Periodic Delta Function and Fourier Expansion in Bessel Functions

H. Vic Dannon
vic0@comcast.net
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Abstract Let $f(x)$ be integrable on $[0,1]$. The zeros of the Bessel Function $J_0(x)$.

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < ...$$

define the orthogonal sequence of functions

$$J_0(\lambda_1 x),\ J_0(\lambda_2 x),\ J_0(\lambda_3 x),...$$

The Bessel Series associated with $f(x)$ is

$$a_1 J_0(\lambda_1 x) + a_2 J_0(\lambda_2 x) + a_3 J_0(\lambda_3 x) + ...$$

where

$$a_n = \frac{2}{[J_0\,'(\lambda_n)]^2} \int_{\xi=0}^{\xi=1} \xi f(\xi) J_0(\lambda_n \xi) d\xi = \frac{2}{[J_1(\lambda_n)]^2} \int_{\xi=0}^{\xi=1} \xi f(\xi) J_0(\lambda_n \xi) d\xi$$

are the Bessel coefficients.

The Bessel Series Theorem supplies the conditions under which the Bessel Series associated with $f(x)$ equals $f(x)$.

It is believed to hold in the Calculus of Limits under Hobson's Conditions. In fact,

The Theorem cannot be proved in the Calculus of Limits under any conditions,

because the summation of the Bessel Series requires integration over the singular Bessel Kernel.

Plots of partial sums of the Bessel Series speak volumes about the sensibility of the claims to have infinity bound by epsilon.

In Infinitesimal Calculus, the Bessel Kernel

$$2\xi\left\{\frac{1}{J_1^2(\lambda_1)}J_0(\lambda_1\xi)J_0(\lambda_1 x)+\frac{1}{J_1^2(\lambda_2)}J_0(\lambda_2\xi)J_0(\lambda_2 x)+...\right\}$$

is the Periodic Delta,

$$\delta_{Periodic}(\xi-x)=\sqrt{\tfrac{\xi}{x}}\left\{...+\delta(\xi-x+2)+\delta(\xi-x)+\delta(\xi-x)+...\right\}$$

$$=\sqrt{\tfrac{\xi}{x}}\left\{\tfrac{1}{2}+\cos\pi(\xi-x)+\cos 2\pi(\xi-x)+...\right\}$$

$$=\sqrt{\tfrac{\xi}{x}}\tfrac{1}{2}\left\{...+e^{-2i\pi(\xi-x)}+e^{-i\pi(\xi-x)}+1+e^{i\pi(\xi-x)}+e^{2i\pi(\xi-x)}+...\right\}.$$

The Periodic Delta equals its Bessel Series, and the Bessel Series associated with any hyper-real integrable $f(x)$, equals $f(x)$

For Bessel's Function $J_\nu(x)$, where ν may be a fraction, the zeros

$$0<\lambda_{\nu 1}<\lambda_{\nu 2}<\lambda_{\nu 3}<...,$$

define the orthogonal sequence of functions

$$J_\nu(\lambda_{\nu 1}x),\ J_\nu(\lambda_{\nu 2}x),\ J_\nu(\lambda_{\nu 3}x),...$$

The ν-Bessel Series associated with $f(x)$ is

$$a_1 J_\nu(\lambda_{\nu 1} x) + a_2 J_\nu(\lambda_{\nu 2} x) + a_3 J_\nu(\lambda_{\nu 3} x) + \ldots.$$

where

$$a_k = \frac{2}{[J_\nu{}'(\lambda_{\nu k})]^2} \int_{\xi=0}^{\xi=1} \xi f(\xi) J_\nu(\lambda_{\nu k}\xi) d\xi = \frac{2}{[J_{\nu+1}(\lambda_{\nu k})]^2} \int_{\xi=0}^{\xi=1} \xi f(\xi) J_\nu(\lambda_{\nu k}\xi) d\xi$$

Then, the ν-Bessel Kernel

$$2\xi \left\{ \frac{J_\nu(\lambda_{\nu 1}\xi) J_\nu(\lambda_{\nu 1} x)}{J_{\nu+1}^2(\lambda_{\nu 1})} + \frac{J_\nu(\lambda_{\nu 2}\xi) J_\nu(\lambda_{\nu 2} x)}{J_{\nu+1}^2(\lambda_{\nu 2})} + \ldots \right\}$$

is the Periodic Delta,

$$\delta_{\mathrm{P}eriodic}(\xi - x) = \sqrt{\tfrac{\xi}{x}} \left\{ \tfrac{1}{2} + \cos \pi(\xi - x) + \cos 2\pi(\xi - x) + \ldots \right\}.$$

and the ν-Bessel Series associated with any hyper-real integrable $f(x)$, equals $f(x)$.

Keywords: Infinitesimal, Infinite-Hyper-Real, Hyper-Real, infinite Hyper-real, Infinitesimal Calculus, Delta Function, Periodic Delta, Bessel Functions, Bessel Coefficients, Fourier-Bessel Series, Bessel Kernel, Fourier-Bessel Expansion.

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Contents

The Origin of the Bessel Series Theorem

For Bessel's Function $J_\nu(x)$, where ν may be a fraction, the zeros

$$0 < \lambda_{\nu 1} < \lambda_{\nu 2} < \lambda_{\nu 3} < ...,$$

define the orthogonal sequence of functions

$$J_\nu(\lambda_{\nu 1} x),\ J_\nu(\lambda_{\nu 2} x),\ J_\nu(\lambda_{\nu 3} x),...$$

on $x \geq 0$, so that

$$\int_{x=0}^{x=1} x J_\nu(\lambda_m x) J_\nu(\lambda_k x) dx = \tfrac{1}{2} J_{\nu+1}^2(\lambda_k)\delta_{mk}.$$

The Functions may be generated by expanding

$$\sum_{n=-\infty}^{n=\infty} J_n(x)\alpha^n = e^{\frac{1}{2}x(\alpha - \frac{1}{\alpha})}$$

$$= 1 + \tfrac{1}{2}x(\alpha - \tfrac{1}{\alpha}) + \tfrac{1}{2}\Big[\tfrac{1}{2}x(\alpha - \tfrac{1}{\alpha})\Big]^2 + \tfrac{1}{2\cdot 3}\Big[\tfrac{1}{2}x(\alpha - \tfrac{1}{\alpha})\Big]^3 + \tfrac{1}{2\cdot 3\cdot 4}\Big[\tfrac{1}{2}x(\alpha - \tfrac{1}{\alpha})\Big]^4 ...,$$

the coefficient of α^0 is

$$J_0(x) = 1 - \tfrac{1}{2^2}x^2 + \tfrac{1}{2^6}x^4 - ...,$$

the coefficient of α^1 is

$$J_1(x),$$

the coefficient of α^{-1} is

$$J_{-1}(x),$$

....

0.1 Euler (1766)

The problem of the vibrations of a stretched membrane, led Euler to the derivation of the Bessel Equation, and to an infinite series solution which is a Bessel Function.

0.2 Fourier (1822)

Fourier assumed the Bessel Series Theorem in the Theory of Heat conduction, and from

$$f(x) = a_1 J_0(\lambda_1 x) + a_2 J_0(\lambda_2 x) + a_3 J_0(\lambda_3 x) + ...$$

derived

$$a_m = \frac{2}{[J_1(\lambda_{0m})]^2} \int_{x=0}^{x=1} x f(x) J_0(\lambda_{0m}\xi) dx$$

0.3 Bessel Equation for a particle trapped in a sphere

A subatomic particle with mass m, is trapped in a sphere of radius a, under the potential

$$V(r) = \begin{cases} 0, & 0 \le r < a \\ N, & r \ge a \end{cases},$$

where N is an infinite Hyper-real number.

De Broglie associated with the particle a wave of length

$$\lambda = \frac{h}{mv},$$

where v is the velocity of the particle, and h is Planck's constant.

The wave's frequency is

$$\nu = \frac{v}{\lambda} = \frac{v}{\frac{h}{mv}} = \frac{mv^2}{h}$$

The wave's angular frequency is

$$\omega = 2\pi\nu = 2\pi\frac{mv^2}{h}$$

In terms of the De Broglie wave, the wave's energy is a multiple of Planck's radiation energy. That is,

$$E = \varepsilon h\nu = \varepsilon\hbar\omega, \quad \hbar = \tfrac{h}{2\pi}, \ \varepsilon \text{ is the multiplier.}$$

The kinetic energy of the particle is

$$\tfrac{1}{2}mv^2 = E.$$

Hence,

$$mv = \sqrt{2mE},$$

$$\lambda = \frac{h}{\sqrt{2mE}},$$

$$v = \lambda \underbrace{\nu}_{\frac{1}{2\pi}\omega} = \frac{\hbar\omega}{\sqrt{2mE}}.$$

$$\frac{1}{v^2} = \frac{2mE}{\hbar^2\omega^2} = \frac{2m\varepsilon}{\hbar\omega}$$

Schrodinger postulated a complex valued potential

$$\Psi(r,\theta,\phi,t) = \psi(r,\theta,\phi)e^{i\omega t}$$

that satisfies the wave equation

$$\nabla^2\Psi(r,\theta,\phi,t) = \frac{1}{v^2}\partial_t^2\Psi(r,\theta,\phi,t).$$

