The Continuum Hypothesis

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Abstract

We prove that the Continuum Hypothesis is equivalent to the Axiom of Choice. Thus, the Negation of the Continuum Hypothesis, is equivalent to the Negation of the Axiom of Choice.

The Non-Cantorian Axioms impose a Non-Cantorian definition of cardinality, that is different from Cantor's cardinality imposed by the Cantorian Axioms.

The Non-Cantorian Theory is the Zermelo-Fraenkel Theory with the Negation of the Axiom of Choice, and with the Negation of the Continuum Hypothesis. This Theory has distinct infinities.

Keywords: Continuum Hypothesis, Axiom of Choice, Cardinal, Ordinal, Non-Cantorian, Countability, Infinity.

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Preface

The Continuum Hypothesis says that there is no infinity between the infinity of the natural numbers, and the infinity of the real numbers.

The account here, follows my attempts to understand the Hypothesis.

In 2004, I thought that I found a proof for the Hypothesis. That turned out to be an equivalent statement to the Continuum Hypothesis.

That condition is the key to the Non-Cantorian Theory, but it took me until 2007 to comprehend its meaning, and to apply it.

The first hurdle is to comprehend that the condition holds in Non-Cantorian Cardinality.

The second hurdle is to realize that the Non-Cantorian cardinality is imposed by an Axiom of the Non-Cantorian Theory.

The Non-Cantorian Theory is essential because Cantor's Theory replaces facts with wishes, attempts to prove Axioms as if they were

Theorems, and borrows from the Non-Cantorian Theory Axioms that do not hold in Cantor's Theory.

At the end, Cantor's Theory produces only one infinity.

To obtain distinct infinities, we need the Non-Cantorian Theory.

The equivalence between the Hypothesis and the Axiom of Choice, renders the consistency result of Godel trivial.

The equivalence between the Negation of the Hypothesis, and the Negation of the Axiom of Choice says that Cohen's result must be wrong. The mixing of the Axiom of Choice with its Negation must lead to inconsistency.

The Hypothesis is the most illusive statement of the Axiom of Choice.

1

Hilbert's 1st problem: The Continuum Hypothesis

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Abstract: There is no set X with $\aleph_0 < cardX < 2^{\aleph_0}$.

Introduction

The continuum hypothesis reflects Cantor's inability to construct a set with cardinality between that of the natural numbers and that of the real numbers. His approach was constructive, But if he was right, such set cannot be constructed, and he needed a proof by contradiction.

That contradiction remained out of reach at Cantor's time. Even when Hilbert presented Cantor's Continuum Hypothesis as his first problem, the tools for the solution did not exist. Tarski obtained the essential lemma only in 1948. But it was not utilized and the problem remained open.

In 1963, Cohen proved that if the commonly accepted postulates of set theory are consistent, then adding the negation of the hypothesis does not result in inconsistency. [1, p.97]. That left the impression that Hilbert's first problem was either solved, or is unsolvable. But it became commonly accepted that the problem was closed.

Not that the question was settled. According to [2,p.189], "Mathematicians do not tend to assume the Continuum Hypothesis as an additional axiom of set theory mostly since they cannot convince themselves that this statement is "true" as many of them have done for the axioms of ZFC including the axiom of choice. However, a mathematician trying to prove a theorem will usually regard a proof of the theorem from the generalized continuum hypothesis as a partial success".

Cohen result was interpreted to mean that there is another set theory that utilizes the negation of the continuum hypothesis. However, the alternative set theory was never developed, and we shall show here that the Continuum Hypothesis can be proved under the assumptions of Cantor's set theory.

1. Proof

To understand the gap between cardinal numbers, it is natural to examine the convergence $\aleph_0^n \to \aleph_0^{\aleph_0}$, and try to pinpoint where the jump from $\aleph_0 (= \aleph_0^n)$ to $2^{\aleph_0} (= \aleph_0^{\aleph_0})$ occurs. This idea[1] motivates our proof.

By [3, p.173], the sequence

$$\alpha_n \equiv \aleph_0 + \aleph_0^2 + ...\aleph_0^n,$$

converges to the series

$$\aleph_0 + \aleph_0^2 + ... + \aleph_0^n + ... = \sum_{n=1}^{\infty} \aleph_0^n .$$

The series has a well-defined sum α, which is a cardinal number that does not depend on the order of the summation.

By [3, p.174],

α is $\geq$ any component of the series.

[1] We actually use sequences of partial sums of series of cardinal numbers

In particular,

$$\alpha \geq \aleph_0 \aleph_0 ... = \prod_{n=1}^{\infty} \aleph_0 = \prod_{n \in N} \aleph_0 .$$

By [2, p.106], (or [3, p.183])

$$\prod_{n \in N} \aleph_0 = \aleph_0^{CardN} .$$

Therefore,

$$\alpha \geq \aleph_0^{CardN} = 2^{\aleph_0} .$$

That is,

$$\alpha \geq 2^{\aleph_0} .$$

Now, suppose that there is a set X with

$$\aleph_0 < cardX < 2^{\aleph_0} .$$

Then, for any finite n,

$$\alpha_n = \aleph_0 + \aleph_0^2 + \aleph_0^3 + ...\aleph_0^n \leq CardX .$$

Tarski ([4], or [3, p.174]) proved that for any cardinal numbers, and for $n = 1,2,...n...$ the inequalities

$$m_1 + m_2 + ... + m_n \leq m,$$

imply the inequality

$$m_1 + m_2 + ... + m_n + ... \leq m.$$

Since for any n

$$\alpha_n \leq cardX,$$

by Tarski result

$$\alpha \leq cardX.$$

Combining this with

$$cardX < 2^{\aleph_0},$$

yields by transitivity of cardinal inequalities [3, p. 147],

$$\alpha < 2^{\aleph_0},$$

which contradicts $\alpha \geq 2^{\aleph_0}$.

2. Discussion:

Why did Cantor fail to prove his hypothesis? Tarski result was not available to Cantor. Furthermore, Cantor was familiar with the partial sums

$$\aleph_0 + \aleph_0^2 + ... + \aleph_0^n,$$

that are the cardinality of all the roots of all the polynomials in integer coefficients of degree $\leq n$. In that context, n must be always finite, and the partial sum never exceeds $\aleph_0$. Only when we take infinite n, we obtain terms such as $\aleph_0^{\aleph_0} = 2^{\aleph_0} > \aleph_0$.

The infinite summation over all n, is the key to our proof. In spite of the degenerate character of $\aleph_0^n$, there is a jump to $2^{\aleph_0}$ that forbids a cardinal number between $CardN$, and $CardR$.

The gap between cardinal numbers may be traced to the jump between finite cardinal numbers n, and the first infinite cardinal $\aleph_0$. That incomprehensible jump may be the reason for the jump from the cardinal number

$$\aleph_0 + \aleph_0^2 + \aleph_0^3 + ...\aleph_0^n = \aleph_0^n = \aleph_0$$

to the cardinal

$$\sum_{n \in N} \aleph_0^n = \aleph_0^{\aleph_0} = 2^{\aleph_0}.$$

Perhaps, there is no cardinality between the integers and the real numbers, because there is no infinite cardinal number between n, and $\aleph_0$. But an infinite cardinality between n, and $\aleph_0$, will contradict the definition of $\aleph_0$, as the first infinite cardinal.

Was Cohen right? In earlier stages of this work, we wondered whether our proof for the Continuum Hypothesis proves Cohen wrong. But our further studies affirm Cohen's result. Under the assumptions of Cantor's set theory, we can prove the Continuum Hypothesis. But with one assumption changed, we can construct a Non-Cantorian set theory , where there is a set X with $CardN < CardX < CardR$. Cohen's result predicts the vulnerability of Cantor's set theory, and makes it easier for us to present the NonCantorian set theory [5].

References

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Preface to 2

How can the Continuum Axiom be proved?

It is equivalent to the claim that the rationals cardinality is $CardN$, which Cantor considered a proven fact.

In **7**, we'll see that this proven fact is a Cantorian Axiom.

Therefore, there is no proof that the rationals cardinality is $CardN$.

In other words, if Cantor have proved that the rationals cardinality is $CardN$, then we have proved the Hypothesis.

We exhibit in **7** one Axiom that says that the rationals Non-Cantorian cardinality is greater than $CardN$, and another Axiom, that says that the rationals Cantorian Cardinality is $CardN$.

In **2**, I was not aware of the Non-Cantorian cardinality.

Furthermore, it is not clear in **2** why we must have the Non-Cantorian Set Theory.

In **4**, I proved that in Cantor's Set Theory there is only one infinity.

The different infinities are all Non-Cantorian.

2

Non-Cantorian Cardinal Numbers

by H. Vic Dannon
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Abstract: $\aleph_0^2 = \aleph_0$ is an assumption of Cantor's set theory, and we develop the arithmetic of Non-Cantorian cardinal numbers

Introduction: The Continuum Hypothesis says that there is no set X with $\aleph_0 < cardX < 2^{\aleph_0}$. In 1963, Cohen proved that if the commonly accepted postulates of set theory are consistent, then the addition of the negation of the hypothesis does not result in inconsistency [1].

Cohen's result was interpreted to mean that there is another set theory that utilizes the negation of the Continuum Hypothesis. However, sets with cardinality between the natural numbers and the real numbers were not found, and the alternative set theory was never developed.

Recently, we proved the Continuum Hypothesis in Cantor's set theory [2]. But according to Cohen, the Continuum Hypothesis is an assumption. How can an assumption be proved? Close scrutiny of our proof will reveal that $\aleph_0^2 = \aleph_0 \Rightarrow$ Continuum Hypothesis. We further show that the converse is also true, and that the Continuum Hypothesis is equivalent to $\aleph_0^2 = \aleph_0$.

Thus, Non-Cantorian set theory is founded on the converse assumption that $cardQ > cardN$.

We first show that the continuum Hypothesis and $\aleph_0^2 = \aleph_0$, are equivalent.

1. $\aleph_0^2 = \aleph_0 \Leftrightarrow$ Continuum Hypothesis:

We prove:

Assume $CardN < 2^{CardN}$. *Then*

$$(CardN)^2 = CardN \Leftrightarrow \text{Continuum Hypothesis.}$$

Proof

$(\Rightarrow)$

By [3, p.173], the sequence

$$CardN + (CardN)^2 + (CardN)^3 + \ldots + (CardN)^n$$

sums up[1] [3] to the series

$$CardN + (CardN)^2 + (CardN)^3 + \ldots .$$

By [3], the series has a well defined sum

$$\alpha \equiv CardN + (CardN) \times (CardN)$$

$$+ (CardN) \times (CardN) \times (CardN) + \ldots,$$

which is a cardinal number.

By [3, p.174],

$\alpha \geq$ any component of the series.

In particular, α is greater than the infinite product term

$$\alpha \geq (CardN) \times (CardN) \times (CardN) \times \ldots$$

$$= (CardN)^{1+1+1+\ldots}.$$

By [3, p.183],

$$(CardN)^{1+1+1+\ldots} = (CardN)^{CardN}.$$

[1] This is "convergence" of infinite series of cardinal numbers precisely as defined in Sierpinski.

Therefore,

$$\alpha \geq (CardN)^{CardN}.$$

Or, using the product notation, by [4, p. 106], we obtain similarly,

$$\alpha \geq \prod_{n \in N} CardN = (CardN)^{CardN}.$$

Since

$$(CardN)^{CardN} \geq 2^{CardN},$$

by transitivity of cardinal inequalities [3, p. 147], we conclude that

$$\alpha \geq 2^{CardN}.$$

Now suppose that the continuum hypothesis does not hold, and there is a set X with

$$CardN < CardX < 2^{CardN}.$$

Since we assume $(CardN)^2 = CardN$, then for any $n = 1,2,3...$,

$$CardN + (CardN)^2 + ... + (CardN)^n \leq CardX.$$

Tarski ([5], or [3, p.174]) proved that

If

$$m_1, m_2, ..m_n, \quad \text{and} \quad m$$

are any cardinal numbers so that for any $n = 1,2,3....,$

$$m_1 + m_2 + ... + m_n \leq m,$$

then,

$$m_1 + m_2 + ... + m_n + ... \leq m.$$

By Tarski result,

$$CardN + (CardN)^2 + ... + (CardN)^n + ... \leq CardX.$$

Since we assume $CardX < 2^{CardN}$, by transitivity of cardinal inequalities [3, p. 147],

$$\underbrace{CardN + (CardN)^2 + (CardN)^3 + ...}_{=\alpha} < 2^{CardN}$$

That is,

$$\alpha < 2^{CardN},$$

which contradicts $\alpha \geq 2^{CardN}$.

Therefore, there is no set X with cardinality between $CardN$, and 2^{CardN}, and the continuum Hypothesis holds.

