu_y=\infty}e^{2\pi i\nu_y y}d\nu_y\right),\quad \begin{aligned}\omega_x &= 2\pi\nu_x\\ \omega_y &= 2\pi\nu_y\end{aligned}$$

7.2 2-Dimesional Fourier Transform

$$\mathcal{F}\left\{f(x,y)\right\}=\int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty}f(x,y)e^{-i\omega_x x-i\omega_y y}dxdy$$

$$=\int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty}f(x,y)e^{-2\pi i(\nu_x x+\nu_y y)}dxdy\,,\quad \begin{aligned}\omega_x &= 2\pi\nu_x\\ \omega_y &= 2\pi\nu_y\end{aligned}$$

7.3 2-Dimesional Inverse Fourier Transform

$$\mathcal{F}^{-1}\left\{F(\omega_x,\omega_y)\right\}=\frac{1}{(2\pi)^2}\int_{\omega_y=-\infty}^{\omega_y=\infty}\int_{\omega_x=-\infty}^{\omega_x=\infty}F(\omega_x,\omega_y)e^{i(\omega_x x+\omega_y y)}d\omega_x d\omega_y$$

$$= \int_{\nu_y=-\infty}^{\nu_y=\infty} \int_{\nu_x=-\infty}^{\nu_x=\infty} F(2\pi\nu_x, 2\pi\nu_y) e^{2\pi i(\nu_x x+\nu_y y)} d\nu_x d\nu_y \,, \quad \begin{aligned} \omega_x &= 2\pi\nu_x \\ \omega_y &= 2\pi\nu_y \end{aligned}$$

7.4 2-Dimesional Fourier Integral Theorem

$$f(x,y) = \frac{1}{(2\pi)^2} \int_{\omega_y=-\infty}^{\omega_y=\infty} \int_{\omega_x=-\infty}^{\omega_x=\infty} \left(\int_{\eta=-\infty}^{\eta=\infty} \int_{\xi=-\infty}^{\xi=\infty} f(\xi,\eta) e^{-i\omega_x\xi - i\omega_y\eta} d\xi d\eta \right) e^{i(\omega_x x+\omega_y y)} d\omega_x d\omega_y$$

$$= \int_{\eta=-\infty}^{\eta=\infty} \int_{\xi=-\infty}^{\xi=\infty} f(\xi,\eta) \left(\frac{1}{2\pi} \int_{\omega_x=-\infty}^{\omega_x=\infty} e^{i\omega_x(x-\xi)} d\omega_x \right) d\xi \left(\frac{1}{2\pi} \int_{\omega_y=-\infty}^{\omega_y=\infty} e^{i\omega_y(y-\eta)} d\omega_y \right) d\eta$$

$$= \int_{\eta=-\infty}^{\eta=\infty} \int_{\xi=-\infty}^{\xi=\infty} f(\xi,\eta) \left(\int_{\omega_x=-\infty}^{\omega_x=\infty} e^{2\pi i\nu_x(x-\xi)} d\nu_x \right) d\xi \left(\int_{\omega_y=-\infty}^{\omega_y=\infty} e^{2\pi i\nu_y(y-\eta)} d\nu_y \right) d\eta \,,$$

$$\omega_x = 2\pi\nu_x$$
$$\omega_y = 2\pi\nu_y$$

8.

$\delta(x,y,z)$ and the Fourier Transform

8.1 3-Dimesional Delta Function

$$\delta\left(x,y,z\right)=\left(\frac{1}{2\pi}\int\limits_{\omega_x=-\infty}^{\omega_x=\infty}e^{i\omega_x x}d\omega_x\right)\left(\frac{1}{2\pi}\int\limits_{\omega_y=-\infty}^{\omega_y=\infty}e^{i\omega_y y}d\omega_y\right)\left(\frac{1}{2\pi}\int\limits_{\omega_z=-\infty}^{\omega_z=\infty}e^{i\omega_z z}d\omega_z\right)$$