Then,

$$0 = \nabla^2\Psi(r,\theta,\phi,t) - \frac{1}{v^2}\partial_t^2\Psi(r,\theta,\phi,t)$$

$$= \nabla^2\psi(r,\theta,\phi)e^{i\omega t} - \frac{2m\varepsilon}{\hbar\omega}\psi(r,\theta,\phi)(-\omega^2)e^{i\omega t}.$$

The Schrodinger equation for the trapped particle is

$$\nabla^2\psi(r,\theta,\phi) + \frac{2m\omega}{\hbar}\varepsilon\psi(r,\theta,\phi) = 0.$$

In spherical coordinates

$$\tfrac{1}{r^2}\partial_r(r^2\partial_r\psi) + \tfrac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta\psi) + \tfrac{1}{r^2\sin^2\theta}\partial_\phi^2\psi - \frac{2m\omega}{\hbar}\varepsilon\psi = 0.$$

Assuming that

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi),$$

the Schrodinger equation becomes

$$\tfrac{1}{R}(r^2R')' + \tfrac{1}{\Theta\sin\theta}(\sin\theta\Theta')' + \tfrac{1}{\Phi\sin^2\theta}\Phi'' - \frac{2m\omega}{\hbar}\varepsilon r^2 = 0.$$

Then,

$$\tfrac{1}{R}(r^2R')' - \frac{2m\omega}{\hbar}\varepsilon r^2 = const \equiv C_1.$$

Substituting

$$R(r) = r^l,$$

$$l(l+1) - \frac{2m\omega}{\hbar}\varepsilon r^2 = C_1.$$

Hence,

$$l(l+1) = C_1.$$

Thus, the Schrodinger Radial Equation is

$$\tfrac{1}{R}(r^2R')' + \frac{2m\omega}{\hbar}\varepsilon r^2 = l(l+1)$$

$$r^2R'' + 2rR' + \underbrace{\frac{2m\omega}{\hbar}\varepsilon r^2}_{\rho^2} R - l(l+1)R = 0,$$

Denoting $\rho = r\sqrt{\tfrac{2m\omega}{\hbar}\varepsilon}$,

$$R'(r) = \frac{dR}{d\rho}\frac{d\rho}{dr} = R'(\rho)\sqrt{\tfrac{2m\omega}{\hbar}\varepsilon},$$

$$2rR'(r) = 2\frac{\rho}{\sqrt{\tfrac{2m\omega}{\hbar}\varepsilon}}R'(\rho)\sqrt{\tfrac{2m\omega}{\hbar}\varepsilon} = 2\rho R'(\rho),$$

$$R''(r) = \frac{d}{d\rho}\left\{R'(\rho)\sqrt{\tfrac{2m\omega}{\hbar}\varepsilon}\right\}\underbrace{\frac{d\rho}{dr}}_{\sqrt{\frac{2m\omega}{\hbar}\varepsilon}} = R''(\rho)\tfrac{2m\omega}{\hbar}\varepsilon,$$

$$r^2R''(r) = \frac{\rho^2}{\tfrac{2m\omega}{\hbar}\varepsilon}R''(\rho)\tfrac{2m\omega}{\hbar}\varepsilon = \rho^2R''(\rho)$$

Then, the Radial Schrodinger equation becomes

$$\rho^2R''(\rho) + 2\rho R'(\rho) + \rho^2R - l(l+1)R = 0.$$

To obtain Bessel's Equation put

$$R(\rho) = \rho^{-\frac{1}{2}}J(\rho)$$

$$R'(\rho) = -\frac{1}{2}\rho^{-\frac{3}{2}}J + \rho^{-\frac{1}{2}}J'$$

$$2\rho R'(\rho) = -\rho^{-\frac{1}{2}}J + 2\rho^{\frac{1}{2}}J',$$

$$R''(\rho) = \frac{3}{4}\rho^{-\frac{5}{2}}J - \rho^{-\frac{3}{2}}J' + \rho^{-\frac{1}{2}}J''$$

$$\rho^2 R''(\rho) = \frac{3}{4}\rho^{-\frac{1}{2}}J - \rho^{\frac{1}{2}}J' + \rho^{\frac{3}{2}}J''$$

$$\rho^2 R - l(l+1)R = \rho^{\frac{3}{2}}J - l(l+1)\rho^{-\frac{1}{2}}J$$

Substituting in the equation,

$$\frac{3}{4}\rho^{-\frac{1}{2}}J - \rho^{\frac{1}{2}}J' + \rho^{\frac{3}{2}}J'' - \rho^{-\frac{1}{2}}J + 2\rho^{\frac{1}{2}}J' + \rho^{\frac{3}{2}}J - l(l+1)\rho^{-\frac{1}{2}}J = 0,$$

$$\frac{3}{4}J - \rho J' + \rho^2 J'' - J + 2\rho J' + \rho^2 J - l(l+1)J = 0,$$

$$\rho^2 J'' + \rho J' + [\rho^2 - l(l+1) - \tfrac{1}{4}]J = 0$$

$$\rho^2 J'' + \rho J' + [\rho^2 - (l+\tfrac{1}{2})^2]J = 0$$

The solutions are the Bessel Functions

$$J_{l+\frac{1}{2}}(\rho) = J_{l+\frac{1}{2}}(r\sqrt{\tfrac{2m\omega}{\hbar}\varepsilon})$$

Requiring the wave function to vanish on the sphere, $r = a$,

$$J_{l+\frac{1}{2}}(a\sqrt{\tfrac{2m\omega}{\hbar}\varepsilon}) = 0.$$

That is, there is a sequence ε_n so that

$$J_{l+\frac{1}{2}}(a\sqrt{\tfrac{2m\omega}{\hbar}\varepsilon_n}) = 0.$$

Thus,

$$\lambda_n = a\sqrt{\tfrac{2m\omega}{\hbar}\varepsilon_n}$$

are the zeros of $J_{l+\frac{1}{2}}(r\sqrt{\tfrac{2m\omega}{\hbar}\varepsilon})$.

Therefore, the solutions of the Bessel equation are the orthogonal Bessel Functions

$$J_{l+\frac{1}{2}}(\underbrace{r\sqrt{\tfrac{2m\omega}{\hbar}\varepsilon_n}}_{\lambda_n a}) = J_{l+\frac{1}{2}}(\lambda_n \tfrac{r}{a}).$$

The solution for $J(\rho)$ is the infinite series

$$\alpha_1 J_{l+\frac{1}{2}}(\lambda_1 \tfrac{r}{a}) + \alpha_2 J_{l+\frac{1}{2}}(\lambda_2 \tfrac{r}{a}) + \alpha_3 J_{l+\frac{1}{2}}(\lambda_3 \tfrac{r}{a}) + \ldots.$$

0.4 The Bessel Series Associated with $f(x)$

Let $f(x)$ be defined on $[0,1]$. The zeros of Bessel's Function $J_0(x)$,

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots$$

define the sequence of functions

$$J_0(\lambda_1 x),\ J_0(\lambda_2 x),\ J_0(\lambda_3 x),\ldots$$

The functions are orthogonal,

$$\int_{x=0}^{x=1} x J_0(\lambda_m x) J_0(\lambda_n x)dx = \delta_{mn}\tfrac{1}{2}[J_0{}'(\lambda_m)]^2 = \delta_{mn}\tfrac{1}{2}[J_1(\lambda_m)]^2.$$

If $f(x)$ can be expanded in the Bessel Functions,

$$f(x) = a_1 J_0(\lambda_1 x) + a_2 J_0(\lambda_2 x) + a_3 J_0(\lambda_3 x) + \ldots,$$

Then,

$$\int_{x=0}^{x=1} x f(x) J_0(\lambda_n x)dx = \int_{x=0}^{x=1} x\{a_1 J_0(\lambda_1 x) + a_2 J_0(\lambda_2 x) + \ldots\} J_0(\lambda_n x)dx$$

$$= a_1 \underbrace{\int_{x=0}^{x=1} xf(x)J_0(\lambda_1 x)J_0(\lambda_n x)dx}_{\frac{2}{[J_1^2(\lambda_n)]^2}\delta_{1n}} + a_2 \underbrace{\int_{x=0}^{x=1} xf(x)J_0(\lambda_2 x)J_0(\lambda_n x)dx}_{\frac{2}{[J_1^2(\lambda_n)]^2}\delta_{2n}} + ...$$

$$= a_n \frac{2}{[J_1^2(\lambda_n)]^2}.$$

Thus, the Bessel coefficients are

$$a_n = \frac{2}{[J_0\,'(\lambda_n)]^2} \int_{\xi=0}^{\xi=1} \xi f(\xi) J_0(\lambda_n \xi) d\xi = \frac{2}{[J_1(\lambda_n)]^2} \int_{\xi=0}^{\xi=1} \xi f(\xi) J_0(\lambda_n \xi) d\xi .$$

The Bessel Series associated with $f(x)$ is

$$a_1 J_0(\lambda_1 x) + a_2 J_0(\lambda_2 x) + a_3 J_0(\lambda_3 x) + ...$$

The Bessel Series Theorem supplies the conditions under which the Bessel Series associated with $f(x)$ equals $f(x)$.

