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## Infinity

**Hilbert's 1st problem: Cantor's Continuum Hypothesis**

**H. Vic Dannon**

**Abstract:** *There is no set* $X$ *with* $\aleph_0 < cardX < 2^{\aleph_0}$ **.**

The continuum hypothesis reflects Cantor's inability to construct a set with cardinality between that of the natural numbers and that of the real numbers. His approach was constructive, But if he was right, such set cannot be constructed, and he needed a proof by contradiction.

That contradiction remained out of reach at Cantor's time. Even when Hilbert presented Cantor's Continuum Hypothesis as his first problem, the tools for the solution did not exist. Tarski obtained the essential lemma only in 1948. But it was not utilized and the problem remained open.

In 1963, Cohen proved that if the commonly accepted postulates of set theory are consistent, then adding the negation of the hypothesis does not result in inconsistency. [1, p.97]. That left the impression that Hilbert's first problem was either solved, or is unsolvable. But it became commonly accepted that the problem was closed.

Not that the question was settled. According to [2,p.189], "Mathematicians do not tend to assume the Continuum Hypothesis as an additional axiom of set theory mostly since they cannot convince themselves that this statement is "true" as many of them have done for the axioms of ZFC including the axiom of choice. However, a mathematician trying to prove a theorem will usually regard a proof of the theorem from the generalized continuum hypothesis as a partial success"

Cohen result was interpreted to mean that there is another set theory that utilizes the negation of the continuum hypothesis. However, the alternative set theory was never developed, and we shall show here that the Continuum Hypothesis can be proved under the assumptions of Cantor's set theory.

**Cantor's Proof of Rational Countability** **H. Vic Dannon**

**Abstract** Cantor's original proof that the rational numbers are countable uses a mapping that is not one-one. Thus, the countability of the rationals was not proved by Cantor.

We show by cardinal number methods that the rationals are countable, and supply a second proof by exhibiting a one-one mapping from the rationals into the natural numbers.

**Cantor's Set, The Cardinality of the Reals, and the Continuum Hypothesis** **H. Vic Dannon.**

**Abstract** The Cantor set is obtained from the closed unit interval [0,1] by a sequence of deletions of middle third open intervals.

Apparently, Cantor constructed this set while attempting to find a cardinality between $CardN$, and $2^{CardN}$. The length-less, nowhere dense Cantor set which is almost a void in (0,1), has cardinality equal to $CardR = 2^{CardN}$, and perhaps led Cantor to his Continuum Hypothesis that there is no set $X$ with $CardN < CardX < 2^{CardN}$.

The puzzling Cantor's set establishes that cardinality is unrelated to measure, and that $CardR$ equals the power of a nowhere-dense set.

How can such a meager set have a cardinality greater than $CardN$?

Is it possible that $Card(0,1)$ may not be any greater than $CardN$?

In the following, we attempt to answer these questions.

**Non-Cantorian Set Theory** **H. Vic Dannon.**

**Abstract** In [1] we showed that in Cantor's set theory,

$$Card(0,1) = Card(rationals).$$

Since the rationals are countable,

$$Card(rationals) = \aleph_0^2 = \aleph_0 ,$$

and

$$\aleph_0 = Card(0,1) = 2^{\aleph_0} .$$

This disproves Cantor's claim that

$$\aleph_0 < 2^{\aleph_0} ,$$

and leads to a single infinity

$$\aleph_0 = 2^{\aleph_0} .$$

Will a non-Cantorian set theory allow for more infinities?

The existence of a non-Cantorian set theory was established in 1963 by Cohen's work on Cantor's Continuum Hypothesis that there is no

set $X$ with $\aleph_0 < CardX < 2^{\aleph_0}$.

Cohen proved that if the commonly accepted postulates of set theory are consistent, then the addition of the negation of the hypothesis does not result in inconsistency [2].

Cohen's result was interpreted to mean that there is a set theory where the negation of the Continuum Hypothesis holds. However, non-Cantorian cardinal numbers were not found, and the non-Cantorian set theory was never developed.

To develop a non Cantorian set theory, we will assume the negation of the Continuum Hypothesis, which is based on Cantor's claim that $\aleph_0 < 2^{\aleph_0}$. In [1] we disproved that claim, but here we will need to allow $\aleph_0 < 2^{\aleph_0}$ as an assumption. We aim to show that even with that disproved assumption, non Cantorian set theory does not exist.

To that end, we re-examine our proof of the Continuum Hypothesis in Cantor's set theory [3]. A close scrutiny of that proof reveals that rationals countability is equivalent to the Continuum Hypothesis.

**Cardinality, Measure, and Category** **H. Vic Dannon**

**Abstract** Lebesgue procedure to find the measure of a general set leads to contradictions. In particular, the set of rational numbers does not have measure zero. In fact, by Lebesgue own criteria, the set of rational numbers in $[0,1]$ is not measurable.

**The Continuum Hypothesis, The Axiom Of Choice, and Non-Cantorian Theory** **H. Vic Dannon.**

**Abstract** We prove that the Continuum Hypothesis is equivalent to the Axiom of Choice. Thus, the Hypothesis-Negation is equivalent to the Axiom of No-Choice.

The Non-Cantorian Axioms impose a Non-Cantorian definition of cardinality, that is different from Cantor's cardinality imposed by the Cantorian Axioms.

The Non-Cantorian Theory is the Zermelo-Fraenkel Theory with the No-Choice Axiom, or the Hypothesis-Negation.

This Theory has distinct infinities.

