Cantor's Paradoxical Diagonal Argument, and Transfinite Induction

H. Vic Dannon
vic0@comcast.net

January, 2018

Abstract Cantor believed that the infinity of the real numbers is strictly greater than the infinity of the counting numbers, $Inf\{1,2,3,...\}$.

He argued that if all the reals written in binary number base, in zeros, and ones, are listed in a sequence, one under the other, then a binary number different from each of the numbers in the sequence can be produced from the diagonal.

Cantor believed that the anti-diagonal number proves that the reals cannot be sequenced.

But similarly to the Rationals, the Reals can be placed on an infinite triangle, and be sequenced by a Zig Zag line through that triangle. Then, their number up to the nth raw is less than $2^{n\ 1}$

Since for any $n = 1,2,3,...$, $2^n < \inf\{1,2,3,...\}$, we have

$$2^{\inf\{1,2,3,...\}} \leq \inf\{1,2,3,...\}$$

That is, the infinity of the Reals is at most the infinity of the Counting numbers.

In fact, Cantor's argument does not de-sequence the Reals: If the Reals are assumed to be sequenced, they remain sequenced regardless of any argument.

Actually, Cantor's Argument is paradoxical: To construct the anti-diagonal number, it assumes countable Reals, and to establish that the anti-diagonal is off the listed reals, it assumes Transfinite Induction, hence uncountable Reals.

Consequently, Cantor's anti-diagonal number cannot be constructed:

Due to the equality of the infinities of the Counting numbers, and the Real numbers, Transfinite Induction is reduced to the Induction Theorem. And the anti-diagonal which is different from each of the listed Reals cannot be specified in its entirety.

Keywords: Continuum Hypothesis, Axiom of Choice, Well-Ordering, Transfinite Induction, Cardinal, Ordinal, Countability, Infinity.

2000 Mathematics Subject Classification 03E04; 03E10; 03E17; 03E50; 03E25; 03E35; 03E55.

Contents

1.

Cantor's Diagonal Argument

Cantor believed that the infinity of the reals is strictly greater than the infinity of the counting numbers,

$$Inf\{1,2,3,...\}.$$

He argued that if all the reals written in binary number base, in zeros, and ones, are listed in a sequence, one under the other, then a binary number different from each of the numbers in the sequence can be produced from the diagonal.

For instance, if the first seven reals are

$$\begin{matrix} 1 & * & * & * & * & * & * \\ * & 0 & * & * & * & * & * \\ * & * & 1 & * & * & * & * \\ * & * & * & 1 & * & * & * \\ * & * & * & * & 0 & * & * \\ * & * & * & * & * & 1 & * \\ * & * & * & * & * & * & 0 \end{matrix}$$

Then, the diagonal first seven digits are

1,0,1,1,0,1,0,….

And the anti-diagonal number with the first seven digits

0,1,0,0,1,0,1,….

is different from each of the first seven reals.

Cantor believed that this anti-diagonal number proved that the reals cannot be sequenced.

We first show that Cantor's Argument is irrelevant to the sequencing of the reals.

2.
If The Reals Are Sequenced, they Stay Sequenced Regardless of Cantor's Diagonal Argument

Cantor argued that

if the reals are sequenced, there is

one real number not in the sequence.

Even if Cantor had succeeded to show that, the sequenced reals can be kept sequenced.

place the anti-diagonal number above the first real,

and the reals stay sequenced,

and their infinity will remain $Inf\{1,2,3,...\}$

In fact,

we may repeat this process countably many times,

and after each time

the reals will stay sequenced,

and their infinity will remain $Inf\{1,2,3,...\}$.

Next, we show moreover, that if the reals are assumed sequenced they will stay so regardless of any argument.

3.
If the Reals Are Sequenced, They Stay Sequenced Regardless of Any Argument

Suppose that the real numbers are listed in a sequence. And suppose that changing any digit in the 1^{st} listed binary real number produces a real number which may be out of the listed Reals. The real numbers will remain sequenced when these $Inf\{1,2,3,...\}$ sequenced real numbers -possibly not in the original list- are added to the list.

