Well-Ordering of the Reals, Equality of all Infinities, and the Continuum Hypothesis

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Abstract We Well-Order the Real Numbers in $[0,1]$.

Thus, the Reals can be sequenced, and $Card\mathbb{N} = 2^{Card\mathbb{N}}$.

Consequently, we obtain the equality of all infinities

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ...$$

This resolves by default the Continuum Hypothesis that says that there is no set X with $Card\mathbb{N} < CardX < 2^{Card\mathbb{N}}$.

Since the real numbers in any interval are countable, this disproves that any countable set can has measure zero.

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Introduction

By the Well-Ordering Theorem, the Natural Numbers are ordered in such a way that every subset of them has a first element.

The Well-Ordering Axiom is the guess that every infinite set of numbers can be well-ordered like the Natural Numbers.

The Well-Ordering Axiom is equivalent to the Axiom of Choice.

In 1963, Cohen claimed that it is impossible to prove that the real numbers can be well-ordered.

In fact, the Dictionary Listing of the real numbers in $[0,1]$ as infinite binary sequences, orders the real numbers with repetition. When the repetitions are eliminated, we obtain the Midpoints set, which is well ordered.

The Well Ordering of the Real Numbers sequences the Real Numbers, and renders them countable.

Consequently, all infinities are equal, and all the Axioms that are equivalent to the Axiom of Choice, are guaranteed by the Theorems that hold for the Natural Numbers.

1.

Dictionary Listing of the Reals in $[0,1]$

We list the real numbers in $[0,1]$, using their infinite sequence binary representation.

The 1st row has the 2^1 infinite binary sequences representing 0, and $\frac{1}{2}$,

$$\begin{pmatrix} 0 \\ 0 \\ \dots \end{pmatrix} \leftrightarrow \frac{0}{2}, \qquad \begin{pmatrix} 1 \\ 0 \\ \dots \end{pmatrix} \leftrightarrow \frac{1}{2}$$

The 2nd row has the 2^2 infinite binary sequences

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \dots \end{pmatrix} \leftrightarrow \frac{0}{2^2}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \dots \end{pmatrix} \leftrightarrow \frac{1}{2^2}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \dots \end{pmatrix} \leftrightarrow \frac{2}{2^2}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ \dots \end{pmatrix} \leftrightarrow \frac{3}{2^2} .$$

The 3rd row has the 2^3 sequences

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \dots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \dots \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \dots \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ \dots \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \dots \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \dots \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ \dots \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ \dots \end{pmatrix},$$

where

$$(0,0,0,0,\ldots) \leftrightarrow \tfrac{0}{2^3}$$

$$(0,0,1,0,\ldots) \leftrightarrow \tfrac{1}{2^3},$$

$$(0,1,0,0,\ldots) \leftrightarrow \tfrac{2}{2^3},$$

$$(0,1,1,0,\ldots) \leftrightarrow \tfrac{3}{2^3},$$

$$(1,0,0,0,\ldots) \leftrightarrow \tfrac{4}{2^3},$$

$$(1,0,1,0,\ldots) \leftrightarrow \tfrac{5}{2^3},$$

$$(1,1,0,0,\ldots) \leftrightarrow \tfrac{6}{2^3},$$

$$(1,1,1,0,\ldots) \leftrightarrow \tfrac{7}{2^3}.$$

The n^{th} row lists the 2^n sequences that start with

$$(0,0,0,0,\ldots,0,0,\ldots) \leftrightarrow \tfrac{0}{2^n},$$

and end with

$$(1,1,1,1,\ldots,1,0,\ldots) \leftrightarrow \tfrac{2^n-1}{2^n}.$$

The listing

			0	1			
		00	01	10	11		
000	001	010	011	100	101	110	111

… … … … … … … … … … … … … … … … …

enumerates all the real numbers in $[0,1]$, although with repetitions.

2.

Well-Ordering the Reals in $[0,1]$ with the Midpoints Set

We eliminate the repetitions in the listing of the reals by constructing the rows of the Midpoints Set.

We'll see that the Midpoints Set represents the reals in $[0,1]$, so that every subset of it has a first element.