0.5 The ν-Bessel Series Associated with $f(x)$

For the Bessel Function $J_\nu(x)$, where ν is a fraction, the zeros

$$0 < \lambda_{\nu 1} < \lambda_{\nu 2} < \lambda_{\nu 3} < ...,$$

define the sequence of functions

$$J_\nu(\lambda_{\nu 1} x),\ J_\nu(\lambda_{\nu 2} x),\ J_\nu(\lambda_{\nu 3} x), ...$$

The functions are orthogonal,

$$\int_{x=0}^{x=1} xJ_\nu(\lambda_{\nu k}x)J_\nu(\lambda_{\nu l}x)dx = \delta_{kl}\,\tfrac{1}{2}[J_\nu\,'(\lambda_{\nu k})]^2 = \delta_{kl}\,\tfrac{1}{2}[J_{\nu+1}(\lambda_{\nu k})]^2.$$

The Bessel Series associated with $f(x)$ is

$$a_1 J_\nu(\lambda_{\nu 1}x) + a_2 J_\nu(\lambda_{\nu 2}x) + a_3 J_\nu(\lambda_{\nu 3}x) + \ldots.$$

where

$$a_k = \frac{2}{[J_\nu\,'(\lambda_{\nu k})]^2}\int_{\xi=0}^{\xi=1} \xi f(\xi)J_\nu(\lambda_{\nu k}\xi)d\xi = \frac{2}{[J_{\nu+1}(\lambda_{\nu k})]^2}\int_{\xi=0}^{\xi=1} \xi f(\xi)J_\nu(\lambda_{\nu k}\xi)d\xi$$

The Bessel Series Theorem supplies the conditions under which the Bessel Series associated with $f(x)$ equals $f(x)$.

1.

Divergence of the Bessel Kernel in the Calculus of Limits

Calculus of Limits Conditions for the Bessel Series to equal its function reflect the belief that a smooth enough function equals its Bessel Series.

In fact, in the Calculus of Limits, no smoothness of the function guarantees even the convergence of the Bessel Series.

In the Calculus of Limits, the Bessel Series is the limit of the sequence of Partial Sums

$$\mathcal{B}_{essel}\mathcal{S}_n\left\{f(x)\right\} = a_1 J_0(\lambda_1 x) + a_2 J_0(\lambda_2 x) + ... + a_n J_0(\lambda_n x)$$

$$= \left(\frac{2}{J_1^2(\lambda_1)} \int_{\xi=0}^{\xi=1} \xi f(\xi) J_0(\lambda_1 \xi) d\xi \right) J_0(\lambda_1 x) + .. + \left(\frac{2}{J_1^2(\lambda_n)} \int_{\xi=0}^{\xi=1} \xi f(\xi) J_0(\lambda_n \xi) d\xi \right) J_0(\lambda_n x)$$

$$= \int_{\xi=0}^{\xi=1} f(\xi) 2\xi \left\{ \frac{J_0(\lambda_1\xi)J_0(\lambda_1 x)}{J_1^2(\lambda_1)} + \frac{J_0(\lambda_2\xi)J_0(\lambda_2 x)}{J_1^2(\lambda_2)} + .. + \frac{J_0(\lambda_n\xi)J_0(\lambda_n x)}{J_1^2(\lambda_n)} \right\} d\xi .$$

As $n \to \infty$, the Bessel Sequence

$$2\xi \left\{ \frac{J_0(\lambda_1\xi)J_0(\lambda_1 x)}{J_1^2(\lambda_1)} + \frac{J_0(\lambda_2\xi)J_0(\lambda_2 x)}{J_1^2(\lambda_2)} + .. + \frac{J_0(\lambda_n\xi)J_0(\lambda_n x)}{J_1^2(\lambda_n)} \right\}$$

becomes the Bessel Kernel,

$$2\xi\left\{\frac{2J_0(\lambda_1\xi)J_0(\lambda_1 x)}{J_1^2(\lambda_1)}+\frac{2J_0(\lambda_2\xi)J_0(\lambda_2 x)}{J_1^2(\lambda_2)}+\frac{2J_0(\lambda_3\xi)J_0(\lambda_3 x)}{J_1^2(\lambda_3)}+...\right\},$$

To see that it diverges for $\xi = x$, we apply the Schlafli Asymptotic Summation Formula [Watson, p. 585]: For large n,

$$2\xi\left\{\frac{J_0(\lambda_1\xi)J_0(\lambda_1 x)}{J_1^2(\lambda_1)}+\frac{J_0(\lambda_2\xi)J_0(\lambda_2 x)}{J_1^2(\lambda_2)}+..+\frac{J_0(\lambda_n\xi)J_0(\lambda_n x)}{J_1^2(\lambda_n)}\right\}=$$

$$\sim\sqrt{\frac{\xi}{x}}\left[\frac{\sin(n+\frac{1}{2})\pi(\xi-x)}{2\sin\frac{1}{2}\pi(\xi-x)}-\frac{\sin(n+\frac{1}{2})\pi(\xi+x)}{2\sin\frac{1}{2}\pi(\xi+x)}\right].$$

Since

$$\frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}=\tfrac{1}{2}+\cos\theta+\cos 2\theta+...\cos n\theta,$$

$$2\xi\left\{\frac{J_0(\lambda_1\xi)J_0(\lambda_1 x)}{J_1^2(\lambda_1)}+\frac{J_0(\lambda_2\xi)J_0(\lambda_2 x)}{J_1^2(\lambda_2)}+..+\frac{J_0(\lambda_n\xi)J_0(\lambda_n x)}{J_1^2(\lambda_n)}\right\}\sim$$

$$\sim\sqrt{\tfrac{\xi}{x}}\left\{\tfrac{1}{2}+\cos\pi(\xi-x)+\cos 2\pi(\xi-x)+...+\cos n\pi(\xi-x)\right\}$$

$$-\sqrt{\tfrac{\xi}{x}}\left\{\tfrac{1}{2}+\cos\pi(\xi+x)+\cos 2\pi(\xi+x)+...+\cos n\pi(\xi+x)\right\}$$

By [Hardy, p.2, #(1.2.3)],

$$\tfrac{1}{2}+\cos\theta+\cos 2\theta+\cos 3\theta+...=0,\ \text{for any } \theta\neq 0$$

Therefore, for n large enough, and $\xi\neq x\neq 0$,

$$2\xi\left\{\frac{J_0(\lambda_1\xi)J_0(\lambda_1 x)}{J_1^2(\lambda_1)}+\frac{J_0(\lambda_2\xi)J_0(\lambda_2 x)}{J_1^2(\lambda_2)}+..+\frac{J_0(\lambda_n\xi)J_0(\lambda_n x)}{J_1^2(\lambda_n)}\right\}\sim$$

$$\sim \sqrt{\tfrac{\xi}{x}}\left\{\tfrac{1}{2} + \cos\pi(\xi - x) + \cos 2\pi(\xi - x) + ... + \cos n\pi(\xi - x)\right\}$$

For $\xi \to x$,

$$2x\left\{\frac{J_0^2(\lambda_1 x)}{J_1^2(\lambda_1)} + \frac{J_0^2(\lambda_2 x)}{J_1^2(\lambda_2)} + .. + \frac{J_0^2(\lambda_n x)}{J_1^2(\lambda_n)}\right\} \sim [\tfrac{1}{2} + \underbrace{\cos 0 + ... \cos 0}_{n}].$$

Hence,

$$2x\left\{\frac{J_0^2(\lambda_1 x)}{J_1^2(\lambda_1)} + \frac{J_0^2(\lambda_2 x)}{J_1^2(\lambda_2)} + \frac{J_0^2(\lambda_3 x)}{J_1^2(\lambda_3)} + ...\right\} = \tfrac{1}{2} + \lim_{n\to\infty} n\,,$$

and the Bessel Kernel diverges to ∞ at any $\xi = x \neq 0$.

Therefore, while the partial sums of the Bessel Series exist, their limit does not. That is, due to the singularity at $\xi = x$, the Bessel Series does not converge in the Calculus of Limits.

Avoiding the singularity at $\xi = x$, by using the Cauchy Principal Value of the integral does not recover the Theorem, because at any $\xi \neq x \neq 0$, the Bessel Kernel vanishes, and the integral will be identically zero, for any function $f(x)$: For any $\xi \neq x \neq 0$,

$$2\xi\left\{\frac{J_0(\lambda_1\xi)J_0(\lambda_1 x)}{J_1^2(\lambda_1)} + \frac{J_0(\lambda_2\xi)J_0(\lambda_2 x)}{J_1^2(\lambda_2)} + \frac{J_0(\lambda_3\xi)J_0(\lambda_3 x)}{J_1^2(\lambda_3)} + ...\right\} =$$

$$= \sqrt{\tfrac{\xi}{x}}\underbrace{\left\{\tfrac{1}{2} + \cos\pi(\xi - x) + \cos 2\pi(\xi - x) + \cos 3\pi(\xi - x) + ...\right\}}_{0}$$

and the Bessel Kernel vanishes.