This completes the proof that

$$(CardN)^2 = CardN \Rightarrow \text{Continuum Hypothesis.}\ \square$$

$(\Leftarrow)$

By [3, p.155], *For any cardinals* m, n,, m_1, and n_1,

$$m < n \textit{, and } m_1 < n_1 \Rightarrow m \times m_1 < n \times n_1.$$

Since

$$CardN < 2^{CardN},$$

we have

$$CardN \times CardN < 2^{CardN} \times 2^{CardN}.$$

But

$$2^{CardN} \times 2^{CardN} = 2^{CardN + CardN} = 2^{CardN}.$$

Hence,

$$(CardN)^2 < 2^{CardN}.$$

Therefore,

$$CardN < (CardN)^2 \Rightarrow CardN < (CardN)^2 < 2^{CardN}.$$

Namely, if $CardN < (CardN)^2$, the rationals serve as a set X

which cardinality is between $CardN$, and $CardR$, and the Continuum Hypothesis does not hold. That is,

$$CardN < (CardN)^2 \Rightarrow \ Hypotesis\ Negation$$

Thus, the Hypothesis implies $(CardN)^2 = CardN$.□

2. $\aleph_0^2 = \aleph_0$ is an assumption.

The equivalence of the continuum hypothesis with the countability of the rationals, indicates that the countability of the rationals is an assumption, just as the continuum hypothesis is, and as such it cannot be proved.

Consequently, Cantor's proof of it, that does not utilize the Continuum Hypothesis, has to raise doubts.

Indeed, if we could prove that Q is countable, then Cantor's Continuum Hypothesis will be a fact, and Cantor's set theory would be the only valid set theory, in contradiction to Cohen's result.

Cohen's proof of the existence of non-Cantorian set theory, mandates that $cardQ=cardN$ is an assumption that cannot be proved just like the continuum hypothesis.

If the countability of Q is assumed, we obtain Cantor's set theory, and Cantor's rules for cardinal numbers.

The assumption $\aleph_0^2 = \aleph_0$ is an Axiom of cantor's set theory, that can replace the Continuum Hypothesis, similarly to the well-ordering Axiom that can replace the Axiom of Choice.

If it seems believable that the rationals can be arranged in a sequence, note that the Axiom of Choice too seems believable, but has to be assumed.

Non-Cantorian set theory is founded on the assumption that Q is uncountable. and $\aleph_0 < \aleph_0^2$.

Cohen result alludes to a Non-Cantorian set theory where the *negation* of Cantor's Continuum Hypothesis holds. Then, Q must be uncountable, and

$$cardN < cardQ < cardR.$$

That is, the set that eluded Cantor is the set of rational numbers Q.

The Non-Cantorian set theory distinguishes the cardinality of Q

from the cardinality of N, and from the cardinality of R.

3. Cantor's Zig-Zag Proof Well-Orders Q, but does it arrange Q in a Sequence?

Non-Cantorian set theory was indicated in 1963 by Cohen, but was born before 1915 when Cantor well-ordered[2] Q, and believed that he arranged it in a sequence as well, so that he may conclude that $\aleph_0^2 = \aleph_0$.

Given two infinite sequences of numbers

$$a_1, a_2, \ldots a_n, \ldots$$

$$b_1, b_2, \ldots b_n, \ldots$$

we can arrange them in one sequence

$$a_1, b_1, a_2, b_2, \ldots a_n, b_n, \ldots$$

The method applies to n sequences, but not to an infinite number of sequences.

[2] The Well-Ordering Axiom was conjectured by Cantor in 1883. Its equivalence to the Axiom of choice was proved by Zermelo in 1904.

Cantor reasoned [5] that we can zigzag through an infinite triangle of rationals, starting from the vertex of the triangle, and progressing towards the triangle's base.

But Scanning the infinite triangle by zigzag lines, forces us eventually to go through an infinite number of infinite sequences. The zig-zag starts with finite sequences, but there are infinitely many infinite sequences above the infinite triangle base.

```
*  →  *     *  →  *     *  →  …  …  *  …  *
   ↙     ↗     ↙     ↗  …  …  …  ↗  …  ↙
*     *     *     *  …  …  …  *  …  *
↓  ↗     ↙     ↗  …  …  …  ↗  …  ↙
*     *     *  …  …  …  *  …  *
   ↙     ↗  …  …  …  ↗  …  ↙
*     *  …  …  ….  *  …  *
↓  ↗  …  …  …  ↗  …  ↙
*  …  …  …  *  …  *
…  …  …  ↗  …  ↙
*  …  *  …  *
↓  ↗  …  ↙
*  …  *
…  ↙
*  →
```

Cantor's zigzag through infinitely many infinite sequences of rational numbers.

Cantor did not draw the zigzag. Instead, for each rational

$$\frac{\mu}{\nu},$$

he defined a variable

$$\rho = \mu + \nu,$$

that is supposed to remain well defined, but long before the triangles base is reached, $\rho = \infty$, and the correspondence between N, and Q breaks down.

Cantor's zig-zag well-orders Q. Depending on the direction of the zig-zag scan, we have either

$$\frac{\mu+1}{\nu-1} \prec \frac{\mu}{\nu} \prec \frac{\mu-1}{\nu+1},$$

or

$$\frac{\mu+1}{\nu-1} \succ \frac{\mu}{\nu} \succ \frac{\mu-1}{\nu+1}.$$

However, Well-Ordering does not guarantee arranging in a sequence. For instance, the real numbers, may be well-ordered, but not arranged in a sequence.

Specifically, the real numbers between 0, and 1, can be arranged in an infinite triangle using their binary representation. The first row will have 0 and 1, the second will have 00, 01, 10, and 11, …. The *nth* row has 2^n numbers. The well ordering will be defined by zig-zag from one row to the next, towards the base of the infinite triangle.

$$
\begin{array}{c}
* \;\rightarrow\; * \\
\searrow \\
\searrow \\
* \;\leftarrow\; * \;\leftarrow\; * \;\leftarrow\; * \\
\swarrow \\
\swarrow \\
* \;\rightarrow\; * \;\rightarrow\; * \;\rightarrow\; \dots\; \dots\; \dots\; \dots\; \rightarrow\; * \\
\dots\; \dots\; \dots\; \dots\; \dots\; \dots\; \dots\; \dots\; \dots\; \dots\; \dots\; \dots\; \dots\; \dots\; \dots\; \searrow \\
\rightarrow\; \dots\; \dots\; \dots\; \dots\; \rightarrow\; \dots\; \dots\; \dots\; \dots\; \dots\; \dots\; \dots\; \dots\; \rightarrow\; \dots\; \rightarrow \\
\dots\; \dots\; \dots\; \dots\; \dots\; \dots\; \dots\; \dots\; \dots\; \dots\; \dots\; \dots\; \dots\; \dots\; \dots\; \dots\; \dots
\end{array}
$$

A Well-Ordering zig-zag through infinitely many infinite sequences of real numbers

If Cantor's diagonal proof is correct, this Well-Ordering zig-zag does not arrange the real numbers in a sequence. But it does not sequence the real numbers any less than Cantor's zig-zag through the rational numbers.

In conclusion, we maintain that Cantor's zig-zag may not arrange the rationals in a sequence.

On the other hand, the assumption that Q may be sequenced may not be easily disproved.

Consequently, we may assume that the rational numbers can be arranged in a sequence, an assumption that is equivalent to the Continuum Hypothesis, and implies Cantor's set theory, with no inconsistencies or contradictions.

Otherwise, we may assume the negation of the Continuum Hypothesis, and recognize $CardQ$ as a cardinal number different from either $CardN$, or $Card\ R$. Then, the rationals may present the different infinite cardinality that Cantor himself was seeking.

We proceed to develop the arithmetic of non-Cantorian cardinal numbers.

4. Non-Cantorian Cardinal Numbers

Non-Cantorian set theory is founded on the assumption that Q is uncountable. Then,

$$\aleph_0 < \aleph_0^2,$$

and

$$cardN < cardQ < cardR.$$

That is, the set that eluded Cantor is the set of rational numbers Q.

We proceed to develop the arithmetic of non-Cantorian cardinal numbers.

Theorem 1 $\aleph_0^n < \aleph_0^{n+1}$, for $n = 1,2,3\ldots$

Proof

For $n = 1$, $\aleph_0 < \aleph_0^2$, is the non-Cantorian assumption.

By induction on n, similarly to our proof that

$$\aleph_0^2 < \aleph_0^3,$$

we conclude that for all finite *n*,

$$\aleph_0^n < \aleph_0^{n+1}. \square$$

The $\aleph_0^n$ are newly added cardinalities that in the Cantorian set theory degenerated into $\aleph_0$.

Theorem 2. $$\aleph_0^n \uparrow \aleph_0^{\aleph_0} = 2^{\aleph_0}.$$

Proof

For each finite *n*, $\aleph_0^n$ is the cardinality of all the roots of all the polynomials in integer coefficients of degree *n*. Therefore,

$$\aleph_0^n \leq cardR = 2^{\aleph_0}.$$

Since $\aleph_0^n$ is monotonically increasing,

$$\aleph_0^n \uparrow \aleph_0^{\aleph_0}.$$

Since $\aleph_0 > 2$,

$$\aleph_0^{\aleph_0} \geq 2^{\aleph_0}.$$

To show that

$$\aleph_0^{\aleph_0} \leq 2^{\aleph_0},$$

note that for $n = 1,2,...$

$$1 + 2^2 + 3^3 + \ldots + n^n < 2^{\aleph_0}.$$

By Tarski, [5]

$$1 + 2^2 + 3^3 + \ldots + n^n + \ldots \le 2^{\aleph_0}$$

That is,

$$\aleph_0^{cardN} \le \sum_{n \in N} n^n \le 2^{\aleph_0}.$$

Thus,

$$\aleph_0^{\aleph_0} = 2^{\aleph_0},$$

and

$$\aleph_0^n \uparrow \aleph_0^{\aleph_0} = 2^{\aleph_0}.\square$$

Theorem 3 $\aleph^n = \aleph = cardR$

Proof

$$\aleph^n = (2^{\aleph_0})^n = 2^{n\aleph_0} = 2^{\aleph_0}.\square$$

Thus, while $\aleph_0^2 = \aleph_0$ is an assumption, $\aleph^2 = \aleph$ holds in Non-Cantorian Set Theory too.

Theorem 4 $$2^{\aleph_0} = n^{\aleph_0} = cardR,$$

for all finite $n = 2,3,4...$.

proof

$$2^{\aleph_0} \le n^{\aleph_0} \le \aleph_0^{\aleph_0} = 2^{\aleph_0}.\square$$

Theorem 5 $$2^{\aleph_0} < 2^{\aleph_0^2}.$$

Proof

If $2^{\aleph_0} = 2^{\aleph_0^2}$, then

$$2^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = (2^{\aleph_0^2})^{\aleph_0} = (2^{\aleph_0})^{\aleph_0^2} = (2^{\aleph_0^2})^{\aleph_0^2}$$

$$= (2^{\aleph_0})^{\aleph_0^3} = ... = (2^{\aleph_0})^{\aleph_0^n} = ...$$

$$= (2^{\aleph_0})^{2^{\aleph_0}} \ge 2^{\aleph_1} = \aleph_2,$$

since by Theorem 2, $\aleph_0^n \uparrow 2^{\aleph_0}$. This contradicts $c < 2^c$.

Theorem 6 $$2^{\aleph_0^n} < 2^{\aleph_0^{n+1}}.$$

Proof

By the same argument of Theorem 5.

Theorem 7 $2^{\aleph_0^n} \uparrow 2^{\aleph}$.

Proof

$$2^{\aleph_0^n} \uparrow 2^{\aleph_0^{\aleph_0}} = 2^{2^{\aleph_0}} = 2^{\aleph}.\square$$

The $2^{\aleph_0^n}$ are newly added cardinalities, that in the Cantorian set theory degenerated into $2^{\aleph_0}$.

5. Rational and Irrational Numbers:

Theorem 8 $cardR > cardQ$,

for Cantorian or Non-Cantorian cardinals.

Proof

Based on the preceding theorems we have

$$cardR = 2^{\aleph_0} = \aleph_0^{\aleph_0} > \aleph_0^2 = cardQ.\square$$

This proof replaces Cantor's diagonal proof that does not apply to non-Cantorian cardinals because in Non-Cantorian set theory, the rationals are uncountable with

$$cardQ > \aleph_0 .$$

Recall that Cantor's diagonal proof exhibits the diagonal element as the one not counted for, to obtain a contradiction. It leaves a lingering doubt why that one missing element cannot be added to the listing, that will still remain countable. In fact, that listing of the real numbers will remain countable after countably many such elements will be exhibited and added to it, while it seems that only one such element may be produced.

Theorem 9 $Card(Irrational\ \ Numbers) = CardR$,

for Cantorian or Non-Cantorian cardinals.

Proof

By [3, p. 417], *the Axiom of Choice implies for any cardinal numbers* m, *and* n, *so that* $m > n$

$$m - n = m .$$

Since by Theorem 8 $cardR > cardQ$

$$CardR - CardQ = CardR . \square$$

6. Algebraic and Transcendental numbers

Theorem 10 $Card(Algebraic\ Numbers) = CardR$,

for Cantorian or Non-Cantorian cardinals.