The 1st row has the 1 binary sequence representing $\frac{1}{2^1}$,

$$(1,0,0,\dots,0,0,0\dots) \leftrightarrow \tfrac{1}{2}$$

The 2nd row has the two binary sequences,

$$(0,1,0,\dots,0\dots) \leftrightarrow \tfrac{1}{2^2},$$

$$(1,1,0,\dots,0,\dots) \leftrightarrow \tfrac{3}{2^2}$$

The 3rd row has the four binary sequences,

$$(0,0,1,0,\dots) \leftrightarrow \tfrac{1}{2^3},$$

$$(0,1,1,0,\dots) \leftrightarrow \tfrac{3}{2^3},$$

$$(1,0,1,0,\dots) \leftrightarrow \tfrac{5}{2^3},$$

$$(1,1,1,0,\ldots) \leftrightarrow \tfrac{7}{2^3}.$$

The 4th row lists the eight binary sequences that start with

$$(0,0,0,1,0\ldots) \leftrightarrow \tfrac{1}{2^4},$$

$$(0,0,1,1,0\ldots) \leftrightarrow \tfrac{3}{2^4}$$

and end with

$$(1,1,1,1,0,\ldots) \leftrightarrow \tfrac{2^4-1}{2^4}.$$

As in section **1**, the listing

								1								
							01		11							
					001		011		101		111					
	0001		0011		0101		0111		1001		1011		1101		1111	
…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…

enumerates all the real numbers in $[0,1]$, but without repetitions.

The $Card\mathbb{N}$ row has $2^{Card\mathbb{N}}$ infinite binary sequences that represent real numbers in $[0,1]$.

the order follows the rows of the Midpoints Set from left to right.

That is, the first element in this ordering is $\frac{1}{2}$ in the 1st row.

The second element in this ordering is $\frac{1}{2^2}$, and the third is $\frac{3}{2^2}$. Both are in the 2^{nd} row.

The fourth is $\frac{1}{2^3}$, the fifth is $\frac{3}{2^3}$, the sixth is $\frac{5}{2^3}$, the seventh is $\frac{7}{2^3}$. All four are in the 3^{rd} row.

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Example 2.1 To determine the first element in say, $[0, \frac{1}{1000}]$, we note that

$$\frac{1}{1000} < \frac{1}{2^9}.$$

Therefore, no midpoints appear in $[0, \frac{1}{1000}]$ till the 10^{th} row

The 10^{th} row has the midpoints

$$\frac{1}{2^{10}}, \frac{3}{2^{10}}, \frac{5}{2^{10}}, \ldots. \frac{1023}{2^{10}}.$$

Since

$$\frac{1}{2^{10}} < \frac{1}{1000},$$

the first element of $[0, \frac{1}{1000}]$ is $\frac{1}{2^{10}}$. $\square$

Example 2.2 To find the first element in $(\frac{1}{16}, \frac{1}{8})$, we note that no midpoints of the 4th row appear in $(\frac{1}{16}, \frac{1}{8})$. Both

$$\frac{1}{2^4} = \frac{1}{16} \text{ , and } \frac{3}{2^4} = \frac{3}{16},$$

are not in $(\frac{1}{16}, \frac{1}{8})$.

The fifth row has the midpoints

$$\frac{1}{2^5}, \frac{3}{2^5}, \frac{5}{2^5}, \ldots. \frac{31}{2^5}$$

$\frac{1}{2^5} = \frac{1}{32}$ is not in the interval $(\frac{1}{16}, \frac{1}{8})$.

But $\frac{3}{2^5}$ is in it, and it is the first element of the real numbers interval $(\frac{1}{16}, \frac{1}{8})$.$\square$

3.