Plots of the Bessel Sequence confirm that

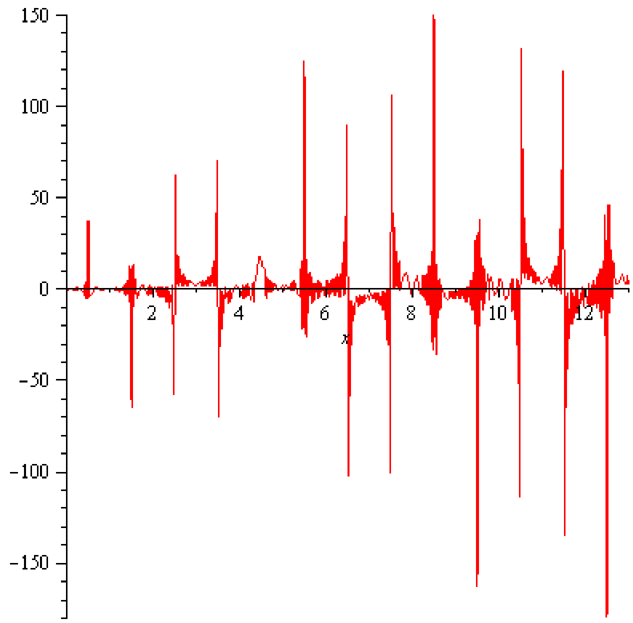
In the Calculus of Limits,

the Bessel Kernel is either singular or zero

1.2 Plots of $2\xi\left\{\dfrac{J_0(\lambda_1\frac{1}{2})J_0(\lambda_1\xi)}{J_1^2(\lambda_1)}+..+\dfrac{J_0(\lambda_k\frac{1}{2})J_0(\lambda_k\xi)}{J_1^2(\lambda_k)}\right\}$

In Maple,

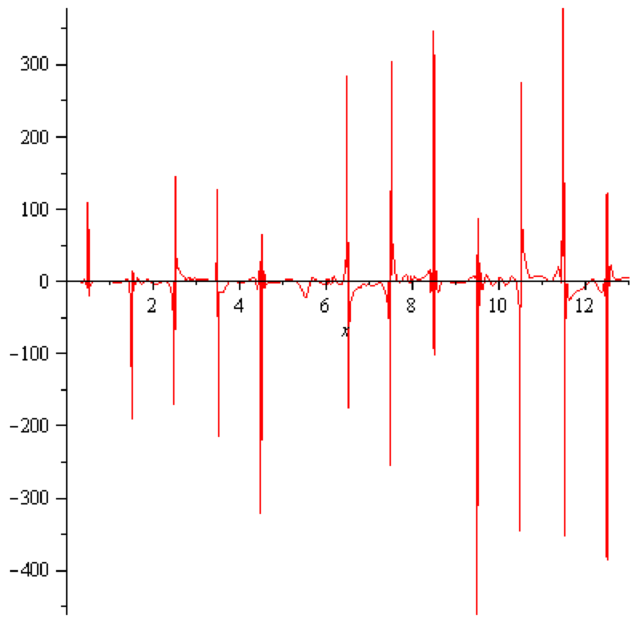
$$plot\left(\sum_{i=1}^{37}2x\frac{\mathrm{BesselJ}(0,\frac{\mathrm{BesselJZeros}(0,i)}{2})}{\left[\mathrm{BesselJ}(1,\mathrm{BesselJZeros}(0,i))\right]^2}\cdot\mathrm{BesselJ}(0,\mathrm{BesselJZeros}(0,i)*x),x=0..13\right)$$



The pulses narrow, and the vanishing between them increases, with more terms:

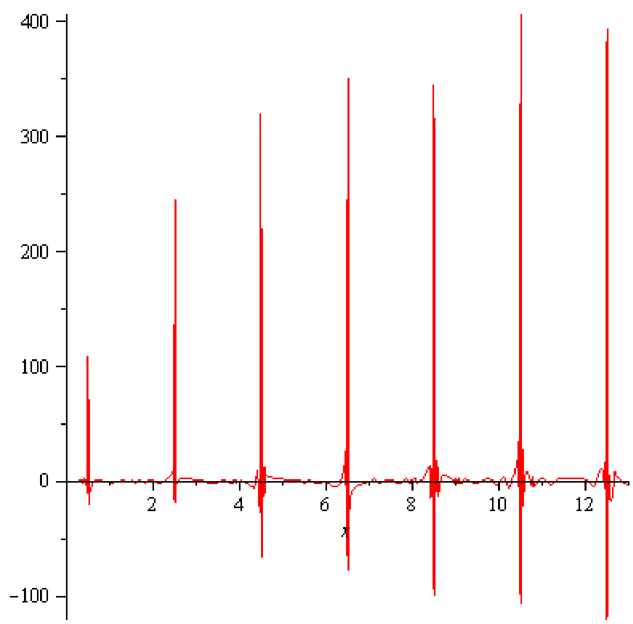
In Maple,

$$plot\left(\sum_{i=1}^{111}2x\frac{\mathrm{BesselJ}(0,\frac{\mathrm{BesselJZeros}(0,i)}{2})}{\left[\mathrm{BesselJ}(1,\mathrm{BesselJZeros}(0,i))\right]^2}\cdot\mathrm{BesselJ}(0,\mathrm{BesselJZeros}(0,i)*x),x=0..13\right)$$



Plotting with the asymptotic sequence

In Maple, $plot\left(\sqrt{2x}\cdot\left(\frac{1}{2}+\sum_{i=1}^{111}\cos\left(i\cdot\pi\cdot\left(x-\frac{1}{2}\right)\right)\right), x = 0..13\right)$



The plots confirm that the Bessel Series Theorem cannot be proved in the Calculus of Limits.

1.3 ν-Bessel Kernel divergence in the Calculus of Limits

For the Bessel Function $J_\nu(x)$, where ν is a fraction,

the ν-Bessel Series is the limit of the sequence of Partial Sums

$$\nu\mathcal{B}_{essel}\mathcal{S}_k\left\{f(x)\right\} = a_1 J_\nu(\lambda_{\nu 1}x) + a_2 J_\nu(\lambda_{\nu 2}x) + ... + a_k J_\nu(\lambda_{\nu k}x)$$

$$= \int_{\xi=0}^{\xi=1} f(\xi)2\xi\left\{\frac{J_\nu(\lambda_{\nu 1}\xi)J_\nu(\lambda_{\nu 1}x)}{J_{\nu+1}^2(\lambda_{\nu 1})} + ... + \frac{J_\nu(\lambda_{\nu k}\xi)J_\nu(\lambda_{\nu k}x)}{J_{\nu+1}^2(\lambda_{\nu k})}\right\}d\xi .$$

As $n \to \infty$, the ν-Bessel Sequence becomes the ν-Bessel Kernel,

$$2\xi\left\{\frac{J_\nu(\lambda_{\nu 1}\xi)J_\nu(\lambda_{\nu 1}x)}{J_{\nu+1}^2(\lambda_{\nu 1})} + \frac{J_\nu(\lambda_{\nu 2}\xi)J_\nu(\lambda_{\nu 2}x)}{J_{\nu+1}^2(\lambda_{\nu 2})} + \frac{J_\nu(\lambda_{\nu 3}\xi)J_\nu(\lambda_{\nu 3}x)}{J_{\nu+1}^2(\lambda_{\nu 3})} + ...\right\},$$

To see divergence at $\xi = x$, apply the Schlafli Formula [Watson, p. 585]: For large k, we choose $\lambda_{\nu,k} < A_k < \lambda_{\nu,k+1}$ so that

$$2\xi\left\{\frac{J_\nu(\lambda_{\nu 1}\xi)J_\nu(\lambda_{\nu 1}x)}{J_{\nu+1}^2(\lambda_{\nu 1})} + ... + \frac{J_\nu(\lambda_{\nu k}\xi)J_\nu(\lambda_{\nu k}x)}{J_{\nu+1}^2(\lambda_{\nu k})}\right\} \sim$$

$$\sim \sqrt{\frac{\xi}{x}}\left[\frac{\sin A_k(\xi - x)}{2\sin\frac{1}{2}\pi(\xi - x)} - \frac{\sin A_k(\xi + x)}{2\sin\frac{1}{2}\pi(\xi + x)}\right]$$

By [Watson, p.584], we may set $A_k = k\pi + \frac{1}{2}(\nu + \frac{1}{2})\pi$. And following Dini, [Watson, p.577], Watson chose also $\nu + \frac{1}{2} \geq 0$, We will set $A_k = (K + \frac{1}{2})\pi$, where $K \geq k$ is a natural number

Since

$$\frac{\sin(K + \frac{1}{2})\theta}{2\sin\frac{1}{2}\theta} = \frac{1}{2} + \cos\theta + \cos 2\theta + ...\cos K\theta$$

$$2\xi\left\{\frac{J_\nu(\lambda_{\nu 1}\xi)J_\nu(\lambda_{\nu 1}x)}{J_{\nu+1}^2(\lambda_{\nu 1})} + \frac{J_\nu(\lambda_{\nu 2}\xi)J_\nu(\lambda_{\nu 2}x)}{J_{\nu+1}^2(\lambda_{\nu 2})} + ...\frac{J_\nu(\lambda_{\nu k}\xi)J_\nu(\lambda_{\nu k}x)}{J_{\nu+1}^2(\lambda_{\nu k})}\right\} \sim$$

$$\sim \sqrt{\tfrac{\xi}{x}}\left\{\tfrac{1}{2} + \cos\pi(\xi - x) + \cos 2\pi(\xi - x) + ...\cos K\pi(\xi - x)\right\}$$

$$-\sqrt{\tfrac{\xi}{x}}\left\{\tfrac{1}{2} + \cos\pi(\xi + x) + \cos 2\pi(\xi + x) + ...\cos K\pi(\xi + x)\right\}$$