Proof

The cardinality of the set of all the roots of all the polynomials in integer coefficients of degree at most *n* is

$$\aleph_0 + \aleph_0^2 + \aleph_0^3 + ... + \aleph_0^n.$$

Therefore, for any $n \in N$,

$$\aleph_0 + \aleph_0^2 + \aleph_0^3 + ... + \aleph_0^n \leq Card(Algebraic\ Numbers).$$

By Tarski [5],

$$\aleph_0 + \aleph_0^2 + ... + \aleph_0^n + ... \leq Card(Algebraic\ Numbers).$$

Since

$$\aleph_0 + \aleph_0^2 + ... + \aleph_0^n + ... = \aleph_0^{\aleph_0} = 2^{\aleph_0},$$

by transitivity of cardinal inequalities,

$$2^{\aleph_0} \leq Card(Algebraic\ Numbers).$$

On the other hand,

$$Card(Algebraic\ \ Numbers) \leq CardR = 2^{\aleph_0}.$$

Consequently,

$$Card(Algebraic\ \ Numbers) = CardR = 2^{\aleph_0}.\square$$

Cantor had $\aleph_0 + \aleph_0^2 + ... + \aleph_0^n + ... = \aleph_0$. But In chapter 4

we show that for Cantor's Cardinals, $2^{\aleph_0} = \aleph_0$.

Theorem 12 $Card(transcendental\ \ numbers) = CardR$,

for Cantorian or Non-Cantorian cardinals.

Proof

By [7], *if a is non-zero, real algebraic number, then e^a is a transcendental number.* The function

$$a \to e^a$$

is an injection from the algebraic numbers into the transcendental numbers.

Therefore,

$$CardR \geq Card(Transcendental\ \ Numbers)$$

$$\geq Card(Algebraic\ Numbers) = CardR.$$

Thus,

$$Card(transcendental\ numbers) = CardR.\square$$

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Preface to 3

The response to my claim in **2** that the rationals cardinality need not be $CardN$, was invariably

"But Cantor's mapping is a bijection…"

One critique pointed out to me that

"$(i, j) \to \dfrac{1}{i+j}$ is a one-one mapping…."

I found out that Cantor did not exhibit a one-one mapping.

Later generations drew the famous Zig-Zag, and everyone believed in the Zig-Zag, and in the effective countability of the rationals.

In **3**, I supply two proofs for the sequencing of the rationals. The first proof is based on Tarski. The second proof constructs an injection, that Cantor failed to find, from the rational numbers into the natural numbers.

Indeed, we can sequence the rationals, but we cannot prove what their cardinality is.

In **7** we'll see that by one Axiom, the Cantorian cardinality of the rationals numbers is $CardN$, and by another Axiom, the Non-Cantorian cardinality of the Rationals is greater than $CardN$.

It is plain, that Cantor was not aware of any of that.

3

Rationals Countability and Cantor's Proof

H. Vic Dannon
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Abstract: Cantor's proof that the rational numbers are countable uses a mapping that is not one-one. Thus, the countability of the rationals was not proved by Cantor.

We prove by cardinal number methods, using a result of Tarski, that the rationals are countable. We confirm this by exhibiting a one-one mapping from the rationals into the natural numbers.

1. Cantor's Mapping is not One-One.

Cantor's proof appears in [1]. Cantor wrote

> *"By (6) of* $\S$ *3,* $\aleph_0 \bullet \aleph_0$ *is the cardinal number of the aggregate of bindings*
>
> $$\{(\mu,\nu)\},$$

where μ *and* ν *are any finite cardinal numbers which are independent of one another. If also* λ *represents any finite cardinal number, so that* $\{\lambda\}$, $\{\mu\}$, *and* $\{\nu\}$ *are only different notations for the same aggregate of all finite numbers, we have to show that*

$$\{(\mu,\nu)\} \sim \{\lambda\}.$$

Let us denote $\mu + \nu$ *by* ρ*; then* ρ *takes all the numerical values* $2,3,4,\ldots$ *and there are in all* $\rho - 1$ *elements* (μ,ν) *for which* $\mu + \nu = \rho$*, namely:*

$$(1,\rho - 1), (2,\rho - 2),\ldots,(\rho - 1,1).$$

In this sequence imagine first the element $(1,1)$, *for which* $\rho = 2$, *put , then the two elements for which* $\rho = 3$, *then the three elements for which* $\rho = 4$,

and so on. Thus, we get all the elements (μ,ν) *in a simple series:*

$$(1,1);\ (1,2),\ (2,1);\ (1,3),\ (2,2),\ (3,1);\ (1,4),\ (2,3),\ldots,$$

and here, as we easily see, the element (μ,ν) *comes at the* λ *th place, where*

$$(9) \qquad \lambda = \mu + \frac{(\mu + \nu - 1)(\mu + \nu - 2)}{2}.$$

The variable λ *takes every numerical value* $1,2,3,\ldots$, *once. Consequently, by means of (9), a reciprocally univocal relation subsists between the aggregates* $\{\nu\}$ *and* $\{(\mu,\nu)\}$ *.”*

Clearly, the variable λ takes several times some of numerical value $1,2,3,\ldots$, The mapping is not one-one.

We have,

$$\mu = 1, \ \nu = 1, \ \lambda = 1, \ \text{ and } (1,1) \to 1.$$

$$\mu = 1, \ \nu = 2, \ \lambda = 2, \ \text{ and } (1,2) \to 2.$$

$$\mu = 2, \ \nu = 1, \ \lambda = 2, \ \text{ and } (2,1) \to 2.$$

$$\mu = 1, \ \nu = 3, \ \lambda = 4, \ \text{ and } (1,3) \to 4.$$

$$\mu = 2, \ \nu = 2, \ \lambda = 4, \ \text{ and } (2,2) \to 4.$$

$$\mu = 3, \ \nu = 1, \ \lambda = 4, \ \text{ and } (3,1) \to 4.$$

$$\mu = 1, \ \ \nu = 4, \ \lambda = 7, \ \text{ and } (1,4) \to 7.$$

$$\mu = 2, \ \ \nu = 3, \ \lambda = 7, \ \text{ and } (2,3) \to 7.$$

$$\mu = 3, \ \ \nu = 2, \ \lambda = 7, \ \text{ and } (3,2) \to 7.$$

$$\mu = 4, \ \ \nu = 1, \ \lambda = 7, \ \text{ and } (4,1) \to 7.$$

Apparently, Cantor expected his mapping to be a bijection, did not check it, and did not see that it is not even one-one.

It is well known [2] that a one-one mapping is required to establish that $cardQ \leq cardN$. Thus, Cantor's claim is unfounded, and his use of $\aleph_0^2 = \aleph_0$, amounts to adding another axiom to his set theory.

2. Proof of Rationals Countability by Tarski result.

We use the graphical interpretation of Cantor's proof by a zig-zag through an infinite triangular matrix of rationals. While the zig-zag by itself does not prove the countability, it is useful to clarify our argument:

The first line in the zig-zag has one rational

1/1.

The second line has two rationals,

1/2, and 2/1.

………………………………………………

The n-th line has n rationals,

1/n, 2/(n-1),…,(n-1)/2, n/1,

$$
\begin{array}{ccccccccccccccc}
\frac{1}{1} & \rightarrow & \frac{1}{2} & & \frac{1}{3} & \rightarrow & \frac{1}{4} & & \frac{1}{5} & \rightarrow & \dots & \dots & * & \dots & * \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & \dots & \dots & \dots & \nearrow & \dots & \swarrow & \\
\frac{2}{1} & & \frac{2}{2} & & \frac{2}{3} & & \frac{2}{4} & \dots & \dots & \dots & * & \dots & * & & \\
\downarrow & \nearrow & & \swarrow & & \nearrow & \dots & \dots & \dots & \nearrow & \dots & \swarrow & & & \\
\frac{3}{1} & & \frac{3}{2} & & \frac{3}{3} & \dots & \dots & \dots & * & \dots & * & & & & \\
 & \swarrow & & \nearrow & \dots & \dots & \dots & \nearrow & \dots & \swarrow & & & & & \\
\frac{4}{1} & & \frac{4}{2} & \dots & \dots & \dots. & * & \dots & * & & & & & & \\
\downarrow & \nearrow & \dots & \dots & \dots & \nearrow & \dots & \swarrow & & & & & & & \\
\frac{5}{1} & \dots & \dots & \dots & * & \dots & * & & & & & & & & \\
\dots & \dots & \dots & \nearrow & \dots & \swarrow & & & & & & & & & \\
* & \dots & * & \dots & * & & & & & & & & & & \\
\downarrow & \nearrow & \dots & \swarrow & & & & & & & & & & & \\
* & \dots & * & & & & & & & & & & & & \\
\dots & \swarrow & & & & & & & & & & & & & \\
* & \rightarrow & & & & & & & & & & & & &
\end{array}
$$

Summing the number of the rationals along the zig-zag, for $n = 1,2,3....,$

$$1 + 2 + 3 + ... + n < \aleph_0. \quad (1)$$

Thus,

$$2(1 + 2 + 3 + ... + n) < \aleph_0.$$

That is,

$$n + n^2 < \aleph_0. \qquad (2)$$

Tarski ([3], or [4, p.174]) proved that

for any sequence of cardinal numbers, m_1, m_2, m_3,..., *and a cardinal* m, *the partial sums inequalities*

$$m_1 + m_2 + ... + m_n \leq m, \qquad \textit{for} \quad n = 1,2,3....$$

imply the series inequality

$$m_1 + m_2 + ... + m_n + ... \leq m.$$

Applying to (1),

$$1 + 2 + 3 + ... + n + ... \leq \aleph_0.$$

Regarding (2), this says

$$\aleph_0 + \aleph_0^2 \leq \aleph_0.$$

Since $\aleph_0^2 \leq \aleph_0 + \aleph_0^2$, by transitivity of cardinal inequalities [3, p. 147],

$$\aleph_0^2 \leq \aleph_0 .$$

Since $\aleph_0 \leq \aleph_0^2$,

$$\aleph_0^2 = \aleph_0 .$$

3. Proof of Rationals Cantorian Countability by injection

Aided by the zig-zag listing of the rationals, we produce a one-one mapping from the rationals into the natural numbers. We construct our mapping with numerical examples. Then we give the general formula.

The first line in the zig-zag has one rational

$$1/1$$

which we assign as follows

$$\frac{1}{1} \to 1 + 2^1 = 3 .$$

The second line in the zig-zag has two rationals,

1/2, and 2/1,

which we assign as follows

$$\frac{1}{2} \to 1 + 2^2 = 5,$$

$$\frac{2}{1} \to 2 + 2^2 = 6.$$

The third line in the zig-zag has three rationals

3/1, 2/2, 1/3,

which are assigned as follows

$$\frac{3}{1} \to 1 + 2^3 = 9,$$

$$\frac{2}{2} \to 2 + 2^3 = 10,$$

$$\frac{1}{3} \to 3 + 2^3 = 11.$$

The fourth line in the zig-zag has four rationals

1/4, 2/3, 3/2, 4/1

which are assigned as follows

$$\frac{1}{4} \to 1 + 2^4 = 17,$$

$$\frac{2}{3} \to 2 + 2^4 = 18,$$

$$\frac{3}{2} \rightarrow 3 + 2^4 = 19.$$

$$\frac{4}{1} \rightarrow 4 + 2^4 = 20.$$

The fifth line in the zig-zag has five rationals

5/1, 4/2, 3/3, 2/4, 1/5,

which are assigned as follows

$$\frac{5}{1} \rightarrow 1 + 2^5 = 33,$$

$$\frac{2}{4} \rightarrow 2 + 2^5 = 34,$$

$$\frac{3}{3} \rightarrow 3 + 2^5 = 35.$$

$$\frac{2}{4} \rightarrow 4 + 2^5 = 36.$$

$$\frac{1}{5} \rightarrow 5 + 2^5 = 37.$$

If

$$m + n - 1 = even = 2k,$$

the $m + n - 1 = 2k$ zig-zag line has the $m + n - 1 = 2k$

rationals

1/n, 2/(n-1),…(n-1)/2, n/1,

which are assigned as follows

$$\frac{1}{n} \to 1 + 2^{2k},$$

$$\frac{2}{n-1} \to 2 + 2^{2k},$$

……………………..

$$\frac{n-1}{2} \to 2k - 1 + 2^{2k},$$

$$\frac{n}{1} \to 2k + 2^{2k}.$$

If

$$m + n - 1 = odd = 2k + 1,$$

the $m + n - 1 = 2k + 1$ zig-zag line has the $m + n - 1 = 2k + 1$ rationals

m/1, (m-1)/2,…, 2/(m-1),1/m,

which are assigned as follows

$$\frac{m}{1} \to 1 + 2^{2k+1},$$

$$\frac{m-1}{2} \to 2 + 2^{2k+1},$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\frac{2}{m-1} \to 2k + 2^{2k+1},$$

$$\frac{1}{m} \to 2k + 1 + 2^{2k+1}.$$

This defines a one-one function from the rationals into the natural numbers.

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Preface to 4

The conclusion of **3** was that the rationals can be sequenced.

Not knowing that the Cantorian Cardinality of the rationals is $CardN$ by an Axiom, I accepted that the rationals cardinality was proven to be $CardN$.

Thus, I had to conclude that the Hypothesis was a proven fact, and there was neither Non-Cantorian Theory, nor Non-Cantorian cardinals.

However, Cantor's claim that his mapping was one-one puzzled me, and the question on my mind was

"What else did he do wrong…"

Taking a second look at the Cantor set, I found out that its properties were as advertised. But Cantor did not take full advantage of them.

The construction of the Cantor set produces a set of rational numbers with cardinality $2^{\aleph_0}$. These numbers can serve as a range for an injection from the real numbers to conclude that there are no more reals than rationals. Hence, the reals too are countable.