For each of the $Inf\{1,2,3,...\}$ variants of the 1^{st} listed number, there are $Inf\{1,2,3,...\}$ many variants of the 2^{nd} listed real number, each produced by changing one digit in the 2^{nd} listed number. There are $(Inf\{1,2,3,...\})^2$ such numbers. Suppose that all of them are out of the listed reals. These numbers are sequenced because

$$(Inf\{1,2,3,...\})^2 = Inf\{1,2,3,...\}.$$

The real numbers will remain sequenced when these $(Inf\{1,2,3,...\})^2$ sequenced real numbers are added to the list.

For each of the $(Inf\{1,2,3,...\})^2$ variants of the 2nd listed real number, there are $Inf\{1,2,3,...\}$ many variants of the 3rd listed real number, each produced by changing one digit in the 3rd listed number. There are $(Inf\{1,2,3,...\})^3$ such numbers. These numbers are sequenced because

$$(Inf\{1,2,3,...\})^3 = Inf\{1,2,3,...\}.$$

Suppose that some of them are out of the listed reals. The real numbers will remain sequenced when these $(Inf\{1,2,3,...\})^3$ sequenced real numbers are added to the list.

After n such steps, the real numbers will remain sequenced when

$$(Inf\{1,2,3,...\})^n = Inf\{1,2,3,...\}$$

sequenced real numbers are added to the list.

In all, the reals will remain sequenced after

$$Inf\{1,2,3,...\} + \underbrace{(Inf\{1,2,3,...\})^2}_{Inf\{1,2,3,...\}} + ... + \underbrace{(Inf\{1,2,3,...\})^n}_{Inf\{1,2,3,...\}} + ...$$

$$= (Inf\{1,2,3,...\}) \times (Inf\{1,2,3,...\})$$

$$= Inf\{1,2,3,...\}$$

real numbers are added to the original listing of the reals.

In any event, Cantor's claim is false: The reals can be sequenced because they are countable.

The Reals can b e sequenced and are shown countable , by placing them on an infinite triangle, similarly to the sequencing of the Rational numbers.

This arrangement of the Rationals known as Cantor Zig Zag is the basis for the sequencing of the Reals.

We describe it next.

4.

The Zig Zag for the Rationals

All the Rationals are on an infinite triangular grid with sides $Inf\{1,2,3,\ldots\}$.

$$
\begin{array}{ccccccccccccccc}
\frac{1}{1} & & \frac{1}{2} & & \frac{1}{3} & & \frac{1}{4} & & \frac{1}{5} & & \ldots & \ldots & * & \ldots & * \\
 & & & & & & & & \ldots & \ldots & \ldots & & \ldots & & \\
\frac{2}{1} & & \frac{2}{2} & & \frac{2}{3} & & \frac{2}{4} & \ldots & \ldots & \ldots & * & \ldots & * & & \\
 & & & & & & \ldots & \ldots & \ldots & & \ldots & & & & \\
\frac{3}{1} & & \frac{3}{2} & & \frac{3}{3} & \ldots & \ldots & \ldots & * & \ldots & * & & & & \\
 & & & & \ldots & \ldots & \ldots & & \ldots & & & & & & \\
\frac{4}{1} & & \frac{4}{2} & \ldots & \ldots & \ldots. & * & \ldots & * & & & & & & \\
 & & \ldots & \ldots & \ldots & & \ldots & & & & & & & & \\
\frac{5}{1} & \ldots & \ldots & \ldots & * & \ldots & * & & & & & & & & \\
\ldots & \ldots & \ldots & & \ldots & & & & & & & & & & \\
* & \ldots & * & \ldots & * & & & & & & & & & & \\
 & & \ldots & & & & & & & & & & & & \\
* & \ldots & * & & & & & & & & & & & & \\
\ldots & & & & & & & & & & & & & & \\
* & & & & & & & & & & & & & &
\end{array}
$$