By [Hardy, p.2, #(1.2.3)],

$$\tfrac{1}{2} + \cos\theta + \cos 2\theta + \cos 3\theta + ... = 0, \text{ for any } \theta \neq 0$$

Therefore, for k large enough, and $\xi \neq x \neq 0$,

$$2\xi\left\{\frac{J_\nu(\lambda_{\nu 1}\xi)J_\nu(\lambda_{\nu 1}x)}{J_{\nu+1}^2(\lambda_{\nu 1})} + \frac{J_\nu(\lambda_{\nu 2}\xi)J_\nu(\lambda_{\nu 2}x)}{J_{\nu+1}^2(\lambda_{\nu 2})} + ...\frac{J_\nu(\lambda_{\nu k}\xi)J_\nu(\lambda_{\nu k}x)}{J_{\nu+1}^2(\lambda_{\nu k})}\right\} \sim$$

$$\sim \sqrt{\tfrac{\xi}{x}}\left\{\tfrac{1}{2} + \cos\pi(\xi - x) + \cos 2\pi(\xi - x) + ...\cos K\pi(\xi - x)\right\}$$

For $\xi \to x$,

$$2x\left\{\frac{J_\nu^2(\lambda_{\nu,1}x)}{J_{\nu+1}^2(\lambda_{\nu,1})} + \frac{J_\nu^2(\lambda_{\nu,2}x)}{J_{\nu+1}^2(\lambda_{\nu,2})} + .. + \frac{J_\nu^2(\lambda_{\nu,k}x)}{J_{\nu+1}^2(\lambda_{\nu,k})}\right\} \sim [\tfrac{1}{2} + \underbrace{\cos 0 + ...\cos 0}_{K}].$$

Hence,

$$2x\left\{\frac{J_0^2(\lambda_1 x)}{J_1^2(\lambda_1)} + \frac{J_0^2(\lambda_2 x)}{J_1^2(\lambda_2)} + \frac{J_0^2(\lambda_3 x)}{J_1^2(\lambda_3)} + ...\right\} = \tfrac{1}{2} + \lim_{k\to\infty} K,$$

and the ν-Bessel Kernel diverges to ∞ at any $\xi = x \neq 0$.

Therefore, while the partial sums of the ν-Bessel Series exist, their limit does not. That is, due to the singularity at $\xi = x$, the ν-Bessel Series does not converge in the Calculus of Limits.

Avoiding the singularity at $\xi = x$, by using the Cauchy Principal Value of the integral does not recover the Theorem, because at any $\xi \neq x \neq 0$, the ν-Bessel Kernel vanishes, and the integral will be identically zero, for any function $f(x)$: For any $\xi \neq x \neq 0$,

$$2\xi\left\{\frac{J_\nu(\lambda_{\nu 1}\xi)J_\nu(\lambda_{\nu 1}x)}{J_{\nu+1}^2(\lambda_{\nu 1})} + \frac{J_\nu(\lambda_{\nu 2}\xi)J_\nu(\lambda_{\nu 2}x)}{J_{\nu+1}^2(\lambda_{\nu 2})} + \frac{J_\nu(\lambda_{\nu 3}\xi)J_\nu(\lambda_{\nu 3}x)}{J_{\nu+1}^2(\lambda_{\nu 3})} + ...\right\} =$$

$$= \sqrt{\frac{\xi}{x}}\underbrace{\left\{\frac{1}{2} + \cos\pi(\xi - x) + \cos 2\pi(\xi - x) + \cos 3\pi(\xi - x) + ...\right\}}_{0},$$

and the ν-Bessel Kernel vanishes.

Thus, the ν-Bessel Series Theorem cannot be proved in the Calculus of Limits.

1.4 Infinitesimal Calculus Solution

By resolving the problem of the infinitesimals [Dan2], we obtained the Infinite Hyper-reals that are strictly smaller than ∞, and constitute the value of the Delta Function at the singularity.

The controversy surrounding the Leibnitz Infinitesimals derailed the development of the Infinitesimal Calculus, and the Delta Function could not be defined and investigated properly.

In Infinitesimal Calculus, [Dan3], we can differentiate over jump discontinuities, and integrate over singularities.

The Delta Function, the idealization of an impulse in Radar circuits, is a Discontinuous Hyper-Real function which definition requires Infinite Hyper-reals, and which analysis requires Infinitesimal Calculus.

In [Dan5], we show that in infinitesimal Calculus, the hyper-real

$$\delta(x) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x} d\omega$$

- is zero for any $x \neq 0$,
- it spikes at $x = 0$, so that its Infinitesimal Calculus integral is $\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1$,
- and $\delta(0) = \frac{1}{dx} < \infty$.

Here, we show that in Infinitesimal calculus, the Bessel Kernel is a hyper-real Periodic Delta: A train of Delta Functions.

And the Bessel Series $\mathcal{B}_{essel}\mathcal{S}\left\{f(x)\right\}$ associated with a Hyper-real function $f(x)$, equals $f(x)$.

2.

Hyper-real Line

Each real number α can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of infinitesimal hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the infinite hyper-reals.
4. The infinite hyper-reals are greater than any real number, yet strictly smaller than infinity.
5. The infinite hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a non-constant hyper-real.
7. The Hyper-reals are the totality of constant hyper-reals, a family of infinitesimals, a family of infinitesimals with

negative sign, a family of infinite hyper-reals, a family of infinite hyper-reals with negative sign, and non-constant hyper-reals.

8. The hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.
9. That line includes the real numbers separated by the non-constant hyper-reals. Each real number is the center of an interval of hyper-reals, that includes no other real number.
10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.
11. Zero is not an infinitesimal, because zero is not strictly greater than zero.
12. We do not add infinity to the hyper-real line.
13. The infinitesimals, the infinitesimals with negative signs, the infinite hyper-reals, and the infinite hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.
14. The hyper-real line is embedded in $\mathbb{R}^\infty$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the hyper-real onto the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal hyper-reals, or to the infinite hyper-reals, or to the non-constant hyper-reals.

16. No neighbourhood of a hyper-real is homeomorphic to an $\mathbb{R}^n$ ball. Therefore, the hyper-real line is not a manifold.

17. The hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

3.

Integral of a Hyper-real Function

In [Dan3], we defined the integral of a Hyper-real Function.

Let $f(x)$ be a hyper-real function on the interval $[a,b]$.

The interval may not be bounded.

$f(x)$ may take infinite hyper-real values, and need not be bounded.

At each

$$a \le x \le b,$$

there is a rectangle with base $[x - \frac{dx}{2}, x + \frac{dx}{2}]$, height $f(x)$, and area

$$f(x)dx.$$

We form the **Integration Sum** of all the areas for the x's that start at $x = a$, and end at $x = b$,

$$\sum_{x \in [a,b]} f(x)dx.$$

If for any infinitesimal dx, the Integration Sum has the same hyper-real value, then $f(x)$ is integrable over the interval $[a,b]$.

Then, we call the Integration Sum the integral of $f(x)$ from $x = a$, to $x = b$, and denote it by

$$\int_{x=a}^{x=b} f(x)dx\,.$$

If the hyper-real is infinite, then it is the integral over $[a,b]$,

If the hyper-real is finite,

$$\int_{x=a}^{x=b} f(x)dx = \text{ real part of the hyper-real}\,.\square$$

3.1 The countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers,

$Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty.$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[a,b]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(x)dx$.

The Lower Integral is the Integration Sum where $f(x)$ is replaced by its lowest value on each interval $[x-\frac{dx}{2}, x+\frac{dx}{2}]$

3.2 $$\sum_{x\in[a,b]}\left(\inf_{x-\frac{dx}{2}\le t\le x+\frac{dx}{2}} f(t)\right)dx$$

The Upper Integral is the Integration Sum where $f(x)$ is replaced by its largest value on each interval $[x-\frac{dx}{2}, x+\frac{dx}{2}]$

3.3 $$\sum_{x\in[a,b]}\left(\sup_{x-\frac{dx}{2}\le t\le x+\frac{dx}{2}} f(t)\right)dx$$

If the integral is a finite hyper-real, we have

3.4 *A hyper-real function has a finite integral if and only if its upper integral and its lower integral are finite, and differ by an infinitesimal.*

4.

Delta Function

In [Dan5], we have defined the Delta Function, and established its properties

1. The Delta Function is a hyper-real function defined from the hyper-real line into the set of two hyper-reals $\left\{0, \frac{1}{dx}\right\}$. The hyper-real 0 is the sequence $\langle 0, 0, 0, ... \rangle$. The infinite hyper-real $\frac{1}{dx}$ depends on our choice of dx.
2. We will usually choose the family of infinitesimals that is spanned by the sequences $\left\langle \frac{1}{n} \right\rangle, \left\langle \frac{1}{n^2} \right\rangle, \left\langle \frac{1}{n^3} \right\rangle, \ldots$ It is a semigroup with respect to vector addition, and includes all the scalar multiples of the generating sequences that are non-zero. That is, the family includes infinitesimals with negative sign. Therefore, $\frac{1}{dx}$ will mean the sequence $\langle n \rangle$. Alternatively, we may choose the family spanned by the sequences $\left\langle \frac{1}{2^n} \right\rangle, \left\langle \frac{1}{3^n} \right\rangle, \left\langle \frac{1}{4^n} \right\rangle, \ldots$ Then, $\frac{1}{dx}$ will mean the

sequence $\left\langle 2^n \right\rangle$. Once we determined the basic infinitesimal dx, we will use it in the Infinite Riemann Sum that defines an Integral in Infinitesimal Calculus.