Consequently, in Cantor's Theory the only infinity is $CardN$.

Cantor's definition of cardinality that does not distinguish between the infinities of integers and rationals, does not distinguish between the infinities of the integers and the reals.

In **7** we show that the existence of unique infinity in Cantor's Theory is an Axiom equivalent to the Hypothesis.

Therefore, the inequality $CardN < CardR$ is an Axiom equivalent to the Negation of the Hypothesis.

Thus, Cantor's "Diagonal Argument" is a "proof" of a Non-Cantorian Axiom.

4

Cantor's Set and the Cardinality of the Reals

H. Vic Dannon
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Introduction:

The Cantor set is obtained from the closed unit interval [0,1] by a sequence of deletions of middle third open intervals. The remaining set contains no interval, but has cardinality equal to $CardR = 2^{CardN}$.

If Cantor was attempting to find a cardinality between $CardN$, and 2^{CardN}, that set might have led him to his Continuum Hypothesis that there is no set X with $CardN < CardX < 2^{CardN}$.

But if the cardinality of such a meager set equals $Card(0,1)$, how large is $Card(0,1)$?

To motivate our discussion, we point out some unsettled issues regarding infinite cardinalities:

1st issue: Card(N)<Card(R)

To prove that

$$CardN < CardR,$$

Cantor listed the reals in $(0,1)$, presumed to be countably many, and exhibited the diagonal element as the one not counted for, in the list.

By [1, p. 57], every real number between 0, and 1, has a unique infinite decimal representation

$$c_1^{(n)}c_2^{(n)}c_3^{(n)}....$$

Missing from that listing is the real number

$$c_1c_2c_3....$$

where

$$c_n = 0, \text{ if } \quad c_n^{(n)} \neq 0, \text{ and } c_n = 1, \text{ if } \quad c_n^{(n)} = 0.$$

What is so crucial about one missing element out of infinitely many? Why cannot we add the one missing diagonal element to the listing?

The listing will remain countable after countably many such additions.

Two infinite sets have the same power, even if one set is “half” of the other. For instance, the natural numbers, and the odd natural numbers have the same power. For infinite sets, we can tolerate the missing of countably many elements long before we conclude a contradiction. In other words, the diagonal proof does not apply credibly to infinite sets.

In this paper we disprove Cantor’s claim that $CardN < CardR$.

2nd issue: Card(A)<Card(P(A)).

The inequality

$$CardN < 2^{CardN},$$

seems like a wish modeled after the finite case

$$n < 2^n.$$

But as $n \to \infty$, we have

$$\lim n \leq \lim 2^n.$$

Is the inequality in

$$CardN < 2^{CardN}$$

indeed strict?

By [1, p. 87], Cantor proved that

The set of all subsets of any given set A is of greater power than the set A

The proof aims to show that

$$Card(A) < Card(P(A)),$$

in order to conclude that for every cardinal number there is a greater cardinal number, and to establish

$$Card(N) < Card(P(N)) = Card(R).$$

But Cantor's proof uses the concept of set of sets which for infinite sets is not well-understood, since it may lead to the Russell paradoxial set.

Russell (1903) defined his set y by

$$x \in y \leftrightarrow x \notin x.$$

Then, in particular,

$$y \in y \leftrightarrow y \notin y$$

which is a contradiction.

In fact, as pointed out in [2, p.87], if we apply Cantor's theorem to the universal class of all objects V, every subset of V is also a member of V, and we have

$$Card(V) = Card(P(V)).$$

Avoiding this fact by claiming that V is not a set, while leaving the definition of what is a set vague enough to suit other results, does not establish the credibility of Cantor's claim. Perhaps, for infinite sets strict inequality does not exist, and we have only

$$Card(A) \le Card(P(A)).$$

In this paper we prove that $CardN = Card(P(N))$.

3rd issue: Card(Cantor set)=Card(R)

The association of *CardR* with the *"Power of the continuum"*, gives the impression that *CardR* must be greater than *CardN*.

But Cantor's set, which cardinality equals to *CardR*, is no continuum whatsoever. It is almost a void in $(0,1)$.

Cantor's set demonstrates that cardinality fails to distinguish between intervals, and sets with no intervals. The cardinality of $(0,1)$ equals that of the meager Cantor set. Thus, it is conceivable that *Card*$(0,1)$ may not be any greater than *Card(Q)*.

We present six different proofs of the equality $Card(0,1) = Card(Q)$. Our first proof constructs an injection form the real numbers into the rationals numbers.

1. Proof by injection that Card(0,1)=Card(Q)

We list the real numbers in $[0,1]$, in the dictionary order, using their binary representation.

The first row has the sequence that starts with 0, and the sequence that starts with 1, each followed by infinitely many zeros.

$$(0,0,0,..., 0,0,0...) \equiv x_{1,1},$$

$$(1,0, 0,...,0, 0,0...) \equiv x_{1,2}$$

The second has the 4 sequences

$$(0,0,0,..., 0,...) \equiv x_{2,1},$$

$$(0,1,0,\ldots,0\ldots) \equiv x_{2,2},$$

$$(1,0,0,\ldots,0,\ldots) \equiv x_{2,3}$$

$$(1,1,0,\ldots,0,\ldots) \equiv x_{2,4}$$

The third row has the 8 sequences

$$(0,0,0,0,\ldots) \equiv x_{3,1},$$

$$(0,0,1,0,\ldots) \equiv x_{3,2},$$

$$(0,1,0,0,\ldots) \equiv x_{3,3},$$

$$(0,1,1,0,\ldots) \equiv x_{3,4},$$

$$(1,0,0,0,\ldots) \equiv x_{3,5},$$

$$(1,0,1,0,\ldots) \equiv x_{3,6},$$

$$(1,1,0,0,\ldots) \equiv x_{3,7},$$

$$(1,1,1,0,\ldots) \equiv x_{3,8}.$$

The n-th row lists the 2^n sequences that start with

$$(0,0,0,0,\ldots) \equiv x_{n,1},$$

and end with

$$(1,1,1,1,...1,0,...) \equiv x_{n,2^n}.$$

The $\aleph_0$ row will list the $2^{\aleph_0}$ sequences that represent the binary expansions of all the real numbers in the interval $(0,1)$.

The sequences in each row are rational numbers, but the infinitely many rows contain all the real numbers. Denote

$$X_n \equiv \{x_{n,1}, x_{n,2}, x_{n,3}, ... x_{n,2^n}\}$$

Then,

$$X_1 \subset X_2 \subset ... \subset X_n \subset X_{n+1} \subset ...$$

We list the sequences=numbers as follows

0 1

00 01 10 11

000 001 010 011 100 101 110 111

... ...

We are guaranteed that this listing will enumerate all the real numbers between 0, and 1.

$$[0,1] \subseteq \bigcup_{n=1}^{\infty} X_n$$

We locate the rational truncation of $\sqrt{2}/2$, up to specified number of digits, in one of the rows. But the irrational number $\sqrt{2}/2$ is only in the last row.

The last row has

infinitely many sequences with a single 1, and infinitely many zeros,

infinitely many sequences with two 1's, and infinitely many zeros,

infinitely many sequences with three 1's, and infinitely many zeros,

……………………………………………………………

as well as

infinitely many sequences with one, and infinitely many 1's,

infinitely many sequences with two zeros, and infinitely many 1's,

infinitely many sequences with three zeros and infinitely many 1's

………………………………………………………………

We proceed to exhibit an injection from the real numbers into the rationals:

We want to map each real number one-one to a distinct rational number. All the rationals in the range of the map, have to be different from each other. Such rationals are the rational endpoints produced in the generation of the Cantor set [3].

The Cantor set is obtained from the closed unit interval [0,1] by a sequence of deletions of middle third open intervals.

First, we delete the open interval (1/3,2/3).

The numbers left in the intervals [0,1/3], and [2/3,1]

have either 0, or 2 of the fraction 1/3 in their expansion in base 3.

The two rational endpoints

$$\frac{1}{3} = y_{1,1} \text{ , and } \frac{2}{3} = y_{1,2}$$

remain in the Cantor set after indefinitely many deletions.

We denote

$$Y_1 \equiv \{y_{1,1}, y_{1,2}\}.$$

Second, we remove the open intervals (1/9,2/9) and (7/9,8/9).

The numbers left in the intervals

[0,1/9], [2/9,1/3], [2/3,7/9], and [8/9,1],

have either 0, or 2 of the fraction 1/9 in their expansion in base 3.

The four rational endpoints

$$\frac{1}{9} = y_{2,1}, \quad \frac{2}{9} = y_{2,2}, \quad \frac{7}{9} = y_{2,3}, \frac{8}{9} = y_{2,4},$$

remain in the Cantor set after indefinitely many deletions.

$$Y_2 \equiv \{ y_{2,1}, y_{2,2}, y_{2,3}, y_{2,4} \}$$

Third, we remove the open intervals

(1/27,2/27), (7/27,8/27), (19/27,20/27), (25/27,26/27).

The numbers left in the 8 closed intervals have either 0, or 2 of the fraction 1/27 in their expansion in base 3.

The eight rational endpoints

$$\frac{1}{27} = y_{3,1}, \frac{2}{27} = y_{3,2},$$

$$\frac{7}{27} = y_{3,3}, \frac{8}{27} = y_{3,4},$$

$$\frac{19}{27} = y_{3,5}, \frac{20}{27} = y_{3,6},$$

$$\frac{25}{27} = y_{3,7}, \frac{26}{27} = y_{3,8},$$

remain in the Cantor set after indefinitely many deletions.

$$Y_3 \equiv \{ y_{3,1}, y_{3,2}, y_{3,3}, y_{3,4}, y_{3,5}, y_{3,6}, y_{3,7}, y_{3,8} \}$$

In the n-th step, we remove 2^{n-1} open intervals leaving 2^n closed intervals, which numbers have either 0, or 2 of the fraction $1/3^n$ in their expansion in base 3.

The 2^n rational endpoints of the removed open intervals, remain in the Cantor set after indefinitely many deletions.

$\bigcup_{n=1}^{\infty} Y_n$ is a subset of the rationals in [0,1] because the midpoints of the removed intervals, ½, 1/6, 5/6,.. are not in any Y_n.

We define a one-one function from the real numbers listed in dictionary order, onto the rational endpoints of the Cantor set.

The two reals in X_1 are assigned by their listing order to each of the two rational endpoints produced in the first deletion 1/3, and 2/3, by the bijection

$$f_1 : X_1 \to Y_1$$

$$(0,0,0,..., 0,0,0,....) \to 1/3,$$

$$(1,0,0,0....0,0,0....) \to 2/3.$$

The four reals in X_2 are assigned by their listing order to each of the four endpoints of the two deleted open intervals,

1/9, 2/9, 7/9, 8/9.

by the bijection

$$f_2 : X_2 \to Y_2.$$

The eight reals in the X_3 are assigned by their listing order to each of the eight endpoints of the four deleted open intervals,

1/27, 2/27, 7/27, 8/27, 19,27, 20/27, 25/27, 26/27.

by the bijection

$$f_3 : X_3 \to Y_3$$

The 2^n reals in X_n are assigned by their order to each of the 2^n endpoints of the 2^{n-1} deleted open intervals, by the bijection

$$f_n : X_n \to Y_n.$$

Now define

$$f : \bigcup_{n=1}^{\infty} X_n \to \bigcup_{n=1}^{\infty} Y_n \quad \text{so that } f|_{X_n} = f_n$$

The mapping f is one-one from the real numbers in $\bigcup_{n=1}^{\infty} X_n$ onto the rational numbers in $\bigcup_{n=1}^{\infty} Y_n$.

Consequently,

$$CardR = Card[0,1] \leq Card \bigcup_{n=1}^{\infty} X_n$$

$$= Card \bigcup_{n=1}^{\infty} Y_n \leq CardQ \leq CardR$$

Hence

$$Card[0,1] = CardQ = \aleph_0^2 . \square$$

2. Proof by Tarski result for Cardinals.

The 1st row in the dictionary listing of the reals has 2^1 reals.

The 2nd row has 2^2 reals.

The 3rd row has 2^3 reals.

....

The n-th row has 2^n reals.

………………………………..

Summing the number of the reals along the listing, for $n = 1,2,3....,$

$$2 + 2^2 + 2^3 + ... + 2^n \leq \aleph_0 + \aleph_0 + \aleph_0 + ... = \aleph_0 \times \aleph_0.$$

That is,

$$2 + 2^2 + 2^3 + ... + 2^n \leq \aleph_0^2$$

Tarski ([4], or [1, p.174]) proved that for any sequence of cardinal numbers, m_1, m_2, m_3,…, and a cardinal m, the partial sums inequalities

$$m_1 + m_2 + ... + m_n \leq m,$$

for $n = 1,2,3....$ imply the series inequality

$$m_1 + m_2 + ... + m_n + ... \leq m.$$

Applying Tarski result, we obtain

$$2 + 2^2 + 2^3 + ... + 2^n + ... \leq \aleph_0^2.$$

Now,

$$2 + 2^2 + ... + 2^n + ... = \lim_{n\to\infty}\left(2^{n+1} - 2\right) = \lim_{n\to\infty} 2^n = 2^{\aleph_0}.$$

Hence,

$$2^{\aleph_0} \leq \aleph_0^2.$$

On the other hand,

$$2^{\aleph_0} = CardR \geq CardQ = \aleph_0^2,$$

Consequently,

$$2^{\aleph_0} = \aleph_0^2. \square$$

3. Proof by Cardinals and Ordinals

By [1, p.277], every ordinal number α has a next ordinal number

$$\alpha + 1 > \alpha,$$

and no intermediate ordinal number ξ with $\alpha + 1 > \xi > \alpha$.