The Rationals can be sequenced by the Zig Zag line

$$
\begin{array}{ccccccccccccccc}
\frac{1}{1} & \rightarrow & \frac{1}{2} & & \frac{1}{3} & \rightarrow & \frac{1}{4} & & \frac{1}{5} & \rightarrow & \ldots & \ldots & * & \ldots & * \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & \ldots & \ldots & \ldots & \nearrow & \ldots & \swarrow & \\
\frac{2}{1} & & \frac{2}{2} & & \frac{2}{3} & & \frac{2}{4} & \ldots & \ldots & \ldots & * & \ldots & * & & \\
\downarrow & \nearrow & & \swarrow & & \nearrow & \ldots & \ldots & \ldots & \nearrow & \ldots & \swarrow & & & \\
\frac{3}{1} & & \frac{3}{2} & & \frac{3}{3} & \ldots & \ldots & \ldots & * & \ldots & * & & & & \\
 & \swarrow & & \nearrow & \ldots & \ldots & \ldots & \nearrow & \ldots & \swarrow & & & & & \\
\frac{4}{1} & & \frac{4}{2} & \ldots & \ldots & \ldots. & * & \ldots & * & & & & & & \\
\downarrow & \nearrow & \ldots & \ldots & \ldots & \nearrow & \ldots & \swarrow & & & & & & & \\
\frac{5}{1} & \ldots & \ldots & \ldots & * & \ldots & * & & & & & & & & \\
\ldots & \ldots & \ldots & \nearrow & \ldots & \swarrow & & & & & & & & & \\
* & \ldots & * & \ldots & * & & & & & & & & & & \\
\downarrow & \nearrow & \ldots & \swarrow & & & & & & & & & & & \\
* & \ldots & * & & & & & & & & & & & & \\
\ldots & \swarrow & & & & & & & & & & & & & \\
* & \rightarrow & & & & & & & & & & & & &
\end{array}
$$

At the top of the triangle, there is one rational, the number 1.

At the 2nd raw, there are two Rationals.

At the 3rd raw, there are three Rationals.

At the 4th raw, four Rationals.

At the 5th raw, five Rationals.

....

At the n-th raw, n Rationals.

The total number of Rationals in the first n Zig Zag lines is

$$1 \quad 2 \quad 3 \quad \ldots \quad n \quad \frac{1 \quad n}{2} n \quad (n \quad 1)^2$$

For any n $\quad 1, 2, 3, \ldots$

$$(n \quad 1)^2 \quad \inf\{1, 2, 3, \ldots\}$$

Therefore,

$$(\inf\{1,2,3,...\})^2 \qquad \inf\{1,2,3,...\}$$

That is, the infinity of the Rationals is at most the infinity of the Counting numbers, and the Rationals are countable.

Similarly, we place the Real numbers on an infinite triangle, and show by a similar inequality that the are countable.

5.

The Zig Zag for the Reals

The reals can be placed on an infinite triangle by the following algorithm:

We represent the real numbers between 0, and 1 in binary vectors, and arrange them in rows.

The first row has the binary infinite vectors

$$\left\langle 0,0,0,0,0,....0,....\right\rangle \leftrightarrow 0$$

and

$$\left\langle 1,0,0,0,0,....0,....\right\rangle \leftrightarrow \frac{1}{2}$$

Next, we vary the second digit in the representing infinite vector, and we obtain four fractions

$$\left\langle 0,0,0,0,0,....0,....\right\rangle \leftrightarrow 0$$

$$\left\langle 0,1,0,0,0,....0,....\right\rangle \leftrightarrow \frac{1}{4}$$

$$\left\langle 1,0,0,0,0,....0,....\right\rangle \leftrightarrow \frac{2}{4}$$

$$\left\langle 1,1,0,0,0,....0,....\right\rangle \leftrightarrow \frac{3}{4}$$

Next, we vary the third digit in the representing infinite vectors, and we have eight fractions

