THE EQUALITY OF

ALL INFINITIES

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FROM THE NATURAL NUMBERS TO THE CONTINUUM

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PREFACE

Infinity is the number of all the Counting Numbers, 1, 2, 3,… And it is the number of the all Quotients of Counting Numbers, such as 3/17,22/7,…

And it is the number of all the Real numbers between 0, and 1, such as 0.1234567891011121314151617….

It seems that there are more Real numbers than Quotients, and more Quotients than Counting Numbers.

But we can arrange the Quotients in a sequence, and count them as in 1,2,3,…

So the infinity of Quotients is no larger than the infinity of the Counting numbers.

Can we arrange the real numbers between 0, and 1 in a sequence, and count them as in 1,2,3,…?

Here, we sequence the real numbers, and conclude that all infinities are equal.

1
THE INFINITY OF THE COUNTING NUMBERS

To conceive the idea of infinity, we consider the sequence of Counting numbers, as it evolves to infinity.

Namely, the sequence

$$1, 2, 3, 4, \ldots\ldots$$

that continues endlessly, has infinity at its "no-end".

The

Infinity of the Counting numbers

is an infinity that we denote by

$$Inf\{1, 2, 3\ldots\}.$$

2
THE INFINITY OF QUOTIENTS

The Quotients of Counting numbers such as

$$\frac{3}{7}, \frac{17}{5}, \frac{777}{11}, ...$$

can be listed in an infinite square

Each side of the square has

$$Inf\{1, 2, 3, ...\}$$

Quotients.

Therefore, the infinity of Quotients is

$$Inf(Quotients) = Inf\left\{1, 2, 3, ...\right\} \times Inf\left\{1, 2, 3, ...\right\}$$

$$
\begin{array}{ccccccccccccccc}
\frac{1}{1} & & \frac{1}{2} & & \frac{1}{3} & & \frac{1}{4} & & \frac{1}{5} & & \dots & \dots & * & \dots & * \\
 & & & & & & & & \dots & \dots & \dots & & \dots & & \\
\frac{2}{1} & & \frac{2}{2} & & \frac{2}{3} & & \frac{2}{4} & \dots & \dots & \dots & * & \dots & * & & \\
 & & & & & & \dots & \dots & \dots & & \dots & & & & \\
\frac{3}{1} & & \frac{3}{2} & & \frac{3}{3} & \dots & \dots & \dots & * & \dots & * & & & & \\
 & & & & \dots & \dots & \dots & & \dots & & & & & & \\
\frac{4}{1} & & \frac{4}{2} & \dots & \dots & \dots. & * & \dots & * & & & & & & \\
 & & \dots & \dots & \dots & & \dots & & & & & & & & \\
\frac{5}{1} & \dots & \dots & \dots & * & \dots & * & & & & & & & & \\
\dots & \dots & \dots & & \dots & & & & & & & & & & \\
* & \dots & * & \dots & * & & & & & & & & & & \\
 & & \dots & & & & & & & & & & & & \\
* & \dots & * & & & & & & & & & & & & \\
\dots & & & & & & & & & & & & & & \\
* & & & & & & & & & & & & & &
\end{array}
$$

3
THE INFINITY OF REAL NUMBERS

By the “real line”, we mean a line on which we marked some point as zero. Then at equal intervals to the right of 0, we mark the counting numbers

$$1, 2, 3...$$

and at mirror images of these points to the left of 0, we mark the negative counting numbers

$$.... -3, -2, -1,$$

To every point on the line, in between our marks of the counting numbers, we assign a number that measures its distance from 0.

We get many numbers, a continuum of numbers, all the real numbers.

The real numbers, have the same property as the Quotients that between any two real numbers there is another real number.

Thus, finding the infinity of the real numbers requires a specific arrangement of them.

A quotient is determined by two counting numbers: its numerator, and its denominator. Each pair of counting numbers determines a quotient, and all the pairs of counting numbers represent all the Quotients.

Similarly, a real number can be determined by using the two digits, 0, and 1. Each infinite sequence of 0's and 1's represents a real number, and all the infinite sequences of 0's and 1's represent all the real numbers.

We will attempt to find the Infinity of the real numbers based on this representation.

We will express the Infinity of the real numbers in terms of the infinity of the Counting numbers.

The Infinity that we aim to obtain is

$$Inf(Reals) = 2^{Inf\{1,2,3,...\}}.$$

We start by considering the real numbers that are fractions, that are at least 0, and at most 1. These fractions may be given by decimal expansions such as

$$0.137690378354......$$

We will see later that our arguments apply to all the real numbers.

We will construct the fractions between 0, and 1, by a process that starts with two fractions, and keeps doubling the numbers of fractions indefinitely.

We will have 2 fractions, 4 fractions, 8 fractions, 16 fractions,….

and after indefinitely many steps, $2^{Inf\{1,2,3,...\}}$ fractions.