The ordinal $\alpha + 7$ is preceded by $\alpha + 6$, and is classified as 1st kind.

The smallest ordinal number that is not preceded by any ordinal is classified as 2nd kind, and is denoted by

$$\omega.$$

By [1, p.288] any 2nd kind ordinal number is the limit of an increasing transfinite sequence of ordinal numbers. In particular,

$$\omega = \lim_{n<\omega} 2^n .$$

By [1, p. 318, Theorem 1], the function

$$f(n) = 2^n$$

is continuous in n, and

$$\omega = \lim_{n<\omega} 2^n = 2^{\lim\limits_{n<\omega} n} = 2^{\omega}.$$

That is,

$$\omega = 2^{\omega}$$

Now, by [2, p. 88, Corollary 2.19]

ω is a cardinal number.

By [2, p.90, Corollary 2.33],

$$\aleph_0 = \omega .$$

Thus, $\omega = 2^{\omega}$ says

$$\aleph_0 = 2^{\aleph_0} . \square$$

4. Proof by Cardinality of Ordinals

By a Theorem of Schonflies (1913) [2, p. 126, Theorem 2.11], for ordinal numbers α, and β

$$Card(\alpha^{\beta}) = \max(Card(\alpha), Card(\beta)).$$

Therefore,

$$Card(2^{\omega}) = \max(Card(2), Card(\omega)) = \aleph_0.$$

On the other hand, by [2, p.126, (2.9)], exponentiation is a repeated multiplication , and for all α, and β

$$\alpha^{\beta} = \prod_{\gamma<\beta} \alpha.$$

Hence,

$$2^{\omega} = \prod_{n<\omega} 2.$$

Therefore,

$$Card(2^{\omega}) = Card \prod_{n<\omega} 2.$$

Now, by [2, p. 106, proposition 4.15], if a is a well ordered cardinal,

$$\prod_{x\in u} a = a^{Card(u)}.$$

Therefore,

$$\prod_{n<\omega} 2 = 2^{Card(\omega)} = 2^{\aleph_0}.$$

In conclusion,

$$\aleph_0 = 2^{\aleph_0}.\Box$$

5. Proof by Effective Countability of Ordinals

By the axiom of Choice, the set of ordinals up to the ordinal 2^{ω} is well ordered, by magnitude, and is sequenced. Therefore, it is well known to be effectively countable [5], and its power is $\aleph_0$.$\Box$

6. Proof by Tarski result applied to Algebraic Numbers

$$\begin{aligned} 2^{CardN} &\le (CardN)^{CardN} \\ &= CardN \times CardN \times CardN \times \dots \\ &\le CardN + (CardN)^2 + (CardN)^3 + \dots \end{aligned}$$

Now, since

$$CardN = (CardN)^2 = (CardN)^3 = \dots,$$

then, for any $n = 1,2,3....$

$$CardN + (CardN)^2 + ... + (CardN)^n \leq cardN.$$

Therefore, By Tarski result,

$$CardN + (CardN)^2 + (CardN)^3 + ... \leq CardN.$$

Thus,

$$2^{CardN} = CardN. \square$$

7. Transcendental and Irrational numbers

The rationals are effectively countable, and $\aleph_0^2 = \aleph_0$.

For the algebraic numbers,

$$\aleph_0 = CardN \leq Card(Algebraic\ Numbers) \leq CardR = \aleph_0$$

.

Hence,

$$Card(Algebraic\ Numbers) = \aleph_0.$$

By [6], *if* a *is non-zero, real algebraic number, then* e^a *is a transcendental number*. The mapping

$$a \rightarrow e^a$$

is an injection from the algebraic numbers into the transcendental numbers. Therefore,

$$\aleph_0 = Card R \geq Card(Transcendental\ \ Numbers)$$

$$\geq Card(Algebraic\ \ Numbers) = \aleph_0 .$$

Thus,

$$Card(Transcendental\ \ Numbers) = \aleph_0 .$$

Finally,

$$\aleph_0 = Card R \geq Card(Irrational\ \ Numbers)$$

$$\geq Card(Transcendental\ \ Numbers) = \aleph_0 .$$

Hence,

$$Card(Irrational\ \ Numbers) = \aleph_0 .$$

8. Non-Cantorian Set Theory

Since Cantor’s set theory leads to a single infinity $\aleph_0 = 2^{\aleph_0}$, we may wish to develop non-Cantorian set theory with more infinities.

Such non-Cantorian set theory was implicated in 1963 by Cohen's work on Cantor's Continuum Hypothesis that there is no set X with

$\aleph_0 < CardX < 2^{\aleph_0}$.

Cohen proved that if the commonly accepted postulates of set theory are consistent, then the addition of the negation of the hypothesis does not result in inconsistency [7].

Cohen's result was interpreted to mean that there is a set theory where the negation of the Continuum Hypothesis holds. However, Non-Cantorian cardinal numbers were not found, and the Non-Cantorian set theory was never developed.

In a coming paper we show that non-Cantorian set theory cannot be constructed.

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Preface to 5

The failure of Cantor's set theory to distinguish between infinities, led me again to try to establish the non-Cantorian Set Theory, that is founded on the Negation of the Hypothesis.

But we need a cardinality that distinguishes between sequences of integers, and rationals.

A Non-Cantorian Theory with Cantor's Cardinality is impossible.

5

Non-Cantorian Set Theory

by H. Vic Dannon
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Introduction:

In [1] we showed that in Cantor's set theory,

$$Card(0,1) = Card(rationals).$$

Since the rationals are countable,

$$Card(rationals) = \aleph_0^2 = \aleph_0,$$

and

$$\aleph_0 = Card(0,1) = 2^{\aleph_0}.$$

This disproves Cantor's claim that

$$\aleph_0 < 2^{\aleph_0},$$

and leads to a single infinity

$$\aleph_0 = 2^{\aleph_0}.$$

Will a non-Cantorian set theory allow for more infinities?

The existence of a non-Cantorian set theory was established in 1963 by Cohen's work on Cantor's Continuum Hypothesis that there is no set X with $\aleph_0 < CardX < 2^{\aleph_0}$.

Cohen proved that if the commonly accepted postulates of set theory are consistent, then the addition of the negation of the hypothesis does not result in inconsistency [2].

Cohen's result was interpreted to mean that there is a set theory where the negation of the Continuum Hypothesis holds. However, non-Cantorian cardinal numbers were not found, and the non-Cantorian set theory was never developed.

To develop a non Cantorian set theory, we will assume the negation of the Continuum Hypothesis, which is based on Cantor's claim that $\aleph_0 < 2^{\aleph_0}$. In [1] we disproved that claim, but here we will need to allow $\aleph_0 < 2^{\aleph_0}$ as an assumption. We aim to show that even with that disproved assumption, non Cantorian set theory does not exist.

To that end, we re-examine our proof of the Continuum Hypothesis in Cantor's set theory [3]. A close scrutiny of that proof reveals that rationals countability is equivalent to the Continuum Hypothesis.

1. Rationals countability $\equiv$ Continuum Hypothesis.

<u>*Theorem*</u> *Assume that* $\aleph_0 < 2^{\aleph_0}$. *Then*

$$\aleph_0^2 = \aleph_0 \Leftrightarrow \text{Continuum Hypothesis.}$$

Proof:

$(\Rightarrow)$

By [4, p.173], the sequence

$$\aleph_0 + \aleph_0^2 + \aleph_0^3 + \ldots + \aleph_0^n$$

sums up [4] to the series

$$\aleph_0 + \aleph_0^2 + \aleph_0^3 + \ldots + \aleph_0^n + \ldots = \sum_{n=1}^{\infty} \aleph_0^n .$$

By [4], the series has a well defined sum

$$\alpha \equiv \aleph_0 + \aleph_0^2 + \aleph_0^3 + \ldots + \aleph_0^n + \ldots,$$

which is a cardinal number.

By [4, p.174],

$$\alpha \geq \text{any component of the series.}$$

In particular, α is greater than the infinite product term

$$\alpha \geq \aleph_0 \cdot \aleph_0 \cdot \aleph_0 \cdot ... = \aleph_0^{1+1+1+\cdots}.$$

By [4, p.183],

$$\aleph_0^{1+1+1+\cdots} = \aleph_0^{\aleph_0}.$$

Therefore,

$$\alpha \geq \aleph_0^{\aleph_0}.$$

Or, using the product notation, by [5, p. 106], we obtain similarly,

$$\alpha \geq \prod_{n \in N} \aleph_0 = \aleph_0^{CardN} = \aleph_0^{\aleph_0}.$$

Since

$$\aleph_0^{\aleph_0} \geq 2^{\aleph_0},$$

by transitivity of cardinal inequalities [4, p. 147], we conclude that

$$\alpha \geq 2^{\aleph_0}.$$

Now suppose that the continuum hypothesis does not hold, and there is a set X with

$$\aleph_0 < CardX < 2^{\aleph_0}.$$

Since $\aleph_0^2 = \aleph_0$, then for any $n = 1,2,3...$,

$$\aleph_0 + \aleph_0^2 + \aleph_0^3 + ... + \aleph_0^n \leq CardX.$$

Tarski ([6], or [4, p.174]) proved that

If

$$m_1, m_2, ..m_n, \quad and \ m$$

are any cardinal numbers so that for any $n = 1,2,3....$,

$$m_1 + m_2 + ... + m_n \leq m,$$

then,

$$m_1 + m_2 + ... + m_n + ... \leq m.$$

By Tarski result

$$\aleph_0 + \aleph_0^2 + \aleph_0^3 + ... + \aleph_0^n + ... \leq CardX.$$

Since $CardX < 2^{\aleph_0}$, by transitivity of cardinal inequalities [4, p. 147],

$$\underbrace{\aleph_0 + \aleph_0^2 + \aleph_0^3 + ... + \aleph_0^n + ...}_{\alpha} < 2^{\aleph_0}$$

That is,

$$\alpha < 2^{\aleph_0},$$

which contradicts $\alpha \geq 2^{\aleph_0}$.

Therefore, there is no set X with $\aleph_0 < CardX < 2^{\aleph_0}$, and the continuum Hypothesis holds.

This completes the proof that $\aleph_0^2 = \aleph_0 \Rightarrow$ Continuum Hypothesis.$\square$

$(\Leftarrow)$

By [4, p.155], *For any cardinals* m, n,, m_1, and n_1,

$$m < n, \textit{ and } m_1 < n_1 \Rightarrow mm_1 < nn_1.$$

Since

$$\aleph_0 < 2^{\aleph_0},$$

we have

$$\aleph_0^2 < (2^{\aleph_0})^2.$$

But

$$(2^{\aleph_0})^2 = 2^{2\aleph_0} = 2^{\aleph_0}.$$

Hence,

$$\aleph_0^2 < 2^{\aleph_0}.$$

Therefore,

$$\aleph_0 < \aleph_0^2 \Rightarrow \aleph_0 < \aleph_0^2 < 2^{\aleph_0}.$$

Namely, if $\aleph_0 < \aleph_0^2$, the rationals serve as a set X which cardinality is between $CardN$,and $CardR$, and the Continuum Hypothesis does not hold. That is,

Negation of $\aleph_0^2 = \aleph_0 \Rightarrow$ Negation of the Continuum Hypothesis.

This says,

Continuum Hypothesis $\Rightarrow \aleph_0^2 = \aleph_0.\square$

In conclusion, the countability of the rationals is equivalent to the

Continuum Hypothesis, and the uncountability of the rationals is equivalent to the negation of the Continuum Hypothesis.

Since the rationals are countable, non-Cantorian set theory does not exist. Consequently, the interpretation of Cohen's work, that there is a set theory in which the Continuum Hypothesis does not hold, is erroneous.

2. Non Cantorian Set Theory does not exist

Under Cantor's claim that

$$\aleph_0 < 2^{\aleph_0},$$

Cohen proved that the continuum hypothesis is an independent axiom of set theory.

Since by our Theorem the continuum Hypothesis is equivalent to rationals' countability, then according to Cohen, rationals' countability is an independent axiom of set theory.

Could the rationals be assumed uncountable?

If you seek a clue to the answer in Cantor's Zig-Zag proof of

Rationals' countability, you'll find that the Zig-Zag proof is flawed.

Cantor's Zig-Zag aims to avoid following through infinitely many infinite sequences. Is it possible that towards its end, the Zig-Zag can not avoid the infinitely many infinite sequences?

Cantor's mapping in his Zig Zag proof that the rational numbers are countable has to be one-one. But in [7] we pointed out that it is not one-one.

However, we exhibited in [7] a one-one mapping from the rationals into the natural numbers, and established the effective countability of the rationals.

Hence, by our Theorem above, the Continuum Hypothesis is a fact, as much as the countability of the rationals is.

The Continuum Hypothesis is not an independent axiom of set theory.

The negation of the Continuum Hypothesis is equivalent to the negation of rationals' countability.