$$=\left(\int\limits_{\nu_x=-\infty}^{\nu_x=\infty}e^{2\pi i\nu_x x}d\nu_x\right)\left(\int\limits_{\nu_y=-\infty}^{\nu_y=\infty}e^{2\pi i\nu_y y}d\nu_y\right)\left(\int\limits_{\nu_z=-\infty}^{\nu_z=\infty}e^{2\pi i\nu_z z}d\nu_z\right),\quad \begin{aligned}\omega_x&=2\pi\nu_x\\ \omega_y&=2\pi\nu_y\\ \omega_z&=2\pi\nu_z\end{aligned}$$

8.2 3-Dimesional Fourier Transform

$$\mathcal{F}\left\{f(x,y,z)\right\}=\int\limits_{z=-\infty}^{z=\infty}\int\limits_{y=-\infty}^{y=\infty}\int\limits_{x=-\infty}^{x=\infty}f(x,y,z)e^{-i\omega_x x-i\omega_y y-i\omega_z z}dxdydz$$

$$=\int\limits_{z=-\infty}^{z=\infty}\int\limits_{y=-\infty}^{y=\infty}\int\limits_{x=-\infty}^{x=\infty}f(x,y,z)e^{-2\pi i(\nu_x x+\nu_y y+\nu_z z)}dxdydz\,,\quad \begin{aligned}\omega_x&=2\pi\nu_x\\ \omega_y&=2\pi\nu_y\\ \omega_z&=2\pi\nu_z\end{aligned}$$

8.3 3-Dimesional Inverse Fourier Transform

$$\mathcal{F}^{-1}\left\{F(\omega_x,\omega_y,\omega_z)\right\} = \frac{1}{(2\pi)^3}\int\limits_{\omega_z=-\infty}^{\omega_z=\infty}\int\limits_{\omega_y=-\infty}^{\omega_y=\infty}\int\limits_{\omega_x=-\infty}^{\omega_x=\infty} F(\omega_x,\omega_y,\omega_z)e^{i(\omega_x x+\omega_y y+\omega_z z)}d\omega_x d\omega_y d\omega_z$$

$$= \int\limits_{\nu_z=-\infty}^{\nu_z=\infty}\int\limits_{\nu_y=-\infty}^{\nu_y=\infty}\int\limits_{\nu_x=-\infty}^{\nu_x=\infty} F(2\pi\nu_x,2\pi\nu_y,2\pi\nu_z)e^{2\pi i(\nu_x x+\nu_y y+\nu_z z)}d\nu_x d\nu_y d\nu_z\,, \quad \begin{aligned}\omega_x &= 2\pi\nu_x\\ \omega_y &= 2\pi\nu_y\\ \omega_z &= 2\pi\nu_z\end{aligned}$$

8.4 3-Dimesional Fourier Integral Theorem

$$f(x,y,z) = \frac{1}{(2\pi)^3}\int\limits_{\omega_z=-\infty}^{\omega_z=\infty}\int\limits_{\omega_y=-\infty}^{\omega_y=\infty}\int\limits_{\omega_x=-\infty}^{\omega_x=\infty}\left(\int\limits_{\zeta=-\infty}^{\zeta=\infty}\int\limits_{\eta=-\infty}^{\eta=\infty}\int\limits_{\xi=-\infty}^{\xi=\infty} f(\xi,\eta,\zeta)e^{-i\omega_x\xi-i\omega_y\eta-i\omega_z\zeta}d\xi d\eta d\zeta\right)\times$$

$$\times e^{i(\omega_x x+i\omega_y y+i\omega_z z)}d\omega_x d\omega_y d\omega_z$$