The fraction

$$0.875000.....0....$$

equals

$$8\times\frac{1}{10}+7\times\frac{1}{10^2}+5\times\frac{1}{10^3}+0\times\frac{1}{10^4}+0\times\frac{1}{10^5}+....$$

Therefore, we can represent it by the coefficients of the powers of $\frac{1}{10}$, as

$$\langle 8,7,5,0,0,0,...0,....\rangle.$$

If, instead of powers of $\frac{1}{10}$, we use powers of $\frac{1}{2}$, then

$$0.875000.....0....$$

equals

$$1\times\frac{1}{2}+1\times\frac{1}{2^2}+1\times\frac{1}{2^3}+0\times\frac{1}{2^4}+0\times\frac{1}{2^5}+....$$

Then, in terms of powers of $\frac{1}{2}$, we can represent it by

$$\langle 1,1,1,0,0,....0,....\rangle$$

Any fraction between 0, and 1, can be written as sums of powers of $\frac{1}{2}$.

The number 0 is

$$0 \times \frac{1}{2} + 0 \times \frac{1}{2^2} + 0 \times \frac{1}{2^3} + 0 \times \frac{1}{2^4} + \dots$$

Therefore, we can represent the number 0 by a sequence of zeros

$$\langle 0, 0, 0, 0, 0, \dots .0, \dots . \rangle.$$

The number $\frac{1}{2}$ is

$$1 \times \frac{1}{2} + 0 \times \frac{1}{2^2} + 0 \times \frac{1}{2^3} + 0 \times \frac{1}{2^4} + \dots$$

Therefore, we can represent the number $\frac{1}{2}$ by

$$\langle 1, 0, 0, 0, 0, \dots .0, \dots . \rangle.$$

We see that

$$\langle 0, 0, 0, 0, 0, \dots .0, \dots . \rangle \leftrightarrow 0$$

$$\langle 1, 0, 0, 0, 0, \dots .0, \dots . \rangle \leftrightarrow \frac{1}{2}$$

Next, we vary the second digit in the representing sequence, and we have four fractions

$$\langle 0, 0, 0, 0, 0, \dots .0, \dots . \rangle \leftrightarrow 0$$

$$\langle 0,1,0,0,0,....0,....\rangle \leftrightarrow \frac{1}{4}$$

$$\langle 1,0,0,0,0,....0,....\rangle \leftrightarrow \frac{2}{4}$$

$$\langle 1,1,0,0,0,....0,....\rangle \leftrightarrow \frac{3}{4}$$

Next, we vary the third digit in the representing sequence, and we have eight fractions

$$\langle 0,0,0,0,0,....0,....\rangle \leftrightarrow 0$$

$$\langle 0,0,1,0,0,....0,....\rangle \leftrightarrow \frac{1}{8}$$

$$\langle 0,1,0,0,0,....0,....\rangle \leftrightarrow \frac{2}{8}$$

$$\langle 0,1,1,0,0,....0,....\rangle \leftrightarrow \frac{3}{8}$$

$$\langle 1,0,0,0,0,....0,....\rangle \leftrightarrow \frac{4}{8}$$

$$\langle 1,0,1,0,0,....0,....\rangle \leftrightarrow \frac{5}{8}$$

$$\langle 1,1,0,0,0,....0,....\rangle \leftrightarrow \frac{6}{8}$$

$$\langle 1,1,1,0,0,....0,....\rangle \leftrightarrow \frac{7}{8}$$

Next, we vary the fourth digit in the representing sequence, and we have sixteen fractions

$$\langle 0,0,0,0,0,....0,....\rangle \leftrightarrow 0$$

$$\langle 0,0,0,1,0,....0,....\rangle \leftrightarrow \frac{1}{16}$$

$$\langle 0,0,1,0,0,....0,....\rangle \leftrightarrow \frac{2}{16}$$

$$\langle 0,0,1,1,0,....0,....\rangle \leftrightarrow \frac{3}{16}$$

$$\langle 0,1,0,0,0,....0,....\rangle \leftrightarrow \frac{4}{16}$$

$$\langle 0,1,0,1,0,....0,....\rangle \leftrightarrow \frac{5}{16}$$

$$\langle 0,1,1,0,0,....0,....\rangle \leftrightarrow \frac{6}{16}$$

$$\langle 0,1,1,1,0,....0,....\rangle \leftrightarrow \frac{7}{16}$$

$$\langle 1,0,0,0,0,....0,....\rangle \leftrightarrow \frac{8}{16}$$

$$\langle 1,0,0,1,0,....0,....\rangle \leftrightarrow \frac{9}{16}$$

$$\langle 1,0,1,0,0,....0,....\rangle \leftrightarrow \frac{10}{16}$$

$$\langle 1,0,1,1,0,....0,....\rangle \leftrightarrow \frac{11}{16}$$

$$\langle 1,1,0,0,0,....0,....\rangle \leftrightarrow \frac{12}{16}$$

$$\langle 1,1,0,1,0,....0,....\rangle \leftrightarrow \frac{13}{16}$$

$$\langle 1,1,1,0,0,....0,....\rangle \leftrightarrow \frac{14}{16}$$

$$\langle 1,1,1,1,0,....0,....\rangle \leftrightarrow \frac{15}{16}$$

Each step of this construction of the fractions between 0, and 1, doubles the number of the fractions that we obtain.

After indefinitely many such steps, we obtain all the sequences which elements are 0's , and 1's.

The number of such sequences is

$$\underbrace{2 \times 2 \times 2 \times ...... \times 2 \times ....}_{Inf\{1,2,3,....\}\text{ many times}} = 2^{Inf\{1,2,3,....\}}.$$

Consequently, the Infinity of the real numbers between 0, and 1, is $2^{Inf\{1,2,3,....\}}$.

But this is also the Infinity of all the real numbers.