A non-Cantorian set theory based on the negation of the Continuum Hypothesis was not found so far, and will never be found, because it does not exist.

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Preface to 6

According to Lebesgue measure theory, countable sets have measure zero.

Since the reals are countable in Cantor's Set Theory, in any interval on the line, the measure of any interval should be zero.

But it is not.

Therefore, countability does not imply measure zero.

With that in mind, I examined the definition of measure by Lebesgue, and his proof that the rational numbers in $[0,1]$ have measure zero.

I believe that Lebesgue's measure theory is founded on non-credible arguments.

I suspect that his measure does not apply to any pathological sets, and his integral does not go any further than the Riemann integral.

Although his integral does relieve us from worrying about evaluating it…

6

Cardinality, Measure, Category

H. Vic Dannon,
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Abstract Lebesgue procedure to find the measure of a general set leads to contradictions. In particular, the set of rational numbers does not have measure zero. In fact, by Lebesgue own criteria, the set of rational numbers in $[0,1]$ is not measurable.

Introduction

Lebesgue defined the measure of an interval to be its length. He defined the measure of the union of infinitely many disjoint intervals (a_i, b_i) in $[0,1]$ to be the sum of the intervals' lengths

$$\mathrm{m}[(a_1,b_1)\cup(a_2,b_2)\cup(a_3,b_3)\cup\ldots] = (b_1 - a_1) + (b_2 - a_2) + (b_3 - a_3) + \ldots.$$

For a general set E in the interval $[0,1]$, he wrote [1, p. 182]

Cover E *by finitely many, or countably many intervals of* *lengths* $l_1, l_2, \ldots$

We wish to have

$$m(E) \leq l_1 + l_2 + \ldots$$

Then,

$$\inf_{\text{all covers of E}} \{l_1 + l_2 + \ldots\}$$

is an upper bound of $m(E)$, *that we denote* $\overline{m(E)}$,

and we have

$$m(E) \leq \overline{m(E)}.$$

Similarly, we have

$$m(E^c) \leq \overline{m(E^c)}.$$

We want to have

$$m(E) + m(E^c) = m([0,1]) = 1.$$

Hence, we must have

$$m(E) = 1 - m(E^c)$$

$$\geq 1 - \overline{m(E^c)}$$

In all we need to have

$$1-\overline{m(E^c)} \leq m(E) \leq \overline{m(E)}$$

When

$$1-\overline{m(E^c)} = \overline{m(E)},$$

then $m(E)$ *is defined, and we say that* E *is measurable.*

Lebesgue applied his procedure to determine the measure of the set of the rational numbers in the interval $[0,1]$ [2, p.35].

He sequenced the rationals

$$\{r_1, r_2, r_3, \ldots\}$$

and covered them by the intervals

$$(r_1 - \tfrac{1}{4}\varepsilon, r_1 + \tfrac{1}{4}\varepsilon),$$

$$(r_2 - \tfrac{1}{8}\varepsilon, r_2 + \tfrac{1}{8}\varepsilon),$$

$$(r_3 - \tfrac{1}{16}\varepsilon, r_3 + \tfrac{1}{16}\varepsilon)\ldots$$

of lengths

$$\frac{1}{2}\varepsilon, \frac{1}{2^2}\varepsilon, \frac{1}{2^3}\varepsilon, \ldots$$

Then,

$$\overline{m(E)} \leq \frac{1}{2}\varepsilon + \frac{1}{2^2}\varepsilon + \frac{1}{2^3}\varepsilon + ... = \varepsilon.$$

Taking the infimum on $\varepsilon > 0$, he effectively set ε to zero, and concluded that $\overline{m(E)} = 0$.□

We aim to show that

- Lebesgue procedure to find the measure of a general set leads to contradictions. In particular, the set of rational numbers does not have measure zero.
- By Lebesgue's own criteria, the set of the rational numbers in $[0,1]$ is not measurable.

We conclude with Lebesgue integration, and Category.

1. First Critique of the Lebesgue procedure

If ε is set to zero, the summation

$$l_1 + l_2 + ...$$

is over infinitely many degenerate intervals of length zero, and the sum of their lengths is of the form

$$\frac{1}{2}\varepsilon + \frac{1}{2^2}\varepsilon + \frac{1}{2^3}\varepsilon + \ldots$$

$$= 0 + 0 + 0 + \ldots$$

$$= 0 \bullet \infty ,$$

and no one knows what $0 \bullet \infty$ means.

$0 \bullet \infty$ may equal any number a:

$$\underbrace{\frac{a}{n} \times n}_{= a \to a} \to 0 \times \infty$$

In particular, $0 \bullet \infty$ may equal ∞:

$$\underbrace{\frac{1}{n} \times n^2}_{= n \to \infty} \to 0 \times \infty$$

Or, it may equal 0:

$$\underbrace{\frac{1}{n^2} \times n}_{= \frac{1}{n} \to 0} \to 0 \times \infty$$

Lebesgue perceived a countable set in terms of the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.$$

that has in $[0,1]$ a simple distribution compared with the intricate distribution of the rational numbers.

Then, the complement of the sequence in $[0,1]$

$$\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},....\right\}^c = (\frac{1}{2},1)\cup(\frac{1}{3},\frac{1}{2})\cup(\frac{1}{4},\frac{1}{3})\cup....$$

has the length

$$(1-\tfrac{1}{2})+(\tfrac{1}{2}-\tfrac{1}{3})+(\tfrac{1}{3}-\tfrac{1}{4})+...=1.$$

Therefore, $\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},....\right\}$ is measurable, and its measure is

$$m\left(\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},....\right\}\right) = m[0,1] - m\left[(\frac{1}{2},1)\bigcup(\frac{1}{3},\frac{1}{2})\bigcup(\frac{1}{4},\frac{1}{3})\cup....\right] = 0.$$

The effect of the set-elements' distribution in $[0,1]$, in determining the measure of a set, is evident in the construction of the Cantor set [3].

The Cantor set has the same cardinality as $[0,1]$, but it is constructed in such a way that its complement

$$(\frac{1}{3},\frac{2}{3})\cup(\frac{1}{3^2},\frac{2}{3^2})\cup(\frac{7}{3^2},\frac{8}{3^2})\cup....$$

has length 1.

Therefore, the Cantor set is measurable, and its measure is

$$m(CantorSet) = m[0,1] - m\left[(\frac{1}{3},\frac{2}{3}) \cup (\frac{1}{3^2},\frac{2}{3^2}) \cup (\frac{7}{3^2},\frac{8}{3^2}) \cup \ldots\right] = 0 .$$

2. Second Critique of the Lebesgue procedure

There are no rational-only intervals, or irrationals-only intervals. In any interval with irrational endpoints, there are infinitely many rational numbers, and in any interval with rational endpoints, there are infinitely many irrational numbers.

The sequencing of the rationals does not alter their dense distribution in the irrationals. We can sequence the rationals, but we cannot squeeze them into any subinterval of $[0,1]$. Not even into a subinterval of size $1-\delta$, for any $\delta > 0$. Similarly, the irrationals are dense in the rationals.

The cardinality of the rationals and irrationals is irrelevant to the density of each set in the other.

Recall Lebesgue's cover of the rationals in $[0,1]$

$$(r_1 - \tfrac{1}{4}\varepsilon, r_1 + \tfrac{1}{4}\varepsilon),(r_2 - \tfrac{1}{8}\varepsilon, r_2 + \tfrac{1}{8}\varepsilon),(r_3 - \tfrac{1}{16}\varepsilon, r_3 + \tfrac{1}{16}\varepsilon)\ldots$$

with length tailored to be $< \varepsilon$.

Its complement in $[0,1]$ is a union of intervals with length $> 1 - \varepsilon$.

And according to Lebesgue, there are no rational numbers in those non-degenerate intervals…

Can there be a non-degenerate interval void of rational numbers?

Lebesgue's claim to be able to keep rationals out of infinitely many intervals in $[0,1]$ is not credible.

There is no open cover of the rationals in $[0,1]$ of length $\varepsilon < 1$ that contains all the rational numbers in $[0,1]$.

Thus, the Lebesgue procedure to extend the definition of measure to a general set is based on an impossibility, and is invalid.

Perhaps, the concept of length in $[0,1]$ can not be extended to sets more general than the union of disjoint open intervals and a real sequence, or a Cantor-like set.

3. Third Critique of the Lebesgue procedure

Actually, Lebesgue's procedure ignores his own characterization of a measurable set. We quote him from [4, p.1051]

> *"A set E is measurable if and only if*
>
> *for as small as we wish $\varepsilon > 0$, E has a cover by $\alpha(\varepsilon)$*
>
> *open intervals, and E^c has a cover by $\beta(\varepsilon)$ open intervals*
>
> *so that the sum of the lengths of the intervals of intersection*
>
> *of the covers is $< \varepsilon$"*

Clearly, this characterization has in mind the simple structures of a real sequence, or a Cantor-like set, where the complement E^c is the union of disjoint open intervals. Then, the open covers may be refined so that their common intersection shrinks and is $< \varepsilon$.

But rational numbers cannot be separated from each other by open intervals of irrational numbers.

The density of the rationals in $[0,1]$ guarantees their presence in any subinterval of any interval in a cover of the irrationals in $[0,1]$.

Therefore, given any $\varepsilon > 0$, there are no refined open covers, so that the sum of the lengths of the intervals that belong to the intersection of the covers is $< \varepsilon$.

That is, by Lebesgue's characterization, both the rationals and the irrationals in $[0,1]$ are non-measurable.

This characterization strengthens the impression that the most general measurable set that is characterized by the Lebesgue criteria, is a union of disjoint open intervals and a real sequence or a Cantor-like set.

4. Integration

Since the set of rational numbers, and the set of irrational numbers in $[0,1]$, are non-measurable, the characteristic functions

$$\chi_{\{Q\bigcap[0,1]\}},$$

and

$$\chi_{\{\{\text{irrationals}\}\subseteq[0,1]\}}$$

are non-measurable, and their Lebesgue integrals do not exist.

Since many general sets may be unmeasurable, the Lebesgue integral may not be the general integral that it purports to be. It may not deliver more than the Riemann integral. Rotating the page for the sake of integration need not resolve essential difficulties.

4. Baire Category

Baire defined a set of numbers to be of *1^{st} category* if the set can be represented as a countable union of nowhere dense sets.

This definition was meant to characterize the set of the rational numbers, and distinguish it from the reals which, according to Cantor's claim, have strictly greater cardinality.

Baire's Theorem [5, p. 2] concludes that

- The complement of any set of first category on the line is dense in the real numbers
- No interval of real numbers is of first category

In a recent paper [6], we proved that in Cantor's set theory, there is only a single cardinality

$$\aleph_0 = 2^{\aleph_0}.$$

Hence,

- Intervals are of first category,

In fact,

- All sets of real numbers are of first category.

Therefore,

- The complement of a set of first category on the line need not be dense in the real numbers.

Consequently,

- Category does not distinguish between sets of real numbers

5. Baire Functions

The continuous functions are Baire class 0 functions [7, p. 137]. Functions that are limits of sequences of continuous functions are Baire class 1 functions. Functions that are limits of sequences of Baire class 1 functions, are Baire class 2 functions.

Thus, Baire class n functions are defined for any $n = 0,1,2,3...$ [7, p. 137]

By [7, p.140, THEOREM 3], the cardinality of the family of Baire functions is $2^{\aleph_0}$.

Therefore, there are countably many Baire functions.

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Preface to 7

Here, I finally understood that the rationals have a Non-Cantorian cardinality greater than $CardN$, and their Cantorian Cardinality is $CardN$.

The Non-Cantorian Axiom

$$CardN \times CardN > CardN$$

links the Negation of the Hypothesis with the Negation of the Axiom of Choice, to deliver the equivalence between them.

The Hypothesis may be the most elusive form of the Axiom of Choice.

The Axiom that defines the Cantorian Cardinality, and Its Negation that defines the Non-Cantorian Cardinality, are fundamental in the theories.

Cantor's "Zig-Zag Argument" that the rationals cardinality is $CardN$, is an attempt to prove an Axiom.

Cantor's "Diagonal Argument" that $CardN < 2^{CardN}$, is an attempt to prove a Non-Cantorian Axiom.

Godel's Consistency turns out to be trivial, since the Hypothesis is equivalent to the Axiom of Choice.

Cohen's Consistency must be wrong, because mixing the Axiom of Choice with its Negation must produce inconsistency

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Continuum Hypothesis, Axiom of Choice, and Non-Cantorian Theory

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Abstract We prove that the Continuum Hypothesis is equivalent to the Axiom of Choice. Thus, the Hypothesis-Negation is equivalent to the Axiom of No-Choice.

The Non-Cantorian Axioms impose a Non-Cantorian definition of cardinality, that is different from Cantor's cardinality imposed by the Cantorian Axioms.

The Non-Cantorian Theory is the Zermelo-Fraenkel Theory with the No-Choice Axiom, or the Hypothesis-Negation.

This Theory has distinct infinities.

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Introduction

The Continuum Hypothesis says that there is no set X with cardinality that is strictly between $CardN$, and $CardR = 2^{CardN}$.