3. The Delta Function is strictly smaller than ∞

4. We define, $\delta(x) \equiv \frac{1}{dx} \chi_{[-\frac{dx}{2},\frac{dx}{2}]}(x)$,

$$\text{where} \quad \chi_{[-\frac{dx}{2},\frac{dx}{2}]}(x) = \begin{cases} 1, x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right] \\ 0, \text{otherwise} \end{cases}.$$

5. Hence,

   - for $x < 0$, $\delta(x) = 0$
   - at $x = -\frac{dx}{2}$, $\delta(x)$ jumps from 0 to $\frac{1}{dx}$,
   - for $x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right]$, $\delta(x) = \frac{1}{dx}$.
   - at $x = 0$, $\delta(0) = \frac{1}{dx}$
   - at $x = \frac{dx}{2}$, $\delta(x)$ drops from $\frac{1}{dx}$ to 0.
   - for $x > 0$, $\delta(x) = 0$.
   - $x\delta(x) = 0$

6. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle \chi_{[-\frac{1}{2},\frac{1}{2}]}(x), 2\chi_{[-\frac{1}{4},\frac{1}{4}]}(x), 3\chi_{[-\frac{1}{6},\frac{1}{6}]}(x) ... \right\rangle$

7. If $dx = \left\langle \frac{2}{n} \right\rangle$, $\delta(x) = \left\langle \frac{1}{2\cosh^2 x}, \frac{2}{2\cosh^2 2x}, \frac{3}{2\cosh^2 3x}, ... \right\rangle$

8. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle e^{-x}\chi_{[0,\infty)}, 2e^{-2x}\chi_{[0,\infty)}, 3e^{-3x}\chi_{[0,\infty)}, ... \right\rangle$

9. $$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1.$$

10. $$\delta(\xi - x) = \frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} e^{-ik(\xi - x)} dk$$

5.

Periodic Delta $\delta_{Periodic}(\xi - x)$

5.1 Periodic Delta Definition

$$\delta_{Periodic}(\xi - x) = \sqrt{\tfrac{\xi}{x}}\left\{... + \delta(\xi - x + 2) + \delta(\xi - x) + \delta(\xi - x) + ...\right\}$$

$$= \sqrt{\tfrac{\xi}{x}}\left\{\tfrac{1}{2} + \cos\pi(\xi - x) + \cos 2\pi(\xi - x) + ...\right\}$$

$$= \sqrt{\tfrac{\xi}{x}}\,\tfrac{1}{2}\left\{... + e^{-2i\pi(\xi - x)} + e^{-i\pi(\xi - x)} + 1 + e^{i\pi(\xi - x)} + e^{2i\pi(\xi - x)} + ...\right\}.$$

is a train of hyper-real Delta functions.

5.2 *Periodic Delta peaks at* $\xi - x = 2m$*, for any* $m = 0,1,2,...$

and has the sifting property over $[0,1]$

Proof of Sifting:

$$\int_{\xi=0}^{\xi=1} \sqrt{\tfrac{\xi}{x}}\left\{\tfrac{1}{2} + \cos\pi(\xi - x) + \cos 2\pi(\xi - x) + ...\right\} d\xi =$$

$$= \sum_{\xi=0}^{\xi=1} \sqrt{\tfrac{\xi}{x}}\left\{\tfrac{1}{2} + \cos\pi(\xi - x) + \cos 2\pi(\xi - x) + ...\right\} d\xi$$

By [Hardy, p.2, #(1.2.3)],

$$\tfrac{1}{2} + \cos\theta + \cos 2\theta + \cos 3\theta + ... = 0, \text{ for any } \theta \neq 0$$

Therefore, for any $\xi \neq x$, the summation terms,

$$\sqrt{\tfrac{\xi}{x}}\left\{\tfrac{1}{2}+\cos\pi(\xi-x)+\cos 2\pi(\xi-x)+...\right\}d\xi\,,$$

vanish.

For $\xi = x$, the summation yields

$$\left\{\tfrac{1}{2}+\cos\pi(0)+\cos 2\pi(0)+...\right\}dx\,.$$

The Hyper real number

$$\tfrac{1}{2}+\cos\pi(0)+\cos 2\pi(0)+...$$

is the sequence

$$\left\langle \tfrac{1}{2}+n \right\rangle = \tfrac{1}{dx}.$$

Consequently, the integral equals

$$\tfrac{1}{dx}dx = 1,$$

and the Periodic Delta has the sifting property.□

6.

Convergent Series

In [Dan8], we defined convergence of infinite series in Infinitesimal Calculus

6.1 Sequence Convergence to a finite hyper-real a

$$a_n \to a \;\; \textit{iff} \;\; \langle a_n \rangle - \langle a \rangle = \text{infinitesimal}.$$

6.2 Sequence Convergence to an infinite hyper-real A

$$a_n \to A \;\; \textit{iff} \;\; \langle a_n \rangle \text{ represents the infinite hyper-real } A.$$

6.3 Series Convergence to a finite hyper-real s

$$a_1 + a_2 + ... \to s \;\; \textit{iff} \;\; \langle a_1 + ... + a_n \rangle - \langle s \rangle = \text{infinitesimal}.$$

6.4 Series Convergence to an Infinite Hyper-real S

$$a_1 + a_2 + ... \to S \;\; \textit{iff}$$

$$\langle a_1 + ... + a_n \rangle \text{ represents the infinite hyper-real } S.$$

7.

Bessel Sequence and $\delta_{Periodic}(\xi - x)$

7.1 Bessel Sequence Definition

The Bessel Series partial sums

$$\mathcal{B}_{essel}\mathcal{S}_k\left\{f(x)\right\} == \int_{\xi=0}^{\xi=1} f(\xi)2\xi\left\{\frac{J_0(\lambda_1\xi)J_0(\lambda_1 x)}{J_1^2(\lambda_1)} + .. + \frac{J_0(\lambda_k\xi)J_0(\lambda_k x)}{J_1^2(\lambda_k)}\right\}d\xi\,.$$

give rise to the Bessel Sequence

$$2\xi\left\{\frac{J_0(\lambda_1\xi)J_0(\lambda_1 x)}{J_1^2(\lambda_1)} + .. + \frac{J_0(\lambda_k\xi)J_0(\lambda_k x)}{J_1^2(\lambda_k)}\right\}.$$

7.2 *The Bessel Sequence* $2\xi\left\{\frac{J_0(\lambda_1\xi)J_0(\lambda_1 x)}{J_1^2(\lambda_1)} + .. + \frac{J_0(\lambda_k\xi)J_0(\lambda_k x)}{J_1^2(\lambda_k)}\right\}$

is asymptotically the Periodic Delta Sequence

$$\sqrt{\tfrac{\xi}{x}}\left\{\tfrac{1}{2} + \cos\pi(\xi - x) + \cos 2\pi(\xi - x) + ... + \cos k\pi(\xi - x)\right\}$$

Proof: In 1.1, it follows that for large k,

$$2\xi\left\{\frac{J_0(\lambda_1\xi)J_0(\lambda_1 x)}{J_1^2(\lambda_1)} + \frac{J_0(\lambda_2\xi)J_0(\lambda_2 x)}{J_1^2(\lambda_2)} + .. + \frac{J_0(\lambda_k\xi)J_0(\lambda_k x)}{J_1^2(\lambda_k)}\right\} \sim$$

$$\sim \sqrt{\tfrac{\xi}{x}}\left\{\tfrac{1}{2} + \cos\pi(\xi - x) + \cos 2\pi(\xi - x) + ... + \cos k\pi(\xi - x)\right\}.\square$$

7.3 *The Bessel Sequence* $2\xi\left\{\frac{J_\nu(\lambda_{\nu 1}\xi)J_\nu(\lambda_{\nu 1}x)}{J_{\nu+1}^2(\lambda_{\nu 1})}+..+\frac{J_\nu(\lambda_{\nu k}\xi)J_\nu(\lambda_{\nu k}x)}{J_{\nu+1}^2(\lambda_{\nu k})}\right\}$

is asymptotically the Periodic Delta Sequence

$$\sqrt{\tfrac{\xi}{x}}\left\{\tfrac{1}{2}+\cos\pi(\xi-x)+\cos 2\pi(\xi-x)+...+\cos k\pi(\xi-x)\right\}$$

Proof: same as 7.2.□

8.