Indeed, the sequences which elements are 0's , and 1's, represent all the real numbers.

For instance, if we use the notations

$$2^0 = 1,$$

$$2^{-3} = \frac{1}{2^3}$$

then, the number 67.5 in powers of 2, is

$$1 \times 2^6 + 0 \times 2^5 + 0 \times 2^4 + 0 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$$
$$+1 \times 2^{-1} + 0 \times 2^{-2} + 0 \times 2^{-3} + ...$$

and it is represented by a sequence of 0's , and 1's.

4

COMPARING INFINITIES

How do we compare the infinities of the Counting numbers, the Quotients, and the reals?

Observing that there are more reals than Quotients and more Quotients than Counting numbers, gives us three infinities, each greater than the other. But we may not be sure whether these infinities may equal each other.

Thus, we have,

$$Inf\left\{1,2,3,...\right\} \leq Inf(Quotients) \leq Inf(\mathrm{Re}\,als).$$

We may be tempted to say that the three infinities are strictly greater than each other, but a set smaller than the Counting numbers can have Infinity $Inf\{1,2,3,\ldots\}$.

For instance, the Infinity of the Even Counting numbers is $Inf\{1,2,3,...\}$.

Could the Quotients too have Infinity that equals $Inf\{1,2,3,...\}$?

Indeed, the Infinity of the Quotients equals $Inf\{1,2,3,...\}$.

By the *Effective Countability Axiom*

> *Any infinite sequence of distinct numbers has infinity that equals* $Inf\{1,2,3,....\}$

Cantor saw that the Quotients can be sequenced, and that their Infinity is $Inf\{1,2,3,....\}$.

We show here that the Real numbers too can be sequenced, and their Infinity too is $Inf\{1,2,3,...\}$.

This will say that all Infinities are equal.

5

THE QUOTIENTS CAN BE SEQUENCED

The Quotients have the property that between any two Quotients there is another quotient.

For instance, between the two Quotients,

$$\frac{7}{17}, \text{ and } \frac{77}{5},$$

there is the quotient

$$\frac{1}{2}\left(\frac{7}{17}+\frac{77}{5}\right).$$

This may interfere with the sequencing of the Quotients, but as we have seen, the Quotients can be arranged in an infinite square.

Then, by following a Zig-Zag line through the square, the Quotients can be sequenced.

Here are the Quotients sequenced by the Zig-Zag line.

$$
\begin{array}{ccccccccccccccc}
\frac{1}{1} & \rightarrow & \frac{1}{2} & & \frac{1}{3} & \rightarrow & \frac{1}{4} & & \frac{1}{5} & \rightarrow & \ldots & \ldots & * & \ldots & * \\
& \swarrow & & \nearrow & & \swarrow & & \nearrow & \ldots & \ldots & \ldots & \nearrow & \ldots & \swarrow & \\
\frac{2}{1} & & \frac{2}{2} & & \frac{2}{3} & & \frac{2}{4} & \ldots & \ldots & \ldots & * & \ldots & * & & \\
\downarrow & \nearrow & & \swarrow & & \nearrow & \ldots & \ldots & \ldots & \nearrow & \ldots & \swarrow & & & \\
\frac{3}{1} & & \frac{3}{2} & & \frac{3}{3} & \ldots & \ldots & \ldots & * & \ldots & * & & & & \\
& \swarrow & & \nearrow & \ldots & \ldots & \ldots & \nearrow & \ldots & \swarrow & & & & & \\
\frac{4}{1} & & \frac{4}{2} & \ldots & \ldots & \ldots. & * & \ldots & * & & & & & & \\
\downarrow & \nearrow & \ldots & \ldots & \ldots & \nearrow & \ldots & \swarrow & & & & & & & \\
\frac{5}{1} & \ldots & \ldots & \ldots & * & \ldots & * & & & & & & & & \\
\ldots & \ldots & \ldots & \nearrow & \ldots & \swarrow & & & & & & & & & \\
* & \ldots & * & \ldots & * & & & & & & & & & & \\
\downarrow & \nearrow & \ldots & \swarrow & & & & & & & & & & & \\
* & \ldots & * & & & & & & & & & & & & \\
\ldots & \swarrow & & & & & & & & & & & & & \\
* & \rightarrow & & & & & & & & & & & & &
\end{array}
$$

6
COUNTABILITY AXIOM

By the Effective Countability Axiom, the sequencing of the Quotients establishes that their Infinity is $Inf\{1,2,3,....\}$.

That is,

- $Inf\{1,2,3,....\} \times Inf\{1,2,3,....\} = Inf\{1,2,3,....\}$

This equality is the *Countability Axiom.*

The term Axiom means that we may not be sure that we can prove it:

For instance, when we follow the zig-zag through the infinite square of the Quotients, we get to infinitely many diagonals of infinite length.

We have to get through those infinitely many infinite diagonals, before we get the other side and reach the paths with finitely many quotients.

We cannot be sure that going through infinitely many infinite sequences, will be like going through one infinite sequence, and we would rather call this type of sequencing an Axiom.

Not being a fact, the Countability Axiom will have a Negation, the inequality,

■ $Inf\left\{1,2,3,....\right\} \times Inf\left\{1,2,3,....\right\} > Inf\left\{1,2,3,....\right\}$

is the *Non-Countability Axiom.*

The inequality is a basis to a Theory that has infinitely many infinities.