Thus, the Hypothesis statement assumes that

$$CardN < 2^{CardN}.$$

In [1], we proved that in Cantor's Theory,

$$2^{CardN} = CardN.$$

Therefore, Cantor's claim that $CardN < CardR$ is disproved, but the Hypothesis statement is trivially satisfied.

Consequently, Cantor's theory offers precisely one unique infinity, defying its purpose to supply us with many distinct infinities.

To obtain distinct infinities we need to develop the Non-Cantorian Theory.

1. The Continuum Hypothesis, and Cardinality

In [3] we proved that the Hypothesis is equivalent to

$$(CardN)^2 = CardN .$$

Here, we show that it is equivalent to each of the following Axioms,

● **Continuum Hypothesis**

A. There is no set X so that $CardN < CardX < 2^{CardN}$

● **Countability Axiom**

B. $CardN \times CardN = CardN$

Cantor believed that the Countability Axiom was a Theorem, and "proved" it by his "Zig-Zag proof".

But the Countability Axiom cannot be proved. It is equivalent to the Hypothesis, and it holds under Cantorian Cardinality.

The Cantorian Cardinality is established by the Effective Countability Axiom, that is too equivalent to the Hypothesis.

● **Generalized Countability Axiom**

C. For any $n = 1,2,3,...$ $(CardN)^n = (CardN)^{n+1}$

● Diagonal Axiom

D. $2^{CardN} = CardN$

Cantor believed that 2^{CardN} is greater than $CardN$, and "proved" the inequality, which is the ***Non-Diagonal Axiom***, as a Theorem in his Theory.

Actually, the ***Non-Diagonal Axiom*** belongs to Non-Cantorian Theory.

Cantor's "proof" is known as "the Diagonal Argument".

The Cantorian Diagonal Axiom allows only one infinity in Cantor's Theory.

Thus, raising the need for the Non-Cantorian Theory.

● Generalized Diagonal Axiom

E. For any $n = 1,2,3,...$ $2^{CardN} = (CardN)^n$

● Effective Countability Axiom

F. $Card\{a_1, a_2, a_3, ..\} = CardN$, for any $\{a_1, a_2, a_3, ..\}$

Any infinite sequence of distinct numbers has $CardN$.

This Axiom establishes Cantorian Cardinality.

The Effective Countability Axiom guarantees that sequencing is sufficient to establish equal Cantorian cardinalities. All sequences have the same cardinality as the sequence of the natural numbers.

Since the Effective Countability is equivalent to the Hypothesis, the Cantorian Cardinality characterizes the Hypothesis exclusively.

Thus, the Effective-Countability Axiom is the key to the Cantorian Theory.

Proof

$\underline{A \Rightarrow B}$

We prove $Negation\ B \Rightarrow Negation\ A$.

By [2, p.155], *For any cardinals* $m_1,\ n_1,,\ m_2,\ \text{and}\ n_2$,

$$m_1 < n_1, \textit{ and } m_2 < n_2 \Rightarrow m_1 \times m_2 < n_1 \times n_2.$$

Assuming that

$$CardN < 2^{CardN},$$

we have

$$CardN \times CardN < 2^{CardN} \times 2^{CardN}.$$

But

$$2^{CardN} \times 2^{CardN} = 2^{CardN+CardN} = 2^{CardN}.$$

Hence,

$$(CardN)^2 < 2^{CardN}.$$

Therefore,

$$CardN < (CardN)^2 \Rightarrow CardN < (CardN)^2 < 2^{CardN}.$$

Namely, if $CardN < (CardN)^2$, the rationals serve as a set X which cardinality is between $CardN$, and $CardR$, and the Continuum Hypothesis does not hold. That is,

$$Continuum\ Hypothesis \Rightarrow (CardN)^2 = cardN. \square$$

$\underline{B \Rightarrow C}$ is clear.

$\underline{C \Rightarrow D}$

$$2^{CardN} \leq (CardN)^{CardN}$$

$$\leq CardN + (CardN)^2 + (CardN)^3 + \ldots$$

Now, C implies that for any $n = 1,2,3....$

$$CardN + (CardN)^2 + ... + (CardN)^n \leq cardN.$$

Tarski ([4], or [2, p.174]) proved that

If

$$m_1, m_2, ..m_n \quad and \ m$$

are any cardinal numbers so that for any $n = 1,2,3....,$

$$m_1 + m_2 + ... + m_n \leq m,$$

then,

$$m_1 + m_2 + ... + m_n + ... \leq m.$$

By Tarski result

$$CardN + (CardN)^2 + (CardN)^3 + ... \leq CardN,$$

$$\leq 2^{CardN}.$$

Therefore,

$$(CardN)^n = (CardN)^{n+1} \Rightarrow 2^{CardN} = CardN. \square$$

<u>$D \Rightarrow A$</u> is clear.

<u>$C \ \& \ D \Leftrightarrow E$</u>

Therefore,

$$A \Leftrightarrow B \Leftrightarrow C \Leftrightarrow D \Leftrightarrow E.$$

$\underline{D \Rightarrow F}$

For any $\{a_1, a_2, a_3, ...\}$,

$$CardN \le Card\{a_1, a_2, a_3, ...\} \le 2^{CardN} = CardN . \Box$$

$\underline{F \Rightarrow B}$ is clear, since $N \times N$ may be sequenced.

Therefore,

$$A \Leftrightarrow B \Leftrightarrow C \Leftrightarrow D \Leftrightarrow E \Leftrightarrow F . \Box$$

It follows that Cantor's Cardinality, that does not distinguish between sequences of integers, and sequences of rationals, does not distinguish between cardinalities of Natural, and Real Numbers.

Cantor's Cardinality, is too coarse to distinguish between infinities.

Will a Non-Cantorian Cardinality distinguish between infinities?

In [3] we showed that a Non-Cantorian Theory does not exist under Cantor's Cardinality

But the Non-Cantorian Axioms establish a Non-Cantorian Cardinality.

Cantor's Cardinality ignores the property that makes the rationals <u>seem</u> larger than the natural numbers. Namely, that between any two rationals there is another rational.

Non-Cantorian Cardinality, may count that.

The Non-Cantorian Cardinality is established by the Non-Effective-Countability Axiom.

2. The Non-Cantorian Cardinality

The Non-Cantorian cardinality is established by the following Axioms, that are all equivalent to the Hypothesis-Negation.

■ ***Hypothesis-Negation***

a) There is a set X so that $CardN < CardX < CardR$

■ ***Non-Countability Axiom***

b) For any $n = 1,2,3,...$ $CardN \times CardN > CardN$

The Rationals Non-Cantorian Cardinality is greater than $CardN$.

The Non-Countability Axiom Cannot be proved. It is equivalent to the Hypothesis-Negation, and it holds under Non-Cantorian Cardinality.

The Non-Cantorian Cardinality is established by the Non-Effective-Countability Axiom, that is too equivalent to the Hypothesis-Negation.

■ ***Generalized Non-Countability Axiom***

c) For any $n = 1,2,3,...$ $\quad (CardN)^{n+1} > (CardN)^{n}$

This Axiom guarantees infinitely many Non-Cantorian distinct infinities, between the Non-Cantorian cardinalities of the natural and the real numbers.

■ ***Non-Diagonal Axiom***

d) $2^{CardN} > CardN$

Cantor "proved" this Non-Cantorian Axiom as a Theorem in his Theory, by his " Diagonal Argument".

In fact, in Cantorian Theory we have the Diagonal Axiom $2^{CardN} = CardN$.

■ ***Generalized Non-Diagonal Axiom***

e) For any $n = 1,2,3,...$ $2^{CardN} > (CardN)^n$

This Axiom too guarantees many Non-Cantorian distinct infinities

■ ***Non-Effective-Countability Axiom***

f) $Card\{a_1, a_2, a_3, ..\} > CardN$, *for some* $\{a_1, a_2, a_3, ...\}$

There are infinite sequences of distinct numbers with Non-Cantorian cardinalities greater than $CardN$. For instance, the rational numbers, and the real numbers.

This Axiom establishes Non-Cantorian cardinality.

The Non-Effective-Countability Axiom guarantees that sequencing is not sufficient to establish equal Non-Cantorian cardinalities.

Not all sequences have the same cardinality as the sequence of the Natural Numbers.

There are sequences with Non-Cantorian cardinality strictly greater than that of the Natural Numbers.

Since the Non-Effective-Countability is equivalent to the

Hypothesis-Negation, the Non-Cantorian cardinality characterizes the Hypothesis-Negation exclusively.

3. The smallest Non-Cantorian Cardinality

The smallest Non-Cantorian cardinality is

$$CardN = Card\{1,2,3,...\}.$$

If $\{a_1, a_2, a_3,....\}$ is an infinite set with distinct elements that are sequenced, so that between none of two consecutive elements there are no other elements of the sequence, then we say that

$$Card\{a_1,a_2,a_3,....\} = CardN\,.$$

There are many sets with this cardinality.

The Odd natural numbers

$$\{1,3,5,7,.....\}$$

The Even natural numbers, which are the multiples of the number 2,

$$\{2,4,6,8,....\}$$

The multiples of the number 3,

$$\{3,6,9,12,15,....\}$$

The powers of the number 2,

$$\{2,2^2,2^3,2^4,....\}$$

The powers of the number 3,

$$\{3,3^2,3^3,3^4,....\}.$$

Theorem $CardN + CardN = CardN$

Proof

Given two infinite disjoint sets,

$$\{a_1, a_2, a_3, ...\},$$

and

$$\{b_1, b_2, b_3, ...\},$$

each with cardinality $CardN$, form the union

$$\{a_1, b_1, a_2, b_2, a_3, b_3, ...\}.$$

Since between any two consecutive elements of the union, there are no elements of the union, we have

$$Card\{a_1, b_1, a_2, b_2, a_3, b_3, ...\} = CardN.$$

We conclude that

$$CardN + CardN = CardN . \square$$

Similarly, for any natural number $n = 1,2,3,...,$

$$\underbrace{CardN + CardN + ... + CardN}_{n \ times} = CardN .$$

If the infinite set is such that between some consecutive elements there is another element, we may expect the Non-Cantorian cardinality of that sequence to be greater than $CardN$.

4. No-Choice Axiom

The **Choice Theorem** says that if for each $n = 1,2,3,...$ there is a non-empty set of numbers A_n, then we can choose from each A_n one number a_n, and obtain a collection of numbers that has a representative from each A_n.

If we replace the index numbers $n = 1,2,3,...$ with an infinite set of numbers I, this choice may not be guaranteed.

There may be an infinite set of numbers I, so that for each index i in it, there is a non-empty set of numbers A_i, with no collection of numbers, that has a representative a_i from each A_i.

The **Axiom of Choice** is the guess that the choice is guaranteed for any infinite set I, and any family of non-empty sets indexed by I.

The ***No-Choice Axiom*** says that there is an infinite index set, and a family of non-empty sets A_i indexed by it, with no collection of numbers, that has a representative a_i from each A_i.

5. No-Well-Ordering Axiom

The Axiom of Choice is equivalent to the Well-Ordering Axiom.

By the **Well-Ordering Theorem**, the Natural Numbers are ordered in such a way that every subset of them has a first element.

The **Well-Ordering Axiom** is the guess that every infinite set of numbers can be well-ordered like the Natural Numbers.

The ***No-Well-Ordering Axiom*** says that there is a set that cannot be well-ordered.

A candidate for such set are the Real Numbers.

In 1963, Cohen claimed that it is not possible to prove that the real numbers can be well-ordered.

6. No-Transfinite-Induction Axiom

The Axiom of Choice is equivalent to the Transfinite Induction Axiom.

The **Induction Theorem** says that

If a property depends on each number $n = 1,2,3,..$, so that

1) The property holds for the first natural number $n = 1$.
2) If the property holds for the natural number k, we can deduct that it holds for the next number $k + 1$.

Then, the property holds for any $n = 1,2,3,\ldots$.

The **Transfinite-Induction Axiom** guesses that the same holds for any infinite index set I.

It says that if I is any well-ordered infinite set of numbers, and if there is any property that depends on each index i from I, so that

1) The property holds for the first element of I,
2) If the property holds for all the $k's$ that precede the index j, we can conclude that the property holds for j,

Then, the property holds for any index i in I.

The ***No-Transfinite-Induction Axiom*** says that

There is a well-ordered infinite set of numbers I, and there is a property that depends on each index i from I, so that

1) The property holds for the first element of I,
2) If the property holds for all the $k's$ that precede the index j, we can conclude that the property holds for j,

But the property does not hold for some index i_0 in I.

7. Continuum Hypothesis $\equiv$ Axiom of Choice

We use the Non-Countability Axiom to link the Continuum Hypothesis with the Axiom of Choice, and prove equivalence between them.

We will show here that the Hypothesis-Negation is equivalent to the No-Choice Axiom.

We use a result that Tarski obtained in 1924. [6, p. 165, #1.17(a)]

Tarski proved that the Axiom of Choice is equivalent to the Axiom

- For any infinite cardinals α, and β,

$$\alpha + \beta = \alpha \times \beta .$$

That is, according to Tarski, the No-Choice Axiom is equivalent to the Axiom

- There are infinite cardinals α, and β, so that

$$\alpha + \beta \neq \alpha \times \beta .$$

If we take

$$\alpha = \beta = CardN .$$

Then,

$$
\begin{aligned}
\alpha + \beta &= CardN + CardN \\
&= CardN \\
&< CardN \times CardN \\
&= \alpha \times \beta
\end{aligned}
$$

That is,

$$\alpha + \beta \neq \alpha \times \beta$$

Thus, the Non-Countability Axiom

$$CardN < CardN \times CardN$$

is equivalent to the No-Choice Axiom.