The Sequencing of the Reals

With the natural numbers $\mathbb{N}$, that is,

$$1, 2, 3, \ldots.$$

we associate the smallest infinity,

$$Card\mathbb{N} \equiv \textit{The Total Number of natural numbers}.$$

The Rational numbers $\mathbb{Q}$, that is fractions such as

$$\frac{3}{7}, \frac{17}{5}, \frac{777}{11}, \ldots.$$

can be listed in an infinite square with side $Card\mathbb{N}$

$$\begin{array}{llllllllllll}
\frac{1}{1} & & \frac{1}{2} & & \frac{1}{3} & & \frac{1}{4} & & \frac{1}{5} & & \ldots & \ldots & * \\
 & & & & & & & & \ldots & \ldots & \ldots & & \ldots \\
\frac{2}{1} & & \frac{2}{2} & & \frac{2}{3} & & \frac{2}{4} & \ldots & \ldots & \ldots & * & \ldots & * \\
 & & & & & & \ldots & \ldots & \ldots & & \ldots \\
\frac{3}{1} & & \frac{3}{2} & & \frac{3}{3} & \ldots & \ldots & \ldots & * & \ldots & * \\
 & & & & \ldots & \ldots & \ldots & & \ldots \\
\frac{4}{1} & & \frac{4}{2} & \ldots & \ldots & \ldots. & * & \ldots & * \\
 & & \ldots & \ldots & \ldots & & \ldots \\
\frac{5}{1} & \ldots & \ldots & \ldots & * & \ldots & * \\
\ldots & \ldots & \ldots & & \ldots \\
* & \ldots & * & \ldots & * \\
 & & \ldots \\
* & \ldots & *
\end{array}$$

Therefore, the total number of Rationals is

$$Card\mathbb{Q} = Card\mathbb{N} \times Card\mathbb{N}$$

The real numbers, $\mathbb{R}$ can be represented in base 2, as all the sequences of $0's$, and $1's$. Therefore, the total number of real numbers is

$$Card\mathbb{R} = 2^{Card\mathbb{N}}$$

Evidently, there are more reals, than rationals, and more rationals then natural numbers. Therefore, we have

$$Card\mathbb{N} \le (Card\mathbb{N})^2 \le 2^{Card\mathbb{N}}.$$

We may be tempted to say that the three infinities are strictly greater than each other, but the Even Numbers, that are a subset of the natural numbers, are sequenced, and their total number equals $Card\mathbb{N}$.

It is not obvious that the set of the rational numbers can be sequenced. In fact, the property that between any two rationals there is another rational number, and consequently, infinitely many real numbers, does not characterize the natural numbers.

It is well-known that the rational numbers can be sequenced by Cantor's Zig-zag.

But Cantor's mapping

$$(i, j) \to \frac{1}{i+j}$$

is not one-one.

In [Dan3], we exhibit an injection from the Rationals into the Natural numbers. Thus,

$$Card\mathbb{Q} = Card\mathbb{N}\,.$$

That is,

$$(Card\mathbb{N})^2 = Card\mathbb{N}\,.$$

Here, our well-ordering of the Real Numbers is a one-one mapping that sequences them, and we have

$$Card\mathbb{R} = Card\mathbb{N}\,,$$

That is,

$$2^{Card\mathbb{N}} = Card\mathbb{N}\,.$$

That is, the infinity that represents the total number of the real numbers equals the infinity that represents the total number of the natural numbers.

We shall present four more proofs that establish firmly the equality $2^{Card\mathbb{N}} = Card\mathbb{N}$.

4.

Proof by Tarski result

For each $n = 1,2,3...$, we have

$$\underbrace{2^{n+1} - 2}_{=2+2^2+2^3+....+2^n} \leq Card\mathbb{N}.$$

That is, for each $n = 1,2,3...$,

$$2 + 2^2 + 2^3 + ... + 2^n \leq Card\mathbb{N}.$$

where the sum of the powers of 2, is the number of the real numbers up to the n^{th} row.

Tarski ([Tars], or [Sierp, p.174]) proved that for any sequence of cardinal numbers, m_1, m_2, m_3,..., and a cardinal m, the partial sums inequalities

$$m_1 + m_2 + ... + m_n \leq m,$$

for $n = 1,2,3....$ imply the series inequality

$$m_1 + m_2 + ... + m_n + ... \leq m.$$

Applying Tarski result, we obtain

$$2 + 2^2 + 2^3 + ... + 2^n + ... \leq Card\mathbb{N}.$$

Now,

$$\begin{aligned} 2 + 2^2 + ... + 2^n + ... &= \lim_{n \to \infty} \left(2^{n+1} - 2\right) \\ &= \lim_{n \to \infty} 2^n \\ &= \lim_{n \to \infty} \underbrace{2 \times 2 \times ... \times 2}_{n \text{ times}} \\ &= \underbrace{2 \times 2 \times 2 \times ...}_{\text{infinitely many times}} \\ &= 2^{Card\mathbb{N}}. \end{aligned}$$

Hence,

$$2^{Card\mathbb{N}} \leq Card\mathbb{N}.$$

Since we have $Card\mathbb{N} \leq 2^{Card\mathbb{N}}$, we conclude

$$Card\mathbb{N} = 2^{Card\mathbb{N}}.\square$$

5.