If we denote $\omega = \inf\{1,2,3,...\}$, we see that in that alternative theory, we will have the infinities

$$\omega \times \omega,$$

$$\omega \times \omega \times \omega,$$

$$\omega \times \omega \times \omega \times \omega,$$

....

But in 2010 we have disproved the inequality, and showed that the Countability Axiom is a FACT, that may not be negated.

Therefore, its negation, the Non-countability Axiom, is a falsehood.

7

CANTOR'S SET

Cantor convinced himself that the infinity of the real numbers is strictly greater than the infinity of the Counting numbers.

That led Cantor to the question

Is there an infinity in between the infinity of the reals and the infinity of the Counting numbers?

The candidate for such in-between-infinity are the Quotients.

But the Quotients can be sequenced, and any set that may be sequenced, is "*effectively countable*", and has the infinity of the Counting numbers.

As Cantor searched for a set with infinity that is in-between, he found the set that bears his name the *Cantor set.*

Cantor's Set can lead to the sequencing of the real numbers, but Cantor never saw that.

To see what Cantor missed, we turn to the construction of Cantor's Set.

The Cantor set is obtained by an inductive process similar to our construction here of the real numbers.

We start with the interval of numbers between 0, and 1.

$$0 \leq x \leq 1,$$

First,

we delete the middle third open interval,

$$\frac{1}{3} < x < \frac{2}{3}$$

This leaves us with the two intervals

$$0 \leq x \leq \frac{1}{3}$$

and

$$\frac{2}{3} \leq x \leq 1.$$

and produces the two quotient endpoints

$$\frac{1}{3}, \quad and \quad \frac{2}{3},$$

that will remain in Cantor's Set after indefinitely many deletions of middle third intervals.

Second,

we delete from $0 \le x \le \frac{1}{3}$, the middle third open interval

$$\frac{1}{9} < x < \frac{2}{9},$$

and we delete from $\frac{2}{3} \le x \le 1$, the middle third open interval

$$\frac{7}{9} < x < \frac{8}{9}.$$

This leaves us with the four intervals

$$0 \le x \le \frac{1}{9}$$

$$\frac{2}{9} \le x \le \frac{1}{3}$$

$$\frac{2}{3} \le x \le \frac{7}{9},$$

$$\frac{8}{9} \le x \le 1,$$

and produces the new four quotient endpoints

$$\frac{1}{9}, \quad \frac{2}{9}, \quad \frac{7}{9}, \quad and \quad \frac{8}{9},$$

that will remain in the Cantor set after indefinitely many deletions of middle third intervals.

Third,

we delete

from $0 \le x \le \frac{1}{9}$ the interval $\frac{1}{27} < x < \frac{2}{27}$,

from $\frac{2}{9} \le x \le \frac{3}{9}$ the interval $\frac{7}{27} < x < \frac{8}{27}$,

from $\frac{2}{3} \le x \le \frac{7}{9}$ the interval $\frac{19}{27} < x < \frac{20}{27}$,

from $\frac{8}{9} \le x \le 1$ the interval $\frac{25}{27} < x < \frac{26}{27}$.

This leaves us with the eight intervals

$$0 \leq x \leq \frac{1}{27},$$

$$\frac{2}{27} \leq x \leq \frac{1}{9},$$

$$\frac{2}{9} \leq x \leq \frac{7}{27},$$

$$\frac{8}{27} \leq x \leq \frac{1}{3},$$

$$\frac{2}{3} \leq x \leq \frac{19}{27},$$

$$\frac{20}{27} \leq x \leq \frac{7}{9},$$

$$\frac{8}{9} \leq x \leq \frac{25}{27},$$

$$\frac{26}{27} \leq x \leq 1,$$

and produces the <u>new</u> eight quotient endpoints,

$$\frac{1}{27}, \quad \frac{2}{27}, \quad \frac{7}{27}, \quad \frac{8}{27}, \quad \frac{19}{27}, \quad \frac{20}{27}, \quad \frac{25}{27}, \quad and \quad \frac{26}{27},$$

that will remain in the Cantor set after indefinitely many deletions of middle third intervals.

In the fourth step of the construction of the Cantor set, we delete eight middle third open intervals.

This produces sixteen intervals, and sixteen new quotient endpoints, that will remain in the Cantor set after indefinitely many deletions of middle third intervals.

Indefinitely many such steps will produce

$$\underbrace{2 \times 2 \times 2 \times ...... \times 2 \times ....}_{Inf\{1,2,3,....\} \text{ many times}} = 2^{Inf\{1,2,3,....\}}$$

new, and distinct quotient endpoints.

We see that the Quotients contain a strictly increasing sequence of $2^{Inf\{1,2,3,....\}}$ distinct Quotients.

This was missed by Cantor, and everyone else whoever looked at the Cantor set.

This leads to the equality of the infinity of the real numbers, and the Counting numbers.

8
Cantor's-Set Sequencing of the Real Numbers

We have seen that the real numbers can be constructed by indefinitely many steps, in each of which the number of real numbers that we obtain is doubled.

We have seen that the Cantor set is constructed by indefinitely many steps, in each of which the number of distinct new quotient endpoints that we obtain is doubled.

That correspondence between the two constructions, enables us to sequence the real numbers.