On the other hand, the Non-Countability Axiom is equivalent to the Hypothesis-Negation.

Therefore, the No-Choice Axiom, and the Hypothesis-Negation are equivalent.

Thus, The Axiom of Choice, and the Hypothesis are equivalent. $\square$

The Continuum Hypothesis is not a stand alone Axiom, independent of the Commonly accepted Axioms of Set Theory.

The Non-Cantorian Theory is based on the Axiom of No-Choice.

8. The Meaning of Godel's Consistency

The failure to identify the Continuum Hypothesis with any of the Axioms of set theory, led Godel in 1938 to confirm the consistency of the Hypothesis with the other Axioms of set theory, and led Cohen in 1963 to confirm the consistency of the Hypothesis-Negation.

Since the Continuum Hypothesis is equivalent to the Axiom of Choice, Godel's Consistency result is self-evident.

The Continuum Hypothesis is consistent with the Axioms of set theory, because it is one of them.

The Continuum Hypothesis is just another statement of the Axiom of Choice.

Therefore, Godel's work amounts to the following:

If the commonly accepted Axioms of Set Theory are consistent, then adding one of them to all of them will cause no inconsistency.

While Godel established a trivial result, his methods enabled Cohen, to establish the Continuum Hypothesis as an independent Axiom of Set Theory.

That stopped work on the Continuum Hypothesis for a long time.

9. Cohen's Consistency Error

Cohen claimed that the addition of the Hypothesis-Negation to the commonly accepted Axioms of set theory, will cause no inconsistency.

But the Hypothesis-Negation is just another statement of the Axiom of No-Choice

Therefore, the addition of the Hypothesis-Negation to the axioms of set theory, means the addition of the Axiom of No-Choice, to the Axiom of Choice.

The mixing of the Axiom of Choice with its Negation, must lead to inconsistency.

Cohen's erroneous consistency result, established the Hypothesis as an independent Axiom of Set Theory.

In fact, the Continuum Hypothesis is equivalent to the Axiom of Choice.

The Hypothesis is one of the commonly accepted Axioms of Set

Theory.

Thus, Non-Cantorian Theory is based on the Axiom of No-Choice.

The Non-Cantorian Theory is the No-Choice Theory of Zermelo and Fraenkel.

10. Non-Cantorian Cardinals

(CardN)×(CardN)

Since the rationals can be listed in an infinite matrix,

$$Card(Rationls) = CardN \times CardN \ .$$

According to the Non-Countability Axiom,

$$CardN < CardN \times CardN \, .$$

That is, the Non-Cantorian cardinality of the rational numbers, is greater than $CardN$.

Since for any $n = 1,2,3,...$,

$$nCardN = CardN \, ,$$

there is a no Non-Cantorian cardinality between the integers, and the rationals.

The Rationals have the smallest Non-Cantorian cardinality that is strictly greater than the cardinality of the Natural numbers.

(CardN)^{3}

The cardinality of the roots of quadratic polynomials in integer coefficients in R is

$$CardN \times CardN \times CardN .$$

By the Generalized Non-Countability Axiom,

$$CardN \times CardN < CardN \times CardN \times CardN .$$

Since for any $n = 1,2,3,...,$

$$nCardN \times CardN = CardN \times CardN ,$$

there is a no Non-Cantorian cardinality between the Rationals, and the roots of the Quadratic Polynomials in integer coefficients.

The Roots of the Quadratic Polynomials in integer coefficients, have the smallest Non-Cantorian cardinality that is strictly greater than the cardinality of the rationals.

(CardN)^n

For any $n = 1,2,3,\ldots,(CardN)^n$ is the cardinality of all the roots of all the polynomials in integer coefficients of degree n.

By the Generalized Non-Countability Axiom, for any $n = 1,2,3,.$

$$(CardN)^n < (CardN)^{n+1}.$$

Furthermore, there is no Non-Cantorian cardinality between the two.

$$(CardN)^{n+1} \quad \textit{is the smallest Non-Cantorian cardinality}$$
$$\textit{that is strictly greater than} \quad (CardN)^n .$$

2^CardN

The cardinality of the real numbers is

$$CardR = 2^{CardN}.$$

By the Generalized Non-Diagonal Axiom,

$$2^{CardN} > (CardN)^n, \ \textit{for any} \ n = 1,2,3,\ldots.$$

Theorem $\underline{(CardN)^{CardN} = 2^{CardN}}$

Proof

$(\geq)$ is clear.

$(\leq)$

$$(cardN)^{CardN} \leq CardN + (CardN)\times(CardN) + (CardN)\times(CardN)\times(CardN) + ...,$$

$$= CardN + (CardN)^2 + (CardN)^3 + ...$$

Now, for $n = 1,2,...$

$$CardN + (CardN)^2 + ... + (CardN)^n \leq 2^{CardN}.$$

Therefore, by Tarski, [4]

$$CardN + (CardN)^2 + (CardN)^3 + ... \leq 2^{CardN}$$

Hence,

$$(cardN)^{CardN} \leq 2^{CardN}.$$

and we conclude

$$(CardN)^{CardN} = 2^{CardN}. \square$$

Theorem for any $n \geq 2$,

$$\underline{(\,cardN\,)^{CardN} = 2^{CardN} = n^{CardN}}$$

Proof

$$n^{CardN} \leq (\,cardN\,)^{CardN} = 2^{CardN} \leq n^{CardN}.\square$$

Theorem $\underline{2^{CardN} \times 2^{CardN} = 2^{CardN}}$

Proof

$$2^{CardN} \times 2^{CardN} = 2^{CardN+CardN} = 2^{CardN}.\square$$

Theorem $\underline{(\,CardN\,)^{n} \uparrow 2^{CardN}}$.

Proof:

$$(\,CardN\,)^{n} \uparrow (\,CardN\,)^{CardN} \geq 2^{CardN} = (\,CardN\,)^{CardN}.\square$$

Algebraic Numbers

For algebraic numbers,

$$(\,CardN\,)^{CardN} = CardN \times CardN \times CardN \times ...$$

$$\leq CardN + CardN \times CardN$$

$$+CardN \times CardN \times CardN + ...$$

$$= CardN + (CardN)^2 + (CardN)^3 + ...$$

$$= Card(Algebraic\ Numbers)$$

$$\leq CardR$$

$$= 2^{CardN}$$

$$= (cardN)^{CardN}$$

Hence,

$$\underline{Card(Algebraic\ Numbers) = 2^{CardN}}.\square$$

Transcendental Numbers

By [5],

If a is non-zero, real algebraic number, then e^a is a transcendental number. The mapping

$$a \to e^a$$

is an injection from the algebraic numbers into the transcendental numbers.

Therefore,

$$2^{CardN} = CardR \geq Card(Transcendental\ \ Numbers)$$

$$\geq Card(Algebraic\ \ Numbers) = 2^{CardN}.$$

Thus,

$$\underline{Card(Transcendental\ \ Numbers) = 2^{CardN}}.$$

Irrational Numbers

$$2^{CardN} = CardR \geq Card(Irrational\ \ Numbers)$$

$$\geq Card(Transcendental\ \ Numbers)$$

$$= 2^{CardN}.$$

Hence,

$$\underline{Card(Irrational\ \ Numbers) = 2^{CardN}}.$$

$2^{(CardN)\times(CardN)}$

Theorem $\underline{2^{CardN} < 2^{(CardN)^2}}$.

Proof

If not, then

$$2^{CardN} = 2^{(CardN)^2} = \left(2^{CardN}\right)^{CardN}$$

$$= \left(2^{(CardN)^2}\right)^{CardN} = 2^{(CardN)^3} = ...$$

Therefore,

$$2^{CardN} = \left(2^{CardN}\right)^{CardN}$$

$$= 2^{CardN} \times 2^{CardN} \times 2^{CardN} \times ...$$

$$= 2^{CardN} \times 2^{(CardN)^2} \times 2^{(CardN)^3} \times ...$$

$$= 2^{CardN + (CardN)^2 + (CardN)^3 + ...}$$

$$= 2^{2^{CardN}}$$

According to [2, p.152, #7], we can prove without the Axiom of Choice that

✚ *There is no cardinal number* m *so that* $2^m = 2^{2^m}$

Thus,

$$2^{2^{CardN}} \neq 2^{CardN},$$

and we conclude that

$$2^{CardN} < 2^{(CardN)^2}. \square$$

$2^{(CardN)^n}$

Theorem $\underline{2^{(CardN)^n} < 2^{(CardN)^{n+1}} \; for \;\; any \;\; n = 1,2,3,...}$

Proof

If we assume that $2^{(CardN)^n} = 2^{(CardN)^{n+1}}$, then by an argument similar to the one used for $2^{CardN} < 2^{(CardN)^2}$, we will have

$$2^{2^{(CardN)^n}} = 2^{(CardN)^n}.$$

Contradicting ✚. □

Theorem $\underline{2^{(CardN)^n} \uparrow 2^{2^{CardN}}}$.

Proof:

$$2^{(CardN)^n} \uparrow 2^{(CardN)^{CardN}} = 2^{2^{CardN}}. \square$$

11. Cantorian Theory, and Cardinals

In Cantor's theory, any set may be sequenced [1], and there is only one infinity. By the Countability, and Diagonal Axioms,

$$CardN = CardN \times CardN = 2^{CardN}.$$

By the Generalized Countability, and Diagonal Axioms,

$$CardN = (CardN)^2 = (CardN)^3 = ...$$

$$= 2^{CardN} = (CardN)^{CardN}$$

$$= 2^{(CardN)^2} = 2^{(CardN)^3} = ...$$

$$= 2^{(CardN)^{CardN}} = 2^{2^{CardN}} = ...$$

The Algebraic Numbers have cardinality

$$CardN + (CardN)^2 + (CardN)^3 + ...$$

$$= CardN + CardN + CardN + ...$$

$$= CardN \times CardN = CardN.$$

The Transcendental, and the Irrational Numbers have cardinality

$$2^{CardN} = CardN.$$

Cantor's theory attempts to prove Axioms, as if they were Theorems, and borrows from the Non-Cantorian Theory Axioms that do not hold in Cantor's Theory.

For instance, it confiscates the Non-Cantorian, Non-Diagonal Axiom

$$CardN < 2^{CardN},$$

and expects it to be compatible with the Cantorian Countability

$$CardN = (CardN)^2,$$

disregarding the Cantorian Generalized Countability, and Diagonal Axioms that guarantee

$$CardN = (CardN)^2 = ... = (CardN)^{CardN} = 2^{CardN}.$$

Cantor's Theory is obtained by augmenting the Zermelo-Fraenkel Theory with the Axiom of Choice, which is equivalent to the Well-Ordering Axiom, and to the Continuum Hypothesis.

Cantor's Theory adds as Axioms, statements that cannot be proved in the Zermelo-Fraenkel Theory.

For instance, according to [6, p. 123], Cohen proved in 1963 that

In the Zermelo-Fraenkel Set Theory, one cannot prove that the set of all real numbers can be well-ordered

This suggests that,

The set of real numbers *<u>may not</u>* *be well-ordered.*

But in defiance, Cantorian Theory adds the Well-Ordering Axiom.

At the end, in-spite of all the patching with added Axioms, Cantor's Theory delivers only one infinity.

12. Non-Cantorian Theory, and Cardinals

The Non-Cantorian Theory is the Zermelo-Fraenkel Theory with the No-Choice Axiom, and the equivalent Hypothesis-Negation, No-Well-Ordering, and No-Transfinite-Induction Axioms.

The Non-Cantorian Cardinality is established with the Non-Effective-Countability Axiom.

To obtain distinct infinities, we have to limit the lowest cardinality to sequences that are like the Natural Numbers, and unlike the Rationals. By the Non-Countability Axiom

$$cardN < CardN \times CardN \, .$$

By the Generalized Non-Countability Axiom,

$$CardN < (CardN)^2 < (CardN)^3 < (CardN)^4 < \ldots.$$

$$< (CardN)^{CardN}$$

$$= 2^{CardN} < 2^{(CardN)^2} < 2^{(CardN)^3} < ...$$

$$< 2^{2^{CardN}} < ...$$

The non-Cantorian cardinality of the Algebraic Numbers is

$$CardN + (CardN)^2 + (CardN)^3 + ... = 2^{CardN}.$$

The Non-Cantorian cardinalities of the Transcendental, and the Irrational Numbers are 2^{CardN}.

The first Non-Cantorian infinities in ascending order are:

$CardN$

$CardN \times CardN$

$CardN \times CardN \times CardN$

$(CardN)^4$

$(CardN)^5$

…………………………………………

$2^{CardN} = (CardN)^{CardN}$

$2^{CardN \times CardN}$

$$2^{(CardN)^3}$$

$$2^{(CardN)^4}$$

$$2^{(CardN)^5}$$

…………………………………………..

$$2^{2^{CardN}}$$

$$2^{2^{(CardN)^2}}$$

$$2^{2^{(CardN)^3}}$$

……………………………………………………………

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