**The Continuum Hypothesis** **H. Vic Dannon**

**Abstract** We prove that the Continuum Hypothesis is equivalent to the Axiom of Choice. Thus, the Negation of the Continuum Hypothesis, is equivalent to the Negation of the Axiom of Choice.

The Non-Cantorian Axioms impose a Non-Cantorian definition of cardinality, that is different from Cantor's cardinality imposed by the Cantorian Axioms.

The Non-Cantorian Theory is the Zermelo-Fraenkel Theory with the Negation of the Axiom of Choice, and with the Negation of the Continuum Hypothesis. This Theory has distinct infinities.

The Continuum Hypothesis says that there is no infinity between the infinity of the natural numbers, and the infinity of the real numbers.

The account here, follows my attempts to understand the Hypothesis.

In 2004, I thought that I found a proof for the Hypothesis. That turned out to be an equivalent statement to the Continuum Hypothesis.

That condition is the key to the Non-Cantorian Theory, but it took me until 2007 to comprehend its meaning, and to apply it.

The first hurdle is to comprehend that the condition holds in Non-Cantorian Cardinality.

The second hurdle is to realize that the Non-Cantorian cardinality is imposed by an Axiom of the Non-Cantorian Theory.

The Non-Cantorian Theory is essential because Cantor's Theory replaces facts with wishes, attempts to prove Axioms as if they were Theorems, and borrows from the Non-Cantorian Theory Axioms that do not hold in Cantor's Theory.

At the end, Cantor's Theory produces only one infinity.

To obtain distinct infinities, we need the Non-Cantorian Theory.

The equivalence between the Hypothesis and the Axiom of Choice, renders the consistency result of Godel trivial.

The equivalence between the Negation of the Hypothesis, and the Negation of the Axiom of Choice says that Cohen's result must be wrong. The mixing of the Axiom of Choice with its Negation must lead to inconsistency.

The Hypothesis is the most illusive statement of the Axiom of Choice.

**Well-Ordering of the Reals, Equality of all Infinities, and the Continuum Hypothesis** **H. Vic Dannon**

**Abstract:** With the natural numbers $\mathbb{N}$, that is,

$$1,2,3,....$$

we associate the smallest infinity,

$$Card\mathbb{N} \equiv \textit{The Total Number of natural numbers.}$$

The Rational numbers $\mathbb{Q}$, that is fractions such as

$$\frac{3}{7},\frac{17}{5},\frac{777}{11},....$$

can be listed in an infinite square with side $Card\mathbb{N}$

$$\begin{array}{ccccccccccccc}
\frac{1}{1} & & \frac{1}{2} & & \frac{1}{3} & & \frac{1}{4} & & \frac{1}{5} & & \cdots & \cdots & * \\
 & & & & & & & & \cdots & \cdots & \cdots & & \cdots \\
\frac{2}{1} & & \frac{2}{2} & & \frac{2}{3} & & \frac{2}{4} & \cdots & \cdots & \cdots & * & \cdots & * \\
 & & & & & & \cdots & \cdots & \cdots & & \cdots & & \\
\frac{3}{1} & & \frac{3}{2} & & \frac{3}{3} & \cdots & \cdots & \cdots & * & \cdots & * & & \\
 & & & & \cdots & \cdots & \cdots & & \cdots & & & & \\
\frac{4}{1} & & \frac{4}{2} & \cdots & \cdots & .... & * & \cdots & * & & & & \\
 & & \cdots & \cdots & \cdots & & \cdots & & & & & & \\
\frac{5}{1} & \cdots & \cdots & \cdots & * & \cdots & * & & & & & & \\
\cdots & \cdots & \cdots & & \cdots & & & & & & & & \\
* & \cdots & * & \cdots & * & & & & & & & & \\
 & & \cdots & & & & & & & & & & \\
* & \cdots & * & & & & & & & & & &
\end{array}$$

Therefore, the total number of Rationals is

$$Card\mathbb{Q} = Card\mathbb{N} \times Card\mathbb{N}$$

The real numbers, $\mathbb{R}$ can be represented in base $2$, as all the sequences of $0's$, and $1's$. Therefore, the total number of real numbers is

$$Card\mathbb{R} = 2^{Card\mathbb{N}}$$

Evidently, there are more reals, than rationals, and more rationals then natural numbers. Therefore, we have

$$Card\mathbb{N} \leq (Card\mathbb{N})^2 \leq 2^{Card\mathbb{N}}.$$

It is well-known that the total number of the even integers, equals the total number of the odd integers, and equals $Card\mathbb{N}$, because there is a one-one, onto correspondence between those sets of numbers.

It is not obvious that such correspondence exists between the sets of the natural numbers and the set of the real numbers. In fact, the property that between any two reals there is another real number, and consequently, infinitely many real numbers, does not characterize the natural numbers. Between two natural numbers, there may be at most finitely many natural numbers.

Nevertheless, we will prove that

$$2^{Card\mathbb{N}} \leq Card\mathbb{N}.$$

That is, the infinity that represents the total number of the natural numbers equals the infinity that represents the total number of the real numbers.

Consequently, we obtain the equality of all infinities

$$Card\mathbb{N} = (Card\mathbb{N})^2 = .... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ...$$

In other words, while infinities may be written differently, based on their source, they are all equal, and indistinguishable from each other.

We can safely use the old symbol

$$\infty$$

because there are no two distinct infinities.

This resolves by default, Cantor's Continuum Hypothesis that says that there is no set $X$ with $Card\mathbb{N} < CardX < 2^{Card\mathbb{N}}$.

We shall present five proofs to establish our claim.