Proof by Cardinals, and Ordinals

By [Sierp, p.277], every ordinal number α has a next ordinal number

$$\alpha + 1 > \alpha,$$

and no intermediate ordinal number ξ with $\alpha + 1 > \xi > \alpha$.

The ordinal $\alpha + 7$ is preceded by $\alpha + 6$, and is classified as 1st kind.

The smallest ordinal number that is not preceded by any ordinal is classified as 2nd kind, and is denoted by

$$\omega.$$

By [Sierp, p.288] any 2nd kind ordinal number is the limit of an increasing transfinite sequence of ordinal numbers. In particular,

$$\omega = \lim_{n<\omega} 2^n.$$

By [Sierp, p. 318, Theorem 1], the function

$$f(n) = 2^n$$

is continuous in n, and

$$\omega = \lim_{n<\omega} 2^n = 2^{\lim_{n<\omega} n} = 2^{\omega}.$$

That is,

$$\omega = 2^{\omega}.$$

Now, by [Lev, p. 88, Corollary 2.19]

ω is a cardinal number,

and by [Lev, p.90, Corollary 2.33],

$$Card\mathbb{N} = \omega.$$

Thus, $\omega = 2^{\omega}$ says

$$Card\mathbb{N} = 2^{Card\mathbb{N}}.\square$$

6.

Proof by Cardinality of Ordinals

By a Theorem of Schonflies (1913) [Lev, p. 126, Theorem 2.11], for ordinal numbers α, and β

$$Card(\alpha^{\beta}) = \max(Card(\alpha), Card(\beta)).$$

Therefore,

$$Card(2^{\omega}) = \max(Card(2), Card(\omega)) = Card\mathbb{N}.$$

On the other hand, by [Lev, p.126, (2.9)], exponentiation is a repeated multiplication , and for all ordinals α, and β

$$\alpha^{\beta} = \prod_{\gamma<\beta} \alpha.$$

Hence,

$$2^{\omega} = \prod_{n<\omega} 2.$$

Therefore,

$$Card(2^{\omega}) = Card \prod_{n<\omega} 2.$$

Now, by [Lev, p. 106, proposition 4.15], if a is a well ordered cardinal,

$$\prod_{x \in u} a = a^{Card(u)}.$$

Therefore,

$$\prod_{n < \omega} 2 = 2^{Card(\omega)} = 2^{Card\mathbb{N}}.$$

In conclusion,

$$Card\mathbb{N} = 2^{Card\mathbb{N}}.\square$$

7.

2^{nd} Proof by Tarski Result

Since $2 \le Card\mathbb{N}$,

$$2^{Card\mathbb{N}} \le (Card\mathbb{N})^{Card\mathbb{N}}$$

By [Lev, p.106], or [Sierp, p.183],

$$(Card\mathbb{N})^{Card\mathbb{N}} = \prod_{n=1}^{\infty} Card\mathbb{N}$$

$$= (Card\mathbb{N}) \times (Card\mathbb{N}) \times \ldots .$$

That infinite product,

$$(Card\mathbb{N}) \times (Card\mathbb{N}) \times \ldots$$

is a component of the infinite series

$$(Card\mathbb{N}) + (Card\mathbb{N})^2 + (Card\mathbb{N})^3 + \ldots$$

which, by [Sierp, p.173], is a well-defined cardinal number.