$$\left\langle 0,0,0,0,0,....0,....\right\rangle \leftrightarrow 0$$

$$\left\langle 0,0,1,0,0,....0,....\right\rangle \leftrightarrow \frac{1}{8}$$

$$\left\langle 0,1,0,0,0,....0,....\right\rangle \leftrightarrow \frac{2}{8}$$

$$\left\langle 0,1,1,0,0,....0,....\right\rangle \leftrightarrow \frac{3}{8}$$

$$\left\langle 1,0,0,0,0,....0,....\right\rangle \leftrightarrow \frac{4}{8}$$

$$\left\langle 1,0,1,0,0,....0,....\right\rangle \leftrightarrow \frac{5}{8}$$

$$\left\langle 1,1,0,0,0,....0,....\right\rangle \leftrightarrow \frac{6}{8}$$

$$\left\langle 1,1,1,0,0,....0,....\right\rangle \leftrightarrow \frac{7}{8}$$

Next, we vary the fourth digit in the representing infinite vectors, and we have sixteen fractions

$$\left\langle 0,0,0,0,0,....0,....\right\rangle \leftrightarrow 0$$

$$\left\langle 0,0,0,1,0,....0,....\right\rangle \leftrightarrow \frac{1}{16}$$

$$\left\langle 0,0,1,0,0,....0,....\right\rangle \leftrightarrow \frac{2}{16}$$

$$\left\langle 0,0,1,1,0,....0,....\right\rangle \leftrightarrow \frac{3}{16}$$

$$\left\langle 0,1,0,0,0,....0,....\right\rangle \leftrightarrow \frac{4}{16}$$

$$\left\langle 0,1,0,1,0,....0,....\right\rangle \leftrightarrow \frac{5}{16}$$

$$\left\langle 0,1,1,0,0,....0,....\right\rangle \leftrightarrow \frac{6}{16}$$

$$\left\langle 0,1,1,1,0,....0,....\right\rangle \leftrightarrow \frac{7}{16}$$

$$\left\langle 1,0,0,0,0,....0,....\right\rangle \leftrightarrow \frac{8}{16}$$

$$\left\langle 1,0,0,1,0,....0,....\right\rangle \leftrightarrow \frac{9}{16}$$

$$\left\langle 1,0,1,0,0,....0,....\right\rangle \leftrightarrow \frac{10}{16}$$

$$\left\langle 1,0,1,1,0,....0,....\right\rangle \leftrightarrow \frac{11}{16}$$

$$\left\langle 1,1,0,0,0,....0,....\right\rangle \leftrightarrow \frac{12}{16}$$

$$\left\langle 1,1,0,1,0,....0,....\right\rangle \leftrightarrow \frac{13}{16}$$

$$\left\langle 1,1,1,0,0,....0,....\right\rangle \leftrightarrow \frac{14}{16}$$

$$\left\langle 1,1,1,1,0,....0,....\right\rangle \leftrightarrow \frac{15}{16}$$

Each step of this construction of the fractions between 0, and 1, doubles the number of the fractions that we obtain.

After indefinitely many such steps, we obtain all the infinite vectors which elements are 0's , and 1's. These infinite binary vectors represent all the real numbers.

This listing of the real numbers as infinite binary vectors of 1's and 0's, sequences the real numbers.

we obtain rows of infinite binary vectors arranged in an infinite triangle. Each row of vectors has double the number of infinite binary vectors, and the $Inf\left\{1,2,3,..\right\}^{\text{th}}$ row has $2^{Inf\{1,2,3,....\}}$ such sequences. We obtain the infinite triangle,

$$\begin{array}{cccccccc} & & & 0 & 1 & & & \\ & & 00 & 01 & 10 & 11 & & \\ 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

The sequencing follows through the rows of the triangle.

At the 1st raw of the triangle, there are 2 Reals

At the 2nd raw, there are 2^2 Reals

At the 3rd raw, 2^3 Reals,

At the 4th raw, 2^4 Reals

....

At the nth raw, 2^n Reals

The total number of Reals in the first n Zig Zag lines is

$$2 \quad 2^2 \quad 2^3 \quad 2^4 \quad \dots \quad 2^n \quad 2\frac{2^n \quad 1}{2 \quad 1} \quad 2^{n \quad 1} \quad 2 \quad 2^{n \quad 1}$$

For any $n \quad 1, 2, 3, \dots$

$$2^{n \quad 1} \quad \inf\{1, 2, 3, \dots\}$$

Therefore,

$$2^{\inf\{1,2,3,\dots\}} \quad \inf\{1, 2, 3, \dots\}$$

That is, the infinity of the Reals is at most the infinity of the Counting numbers, and the Reals are countable.