First,

we assign the two distinct real numbers represented by the sequences

$$\langle 0,0,0,0,0,....0,....\rangle \leftrightarrow 0,$$

$$\langle 1,0,0,0,0,....0,....\rangle \leftrightarrow \frac{1}{2},$$

to the two distinct quotient endpoints that are generated in the first deletion

$$\frac{1}{3}, \quad and \quad \frac{2}{3}.$$

Second,

we assign the four distinct real numbers represented by the sequences

$$\langle 0,0,0,0,0,....0,....\rangle \leftrightarrow 0$$

$$\langle 0,1,0,0,0,....0,....\rangle \leftrightarrow \frac{1}{4}$$

$$\langle 1,0,0,0,0,....0,....\rangle \leftrightarrow \frac{2}{4}$$

$$\langle 1,1,0,0,0,....0,....\rangle \leftrightarrow \frac{3}{4}$$

to the four new distinct quotient endpoints that are generated in the second deletion

$$\frac{1}{9}, \quad \frac{2}{9}, \quad \frac{7}{9}, \quad and \quad \frac{8}{9}.$$

Third,

we assign the eight distinct real numbers represented by the sequences

$$\langle 0,0,0,0,0,....0,....\rangle \leftrightarrow 0$$

$$\langle 0,0,1,0,0,....0,....\rangle \leftrightarrow \frac{1}{8}$$

$$\langle 0,1,0,0,0,....0,....\rangle \leftrightarrow \frac{2}{8}$$

$$\langle 0,1,1,0,0,....0,....\rangle \leftrightarrow \frac{3}{8}$$

$$\langle 1,0,0,0,0,....0,....\rangle \leftrightarrow \frac{4}{8}$$

$$\langle 1,0,1,0,0,....0,....\rangle \leftrightarrow \frac{5}{8}$$

$$\langle 1,1,0,0,0,....0,....\rangle \leftrightarrow \frac{6}{8}$$

$$\langle 1,1,1,0,0,....0,....\rangle \leftrightarrow \frac{7}{8}$$

to the eight new distinct quotient endpoints that are generated in the third deletion

$$\frac{1}{27}, \quad \frac{2}{27}, \quad \frac{7}{27}, \quad \frac{8}{27}, \quad \frac{19}{27}, \quad \frac{20}{27}, \quad \frac{25}{27}, \quad and \quad \frac{26}{27}.$$

Proceeding with these assignments indefinitely, we will have the $2^{Inf\{1,2,3,....\}}$ real numbers assigned to the $2^{Inf\{1,2,3,....\}}$ quotient endpoints that are produced in the deletions that generate the Cantor set.

Since all the Quotients can be sequenced, the $2^{Inf\{1,2,3,....\}}$ quotient endpoints are sequenced, and the corresponding real numbers are sequenced too.

9

Direct Sequencing of the Real Numbers

Our listing of the real numbers as infinite sequences of 1's and 0's, in section 3, sequences the real numbers directly.

we obtain the rows of an infinite triangle. Each row has double the number of infinite binary sequences, and the $Inf\{1,2,3,..\}^{\text{th}}$ row has $2^{Inf\{1,2,3,....\}}$ such sequences.

							0		1							
					00		01		10		11					
	000		001		010		011		100		101		110		111	
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...

The sequencing follows through the rows.

10

COUNTABLE INFINITY OF THE REAL NUMBERS

By the Effective Countability Axiom, the sequencing of the Reals says that their Infinity is $Inf\{1,2,3,....\}$.

That is,

- $$2^{Inf\{1,2,3,...\}} = Inf\{1,2,3,...\}$$

Cantor however, argued that the Cantorian Infinity of the real numbers is strictly greater than $Inf\{1,2,3,....\}$.

In fact, only in a Theory based on the Negation of Cantor's Theory Axioms, the Reals Infinity is strictly greater than $Inf\{1,2,3,..\}$, and we have proved in 2010 that such Non-Cantorian theory is prohibited.

11
THE CONTINUUM HYPOTHESIS

Cantor believed that the infinity of the real numbers $2^{Inf\{1,2,3,....\}}$ is strictly greater than $Inf\{1,2,3,..\}$.

Thus, he wondered

Is there an infinity in-between the infinity of the real numbers and the infinity of Counting numbers?

The Quotients are in-between. But their Infinity equals that of the Counting numbers.

As Cantor searched for a set with infinity that is in-between, he found the *Cantor set.*

As we have seen, Cantor's Set is a very small set, but it has as many numbers as the real numbers.

Consequently, Cantor believed that there is no infinity between the infinity of Counting numbers, and the infinity of the real numbers.

Indeed, there is no infinity between the equal infinities of the real numbers and the Counting numbers.

But Cantor did not know that.

Being unable to find a proof, Cantor postulated it as an Axiom, known as the *Continuum Hypothesis*.

> *Assume that the infinity of the real numbers is strictly greater than the infinity of the Counting number.*
>
> *Then, there is no set with infinity in-between the infinity of the real numbers, and the infinity of the Counting numbers*

Since the infinities of the real numbers, and the Counting numbers are equal, there is no infinity in-between them, and the Continuum Hypothesis is a FACT, and not an Axiom.

Consequently, its negation is a falsehood.

12

CONSISTENCY OF THE HYPOTHESIS

In 1901, Hilbert presented a list of 23 open problems in Mathematics. The problem of proving, or disproving the Continuum Hypothesis was first on that list, and became known as Hilbert's First Problem.