$$= \int\limits_{\zeta=-\infty}^{\zeta=\infty}\int\limits_{\eta=-\infty}^{\eta=\infty}\int\limits_{\xi=-\infty}^{\xi=\infty} f(\xi,\eta,\zeta)\left(\frac{1}{2\pi}\int\limits_{\omega_x=-\infty}^{\omega_x=\infty} e^{i\omega_x(x-\xi)}d\omega_x\right)d\xi\left(\frac{1}{2\pi}\int\limits_{\omega_y=-\infty}^{\omega_y=\infty} e^{i\omega_y(y-\eta)}d\omega_y\right)d\eta\,\times$$

$$\times\left(\frac{1}{2\pi}\int\limits_{\omega_z=-\infty}^{\omega_z=\infty} e^{i\omega_z(z-\zeta)}d\omega_z\right)d\zeta$$

$$= \int\limits_{\zeta=-\infty}^{\zeta=\infty}\int\limits_{\eta=-\infty}^{\eta=\infty}\int\limits_{\xi=-\infty}^{\xi=\infty} f(\xi,\eta,\zeta)\left(\int\limits_{\nu_x=-\infty}^{\nu_x=\infty} e^{2\pi i\nu_x(x-\xi)}d\nu_x\right)d\xi\left(\int\limits_{\nu_y=-\infty}^{\nu_y=\infty} e^{2\pi i\nu_y(y-\eta)}d\nu_y\right)d\eta\,\times$$

$$\times\left(\int\limits_{\nu_z=-\infty}^{\nu_z=\infty} e^{2\pi i\nu_z(z-\zeta)}d\nu_z\right)d\zeta\,, \quad \begin{aligned}\omega_x &= 2\pi\nu_x\\ \omega_y &= 2\pi\nu_y\\ \omega_z &= 2\pi\nu_z\end{aligned}$$

9.

Delta Sequence $\delta_n(x) = \frac{n}{\pi}\left(\frac{\sin(nx)}{nx}\right)^2$

We show that the Hyper-real Delta Function is represented by the Delta Sequence

$$\delta_n(x) = \frac{n}{\pi}\left(\frac{\sin(nx)}{nx}\right)^2.$$

The n[th] component of the Hyper-real Delta is $\frac{n}{\pi}\left(\frac{\sin(nx)}{nx}\right)^2$. That is,

$$\delta(x) = \left\langle \frac{n}{\pi}\left(\frac{\sin(nx)}{nx}\right)^2 \right\rangle.$$

9.1 Each $\delta_n(x) = \frac{n}{\pi}\left(\frac{\sin(nx)}{nx}\right)^2$

- has the sifting property $\int\limits_{x=-\infty}^{x=\infty} \delta_n(x)dx = 1$,
- is a continuous Hyper-real function,
- peaks at $x = 0$ to $\delta_n(0) = \frac{1}{\pi}n$.

Proof:

$$\int_{x=-\infty}^{x=\infty} \delta_n(x)dx = \frac{n}{\pi} \int_{x=-\infty}^{x=\infty} \frac{\sin^2(nx)}{n^2x^2}dx = \frac{1}{\pi n} 2 \underbrace{\int_{x=0}^{x=\infty} \frac{\sin^2(nx)}{x^2}dx}_{\frac{1}{2}\pi n, \text{by } [\text{Spiegel2}]} = 1 . \square$$

To see that $\delta_n(0) = \frac{1}{\pi} n$, we use the infinitesimal $\left\langle \frac{1}{n^2} \right\rangle$. Then,

$$\delta_n(\tfrac{1}{n^2}) = \frac{n}{\pi} \left(\frac{\sin(\frac{1}{n})}{\frac{1}{n}} \right)^2 .$$