Where does Cantor Diagonal Argument fail?

To see that, we need to look at that argument more closely

6.

Cantor's Argument is Paradoxical

Cantor's Argument is paradoxical because to construct the anti-diagonal number, it assumes countable Reals, and to establish that the anti-diagonal is off the listed reals, it assumes Transfinite Induction, hence uncountable Reals.

Cantor's anti-diagonal number is produced by the Induction Theorem. That Theorem says that

If a property depends on each counting number $n = 1, 2, 3, ..$, so that

1) The property holds for the first Counting number $n = 1$.
2) If the property holds for the Counting number k, we can deduct that it holds for the next number $k + 1$.

Then, the property holds for any $n = 1, 2, 3, ...$

Assuming that the reals are countable,

1) the anti-diagonal is different from the first listed real
2) If the anti-diagonal is different from the first k reals, it can be made different from the next $k + 1$ listed real

Therefore, the anti-diagonal is different from any listed number for $n = 1, 2, 3, ...$

Now, to construct the anti-diagonal, n has to be specified. Then, the anti-diagonal is different from the preceding n listed reals.

The finite induction does not tell anything about the infinitely many reals beyond the specified first n reals.

If we assume that the list includes ALL the reals, the anti-diagonal must be one of the unspecified reals, that is not covered by the Finite Induction Theorem.

To tell about infinite indices n, Cantor's Diagonal Argument needs a stronger version of the Induction Theorem, the Transfinite Induction Axiom. In that version the running index can be infinite.

The Transfinite Induction Axiom guesses that

If I is a well-ordered infinite set of numbers, and if there is any property that depends on each index i from I, so that

1) The property holds for the first element of I,
2) If the property holds for all the $k's$ that precede the index j, we can conclude that the property holds for j,

Then, the property holds for any index i in I.

If Transfinite Induction is available, it will make all the Reals specified, and guarantee that the anti-diagonal is different from any of the Reals.

But that requires the Reals to be uncountable. Otherwise, the Transfinite Induction degrades into the Induction Theorem.

Nevertheless, for the anti-diagonal to be different from any of the reals, the Reals must be uncountable.

Thus, Cantor's Diagonal Argument is Paradoxical.

7.
Cantor's Diagonal Argument and Transfinite Induction

The Transfinite Induction Axiom guesses that

If I is a well-ordered infinite set of numbers, and if there is any property that depends on each index i from I, so that

1) The property holds for the first element of I,
2) If the property holds for all the $k's$ that precede the index j, we can conclude that the property holds for j,

Then, the property holds for any index i in I.

If the reals are the index set, Transfinite Induction will make all the reals specified, and guarantee that the anti-diagonal is different from any of the reals.

But when this Axiom was formed, the reals were believed to be not well-ordered, and could not serve as the index set.

The Transfinite Induction is equivalent to the Axiom of Choice. In [Dan7] (and in [Dan9]), we proved that the Axiom of Choice is equivalent to the Continuum Hypothesis. Therefore, the Transfinite Induction is equivalent to the Continuum Hypothesis.

What is the Continuum Hypothesis?

8.

The Continuum Hypothesis

Cantor convinced himself that the infinity of the real numbers is strictly greater than the infinity of the Counting numbers.

That led Cantor to the question

Is there an infinity in between the infinity of the Reals and the infinity of the Counting numbers?

The assumption that there is no such infinity became known as the Continuum Hypothesis.

As we have seen, the real numbers may be sequenced, and their Infinity equals the infinity of the Counting numbers.

Indeed, at the $Inf\{1,2,3,...\}$ row in the triangle, the number of infinite binary vectors representing real numbers is

$$\underbrace{2\times 2\times 2\times ......\times 2\times ....}_{Inf\{1,2,3,....\}\text{ many times}} = 2^{Inf\{1,2,3,....\}}.$$

Consequently, there are $2^{Inf\{1,2,3,....\}}$ real numbers between 0, and 1.