That problem remained unsolved.

To prove the Continuum Hypothesis, it had to follow from the commonly accepted Axioms of Set Theory, or to be equivalent to one of them.

The Commonly accepted Axioms of Set Theory include Axioms stated by Zermelo, and Fraenkel, and the Axiom of Choice that was added later.

The Axiom of Choice has equivalent statements that do not follow from each other trivially. So it was supposed to be the rational candidate to be equivalent to the Hypothesis.

Nevertheless, no such equivalence was found.

In 1938, Godel settled for a proof of the Consistency of the Continuum Hypothesis.

Godel created a theory named Forcing Theory, and with its aid, argued that the Continuum Hypothesis is consistent with the rest of the Axioms of Set Theory.

Namely, that if the commonly accepted axioms of Set Theory are consistent, then adding the Continuum Hypothesis to them will cause no inconsistency.

Godel’s consistency claim did not prove or disprove the Continuum Hypothesis.

It only established that adding the Continuum Hypothesis, does no harm.

It followed that perhaps, the Continuum Hypothesis, like other Axioms, is a stand-alone, independent of the other axioms, and necessary for a proper functioning of Set Theory beyond the Zermelo-Fraenkel-Choice Theory.

Or perhaps, the Continuum Hypothesis is indeed equivalent to one of the commonly accepted Axioms, and that is why it does not negate any of them.

Thus, Godel’s argument led to no resolution, until his methods were applied by Cohen.

13

COHEN'S INDEPENDENCE OF THE HYPOTHESIS

In 1963, Cohen used Godel's Forcing Theory to argue that the Hypothesis-Negation is consistent too.

Cohen claimed that if the Commonly accepted axioms of Set Theory are consistent, then adding the Hypothesis-Negation will cause no inconsistency.

The *Hypothesis-Negation* in Cohen's terms says that

■ *There is an infinite set which infinity is between the infinity of the Counting numbers, and the infinity of the real numbers.*

We have seen that because the two infinities are equal, there is no such set, and the Hypothesis-Negation is a falsehood.

But for generations to come, Cohen's erroneous claim, established the Hypothesis as an independent Axiom of Set Theory. Independent of the rest of the Axioms of Set Theory. Therefore, impossible to be proved, and necessary to be added to the rest of the Axioms, to ensure a complete Set Theory.

Cohen's claim left the impression that Hilbert's First problem was either solved, or is unsolvable. But it became commonly accepted that the problem was closed.

In response to the Hilbert problem that asked for a proof or disproof, Cohen's answer was, there is no proof, the Continuum Hypothesis is a stand-alone Axiom.

According to Cohen, either the Hypothesis can be added to the Zermelo-Fraenkel-Choice Axioms, to obtain Cantor's Set Theory, or its Negation may be added, to obtain a Non-Cantorian Set Theory.

It is safe to say that very few read Cohen's arguments.

It is safe to say that even fewer followed Cohen's arguments.

No one was up to disputing Cohen's arguments, but it was difficult to believe that the Continuum Hypothesis is "true" like any of the Zermelo-Fraenkel Axioms, and the acceptance of the Continuum Hypothesis as an Axiom did not take place.

Cohen's result had to mean that there is another Set Theory that utilizes the Hypothesis-Negation.

But the alternative Set Theory was never developed.

That should have indicated that there may be something wrong with Cohen's claims.

Indeed, we will see that adding the Hypothesis-Negation to the Commonly accepted axioms of Set Theory, has to lead to inconsistency.

14

ONE INFINITY

As we have seen, the real numbers may be sequenced, and their Infinity equals that of the Counting numbers.

Therefore, we have

$$2^{Inf\{1,2,3,\ldots\}} = Inf\{1,2,3,\ldots\}.$$

This equality can be shown to be equivalent to the Continuum Hypothesis.

Cantor's "proof" of an inequality, is unconvincing.

Cantor's "Diagonal Argument" assumes that the real numbers may be listed in a sequence, and exhibits a real number that is not on that list.

But one number missing, means nothing for infinite sets.

For instance, the even Counting numbers have the same infinity as the Counting numbers, although there are infinitely many odd Counting numbers.

The "Diagonal Argument" approach is suitable for finite sets.

It is unsuitable for infinite sets.

The inequality that Cantor attempted to prove with his "Diagonal Argument" holds only in a Theory which axioms are Negations of the Cantor's Axioms.

For instance, such Theory, has the Continuum Hypothesis Negation Axiom that says that there is an infinity between the infinities of the Counting and the Real numbers

But since the Continuum Hypothesis is a fact, the Continuum Hypothesis Negation is a falsehood.

15
THE AXIOM OF CHOICE

The *Choice Theorem* says that

If for each $n = 1, 2, 3, \ldots$ there is a non-empty set of numbers, then we can choose from each of the sets one number, and obtain a collection of numbers that has a representative from each set.

If we replace the index numbers $n = 1, 2, 3, \ldots$ with an infinite set of numbers I, this choice may not be guaranteed.

There may be an infinite set of numbers I, so that for each index i in it, there is a non-empty set of numbers, with no collection of numbers, that has a representative from each set.

The *Axiom of Choice* is the guess that the choice is guaranteed for any infinite set I, and any family of non-empty sets indexed by I.