Since for any infinitesimal o, $\dfrac{\sin(o)}{o} = 1$, we have

$$\delta_n(\tfrac{1}{n^2}) = \frac{n}{\pi},$$

and $\delta_n(0) = \frac{1}{\pi} n . \square$

Therefore,

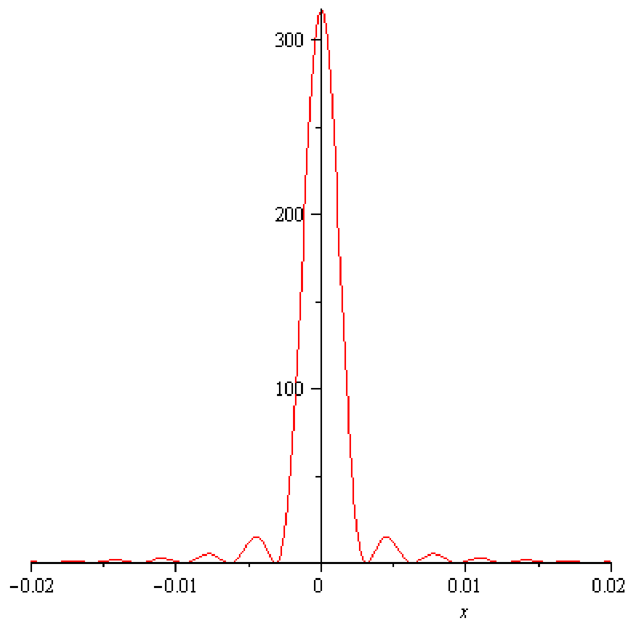
9.2 *The sequence represents the Hyper-real Delta*

$$\delta(x) = \left\langle \frac{\sin^2(x)}{\pi x^2}, \frac{\sin^2(2x)}{2\pi x^2}, \frac{\sin^2(3x)}{3\pi x^2}, \ldots \right\rangle .$$

9.3

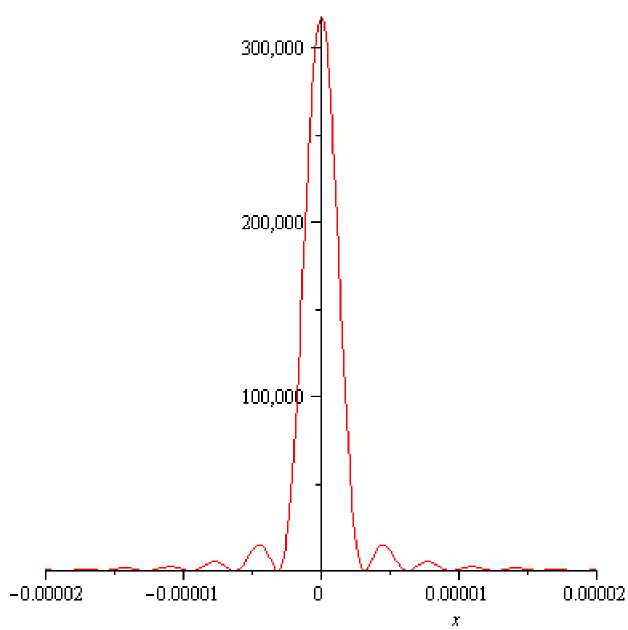
$$plot\left(\frac{1}{1000\,\pi} \frac{\sin^2(1000\,x)}{x^2},\ x = -\frac{2}{100} .. \frac{2}{100} \right)$$

plots in Maple the 1000th component, that peaks at $\dfrac{1000}{\pi} \approx 318$.



9.4 $$plot\left(\frac{1}{10^6\pi}\frac{\sin^2(10^6 x)}{x^2},\ x=-\frac{2}{10^5}..\frac{2}{10^5}\right)$$

plots in Maple the 10^6 component, that peaks at $\dfrac{10^6}{\pi} \approx 318,310$.



To show the relation between the infinitesimal dx, and this Hyper-real $\delta(x)$, we note

9.5 ***<u>If</u>*** dx *is given by* $i_n = \frac{1}{n}$,

<u>Then</u> This Hyper-real $\delta(x)$

★ peaks to $\frac{1}{\pi dx}$.

★ may be written symbolically by $\delta(x) = \frac{1}{\pi dx}\left(\frac{\sin(\frac{x}{dx})}{\frac{x}{dx}}\right)^2$

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