Bessel Kernel and $\delta_{Periodic}(\xi - x)$

8.1 Bessel Kernel in the Calculus of Limits

The Bessel Series partial sums

$$\mathcal{B}_{essel}\mathcal{S}_k\left\{f(x)\right\} = \int\limits_{\xi=0}^{\xi=1} f(\xi)\underbrace{2\xi\left\{\frac{J_0(\lambda_1\xi)J_0(\lambda_1 x)}{J_1^2(\lambda_1)} + .. + \frac{J_0(\lambda_k\xi)J_0(\lambda_k x)}{J_1^2(\lambda_k)}\right\}}_{\text{Bessel Sequence}} d\xi\,.$$

give rise to the Bessel Sequence. Its limit is the Bessel Kernel

$$2\xi\left\{\frac{J_0(\lambda_1\xi)J_0(\lambda_1 x)}{J_1^2(\lambda_1)} + \frac{J_0(\lambda_2\xi)J_0(\lambda_2 x)}{J_1^2(\lambda_2)} + \frac{J_0(\lambda_3\xi)J_0(\lambda_3 x)}{J_1^2(\lambda_3)} + ...\right\}$$

8.2 *In the Calculus of Limits, the Bessel Kernel does not have the sifting property*

<u>*Proof*</u>: for $\xi \to x$,

$$2x\left\{\frac{J_0^2(\lambda_1 x)}{J_1^2(\lambda_1)} + .. + \frac{J_0^2(\lambda_k\xi)}{J_1^2(\lambda_k)}\right\} \sim \left\{\tfrac{1}{2} + \cos\pi(0) + .. + \cos k\pi(0)\right\}$$

$$\geq (\tfrac{1}{2} + k) \underset{k\to\infty}{\to} \infty$$

That is,

$$2\xi\left\{\frac{J_0(\lambda_1\xi)J_0(\lambda_1 x)}{J_1^2(\lambda_1)} + \frac{J_0(\lambda_2\xi)J_0(\lambda_2 x)}{J_1^2(\lambda_2)} + ...\right\} \text{ is not integrable.}\square$$

8.3 Hyper-real Bessel Kernel in Infinitesimal Calculus

$$\mathcal{B}_{essel}\mathcal{K}_{ernel}(\xi,x)=2\xi\left\{\frac{J_0(\lambda_1\xi)J_0(\lambda_1 x)}{J_1^2(\lambda_1)}+\frac{J_0(\lambda_2\xi)J_0(\lambda_2 x)}{J_1^2(\lambda_2)}+...\right\}$$

$$=\sqrt{\tfrac{\xi}{x}}\left\{\tfrac{1}{2}+\cos\pi(\xi-x)+\cos 2\pi(\xi-x)+...\right\}$$

$$=\sqrt{\tfrac{\xi}{x}}\tfrac{1}{2}\left\{...+e^{-2i\pi(\xi-x)}+e^{-i\pi(\xi-x)}+1+e^{i\pi(\xi-x)}+e^{2i\pi(\xi-x)}+...\right\}$$

$$=\begin{cases}\sqrt{\tfrac{\xi}{x}}\left\langle\tfrac{1}{2}+n\right\rangle & ,\xi-x=2m\\ 0 & ,\xi-x\neq 2m\end{cases}$$

$$=\sqrt{\tfrac{\xi}{x}}\left\{...+\delta(\xi-x+2)+\delta(\xi-x)+\delta(\xi-x)+...\right\}$$

$$=\delta_{Periodic}(\xi-x).$$

Proof: By 1.1,

$$\mathcal{B}_{essel}\mathcal{K}_{ernel}(\xi,x)=2\xi\left\{\frac{J_0(\lambda_1\xi)J_0(\lambda_1 x)}{J_1^2(\lambda_1)}+\frac{J_0(\lambda_2\xi)J_0(\lambda_2 x)}{J_1^2(\lambda_2)}+...\right\}$$

$$=\sqrt{\tfrac{\xi}{x}}\left\{\tfrac{1}{2}+\cos\pi(\xi-x)+\cos 2\pi(\xi-x)+...\right\}$$

$$=\begin{cases}\sqrt{\tfrac{\xi}{x}}\left\langle\tfrac{1}{2}+n\right\rangle & ,\xi-x=2m\\ 0 & ,\xi-x\neq 2m\end{cases}$$

Denoting the hyper-real $\left\langle\frac{1}{2}+n\right\rangle$ by $\frac{1}{dx}$,

$$=\sqrt{\tfrac{\xi}{x}}\left\{...+\begin{cases}0,\xi-x\neq-2\\ \frac{1}{dx},\xi-x=-2\end{cases}+\begin{cases}0,\xi-x\neq 0\\ \frac{1}{dx},\xi-x=0\end{cases}+\begin{cases}0,\xi-x\neq 2\\ \frac{1}{dx},\xi-x=2\end{cases}+...\right\}$$

$$=\sqrt{\tfrac{\xi}{x}}\left\{...+\delta(\xi-x+2)+\delta(\xi-x)+\delta(\xi-x+2)+...\right\}.\square$$

8.4 Hyper-real ν-Bessel Kernel in Infinitesimal Calculus

Let $\left\langle \frac{1}{2} + n \right\rangle$ *be an infinite Hyper-real.*

The Hyper-real ν*-Bessel Kernel is*

$$2\xi\left\{\frac{J_\nu(\lambda_{\nu 1}\xi)J_\nu(\lambda_{\nu 1}x)}{J_{\nu+1}^2(\lambda_{\nu 1})} + \frac{J_\nu(\lambda_{\nu 2}\xi)J_\nu(\lambda_{\nu 2}x)}{J_{\nu+1}^2(\lambda_{\nu 2})} + ...\right\} =$$

$$= \sqrt{\tfrac{\xi}{x}}\left\{\tfrac{1}{2} + \cos\pi(\xi - x) + \cos 2\pi(\xi - x) + ...\right\}$$

$$= \sqrt{\tfrac{\xi}{x}}\,\tfrac{1}{2}\left\{... + e^{-2i\pi(\xi-x)} + e^{-i\pi(\xi-x)} + 1 + e^{i\pi(\xi-x)} + e^{2i\pi(\xi-x)} + ...\right\}$$

$$= \begin{cases} \sqrt{\tfrac{\xi}{x}}\left\langle \tfrac{1}{2}+n \right\rangle & ,\xi - x = 2m \\ 0 & ,\xi - x \neq 2m \end{cases}$$

$$= \sqrt{\tfrac{\xi}{x}}\left\{... + \delta(\xi - x + 2) + \delta(\xi - x) + \delta(\xi - x) + ...\right\}$$

$$= \delta_{Periodic}(\xi - x).$$

9.

Bessel Series and $\delta_{Periodic}(\xi - x)$

9.1 Bessel Series of a Hyper-real Function

Let $f(x)$ be a hyper-real function integrable on $[0,1]$.

The zeros of the Bessel Function $J_0(x)$.

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots$$

define the orthogonal sequence of functions

$$J_0(\lambda_1 x),\ J_0(\lambda_2 x),\ J_0(\lambda_3 x),\dots$$

For each $n = 0,1,2,3,\dots$, the hyper-real integrals

$$a_n = \frac{2}{[J_1(\lambda_n)]^2} \int_{x=0}^{x=1} x f(x) J_0(\lambda_n x) dx$$

exist, with finite, or infinite hyper-real values. The a_n are the Bessel Coefficients of $f(x)$.

The Bessel Series associated with $f(x)$ is

$$\mathcal{B}_{essel}\mathcal{S}\left\{f(x)\right\} = a_1 J_0(\lambda_1 x) + a_2 J_0(\lambda_2 x) + a_3 J_0(\lambda_3 x) + \dots.$$

For each x, it may assume finite or infinite hyper-real values.

9.2 $$\mathcal{B}_{essel}\mathcal{S}\left\{\delta_{Periodic}(\xi - x)\right\} = \delta_{Periodic}(\xi - x)$$

<u>*Proof*</u>:

$$\mathcal{B}_{essel}\mathcal{S}\left\{\delta_{Periodic}(\xi - x)\right\} = a_1 J_0(\lambda_1 x) + a_2 J_0(\lambda_2 x) + a_3 J_0(\lambda_3 x) + ...$$

where

$$a_k = \frac{2}{J_1^2(\lambda_k)} \int_{x=0}^{x=1} x\delta_{Periodic}(\xi - x) J_0(\lambda_k x) dx$$

Substituting from 8.3,

$$\delta_{Periodic}(\xi - x) = 2\xi\left\{\frac{J_0(\lambda_1\xi)J_0(\lambda_1 x)}{J_1^2(\lambda_1)} + \frac{J_0(\lambda_2\xi)J_0(\lambda_2 x)}{J_1^2(\lambda_2)} + ...\right\},$$

$$a_k = \frac{2}{J_1^2(\lambda_k)} \int_{x=0}^{x=1} x2\xi\left\{\frac{J_0(\lambda_1\xi)J_0(\lambda_1 x)}{J_1^2(\lambda_1)} + \frac{J_0(\lambda_2\xi)J_0(\lambda_2 x)}{J_1^2(\lambda_2)} + ...\right\} J_0(\lambda_k x) dx$$

$$= J_0(\lambda_1\xi)\frac{2\xi}{J_1^2(\lambda_k)J_1^2(\lambda_1)} 2\int_{x=0}^{x=1} xJ_0(\lambda_1 x)J_0(\lambda_k x)dx + ...$$

$$... + J_0(\lambda_m\xi)\frac{2\xi}{J_1^2(\lambda_k)J_1^2(\lambda_m)} 2\int_{x=0}^{x=1} xJ_0(\lambda_m x)J_0(\lambda_k x)dx + ...$$