By [Sierp, p.174],

Any component of the series $\le (Card\mathbb{N}) + (Card\mathbb{N})^2 + \ldots$

In particular,

$$(Card\mathbb{N}) \times (Card\mathbb{N}) \times \ldots \le (Card\mathbb{N}) + (Card\mathbb{N})^2 + (Card\mathbb{N})^3 + \ldots$$

Therefore,

$$2^{Card\mathbb{N}} \le (Card\mathbb{N}) + (Card\mathbb{N})^2 + (Card\mathbb{N})^3 + ...$$

Since $Card\mathbb{N} = (Card\mathbb{N})^2$, then for each $n = 1,2,3,...$

$$(Card\mathbb{N}) + (Card\mathbb{N})^2 + ... + (Card\mathbb{N})^n =$$

$$= \underbrace{Card\mathbb{N} + Card\mathbb{N} + ... + Card\mathbb{N}}_{n \text{ times}}$$

$$= n \times Card\mathbb{N}$$

$$\le (Card\mathbb{N}) \times (Card\mathbb{N})$$

$$= Card\mathbb{N}$$

Tarski ([Tars], or [Sierp, p.174]) proved that for any sequence of cardinal numbers, m_1, m_2, m_3,..., and a cardinal m, the partial sums inequalities

$$m_1 + m_2 + ... + m_n \le m,$$

for $n = 1,2,3....$ imply the series inequality

$$m_1 + m_2 + ... + m_n + ... \le m.$$

By Tarski, the infinite series is bounded by $Card\mathbb{N}$,

$$(Card\mathbb{N}) + (Card\mathbb{N})^2 + ... \le Card\mathbb{N}$$

Consequently,

$$2^{Card\mathbb{N}} \le Card\mathbb{N}$$

8.

The Equality of all Infinities

From $Card\mathbb{N} = (Card\mathbb{N})^2$, we have for each $n = 1, 2, 3, ...$

8.1 $$Card\mathbb{N} = (Card\mathbb{N})^n$$

Consequently, we obtain the equality of all infinities

8.2 $$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ...$$

In other words, while infinities, depending on their source, may be written differently, they are all equal, and indistinguishable from each other.

In particular, we conclude that

8.3 Cantor's claim that $Card\mathbb{N} < 2^{card\mathbb{N}}$ **and his Diagonal Proof are false.**

We will show that Cantor's failure is rooted in applying Finite Sets arguments to Infinite Sets.

9.

The Finite Set arguments of Cantor's Diagonal Proof

To prove that

$$CardN < CardR,$$

Cantor listed the reals in $(0,1)$, presumed to be countably many, and exhibited the diagonal element as the one not counted for, in the list.

By [Sierp, p. 57], every real number between 0, and 1, has a unique infinite decimal representation

$$c_1^{(n)}c_2^{(n)}c_3^{(n)}....$$

Missing from that listing is the real number

$$c_1c_2c_3....$$

where

$$c_n = 0, \quad \text{if} \quad c_n^{(n)} \neq 0, \quad \text{and} \quad c_n = 1, \quad \text{if} \quad c_n^{(n)} = 0.$$

What is so crucial about one missing element out of infinitely many? Why cannot we add the one missing diagonal element to the listing?

The listing will remain countable after countably many such additions.

Two infinite sets have the same cardinality, even if one set is “half” of the other. For instance, the natural numbers, and the odd natural numbers have the same cardinality. For infinite sets, we can tolerate the missing of countably many elements, long before we conclude a contradiction.

In other words, Cantor’s diagonal argument does not apply credibly to infinite sets.

10.

The ill-defined "Set of sets" in Cantor's $Card(A) < Card(P(A))$

The inequality

$$CardN < 2^{CardN},$$

seems like a wish modeled after the finite case

$$n < 2^n.$$

But as $n \to \infty$, we have

$$\lim n \leq \lim 2^n.$$

Is the inequality in

$$CardN < 2^{CardN}$$

indeed strict?

By [Sierp, p. 87], Cantor proved that

$P(A)$, the set of all subsets of any given set A, has cardinality greater than the cardinality of A.

If indeed $Card(A) < Card(P(A))$, then for every cardinal number there is a greater cardinal number.

And, in particular, $Card(N) < Card(P(N)) = Card(R)$.

But Cantor's proof uses the concept of Set of sets which for infinite sets is not well-understood, since it may lead to the Russell paradoxial set.

Russell (1903) defined his set of sets y by

$$x \in y \leftrightarrow x \notin x .$$

Then, in particular,

$$y \in y \leftrightarrow y \notin y$$

which is a contradiction.