However, since all cardinalities equal $Inf\{1,2,3,...\}$,

$$CardI = Inf\{1,2,3,...\},$$

and the Axiom of Choice is guaranteed by the Choice Theorem.

The *Axiom of No-Choice* says that there is an infinite set, and a family of non-empty sets indexed by it, with no collection of numbers, that has a representative from each of the non-empty sets.

16

THE CONTINUUM HYPOTHESIS EQUIVALENCE TO THE AXIOM OF CHOICE

The existence of many equivalent statements of the Axiom of Choice, suggested that the Continuum Hypothesis may be equivalent to the Axiom of Choice.

Godel, and Cohen failed to prove such equivalence, and settled for their consistency proofs.

To show the equivalence, we use a result that Tarski obtained in 1924.

Tarski proved that the Axiom of Choice is equivalent to the statement

- For any infinite cardinals α, and β,

$$\alpha + \beta = \alpha \times \beta .$$

Therefore, the Axiom of No-Choice is equivalent to the statement

- There are infinite cardinals α, and β, so that

$$\alpha + \beta \neq \alpha \times \beta.$$

If we take

$$\alpha = \beta = Inf\{1,2,3,...\}.$$

Then, in Non-Cantorian Axioms,

$$\begin{aligned} \alpha + \beta &= Inf\{1,2,3...\} + Inf\{1,2,3...\} \\ &= Inf\{1,2,3...\} \\ &< Inf\{1,2,3...\} \times Inf\{1,2,3...\} \\ &= \alpha \times \beta \end{aligned}$$

That is,

$$\alpha + \beta \neq \alpha \times \beta$$

Thus, the Non-Countability Axiom

$$Inf\{1,2,3...\} < Inf\{1,2,3...\} \times Inf\{1,2,3...\}$$

is equivalent to the No-Choice Axiom.

On the other hand, the Non-Countability Axiom is equivalent to the Continuum Hypothesis Negation.

Therefore, the Axiom of No-Choice, and the Continuum Hypothesis Negation are equivalent.

Consequently, the Axiom of Choice, and the Continuum Hypothesis are equivalent.□

17

GODEL'S CONSISTENCY, AND COHEN'S INDEPENDENCE

The failure to identify the Continuum Hypothesis with any of the Axioms of Set Theory, led Godel in 1938 to confirm the consistency of the Continuum Hypothesis with the other Axioms of Set Theory, and led Cohen in 1963 to confirm the consistency of the Continuum Hypothesis Negation.

Since the Hypothesis is equivalent to the Axiom of Choice, Godel's Consistency result is self-evident.

The Continuum Hypothesis is consistent with the Axioms of Set Theory, because it is one of them.

The Continuum Hypothesis is just another statement of the Axiom of Choice.

Therefore, Godel's work amounts to the following

If the commonly accepted Axioms of Set Theory are consistent, then adding one of them to all of them will cause no inconsistency.

That certainly sounds right, albeit trivial.

Cohen claimed that the addition of the Continuum Hypothesis-Negation to the commonly accepted Axioms of Set Theory, will cause no inconsistency.

But the Continuum Hypothesis-Negation is just another statement of the Axiom of No-Choice, and the mixing of the Axiom of Choice with its Negation, must lead to inconsistency.

That is how Cohen established the Continuum Hypothesis as an independent Axiom of Set Theory.

We have seen that the Continuum Hypothesis is not an Axiom, but a fact.

18

WELL-ORDERING THEOREM

The Axiom of Choice is equivalent to the Well-Ordering Axiom.

The Counting Numbers are ordered in such a way that every subset of them has a first element.

That property is called Well-Ordering.

An infinite set of numbers is well-ordered if it is ordered, so that every subset of it has a first element.

The *Well-Ordering Axiom* is the guess that every infinite set of numbers can be well-ordered like the Counting Numbers.

In 1963, Cohen claimed that it is not possible to prove that the real numbers can be well-ordered.

However, our direct sequencing of the real numbers between 0,and 1, defines precisely such ordering, in which any sub-interval of numbers between 0,and 1, will have a first element.

We only need to eliminate the repeating elements in that sequencing.

In the following we describe such well-order.

19
WELL-ORDERING THE REALS IN $[0,1]$ WITH THE MIDPOINTS SET

In a procedure similar to the sequencing of the reals, we construct the rows of the Midpoints Set that represents the reals in $[0,1]$, so that every subset of it has a first element.

The 0th row has one binary sequence representing $\frac{1}{2^1}$,

$$(1,0,0,\ldots,0,0,0\ldots) \leftrightarrow \frac{1}{2}$$

The 1st row has the two binary sequences,

$$(0,1,0,\ldots,0\ldots) \leftrightarrow \frac{1}{2^2},$$

$$(1,1,0,\ldots,0,\ldots) \leftrightarrow \frac{3}{2^2}$$

The 2nd row has the four binary sequences,

$$(0,0,1,0,\ldots) \leftrightarrow \frac{1}{2^3},$$

$$(0,1,1,0,\ldots) \leftrightarrow \tfrac{3}{2^3},$$

$$(1,0,1,0,\ldots) \leftrightarrow \tfrac{5}{2^3},$$

$$(1,1,1,0,\ldots) \leftrightarrow \tfrac{7}{2^3}.$$

The 3rd row lists the eight binary sequences that start with

$$(0,0,0,1,0\ldots) \leftrightarrow \tfrac{1}{2^4},$$

$$(0,0,1,1,0\ldots) \leftrightarrow \tfrac{3}{2^4}$$

and end with

$$(1,1,1,1,0,\ldots) \leftrightarrow \tfrac{2^4-1}{2^4}.$$

This listing

$$\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & 01 & & 11 & & & & & & & \\
 & & & & & 001 & & 011 & & 101 & & 111 & & & & & \\
 & 0001 & & 0011 & & 0101 & & 0111 & & 1001 & & 1011 & & 1101 & & 1111 & \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}$$

enumerates all the real numbers in $[0,1]$, but without repetitions.