By [Spiegel, p.143, #24.95] for any $m \neq k$,

$$\int_{x=0}^{x=1} xJ_0(\lambda_m x)J_0(\lambda_k x)dx = \frac{1}{\lambda_k^2 - \lambda_m^2}\{\lambda_m \underbrace{J_0(\lambda_k)}_{0} J_1(\lambda_m) - \lambda_k \underbrace{J_0(\lambda_m)}_{0} J_1(\lambda_k)\} = 0$$

By [Spiegel, p.143, #24.96] for $m = k$,

$$\int_{x=0}^{x=1} xJ_0(\lambda_k x)J_0(\lambda_k x)dx = \tfrac{1}{2}J_1^2(\lambda_k) + \tfrac{1}{2}\underbrace{J_0^2(\lambda_m)}_{0} = \tfrac{1}{2}J_1^2(\lambda_k).$$

Hence,

$$a_k = J_0(\lambda_k \xi)\frac{2\xi}{J_1^2(\lambda_k)J_1^2(\lambda_k)}\underbrace{2\int_{x=0}^{x=1} xJ_0(\lambda_k x)J_0(\lambda_k x)dx}_{J_1^2(\lambda_k)}$$

$$= J_0(\lambda_k \xi)\frac{2\xi}{J_1^2(\lambda_k)}$$

Therefore,

$$\mathcal{B}_{essel}\mathcal{S}\left\{\delta_{Periodic}(\xi - x)\right\} = 2\xi\frac{J_0(\lambda_1 \xi)}{J_1^2(\lambda_1)}J_0(\lambda_1 x) + 2\xi\frac{J_0(\lambda_2 \xi)}{J_1^2(\lambda_2)}J_0(\lambda_2 x) + ...$$

$$= \delta_{Periodic}(\xi - x). \square$$

9.3 ν-$\mathcal{B}_{essel}\mathcal{S}\left\{\delta_{Periodic}(\xi - x)\right\} = \delta_{Periodic}(\xi - x)$

Proof: same as 8.2. □

10.

Bessel Series Theorem

The Bessel Series Theorem for a hyper-real function, $f(x)$, is the Fundamental Theorem of Bessel Series.

It supplies the conditions under which the Bessel Series associated with $f(x)$ equals $f(x)$.

It is believed to hold in under Hobson's conditions in the Calculus of Limits. In fact,

The Theorem cannot be proved in the Calculus of Limits under any conditions,

because the summation of the Bessel Series requires integration of the singular Bessel Kernel.

10.1 Bessel Series Theorem cannot be proved in the Calculus of Limits

<u>*Proof*</u>: Let $f(x)$ be integrable on $[0,1]$.

In the Calculus of Limits, the Bessel Series is the limit of

$$\mathcal{B}_{essel}\mathcal{S}_n\left\{f(x)\right\} = a_1 J_0(\lambda_1 x) + a_2 J_0(\lambda_2 x) + ... + a_n J_0(\lambda_n x)$$

$$= \left(\frac{2}{J_1^2(\lambda_1)}\int_{\xi=0}^{\xi=1}\xi f(\xi)J_0(\lambda_1\xi)d\xi\right)J_0(\lambda_1 x) + .. + \left(\frac{2}{J_1^2(\lambda_n)}\int_{\xi=0}^{\xi=1}\xi f(\xi)J_0(\lambda_n\xi)d\xi\right)J_0(\lambda_n x)$$

$$= \int_{\xi=0}^{\xi=1} f(\xi)2\xi \underbrace{\left\{ \frac{J_0(\lambda_1\xi)J_0(\lambda_1 x)}{J_1^2(\lambda_1)} + \frac{J_0(\lambda_2\xi)J_0(\lambda_2 x)}{J_1^2(\lambda_2)} + .. + \frac{J_0(\lambda_n\xi)J_0(\lambda_n x)}{J_1^2(\lambda_n)} \right\}}_{\text{Bessel Sequence}} d\xi .$$

As $n \to \infty$, the Bessel Sequence becomes the Bessel Kernel,

$$2\xi \left\{ \frac{J_0(\lambda_1\xi)J_0(\lambda_1 x)}{J_1^2(\lambda_1)} + \frac{J_0(\lambda_2\xi)J_0(\lambda_2 x)}{J_1^2(\lambda_2)} + \frac{J_0(\lambda_3\xi)J_0(\lambda_3 x)}{J_1^2(\lambda_3)} + ... \right\},$$

By 1.1, the Bessel Kernel diverges to infinity at any $\xi = x$.

Therefore, while the partial sums of the Bessel Series exist, their limit does not. Calculus of Limits fails to comprehend the sifting through the values of $f(\xi)$ by the Bessel Kernel, and the picking of $f(\xi)$ at $\xi = x$.

Avoiding the singularity at $\xi = x$, by using the Cauchy Principal Value of the integral does not recover the Theorem, because for any $\xi \neq x$, the Bessel Kernel vanishes, and the integral is identically zero, for any function $f(x)$.

Thus, the Bessel Series Theorem cannot be proved in the Calculus of Limits.□

10.2 Calculus of Limits Conditions are insufficient for the Bessel Series Theorem

Proof: The Hobson Conditions [Watson, p.591] are

1. $f(t)$ defined in $(0,1)$

2. $\sqrt{t}f(t)$, and $\left|\sqrt{t}f(t)\right|$ are integrable on $[0,1]$

3. at $0 < a < x < b < 1$,

   (i) $\sup_{x-\varepsilon<t<x+\varepsilon} f(t) - \inf_{x-\varepsilon<t<x+\varepsilon} f(t)$ is bounded for any $\varepsilon > 0$

   (ii) the limits $f(x-0)$, and $f(x+0)$ exist.

It is clear from 10.1 that these conditions on $f(x)$ do not resolve the singularity of the Bessel kernel, and are insufficient for the Bessel Series Theorem.□

In Infinitesimal Calculus, by 8.3, the Bessel Kernel is the Delta Function, and by 9.2, it equals its Bessel Series.

Then, the Bessel Series Theorem holds for any Hyper-Real Function:

10.3 Bessel Series Theorem for Hyper-real $f(x)$

If $f(x)$ is hyper-real function integrable on $[0,1]$,

Then, $$f(x) = \mathcal{B}_{essel}\mathcal{S}\left\{f(x)\right\}$$

<u>*Proof*</u>:

$$f(x) = \int_{\xi=0}^{\xi=1} f(\xi)\underbrace{\left\{... + \delta(\xi - x + 2) + \delta(\xi - x) + \delta(\xi - x - 2) + ...\right\}}_{\delta_{Periodic}(\xi - x),\text{ where the period of Delta is 2}} d\xi$$

Substituting from 8.3,

$$\delta_{Periodic}(\xi - x) = 2\xi \left\{ \frac{J_0(\lambda_1 \xi) J_0(\lambda_1 x)}{J_1^2(\lambda_1)} + \frac{J_0(\lambda_2 \xi) J_0(\lambda_2 x)}{J_1^2(\lambda_2)} + ... \right\},$$

we have

$$f(x) = \int_{\xi=0}^{\xi=1} f(\xi) 2\xi \left\{ \frac{J_0(\lambda_1 \xi) J_0(\lambda_1 x)}{J_1^2(\lambda_1)} + \frac{J_0(\lambda_2 \xi) J_0(\lambda_2 x)}{J_1^2(\lambda_2)} + ... \right\} d\xi$$

This Hyper-real Integral is the summation,

$$\sum_{\xi=0}^{\xi=1} f(\xi) 2\xi \left\{ \frac{J_0(\lambda_1 \xi) J_0(\lambda_1 x)}{J_1^2(\lambda_1)} + \frac{J_0(\lambda_2 \xi) J_0(\lambda_2 x)}{J_1^2(\lambda_2)} + ... \right\} d\xi$$

which amounts to the hyper-real function $f(x)$,and is well-defined.

Hence, the summation of each term in the integrand exists, and we may write the integral as the sum

$$= \underbrace{\left(\frac{2}{J_1^2(\lambda_1)} \int_{\xi=0}^{\xi=1} f(\xi) \xi J_0(\lambda_1 \xi) d\xi \right)}_{a_1} J_0(\lambda_1 x) + \underbrace{\left(\frac{2}{J_1^2(\lambda_2)} \int_{\xi=0}^{\xi=1} f(\xi) \xi J_0(\lambda_2 \xi) d\xi \right)}_{a_2} J_0(\lambda_2 x) + ...$$

$$= a_1 J_0(\lambda_1 x) + a_2 J_0(\lambda_2 x) + ...$$

$$= \mathcal{B}_{essel} \mathcal{S} \left\{ f(x) \right\}. \square$$

In particular, the Delta Function violates Hobson's Conditions

- *The Hyper-real* $\delta(t)$*,is not defined in the Calculus of Limits, and is not integrable in* $[0,1]$.

But by 9.2, $\delta_{Periodic}(\xi - x)$ satisfies the Bessel Series Theorem.

10.4 ν-Bessel Series Theorem for Hyper-real $f(x)$

If $f(x)$ ***is hyper-real function integrable on*** $[0,1]$***,***

Then,
$$f(x) = \nu\mathcal{B}_{essel}\mathcal{S}\left\{f(x)\right\}$$

Proof: same as 10.3.□

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