In fact, as pointed out in [Lev, p.87], if we apply Cantor's theorem to the universal class of all objects V, every subset of V is also a member of V, and we have

$$Card(V) = Card(P(V)) .$$

Avoiding this fact by claiming that V is not a set, while leaving the definition of what is a Set vague enough to suit other results, does not make Cantor's claim credible.

Indeed, we have seen that for the infinite set of the Natural Numbers we have $Card\mathbb{N} = CardP(\mathbb{N})$.

11.

The Continuum Hypothesis

Cantor's Continuum Hypothesis says that there is no set X with $Card\mathbb{N} < CardX < 2^{Card\mathbb{N}}$.

The equality

$$Card\mathbb{N} = 2^{Card\mathbb{N}},$$

resolves the Continuum Hypothesis by default. That is,

11.1 The Continuum Hypothesis is a fact

In other words,

11.2 No Continuum Hypothesis Negation

12.

Non-Cantorian Theory

Since the Continuum Hypothesis is a fact, that follows from the Axioms of Set Theory, there is no independent Axiom that negates it, and can serve as the basis for a Non-Cantorian Theory.

Alternatively, We have shown in [Dan2] that Non-Cantorian theory, with its distinct infinities is characterized by the strict cardinal inequality $Card\mathbb{N} < Card\mathbb{N} \times Card\mathbb{N}$. Since

$$Card\mathbb{N} = Card\mathbb{N} \times Card\mathbb{N},$$

there is no Non-Cantorian Theory, or distinct Non-Cantorian infinities

On either one of these counts we conclude,

12.1 No Non-Cantorian Theory, and no distinct Non-Cantorian Cardinalities

13.

Axiom of Choice, and Well-Ordering

The **Choice Theorem** says that if for each $n = 1, 2, 3, \ldots$ there is a non-empty set of numbers A_n, then we can choose from each A_n one number a_n, and obtain a collection of numbers that has a representative from each A_n.

If we replace the index numbers $n = 1, 2, 3, \ldots$ with an infinite set of numbers I, this choice may not be guaranteed.

There may be an infinite set of numbers I, so that for each index i in it, there is a non-empty set of numbers A_i, with no collection of numbers, that has a representative a_i from each A_i.

The **Axiom of Choice** is the guess that the choice is guaranteed for any infinite set I, and any family of non-empty sets indexed by I.

In [Dan2], we proved that the Axiom of Choice is equivalent to the Continuum hypothesis.

Since the Continuum hypothesis holds, so does the Axiom of Choice. that is,

13.1 The Axiom of Choice is a fact

Furthermore, since all cardinalities equal $Card\mathbb{N}$, then,

$$CardI = Card\mathbb{N},$$

and we have,

13.2 The Axiom of Choice is guaranteed by the Choice Theorem

Similarly,

13.3 The Well-Ordering Axiom is guaranteed by the Well-Ordering Theorem

14.

Transfinite-Induction Axiom

The **Induction Theorem** says that

If a property depends on each number $n = 1, 2, 3, ..$, so that

1) The property holds for the first natural number $n = 1$.
2) If the property holds for the natural number k, we can deduct that it holds for the next number $k+1$.

Then, the property holds for any $n = 1, 2, 3, ...$.

The **Transfinite-Induction Axiom** guesses that the same holds for any infinite index set I.

It says that if I is any well-ordered infinite set of numbers, and if

there is any property that depends on each index i from I, so that

1) The property holds for the first element of I,

2) If the property holds for all the $k's$ that precede the index j, we can conclude that the property holds for j,

Then, the property holds for any index i in I.

The Transfinite Induction Axiom is equivalent to the Axiom of Choice. Hence,

14.1 The transfinite induction Axiom is a fact

Furthermore, since all cardinalities equal $Card\mathbb{N}$, then,

$$CardI = Card\mathbb{N},$$

and we have,

14.2 The Transfinite Induction Axiom is guaranteed by the Induction Theorem

15.

A Countable Set may have non-zero measure

Since the real numbers in any interval are countable, and their measure=the interval's length, is non-zero, we conclude that a countable set can have a non-zero measure. Thus, the statement that any countable set has measure zero is false.

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