The $Inf\{1,2,3,...\}$ row has $2^{Inf\{1,2,3,...\}}$ infinite binary sequences that represent real numbers in $[0,1]$.

the order follows the rows of the Midpoints Set from left to right.

That is, the first element in this ordering is $\frac{1}{2}$ in the 1^{st} row.

The second element in this ordering is $\frac{1}{2^2}$, and the third is $\frac{3}{2^2}$. Both are in the 2^{nd} row.

The fourth is $\frac{1}{2^3}$, the fifth is $\frac{3}{2^3}$, the sixth is $\frac{5}{2^3}$, the seventh is $\frac{7}{2^3}$. All four are in the 3^{rd} row.

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Now, to determine the first element in say, $[0,\frac{1}{1000}]$, we note that

$$\frac{1}{1000} < \frac{1}{2^9}.$$

Therefore, no midpoints appear in $[0, \frac{1}{1000}]$ till the 10^{th} row. The 10^{th} row has the midpoints

$$\frac{1}{2^{10}}, \frac{3}{2^{10}}, \frac{5}{2^{10}}, \ldots. \frac{1023}{2^{10}}.$$

Since

$$\frac{1}{2^{10}} < \frac{1}{1000},$$

the first element of $[0, \frac{1}{1000}]$ is $\dfrac{1}{2^{10}}$.

Similarly, to find the first element in $(\frac{1}{16}, \frac{1}{8})$, we note that no midpoints of the 4^{th} row appear in $(\frac{1}{16}, \frac{1}{8})$.

Both

$$\frac{1}{2^4} = \frac{1}{16} \text{ , and } \frac{3}{2^4} = \frac{3}{16},$$

are not in $(\frac{1}{16}, \frac{1}{8})$.

The fifth row has the midpoints

$$\frac{1}{2^5}, \frac{3}{2^5}, \frac{5}{2^5}, \ldots. \frac{31}{2^5}$$

$\dfrac{1}{2^5} = \dfrac{1}{32}$ is not in the interval $(\frac{1}{16}, \frac{1}{8})$.

But $\dfrac{3}{2^5}$ is in it, and it is the first element of the real numbers interval $(\frac{1}{16}, \frac{1}{8})$.

20

TRANSFINITE INDUCTION

The Axiom of Choice is equivalent to the Transfinite Induction Axiom.

The *Induction Theorem* says that

If a property depends on each number $n = 1,2,3,..$, so that

1) The property holds for the first Counting number $n = 1$.
2) If the property holds for the Counting number k, we can deduct that it holds for the next number $k+1$.

Then, the property holds for any $n = 1,2,3,...$.

The *Transfinite Induction Axiom* guesses that the same holds for any infinite index set I.

It says that if I is any well-ordered infinite set of numbers, and if there is any property that depends on each index i from I, so that

1) The property holds for the first element of I,
2) If the property holds for all the $k's$ that precede the index j, we can conclude that the property holds for j,

Then, the property holds for any index i in I.

However, since all infinities equal $Inf\{1,2,3,...\}$,

$$Inf\{I\} = Inf\{1,2,3,...\},$$

and the Transfinite Induction Axiom is guaranteed by the Induction Theorem.

21
THE EQUAL INFINITIES

Since

$$Inf\{1,2,3,....\} \times Inf\{1,2,3,....\} = Inf\{1,2,3,....\},$$

we have,

$$(Inf\{1,2,3,....\})^3 =$$

$$= Inf\{1,2,3..\} \times Inf\{1,2,3..\} \times Inf\{1,2,3..\}$$

$$= Inf\{1,2,3..\} \times Inf\{1,2,3..\}$$

$$= Inf\left\{1,2,3..\right\}$$

By induction, we obtain an infinite chain of equalities

$$Inf\{1,2,3,...\} = (Inf\{1,2,3,...\})^2$$

$$= (Inf\{1,2,3,...\})^3$$

$$= (Inf\{1,2,3,...\})^4$$

....

$$= \left(Inf \left\{ 1,2,3,... \right\} \right)^{Inf\{1,2,3,...\}}$$

$$= 2^{Inf\{1,2,3,...\}}$$

$$= 2^{(Inf\{1,2,3,...\})^2}$$

$$= 2^{(Inf\{1,2,3,...\})^3}$$

....

$$= 2^{(Inf\{1,2,3,...\})^{Inf\{1,2,3,...\}}}$$

$$= 2^{2^{Inf\{1,2,3,...\}}}$$

$$= 2^{2^{\left(Inf\left\{1,2,3,...\right\}\right)^2}}$$

$$= 2^{2^{\left(Inf\left\{1,2,3,...\right\}\right)^3}}$$

....

$$= 2^{2^{2^{Inf\left\{1,2,3,...\right\}}}}$$

....

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