But since they are sequenced, and can be counted, there are $Inf\{1,2,3,...\}$ real numbers between 0, and 1.

That is,

$$2^{Inf\{1,2,3,....\}} = Inf\{1,2,3,...\}.$$

Thus, there is no infinity between the infinity of the real numbers, and the infinity of the counting numbers, and the Continuum Hypothesis holds.

Nevertheless, the equality of all infinities reduces the Transfinite Induction Axiom into the Induction Theorem, and disables Cantor's Diagonal Argument.

9.

Cantor's Diagonal Argument and the Equality of All Infinities

The Induction Theorem used by Cantor applies to the counting numbers, but not to infinities.

To be applied to the infinitely many real numbers, Cantor's Argument required Transfinite Induction.

To that end, The infinity of the Reals had to be strictly greater than the infinity of the Counting numbers.

Thus, to construct the anti-diagonal number, Cantor's Argument assumes countable Reals, and to establish that the anti-diagonal exists for all the Reals, it needs to assume uncountable well ordered Reals.

The equality of all infinities reduces the Transfinite Induction Axiom to the Induction Theorem. Thus, for a given n, the anti-diagonal number is different from any of the listed reals up to n.

This does not produce an anti-diagonal which is different from ALL the infinitely many Reals listed beyond n.

The equality of all infinities guarantees that Cantor's anti-diagonal number is amongst the listed Reals.

References

[Cantor] Cantor, Georg, *Contributions to the founding of the theory of Transfinite Numbers,* Open Court Publishing 1915, Dover 1955.

[Dan1] Dannon, H. Vic, **Hilbert's 1st Problem: Cantor's Continuum Hypothesis** Gauge Institute Journal Vol.1 No 1, February 2005

[Dan2] Dannon, H. Vic, **Rationals Countability and Cantor's Proof**
Gauge Institute Journal Vol.2 No 1, February 2006

[Dan3] Dannon, H. Vic, **Cantor's Set and the Cardinality of the Reals**
Gauge Institute Journal Vol.3 No 1, February 2007

[Dan4] Dannon, H. Vic, **Non-Cantorian Set Theory** Gauge Institute Journal Vol.3 No 2, May 2007

[Dan5] Dannon, H. Vic, **Cardinality, Measure, and Category**
Gauge Institute Journal Vol.3, No. 3, August 2007

[Dan6] Dannon, H. Vic, **The Continuum Hypothesis, The Axiom of Choice, and Non-Cantorian Theory** Gauge Institute Journal Vol.3 No 4, November 2007

[Dan7] Dannon, H. Vic, **Well-Ordering of the Reals, Equality of all Infinities, and the Continuum Hypothesis** Gauge Institute Journal Vol.6 No 2, May 2010

[Dan8] Dannon, H. Vic, **Correction to Rationals Countability and Cantor's Proof**

[Dan9] Dannon, H. Vic, **THE EQUALITY OF ALL INFINITIES: From the Natural Numbers to the Continuum** Gauge Institute, April 2010

[Levy] Levy, Azriel, *Basic Set Theory*. Dover, 2002.

[Lipschutz] Lipschutz, Seymour, *Theory and problems of Set Theory and Related Topics*, McGraw-Hill, 1964.

[Machover] Machover, Moshe, *Set theory, logic, and their limitations*. Cambridge U. Press, 1996.

[Sierpinski] Sierpinski, Waclaw, *Cardinal and Ordinal Numbers.* Warszawa, 2nd edition, 1965, (Also in the 1958, 1st edition)

[Tarski] Tarski, Alfred, *Axiomatic and algebraic aspects of two theorems on sums of cardinals*, Fundamenta Mathematicae, Volume 35, 1948, p. 79-104. Reprinted in *Alfred Tarski Collected papers*, edited by Steven R. Givant and Ralph N. McKenzie, Volume 3, 1945-1957, p. 173, Birkhauser, 1986.